TRANSFERING SATURATION, THE FINITE COVER PROPERTY, AND STABILITY *

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Abstract

Saturation is (μ, κ)-transferable in T if and only if there is an expansion T_1 of T with $|T_1| = |T|$ such that if M is a μ-saturated model of T_1 and $|M| \geq \kappa$ then the reduct $M|L(T)$ is κ-saturated. We characterize theories which are superstable without f.c.p., or without f.c.p. as, respectively those where saturation is $(\aleph_0, \lambda)$-transferable or $(\kappa(T), \lambda)$-transferable for all λ. Further if for some $\mu \geq |T|$, $2^\mu > \mu^+$, stability is equivalent to for all $\mu \geq |T|$, saturation is $(\mu, 2^\mu)$-transferable.

1 Introduction

The finite cover property (f.c.p.) is in a peculiar position with respect to the stability hierarchy. Theories without the f.c.p. are stable; but f.c.p. is independent from ω-stability or superstability. We introduce a notion of transferability of saturation which rationalizes this situation somewhat by placing f.c.p. in a natural hierarchy of properties. For countable theories the hierarchy is ω-stable without f.c.p., superstable without f.c.p., not f.c.p., and stable. For appropriate (μ, κ) each of these classes of theories is characterized by (μ, κ)-transferability of saturation in following sense.

Definition 1.1 *Saturation is (μ, κ)-transferable in T if and only if there is an expansion T_1 of T with $|T_1| = |T|$ such that if M is a μ-saturated model of T_1 and $|M| \geq \kappa$, then the reduct $M|L(T)$ is κ-saturated.*

The finite cover property was introduced by Keisler in [Ke] to produce unsaturated ultrapowers. One of his results and a slightly later set theoretic advance by Kunen yield immediately that if for $\lambda > 2^{|T|}$, saturation is $(|T|^+, \lambda)$-transferable then T does not have the finite cover property. The finite cover property was also studied extensively by Shelah in [Sh 10] and chapters II, VI and VII of [Sh c]; those techniques are used here.

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Our notation generally follows [Sh c] with a few minor exceptions: $|T|$ is the number of symbols in $|L(T)|$ plus $\aleph_0$. We do not distinguish between finite sequences and elements, i.e. we write $a \in A$ to represent that the elements of the finite sequence a are from the set A. References of the form IV x.y are to [Sh c].

There are several equivalent formulations of the finite cover property. The following, which looks like a strengthening of the compactness theorem, is most relevant here.

Definition 1.2 *The first order theory T does not have the finite cover property if and only if for every formula $\phi(x; y)$ there exists an integer n depending on ϕ such that for every A contained in a model of T and every subset p of $\{\phi(x, a), \neg\phi(x, a); a \in A\}$ the following implication holds: if every $q \subseteq p$ with cardinality less than n is consistent then p is consistent.*

The two main tools used in this paper are the following consequence of not f.c.p. and a sufficient condition for λ-saturation.

Fact 1.3 (II.4.6) *Let T be a complete first order theory without the f.c.p. and Δ a finite set of $L(T)$-formulas. There is an integer k_Δ such that if $M \models T$ is a saturated model, $A \subseteq M$ with $|A| < |M|$ and $\mathbf{I}$ is a set of Δ-indiscernibles over A with cardinality at least k_Δ then there exists $\mathbf{J} \subseteq M$ a set of Δ-indiscernibles (over A) extending $\mathbf{I}$ of cardinality $|M|$.*

The principal tool for establishing the transfer of saturation is

Fact 1.4 (III.3.10) *If a model M of a stable theory if M is either $F^a_{\kappa(T)}$-saturated or $\kappa(T)+\aleph_1$-saturated and for each set of indiscernibles $\mathbf{I}$ contained in M there is an equivalent set of indiscernibles $\mathbf{J}$ contained in M with $|\mathbf{J}| = \lambda$ then M is λ-saturated.*

We thank Anand Pillay for raising the issue of the superstable case and the referee for the final formulation of Theorem 2.2 which generalizes our earlier version and for correcting an oversight in another argument.

2 The transferability hierarchy

In this section we characterize certain combinations of stability and the finite cover property in terms of transferability of saturation. Extending the notation we write *saturation is* $(0,\kappa)$*-transferable in* T if and only if there is an expansion T_1 of T with $|T_1| = |T|$ such that if $M \models T_1$ and $|M| \geq \kappa$, $M|L(T)$ is κ-saturated. In particular, taking $|M| = \kappa$, $PC(T_1, T)$ is categorical in κ. Using this language we can reformulate an old result of Shelah (Theorems VI.5.4 and VIII.4.1) to provide the first stage of our hierarchy.

