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# Uniformization and Skolem Functions in the Class of Trees.

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ABSTRACT

The monadic second-order theory of trees allows quantification over elements and over arbitrary subsets. We classify the class of trees with respect to the question: does a tree $T$ have definable Skolem functions (by a monadic formula with parameters)? This continues [LiSh539] where the question was asked only with respect to choice functions. Here we define a subclass of the class of tame trees (trees with a definable choice function) and prove that this is exactly the class (actually set) of trees with definable Skolem functions.

## 1. Introduction: The Uniformization Problem

**Definition 1.** The monadic second-order logic is the fragment of the full second-order logic that allows quantification over elements and over monadic (unary) predicates only. The monadic version of a first-order language $L$ can be described as the augmentation of $L$ by a list of quantifiable set variables and by new atomic formulas $t \in X$ where $t$ is a first order term and $X$ is a set variable. The monadic theory of a structure $\mathcal{M}$ is the theory of $\mathcal{M}$ in the extended language where the set variables range over all subsets of $|\mathcal{M}|$ and $\in$ is the membership relation.

**Definition 2.** The *monadic language of order* $L$ is the monadic version of the language of order $\{<\}$. For simplicity, we add to $L$ the predicate sing$(X)$ saying "$X$ is a singleton" and use only formulas with set variables. Thus the meaning of $X < Y$ is: $X = \{x\}$ & $Y = \{y\}$ & $x < y$.

**Definition 3.** Let $T$ be a tree and $\bar{P} \subseteq T$.
(1) $\varphi$ is an $(n, l)$-formula if $\varphi = \varphi(X, Y, \bar{P})$ with $\mathrm{dp}(\varphi) = n$ and $l(\bar{P}) = l$.
(2)$\varphi = \varphi(X, Y, \bar{P})$ is *potentially uniformizable in* $T$ (p.u) if $T \models (\forall Y)(\exists X)\varphi(X, Y, \bar{P})$.

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## 2. Tame Trees

**Definition 2.1.** A tree is a partially ordered set $(T, \triangleleft)$ such that for every $\eta \in T$, $\{\nu : \nu \triangleleft \eta\}$ is linearily orderd by $\triangleleft$.
Note, a chain $(C, <^*)$ and even a set without structure $I$ is a tree.
Branch, Sub-branch, Initial segment.

**Definition 2.2.** (1) $(C, <^*)$ is a scattered chain iff ...
(2) For a scattered chain $(C, <^*)$ Hdeg$(C)$ is defined inductively by:
Hdeg$(C)$=0 iff ...
Hdeg$(C)$=$\alpha$ iff ...
Hdeg$(C)$$\geq \delta$ iff ...

**Theorem 2.3.** *Hdeg$(C)$ exists for every scattered chain $C$.*

**Lemma 2.4.** *Hdeg$(C) < \omega$ then $C$ has a definable well ordering.*

**Proof.** See A1 in the appendix

♡

**Definition 2.5.** $\sim^0_A$, $\sim^1_A$. (from [LiSh539] 4.1)

**Definition 2.6.** (1) A tree $T$ is called *wild* if either
$(i)$ $sup\{|top(A)/\sim^1_A| : A \subseteq T$ an initial segment$\} \geq \aleph_0$ or
$(ii)$ There is a branch $B \subseteq T$ and an embedding $f\colon \mathcal{Q} \to B$ or
$(iii)$ All the branches of $T$ are scattered linear orders but $sup\{\text{Hdeg}(B) : B$ a branch of $T\} \geq \omega$.
$(iv)$ There is an embedding $f\colon {}^{\omega>}2 \to T$
(2) A tree $T$ is *tame for* $(n^*, k^*)$ if the value in $(i)$ is $\leq n^*$, $(ii)$ does not hold and the value in $(iii)$ is $\leq k^*$
(3) A tree $T$ is *tame* if $T$ is tame for $(n^*, k^*)$ for some $n^*, k^* < \omega$.

The following is the content of [LiSh539], (2) $\Rightarrow$ (3) is given in theorem A2 in the appendix.

**Theorem 2.7.** *The following are equivalent:*
1. *$T$ has a definable choice function.*
2. *$T$ has a definable well ordering.*
3. *$T$ is tame.*

♡

## 3. Composition Theorems

**Notations.** $x, y, z$ denote individual variables, $X, Y, Z$ are set variables, $a, b, c$ elements and $A, B, C$ sets. $\bar{a}$, $\bar{A}$ are finite sequences and $\text{lg}(\bar{a})$, $\text{lg}(\bar{A})$ their length. We write e.g. $\bar{a} \in C$ and $\bar{A} \subseteq C$ instead of $\bar{a} \in {}^{\text{lg}(\bar{a})}C$ or $\bar{A} \in {}^{\text{lg}(\bar{A})}\mathcal{P}(C)$

**Definition 3.1.** For any chain $C$, $\bar{A} \in {}^{\text{lg}(\bar{A})}\mathcal{P}(C)$, and a natural number $n$, define by induction

$$t = \text{Th}^n(C; \bar{A})$$

for $n = 0$:

$$t = \{\phi(\bar{X}) :\ \phi(\bar{X}) \in L,\ \phi(\bar{X}) \text{ quantifier free},\ C \models \phi(\bar{A})\}.$$

for $n = m + 1$:

$$t = \{\mathrm{Th}^m(C; \bar{A}\,{}^\wedge B) :\ B \in \mathcal{P}(C)\}.$$

We may regard $\mathrm{Th}^n(C; \bar{A})$ as the set of $\varphi(\bar{X})$ that are boolean combinations of monadic formulas of quantifier depth $\leq n$ such that $C \models \varphi(\bar{A})$.

**Definition 3.2.** $\mathcal{T}_{n,l}$ is the set of all formally possible $\mathrm{Th}^n(C; \bar{P})$ where $C$ is a chain and $\lg(\bar{P}) = l$. $T_{n,l}$ is $|\mathcal{T}_{n,l}|$.

**Fact 3.3.** *(A) For every formula $\psi(\bar{X}) \in L$ there is an $n$ such that from $\mathrm{Th}^n(C; \bar{A})$ we can effectively decide whether $C \models \psi(\bar{X})$. If $n$ is minimal with this property we will write $\underline{\mathrm{dp}(\psi) = n}$.*
*(B) If $m \geq n$ then $\mathrm{Th}^n(C; \bar{A})$ can be effectively computed from $\mathrm{Th}^m(C; \bar{A})$.*
*(C) For every $t \in \mathcal{T}_{n,l}$ there is a monadic formula $\psi_t(\bar{X})$ with $\mathrm{dp}(\psi) = n$ such that for every $\bar{A} \in {}^l\mathcal{P}(C)$, $\quad C \models \psi_t(\bar{A}) \iff \mathrm{Th}^n(C; \bar{A}) = t$.*
*(D) Each $\mathrm{Th}^n(C; \bar{A})$ is hereditarily finite, and we can effectively compute the set $T_{n,l}$ of formally possible $\mathrm{Th}^n(C; \bar{A})$.*

**Proof.** Easy.

♡

**Definition 3.4.** If $C, D$ are chains then $C + D$ is any chain that can be split into an initial segment isomorphic to $C$ and a final segment isomorphic to $D$.
If $\langle C_i : i < \alpha \rangle$ is a sequence of chains then $\sum_{i<\alpha} C_i$ is any chain $D$ that is the concatenation of segments $D_i$, such that each $D_i$ is isomorphic to $C_i$.

**Theorem 3.5 (composition theorem for linear orders).**
*(1) If $\lg(\bar{A}) = \lg(\bar{B}) = \lg(\bar{A}') = \lg(\bar{B}') = l$, and*

$$\mathrm{Th}^m(C; \bar{A}) = \mathrm{Th}^m(C'; \bar{A}') \quad \text{and} \quad \mathrm{Th}^m(D; \bar{B}) = \mathrm{Th}^m(D'; \bar{B}')$$

*then*

$$\mathrm{Th}^m(C + D; A_0 \cup B_0, \ldots, A_{l-1} \cup B_{l-1}) = \mathrm{Th}^m(C' + D'; A'_0 \cup B'_0, \ldots, A'_{l-1} \cup B'_{l-1}).$$

*(2) If for $i < \alpha$, $\mathrm{Th}^m(C_i; \bar{A}_i) = \mathrm{Th}^m(D_i; \bar{B}_i)$ where $\bar{A}_i = \langle A^i_0, \ldots, A^i_{l-1} \rangle$, $\bar{B}_i = \langle B^i_0, \ldots, B^i_{l-1} \rangle$ then*

$$\mathrm{Th}^m\Big(\sum_{i<\alpha} C_i;\ \cup_{i<\alpha} A^i_0, \ldots, \cup_{i<\alpha} A^i_{l-1}\Big) = \mathrm{Th}^m\Big(\sum_{i<\alpha} D_i;\ \cup_{i<\alpha} B^i_0, \ldots, \cup_{i<\alpha} B^i_{l-1}\Big)$$

**Proof.** By [Sh] Theorem 2.4 (where a more general theorem is proved), or directly by induction on $m$.

♡

**Definition 3.6.** (1) $t_1 + t_2 = t_3$ means:
for some $m, l < \omega$, $t_1, t_2, t_3 \in \mathcal{T}_{m,l}$ and if

$$t_1 = \mathrm{Th}^m(C; A_0, \ldots, A_{l-1}) \quad \text{and} \quad t_2 = \mathrm{Th}^m(D; B_0, \ldots, B_{l-1})$$

then

$$t_3 = \mathrm{Th}^m(C + D; A_0 \cup B_0, \ldots, A_{l-1} \cup B_{l-1}).$$

By the previous theorem, the choice of $C$ and $D$ is immaterial.
(2) $\sum_{i<\alpha} \mathrm{Th}^m(C_i; \bar{A}_i)$ is $\mathrm{Th}^m(\sum_{i<\alpha} C_i;\ \cup_{i<\alpha} A^i_0, \ldots, \cup_{i<\alpha} A^i_{l-1})$.