Fact 2.1 *For a countable theory T, the following are equivalent.*

1. *T does not have the finite cover property and is ω-stable.*
2. *For all $\lambda > \aleph_0$, saturation is $(0,\lambda)$-transferable in T.*
3. *For some $\lambda > 2^{\aleph_0}$ such that saturation is $(0,\lambda)$-transferable in T.*

Since the proof of 1) implies 2) is not given in [Sh c] and follows the line of our other arguments we sketch the proof in our discussion after Theorem 2.5. This result holds only for countable languages; the remainder apply to theories of arbitrary cardinality.

We introduce the following special notation to uniformize the statement of the next result.

$$\kappa'(T) = \begin{cases} \kappa(T) & \text{if } T \text{ is stable} \\ |T|^+ & \text{if } T \text{ is unstable} \end{cases}$$

Theorem 2.2 *The following are equivalent for a complete theory T.*

1. *T does not have the finite cover property.*
2. *For all $\lambda \geq |T|^+$, saturation is $(\kappa'(T),\lambda)$-transferable in T.*
3. *For some $\lambda > 2^{|T|}$, saturation is $(\kappa'(T),\lambda)$-transferable in T.*

Proof: It is obvious that (2) implies (3). Now we show (3) implies (1) by showing that saturation is not even $(|T|^+, \lambda)$-transferable (and so certainly not $(\kappa(T)+\aleph_1, \lambda)$-transferable) if T has the f.c.p.. Let T_1 be any extension of T and N_0 an arbitrary model of T_1 with cardinality at least λ. By Kunen's theorem (see [Ku], or Theorem 6.1.4 in [CK]) there exists an $\aleph_1$-incomplete $|T|^+$-good ultrafilter D on $|T|$. Denote by N_1 the ultrapower $N_0^{|T|}/D$. By [Ke] 1.4 and 4.1 or VI.5.3, N_1 is $|T|^+$-saturated but not $(2^{|T|})^+$-saturated.

We now show (1) implies (2). Let T be a theory without the f.c.p.. By II.4.1, T is stable. The proof now splits into two cases depending on whether T is superstable. We begin with the case that T is stable but not superstable. Then $\kappa'(T) \geq \kappa(T) + \aleph_1$ and this inequality will be essential shortly.

Let $L_1 := L(T) \bigcup \{F\}$ where F is a binary function symbol. The theory T_1 consists of T and the following axioms:

1. For each x, the function $F(x, \cdot)$ is injective.
2. For every finite $\Delta \subseteq L(T)$, let k_Δ be the integer from Fact 1.3. If I is a finite set of Δ-indiscernibles of cardinality at least k_Δ then there exists an x_I such that
   (a) the range of $F(x_I, \cdot)$ contains I and
   (b) the range of $F(x_I, \cdot)$ is a set of Δ-indiscernibles.

It should be clear that the above axioms can be formulated in first order logic in the language L_1. To see that T_1 is consistent, we expand a saturated model N of T to a model of T_1. Fix a 1-1 correspondence between finite sets of Δ-indiscernibles $\mathbf{I}$ with $|\mathbf{I}| \geq k_\Delta$ and elements $x_{\mathbf{I}}$ of N. By Fact 1.3, each sufficiently large finite sequence of Δ-indiscernibles $\mathbf{I}$ in N extends to one with $|N|$ elements. Fix a 1-1 correspondence between the universe of N and this sequence. Interpret $F(x_I, x)$ as this correspondence.

Suppose N^* is a $\kappa'(T)$-saturated model of T_1 of cardinality at least λ and denote $N^*|L(T)$ by N. Since $\kappa'(T) \geq \kappa(T) + \aleph_1$ by Fact 1.4 we need only establish the following claim.

Claim 2.3 *Any infinite sequence of indiscernibles* $\mathbf{I}$ *in* N *extends to a sequence* $\mathbf{J}$ *of indiscernibles (over the empty set) with cardinality* $|N|$.