**Notation 3.7.**
(1) $\mathrm{Th}^n(C;\bar{P},\bar{Q})$ is $\mathrm{Th}^n(C;\bar{P}\,\hat{}\,\bar{Q})$.
(2) If $D$ is a subchain of $C$ and $X_1,\dots,X_{l-1}$ are subsets of $C$ then $\mathrm{Th}^m(D;X_0,\dots,X_{l-1})$ abbreviates $\mathrm{Th}^m(D;X_0\cap D,\dots,X_{l-1}\cap D)$.
(3) For $C$ a chain, $a<b\in C$ and $\bar{P}\subseteq C$ we denote by $\mathrm{Th}^n(C;\bar{P})\restriction_{[a,b)}$ the theory $\mathrm{Th}^n([a,b);\bar{P}\cap[a,b))$.
(4) We will use abbreviations as $\bar{P}\cup\bar{Q}$ for $\langle P_0\cup Q_0,\dots\dots\rangle$ and $\cup_i\bar{P}_i$ for $\langle\cup_i P_0^i,\dots\dots\rangle$ (of course we assume that all the involved sequences have the same length).
(5) We shall not always distinguish between $\mathrm{Th}^n(C;\bar{P},\emptyset)$ and $\mathrm{Th}^n(C;\bar{P})$.

**Theorem 3.8.** *For every $n,l<\omega$ there is $m=m(n,l)<\omega$, effectively computable from $n$ and $l$, such that whenever $I$ is a chain, for $i\in I$ $C_i$ is a chain, $\bar{Q}_i\subseteq C_i$ and $\lg(\bar(Q_i)=l$, if $(C;\bar{Q})=\sum_{i\in I}(C_i;\bar{Q}_i):=(\sum_{i\in I}C_i;\cup_{i\in I}\bar{Q}_i)$ and if for $t\in\mathcal{T}_{n,l}$ $P_t:=\{i\in I:\mathrm{Th}^n(C_i;\bar{Q}_i)=t\}$ and $\bar{P}:=\langle P_t:t\in\mathcal{T}_{n,l}\rangle$ then from $\mathrm{Th}^m(I;\bar{P})$ we can effectively compute $\mathrm{Th}^n(C;\bar{Q})$*

**Proof.** By [Sh] Theorem 2.4.

♡

**Definition 3.9.**
(1) Let $T_0$, $T_1$ be disjoint trees with $\eta_0=root(T_0)$. Define a tree $T$ to be the ordered sum of $T_0$ and $T_1$ by:
$T=T_0\bigoplus T_1$ iff $T=T_0\cup T_1$ where the partial order on $T$, $\lhd_T$, is induced by the partial orders of $T_0$ and $T_1$ and the (only) additional rule:

$$\sigma\in T_1\Rightarrow\eta_0\lhd\sigma.$$

(2) If $T_0$ doesn't have a root then $\lhd_T$ is the disjoint union $\lhd_{T_0}\cup\lhd_{T_1}$ (So $[\tau\in T_0\ \&\ \sigma\in T_1]\Rightarrow\tau\perp\sigma$).
(3) When $I$ is a chain and $T_i$ are pairwise disjoint trees for $i\in I$ we define $T=\bigoplus_{i\in I}T_i$ by $T=\cup_{i\in I}T_i$ with similar rules on $\lhd=\lhd_T$ namely

$$\sigma,\tau\in T_i\ \Rightarrow\ [\sigma\lhd\tau\iff\sigma\lhd_{T_i}\tau]$$

$$[\sigma=root(T_i),\ i<_I j,\ \tau\in T_j]\ \Rightarrow\ \sigma\lhd\tau$$

$$[\sigma\in T_i,\ \sigma\neq root(T_i),\ i\neq j,\ \tau\in T_j]\ \Rightarrow\ \sigma\perp\tau$$

**Theorem 3.10 (composition theorem along a complete branch).**
*For every $n<\omega$ there is an $m=m(n)<\omega$, effectively computable from $n$, such that if $I$ is a chain and $T_i$ are trees for $i\in I$ then $\langle\mathrm{Th}^m(T_i):i\in I\rangle$ and $\mathrm{Th}^m(\langle\eta_i:i\in I\rangle)$ (which is a theory of a chain) determine $\mathrm{Th}^n(\bigoplus_{i\in I}T_i)$.*

**Proof.** See theorem 3.14.

♡

Given a tree $T$, we would like to represent it as a sum of subtrees, ordered by a branch $B\subseteq T$. Sometimes however we may have to use a chain $\mathcal{B}$ that embeds $B$.

**Definition 3.11.** Let $T$ be a tree $T$, $B \subseteq T$ a branch $\nu \in T$, $\eta \in B$ and $X \subseteq B$ be an initial segment without a last element.
(a) $\nu$ *cuts* $B$ at $\eta$ if $\eta \triangleleft \nu$ and for every $\tau \in B$, if $\neg \tau \triangleleft \eta$ then $\neg \tau \triangleleft \nu$, (In particular, $\eta$ cuts $B$ at $\eta$). $\nu$ cuts $B$ at $\{\eta\}$ has the same meaning.
(b) $\nu$ *cuts* $B$ at $X$ if $\eta \triangleleft \nu$ for every $\eta \in X$ and $\neg \tau \triangleleft \nu$ for every $\tau \in B \setminus X$.
(c) $\mathcal{B}^+ \subseteq \mathcal{P}(B)$ is defined by $X \in \mathcal{B}^+$ iff $\big[X = \{\eta\}$ for some $\eta \in B\big]$ or $\big[X \subseteq B$ is an initial segment without a last element and there is $\nu \in T \setminus B$ that cuts $B$ at $X\big]$.
(d) Define a linear order $\leq = \leq_{\mathcal{B}^+}$ on $\mathcal{B}^+$ by $X_0 \leq X_1$ iff $\big[X_0 = \{\eta_0\}$, $X_1 = \{\eta_1\}$ and $\eta_0 \triangleleft \eta_1\big]$ or $\big[X_0 \subseteq X_1\big]$.
Note that the statements $X \in \mathcal{B}^+$ and $X_0 \leq_{\mathcal{B}^+} X_1$ are expressible by monadic formulas $\psi_{\in}(X, B)$ and $\psi_{\leq}(X_0, X_1, B)$.
(e) For $X \in \mathcal{B}^+$ define $T_X := \big\{\nu \in T : \nu \text{ cuts } B \text{ at } X\big\}$.

Now $\mathcal{B}^+$ has the disatvantage of not being a subset of $T$ and (at the small cost of adding a new parameter) we shall replace the chain $(\mathcal{B}^+, <_{\mathcal{B}^+})$ by a chain $(\mathcal{B}, <_{\mathcal{B}})$ where $\mathcal{B} \subseteq T$.

**Definition 3.12.** $\mathcal{B} \subseteq T$ is obtained by replacing every $X \in \mathcal{B}^+$ by an element $\eta_{\text{x}} \in T$ in the following way: if $X = \{\eta\}$ then $\eta_{\text{x}} = \eta$ and if $X \subseteq B$ is an initial segment then $\eta_{\text{x}}$ is a favourite element from $T_X$. $\leq_{\mathcal{B}}$ is defined by $\eta_{\text{x}_1} \leq_{\mathcal{B}} \eta_{\text{x}_2} \iff X_1 \leq_{\mathcal{B}^+} X_2$ and $B^c \subseteq T$ will be $\mathcal{B} \setminus \{\eta_X : X = \{\nu\}, \nu \in B\}$, (so $(\mathcal{B} \setminus B^c, \leq_{\mathcal{B}}) \cong (B, \triangleleft)$). For $\eta \in B$ let $T_\eta$ be $T_{\{\eta\}}$ as defined in (e) above, and for $\eta = \eta_{\text{x}} \in B^c$ let $T_\eta = T_X$ as above (in this case $T_\eta$ is $\{\nu \in T : \nu \sim^0_B \eta\}$ as in definition 2.5).

**Fact 3.13.** *$\leq_{\mathcal{B}}$ is definable from $B$ and $B^c$, $T_\eta$ is definable from $\eta, B$ and $B^c$ and $T = \bigoplus_{\eta \in \mathcal{B}} T_\eta$ in accordance with definition 3.9.*