Proof: Let $q(x)$ be the set of formulas which expresses that for each finite Δ the range of $F(x, \cdot)$ is a set of Δ-indiscernibles and $\mathbf{I}$ is contained in the range of $F(x, \cdot)$. If $a \in N$ realizes the type $q(x)$ then, since $F(a, \cdot)$ is 1-1, $\mathbf{J} := \{F(a, b) : b \in N\}$ is as required. We now show $q(x)$ is consistent. Fix a finite $q^* \subseteq q(x)$ and let Δ be a finite subset of $L(T)$ such that all the $L(T)$-formulas from q^* appear in Δ. Let $m < \omega$ be sufficiently large so that all the elements of I appearing in q^* are among $\{b_0, \ldots, b_{m-1}\}$ and $m \geq k_\Delta$. It suffices to show that for some $a \in N$, each b_i for $i < m$ is in the range of $F(a, \cdot)$ and the range of $F(a, \cdot)$ is a set of Δ-indiscernibles. This follows immediately from T_1, by the assumption that $m \geq k_\Delta$. Since q is over a countable set there exists an element $a \in N^*$ satisfying q^* and we finish. $\blacksquare_{2.3}$

We now prove the second case of 1 implies 2: superstable T. The general outline of the proof is the same but we replace $\kappa(T) + \aleph_1$-saturation with $F^a_{\kappa(T)}$-saturation and we must use a different trick to find an equivalent set of indiscernibles. The idea for guaranteeing $F^a_{\kappa(T)}$-saturation is taken from Proposition 1.6 of [Ca]; the referee suggested moving it from a less useful place in the argument to here.

Lemma 2.4 *If* T *does not have the f.c.p. then there is an expansion* T_1 *of* T *such that if* M *is a* $\kappa(T)$*-saturated model of* T_1 *then* $M|L(T)$ *is* $F^a_{\kappa(T)}$*-saturated.*

Proof: Let T be a theory without the f.c.p.. Form L_1 by adding to L new k-ary function symbols $f_i^{\theta,E}$, for $i < m = m(\theta, E)$, for each pair of formulas $\theta(z), E(x, y, z)$ with $\lg(z) = k$ such that for any $M \models T$ and $a \in M$, if $M \models \theta(a)$ then $E(x, y, a)$ is an equivalence relation with m classes. The theory T_1 consists of T and the following axioms: For each k-ary sequence z such that $\theta(z)$, the elements $f_i^{\theta,E}(z)$, $i < m$ provide a complete set of representatives for $E(x, y, z)$. In any model of T, one can choose Skolem functions $f_i^{\theta,E}(z)$ to give sets of representatives for the finite equivalence relations so T_1 is consistent. Now suppose that $N^* \models T_1$ is $\kappa(T)$-saturated.

For any $q = \mathrm{stp}(d/C)$ with $|C| < \kappa(T)$, note that q is equivalent to the L_1-type over C consisting of the formulas $E(x, f_i^{\theta,E}(c))$ for E a finite equivalence relation defined over a finite sequence $c \in C$ such that $E(d, f_i^{\theta,E}(c))$. Since this type is realized, $N = N^*|L(T)$ is $F^a_{\kappa(T)}$-saturated. $\blacksquare_{2.4}$

Now we show finish showing (1) implies (2) in the superstable case. Let $\lambda \geq |T|^+$ be given. We must find a T_2 to witness $(\aleph_0, \lambda)$-transferability. First expand T to T_1 as in Lemma 2.4 so that if M is an $\aleph_0$-saturated model of T_1 then $M|L(T)$ is $F^a_{\aleph_0}$-saturated. Form L_2 by adding to L_1 an $n+2$-ary function symbol F_n for each n. The theory T_2 consists of T_1 and the following axioms:

1. For each x and n-ary sequence z, the function $F_n(x, z, \cdot)$ is injective.
2. For every finite $\Delta \subseteq L(T)$ and n-ary sequence z, let k_Δ be the integer from Fact 1.3. If I is a finite set of Δ-indiscernibles over z of cardinality at least k_Δ then there exists an x_I such that
   (a) the range of $F_n(x_I, z, \cdot)$ contains I,
   (b) the range of $F_n(x_I, z, \cdot)$ is a set of Δ-indiscernibles over z.

The consistency of T_2 is entirely analogous to the similar step in the proof of Theorem 2.2. We just have to interpret each $F_n(x, z, y)$ instead of a single function of two variables. Now suppose that $N^* \models T_2$ is an $\aleph_0$-saturated model of cardinality at least λ. The reduct N of N^* to $L(T)$ is $F^a_{\aleph_0}$-saturated and it suffices by Fact 1.4 to show for each set of indiscernibles I contained in N there is an equivalent set of indiscernibles J with $|J| = \lambda$.