♡

**Theorem 3.14 (Composition theorems for trees).**
*Assume $T$ is a tree, $B \subseteq T$ a branch and $\bar{Q} \subseteq T$ with $\lg(\bar{Q}) = l$. Let $\mathcal{B}$ and $B^c$ be defined as above, for $\eta \in \mathcal{B}$ $T_\eta$ is defined as above (so $T = \bigoplus_{\eta \in \mathcal{B}} T_\eta$) and $S_\eta$ is $T_\eta \setminus B$ (so, abusing notations, $T = B \cup \bigoplus_{\eta \in \mathcal{B}} S_\eta$). Then:*
*1) Composition theorem on a branch: for every $n < \omega$ there is $k = k(n,l) < \omega$, effectively computable from $n$ and $l$, such that $\mathrm{Th}^k(\mathcal{B}; B, B^c, \bar{P})$ determines $\mathrm{Th}^n(T; \bar{Q})$ where for $t \in \mathcal{T}_{n,l}$, $P_t := \{\eta \in \mathcal{B} : \mathrm{Th}^n(T_\eta; \bar{Q} \cap T_\eta) = t\}$ and $\bar{P} := \langle P_t : t \in \mathcal{T}_{n,l}\rangle$.*
*2) Composition theorem along a branch: for every $n < \omega$ there is $k = k(n,l) < \omega$, effectively computable from $n$ and $l$, such that*
*$\mathrm{Th}^k(B; \bar{Q})$ and $\langle \mathrm{Th}^k(S_\eta; B, B^c, \bar{Q}) : \eta \in \mathcal{B}\rangle$ determine $\mathrm{Th}^n(T; \bar{Q})$.*

**Proof.** By Theorem 1 in [GuSh] §2.4.

♡

**Definition 3.15.** Additive colouring....

**Theorem 3.16 (Ramsey theorem for additive colourings).** ...

**Proof.** By [Sh] Theorem 1.1.

♡

## 4. Well Orderings of Ordinals

A chain is *tame* iff it is scattered of Hausdorff degree $< \omega$. We will define for a tame chain $C$, $\mathrm{Log}(C)$ and show later (in proposition 4.8) that this function is well defined.

**Definition 4.1.** Let Log:{tame chains} $\to \omega \cup \{\infty\}$ be defined by:
$\mathrm{Log}(C) = \infty$ iff there is $\varphi(x, y, \bar{P})$ that defines a well ordering on the elements of $C$ of order type $\geq \omega^\omega$,
$\mathrm{Log}(C) = k$ iff there is $\varphi(x, y, \bar{P})$ that defines a well ordering on the elements of $C$ of order type $\alpha$ with $\omega^k \leq \alpha < \omega^{k+1}$.

**Fact 4.2.** *A tame chain $C$ has a reconstrutible well ordering i.e. there is a formula $\varphi(x, y, \bar{P})$ ($\bar{P} \subseteq C$) that defines a well ordering on the elements of $C$ of order type $\alpha$ and there is a formula $\psi(x, y, \bar{Q})$ ($\bar{Q} \subseteq \alpha$) that defines a linear order $<^*$ on the elements of $\alpha$ such that $(\alpha, <^*) \cong (C, <)$.*

**Proof.** By induction on $\mathrm{Hdeg}(\alpha)$, using the proof of Theorem A1 in the appendix.

♡

**Definition 4.3.** Let $\alpha, \beta$ be ordinals. $\alpha \to \beta$ means the following: "there is $\varphi(x, y, \bar{P})$ that defines a well ordering on the elements of $\alpha$ of order type $\beta$".

**Claim 4.4.**
*1)* $\alpha \to \beta \;\&\; \beta \to \gamma \;\Rightarrow\; \alpha \to \gamma$.
*2)* $\alpha \to \gamma \;\&\; \gamma \geq \alpha \cdot \omega \;\Rightarrow\; \alpha \to \alpha \cdot \omega$.

**Proof.** Straightforward.

♡

**Notation.** Suppose $\alpha \to \beta$ holds by $\varphi(x, y, \bar{P})$. Define a bijection $f\colon \alpha \to \beta$ by $f(i) = j$ iff $i$ is the $j$'th element in the well order defined by $\varphi$.