Let $\mathbf{I} = \{b_n : n < \omega\}$ be such an infinite set of indiscernibles in N. Let $p^* = \mathrm{Av}(I, N)$ and, since N is $F^a_{\aleph_0}$-saturated, choose $m < \omega$ such that for $B = \{b_0 \ldots b_{m-1}\}$, $p^*|B$ is stationary and p^* does not fork over B. We show there is a sequence $\mathbf{J}$ of indiscernibles based on $p^*|B$ with $|\mathbf{J}| = |N|$. Let $q_1(x)$ be a type over B that contains $(\forall y)\theta(F_m(x, b_0, \ldots b_{m-1}, y))$ for all $\theta(x) \in p^*|B$, for each $\phi(x_0, \ldots x_{n-1}) \in L(T)$ such that $N \models \phi(b_0, \ldots b_{n-1})$, the formula $(\forall y_1) \ldots (\forall y_n)\phi(F_m(x, b_0, \ldots b_{m-1}, y_1), \ldots F_m(x, b_0, \ldots b_{m-1}, y_n))$ and the assertion that $F_m(x, c, \cdot)$ is injective. The definition of T_2 implies the consistency of q_1. Since q_1 is a type over a finite set, q_1 is realized by an element $a \in N^*$; this guarantees the existence of a set of $|N|$ indiscernibles equivalent to I as required. $\blacksquare_{2.2}$

In the superstable case we can get one slightly stronger result which allows to characterize superstable without f.c.p. by $(\aleph_0, \lambda)$-transferability.

Theorem 2.5 *If for some $\lambda > 2^{|T|}$, saturation is $(\aleph_0, \lambda)$-transferable in T then T is superstable without the f.c.p.*

Proof: By Theorem 2.2 we deduce that T does not have the f.c.p. (using $\lambda > 2^{|T|}$) and so T is stable. Suppose for contradiction there are a stable but not superstable T and a T_1 which witnesses $(\aleph_0, \lambda)$-transferability in T. Apply VIII.3.5 to PC(T_1, T) taking $\kappa = \aleph_0$, $\mu = (2^{|T|})^+$ and $\lambda \geq \mu$. There are 2^μ models of T_1 with cardinality λ, which are $\aleph_0$-saturated, whose reducts to $L(T)$ are nonisomorphic. So some are not λ-saturated. $\blacksquare_{2.5}$.

We were unable to find a uniform argument for 1 implies 2 of Theorem 2.2; there seem to be two quite different ideas for making the large set of indiscernibles equivalent to the given set. The proof of 1) implies 2) of Fact 2.1 proceeds along similar lines with the following variation. Since T is ω-stable every ω-saturated model is F^a_ω-saturated. Again using the ω stability, it easy to Skolemize with countably many functions so that each type over a finite set is realized. Then the same trick as in the superstable case of Theorem 2.2 guarantees the existence of large equivalent indiscernible sets.

The proof of Theorem 2.2 yields somewhat more than is necessary. The theory T_1 which is found in the implication (1) implies (2) does not depend on λ and contains only a single additional function symbol. We could obtain a stronger result than (3) implies (1) with the same proof by demanding in a modified (3) that the model witnessing $(|T|^+, \lambda)$-transferability have cardinality $\lambda = \lambda^{|T|} > 2^{|T|}$.

As pointed out by the referee, we can use the arguments of Theorem 2.2 to characterize $\kappa(T)$ for theories without the finite cover property if

$\kappa(T)$ satisfies the set-theoretic conditions of Theorem VIII.3.5. For example, under the GCH if $\kappa(T)$ is not the successor of a singular cardinal and T does not have the f.c.p. $\kappa(T)$ is the least κ such that there is $\lambda > 2^{|T|}$ for which saturation is (κ, λ)-transferable.

Theorem 2.6 *Suppose that there exists a cardinal $\mu \geq |T|$ such that $2^\mu > \mu^+$. For a complete theory T, the following are equivalent:*

1. *T is stable.*
2. *For all $\mu \geq |T|$, saturation is $(\mu^+, 2^\mu)$-transferable in T.*
3. *For some $\mu \geq |T|$, saturation is (μ^+, μ^{++})-transferable in T.*

The condition $\mu^+ < 2^\mu$ is used only for (2) implies (3) (which is obvious with that hypothesis). In the next two lemmas we prove in ZFC that (1) implies (2), and that (3) implies (1). This shows in ZFC that stability is bracketed between two transferability conditions.

Lemma 2.7 *If T is stable and $\mu \geq |T|$, saturation is $(\mu^+, 2^\mu)$-transferable in T.*

Proof: We must find an expansion T_1 of T such that if M is a μ^+-saturated model of T_1 and $|M| \geq 2^\mu$, $M|L$ is 2^μ-saturated. Form L_1 by adding one additional binary predicate $E(x, y)$ and add axioms asserting that E codes all finite sets. (I.e., for every set of k elements x_i there is a unique y such that $E(z, y)$ if and only if z is one of the x_i.) For any model M_1 of T_1 and any element b of M_1, let $[b] := \{a \in M_1 : M_1 \models E[a, b]\}$.

Now let M_1 be a μ^+-saturated model of T_1 and M the reduct of M_1 to L. Suppose $A \subseteq M$ has cardinality less than 2^μ and $p \in S^1(A)$. We must show p is realized in M. By the definition of $\kappa(T)$ there exists $B \subseteq A$ of cardinality less than $\kappa(T)$ such that p does not fork over B. Since M_1 is $|T|^+$-saturated, we may take $p|B$ to be stationary. Let $\hat{p} \in S(M)$ be an extension of p that does not fork over B. Since $\mu^+ > |T| \geq \kappa(T)$, by μ^+-saturation of

M there exists $I := \{a_n : n < \omega\} \subseteq M$ such that $a_n \models \hat{p}|(B \cup \{a_k : k < n\})$. Since the sequence is chosen over a stationary type, I is a set of indiscernibles.

Define an L_1-type q over I so that if b realizes q, $[b] \bigcup I$ is a set of indiscernibles over the empty set. Since the relation E codes finite sets, and I is a set of indiscernibles q is consistent. By the $\aleph_1$-saturation of M_1 there exists $b \in M$ realizing the type q. If $[b]$ has 2^μ elements we are finished since for each formula $\phi(x, \overline{y})$ and each $\overline{a} \in A$ with $\phi(x, \overline{a}) \in p$, only finitely many elements of $[b]$ satisfy $\neg\phi(x, \overline{a})$. To show $[b]$ is big enough, using the μ^+-saturation of M, we define inductively for $\eta \in 2^{\leq \mu}$ elements $c_\eta \in M$ such that

1. $c_\emptyset = b$
2. For any η, $[c_{\eta \frown 0}]$ and $[c_{\eta \frown 1}]$ are disjoint subsets of $[c_\eta]$.
3. If $\lg(\eta)$ is a limit ordinal α, $[c_\eta] \subseteq \cap_{i<\alpha}[c_{\eta|i}]$

Now for $s \in 2^\mu$, the c_s witness that $|[b]| = 2^\mu$. $\blacksquare_{2.7}$.

Lemma 2.8 *If $\mu \geq |T|$ and saturation is (μ^+, μ^{++})-transferable in T then T is stable.*

Proof: For the sake of contradiction suppose T is an unstable theory and that there is a $T_1 \supseteq T$ such that if M is a μ^+-saturated model of T_1 with cardinality at least μ^{++}, $M|L(T)$ is μ^{++}-saturated. Fix $M_0 \models T_1$ with cardinality at least μ^{++}. Let D be a μ-regular ultrafilter on $I = \mu$. Construct an ultralimit sequence $\langle M_\alpha : \alpha < \mu^+\rangle$ as in VI.6 with $M_{\alpha+1} = M_\alpha^I/D$ and taking unions at limits. By VI.6.1 M_{μ^+} is μ^+-saturated. But by VI.6.2, since T is unstable, M_{μ^+} is not μ^{++}-saturated. $\blacksquare_{2.8}$

The methods and concerns of this paper are similar to those in the recent paper of E. Casanovas [Ca]. He defines a model to be expandable if every consistent expansion of $\mathrm{Th}(M)$ with at most $|M|$ additional symbols can be realized as an expansion of M. His results are orthogonal to those here. He shows for countable stable T that T has an expandable model which is not saturated of cardinality greater than the continuum if and only if T is not superstable or T has the finite cover property.

By varying the parameters in (μ, κ)-transferability of saturation we have characterized four classes of countable theories: ω-stable without f.c.p., superstable without f.c.p., not f.c.p., and stable. For uncountable λ, they correspond respectively to: $(0, \lambda)$-transferability, $(\aleph_0, \lambda)$-transferability, $(\aleph_1, \lambda)$-transferability, $(\aleph_1, 2^{\aleph_0})$-transferability. Although the analogous results for uncountable languages are more cumbersome to summarise, countability of the language is only essential for the ω-stable characterization.

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