**Lemma 4.5.** *For any ordinal $\alpha$, $\alpha \not\to \alpha \cdot \omega$.*

**Proof.** Assume that $\alpha$ is minimal such that $\alpha \to \alpha \cdot \omega$. It follows that:
$(i)$ $\alpha \geq \omega$,
$(ii)$ $\alpha$ is a limit ordinal (by $\alpha \to \alpha + 1$ and 2.7),
$(iii)$ for $\beta < \alpha$, $\{f(i) : i < \beta\}$ does not contain a final segment of $\alpha \cdot \omega$ (otherwise clearly $\beta \to \alpha \cdot \omega$ hence by 2.7 $\beta \to \beta \cdot \omega$ but $\alpha$ is minimal).
So let $\varphi(x, y, \bar{P})$ define a well order of $\alpha$ of order type $\alpha \cdot \omega$ and let $Q \subseteq \alpha$ be the following subset: $x \in Q$ iff for some $k < \omega$, $\alpha \cdot 2k \leq f(x) < \alpha \cdot (2k + 1)$. Let $E$ an equivalence relation on $\alpha$ defined by $xEy$ iff for some $l < \omega$, $f(x)$ and $f(y)$ belong to the segment $[\alpha \cdot l, \alpha \cdot (l+1))$. Clearly there is a monadic formula $\psi(x, y, \bar{P}, Q)$ that defines $E$ moreover, some monadic formula $\theta(X, \bar{P}, Q)$ expresses the statement " $\bigvee_{i<\omega} (X = Q_i)$" where $\langle Q_i : i < \omega \rangle$ are the $E$-equivalence classes.
Let $n := \max\{\mathrm{dp}(\varphi), \mathrm{dp}(\psi), \mathrm{dp}(\theta)\} + 5$, and
$m := |\{\mathrm{Th}^n(C; \bar{X}, Y, Z) : C \text{ a chain },\ \bar{X}, Y, Z \subseteq C,\ \lg(\bar{X}) = \lg(\bar{P})\}|$.
let $\delta = \mathrm{cf}(\alpha)$ and $\{x_i\}_{i<\delta}$ be stricly increasing and cofinal in $\alpha$. By [Sh]Theorem 1.1 applied to the colouring $h(i, j) = \mathrm{Th}^n(\alpha; \bar{P}, Q, x_i, x_j)$ we get a cofinal subsequence $\{\beta_j\}_{j<\delta}$ such that $\mathrm{Th}^n(\alpha; \bar{P}, Q, \beta_{j_1}, \beta_{j_2})$ is constant for $j_1 < j_2 < \delta$. Note that it follows
(†) the theories $\mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[0,\beta_j)}$, $\mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_j,\alpha)}$, and $\mathrm{Th}^n(\alpha; \bar{P}, Q, \beta_{j_1}) \restriction_{[\beta_{j_1},\beta_{j_2})}$ are constant for every $j < \delta$ and for every $j_1 < j_2 < \delta$.

Note that each $E$-equivalence class $Q_i$ is unbounded in $\alpha$ since if some $\beta < \alpha$ contains some $E$-equivalence class $Q_i$ it would easily follow that $\beta \to \alpha$ contradicting fact $(iii)$.
Fix some $1 < j < \delta$ let $x < \beta_j$ and let $Q_{i(x)}$ be the $E$-equivalence class containing $x$. Since $Q_{i(x)}$ is unbounded in $\alpha$ there is some $j < l < \delta$ such that $[\beta_j, \beta_l) \cap Q_{i(x)} \neq \emptyset$. This statement is expressible by $\mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l)$ which is equal to
$\mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l) \restriction_{[\beta_j,\beta_l)} + \mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l) \restriction_{[\beta_l,\alpha)} =$
$\mathrm{Th}^n(\alpha; \bar{P}, Q, x, \emptyset, \emptyset) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha; \bar{P}, Q, \emptyset, \beta_j, \emptyset) \restriction_{[\beta_j,\beta_l)} + \mathrm{Th}^n(\alpha; \bar{P}, Q, \emptyset, \emptyset, \beta_l) \restriction_{[\beta_l,\alpha)}$.
By (†) we may replace the second theory by $\mathrm{Th}^n(\alpha; \bar{P}, Q, \emptyset, \beta_j, \emptyset) \restriction_{[\beta_j,\beta_{j+1})}$
and the third theory by $\mathrm{Th}^n(\alpha; \bar{P}, Q, \emptyset, \emptyset, \beta_{j+1}) \restriction_{[\beta_{j+1},\alpha)}$, and conclude:

$$\mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l) = \mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_{j+1})$$

Therefore for every $x < \beta_j$, $[\beta_j, \beta_{j+1}) \cap Q_{i(x)} \neq \emptyset$.

Finally, let $j < \delta$ be such that the segment $[0, \beta_j)$ intersects $m+1$ different $E$-equivalence classes, say $Q_{i_0}, \ldots, Q_{i_m}$. By the previous argument we have $[\beta_j, \beta_{j+1}) \cap Q_{i_l} \neq \emptyset$ for every $l \leq m$. By the choice of $m$ there are different $a, b \in \{i_0, \ldots, i_m\}$ such that
$(*)$ $\mathrm{Th}^n(\alpha; \bar{P}, Q, Q_a) \restriction_{[\beta_j,\beta_{j+1})} = \mathrm{Th}^n(\alpha; \bar{P}, Q, Q_b) \restriction_{[\beta_j,\beta_{j+1})}$.
Let $R \subseteq \alpha$ be $\big([0,\beta_j) \cap Q_a\big) \cup \big(([\beta_j, \beta_{j+1}) \cap Q_b\big) \cup \big([\beta_{j+1}, \alpha) \cap Q_a\big)$.
Now $\mathrm{Th}^n(\alpha, \bar{P}, Q, R) =$
$\mathrm{Th}^n(\alpha, \bar{P}, Q, R) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha, \bar{P}, Q, R) \restriction_{[\beta_j,\beta_{j+1})} + \mathrm{Th}^n(\alpha, \bar{P}, Q, R) \restriction_{[\beta_{j+1},\alpha)} =$
$\mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_b) \restriction_{[\beta_j,\beta_{j+1})} + \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[\beta_{j+1},\alpha)} =$ (by $(*)$)
$\mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[\beta_j,\beta_{j+1})} + \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[\beta_{j+1},\alpha)} =$
$\mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a)$.
But $Q_a$ is an $E$-equivalence class while $R$ is not. Since $\mathrm{Th}^n(\alpha, \bar{P}, Q, Z)$ computes the statement "$Z$ is $E$-equivalence class" we get a contradiction from $\mathrm{Th}^n(\alpha, \bar{P}, Q, R) = \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a)$.

♡

**Claim 4.6.** *If $\alpha \to \beta$ and $\beta < \alpha$ then $(\exists \gamma_1, \gamma_2)\big((\gamma_1 + \gamma_2 = \alpha) \,\&\, (\gamma_2 + \gamma_1 = \beta)\big)$.*

**Proof.** Let's prove first:
Subclaim: $\omega + \omega \not\to \omega$.
Proof of the subclaim: Assume that $\varphi(x, y, \bar{P})$ well orders $\omega + \omega$ of order type $\omega$ and that $\mathrm{dp}(\varphi) = n$, $l(\bar{P}) = l$. Let $x <^* y$ mean $(\omega + \omega, <) \models \varphi(x, y, \bar{P})$.

→[Insert Ramsey theorems]

Let $\{x_i\}_{i<\omega}$ be increasing and unbounded in $[0, \omega)$ satisfying, for $i < j < \omega$ and for some $s_0 \in \mathcal{T}_{n,l+2}$ and $t_0 \in \mathcal{T}_{n,l+2}$

$$\mathrm{Th}^n(\omega + \omega; x_i, \emptyset, \bar{P}) \restriction_{[x_i,x_j)} = s_0, \quad \mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[x_i,x_j)} = t_0,$$

let $\{y_j\}_{j<\omega}$ increasing and unbounded in $[\omega, \omega+\omega)$ satisfying, for $j < k < \omega$ and for some $s_1 \in \mathcal{T}_{n,l+2}$ and $t_1 \in \mathcal{T}_{n,l+2}$

$$\mathrm{Th}^n(\omega + \omega; \emptyset, y_j, \bar{P}) \restriction_{[y_j,y_k)} = s_1, \quad \mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[y_i,y_k)} = t_1.$$

Using Ramsey Theorem (and as $<^*$ is well founded) we may assume that $i_1 < i_2 \Rightarrow x_{i_1} <^* x_{i_2}$ and $j_1 < j_2 \Rightarrow y_{j_1} <^* y_{j_2}$.
We will show now that for $0 < i < \omega$ and $0 < j < \omega$, $\mathrm{Th}^n(\omega + \omega; x_i, y_j, \bar{P})$ is constant. Indeed,

$t^* := \mathrm{Th}^n(\omega+\omega; x_i, y_j, \bar{P}) =$
$\mathrm{Th}^n(\omega+\omega;\emptyset,\emptyset,\bar{P})\restriction_{[0,x_0)} + \mathrm{Th}^n(\omega+\omega;\emptyset,\emptyset,\bar{P})\restriction_{[x_0,x_i)} +$
$\mathrm{Th}^n(\omega+\omega;x_i,\emptyset,\bar{P})\restriction_{[x_i,x_{i+1})} + \mathrm{Th}^n(\omega+\omega;\emptyset,\emptyset,\bar{P})\restriction_{[x_{i+1},\omega)} + \mathrm{Th}^n(\omega+\omega;\emptyset,\emptyset,\bar{P})\restriction_{[\omega,y_0)} +$
$\mathrm{Th}^n(\omega+\omega;x_i,\emptyset,\bar{P})\restriction_{[y_i,y_j)} + \mathrm{Th}^n(\omega+\omega;\emptyset,y_j,\bar{P})\restriction_{[y_j,y_{j+1})} + \mathrm{Th}^n(\omega+\omega;\emptyset,\emptyset,\bar{P})\restriction_{[y_{j+1},\omega+\omega)}$ .
Call the sum $t^* = r_1 + r_2 + \ldots + r_8$. Now $r_1$ is constant, $r_2 = t_0 \cdot i = t_0$ (check that $t_0 + t_0 = t_0$), $r_3$ is $s_0$, $r_4 = t_0 \cdot \omega$ hence is constant, $r_5$ is constant, $r_6 = t_1 \cdot j = t_1$, $r_7 = s_1$ and $r_8 = t_0 \cdot \omega$ hence is constant. Therefore $t^*$ does not depend on $i$ and $j$.
Now as $\{y_j\}_{j<\omega}$ is unbounded with respect to $<^*$, there is some $j < \omega$ such that $x_1 <^* y_j$. This is expressed by $\mathrm{Th}^n(\omega+\omega; x_1, y_j, \bar{P})$ which we have just seen to be independent of $i$ and $j$ hence

$$(\forall\, 0 < i < \omega)(\forall\, 0 < j < \omega)\big[(\omega+\omega,<) \Rightarrow \varphi(x_i, y_j, \bar{P})\big]$$

it follows that $\mathrm{otp}(\omega+\omega,<^*) \geq \omega+1$, a contradiction. This proves $\omega+\omega \not\to \omega$.

Returning to the proof of the claim, let $\beta$ be the minimal ordinal such that there exists some $\alpha > \beta$ with $\alpha \to \beta$ but there aren't any $\gamma_1, \gamma_2 \leq \alpha$ with $(\gamma_1 + \gamma_2 = \alpha)\,\&\,(\gamma_2 + \gamma_1 = \beta)$. Call such a $\beta$ *weird* and let $\alpha > \beta$ the first ordinal witnessing the weirdness of $\beta$. By transitivity of $\to$ it is easy to see that $\beta$ is limit. Moreover, $\gamma < \beta \Rightarrow \beta \not\to \gamma$ hence if $\beta = \gamma_1 + \gamma_2$ then $\gamma_2 + \gamma_1 \geq \beta$. It follows that there are two possible cases: either $(*)$ $\gamma < \beta \Rightarrow (\gamma + \gamma < \beta)$, hence $\gamma < \beta \Rightarrow (\gamma \cdot \omega \leq \beta)$ and $\gamma < \beta \Rightarrow (\mathrm{otp}([\gamma,\beta)) = \beta)$, or $(**)$ $\beta = \gamma + \gamma$.

First case: $(*)$ holds i.e. $\gamma < \beta \Rightarrow (\gamma + \gamma < \beta)$. Let $\alpha = \beta + \gamma$ what can $\gamma$ be? If $\gamma < \beta$ then by $(*)$ $\gamma + \beta = \beta$ and $\alpha$ does not witness the weirdness of $\beta$, so $\alpha \geq \beta + \beta$.
Let $\varphi(x, y, \bar{P})$ well order $\alpha$ of order type $\beta$ with $\mathrm{dp}(\varphi) = n$ and $l(\bar{P}) = l$. As above $x <^* y$ means $(\alpha,<) \models \varphi(x, y, \bar{P})$ and finally let $\delta = \mathrm{cf}(\beta)$.
Now $\mathrm{otp}(\alpha,<^*) = \beta$ but what is $\mathrm{otp}([0,\beta),<^*\restriction_{[0,\beta)})$? Clearly, as $\mathrm{Th}^n(\alpha,\bar{P}) = \mathrm{Th}^n(\alpha,\bar{P})\restriction_{[0,\beta)}$ $+\mathrm{Th}^n(\alpha,\bar{P})\restriction_{[\beta,\alpha)}$ we have $\beta \to \mathrm{otp}([0,\beta),<^*\restriction_{[0,\beta)})$ hence $\beta = \mathrm{otp}([0,\beta),<^*\restriction_{[0,\beta)})$ (otherwise, by $(*)$, $\mathrm{otp}([0,\beta),<^*\restriction_{[0,\beta)})$ is weird and $<\beta$). Similarly we can show that $\mathrm{otp}([\beta,\beta+\beta),<^*\restriction_{[\beta,\beta+\beta)}) = \beta$.

→[Insert Ramsey theorems]

Now proceed as before: choose $\{x_i\}_{i<\delta} \subseteq [0,\beta)$ and $\{y_j\}_{j<\delta} \subseteq [\beta,\beta+\beta)$ that are homogeneous unbounded and $<^*$ unbounded and use them to show that $\mathrm{otp}(\alpha,<^*) \geq \beta + 1$.

Second case: $(**)$ holds i.e. $\beta = \gamma + \gamma$.
Call $\epsilon$ *quite weird* if for some $k < \omega$ $\epsilon \cdot k$ is weird. Let $\epsilon \leq \gamma$ be the first quite weird ordinal. Let $k_1$ be the first such that $\epsilon \cdot k_1$ is weird. Look at $\gamma$: if $\gamma = \gamma_1 + \gamma_2$ and $\gamma_2 + \gamma_1 < \gamma$ we would have $\alpha \to \beta = \gamma + \gamma \to \gamma + \gamma_2 + \gamma_1 < \beta$ and a contradiction. Hence either $\gamma_1 < \gamma \Rightarrow (\gamma_1 + \gamma_1 < \gamma)$ and in this case $\gamma = \epsilon$ or $\gamma = \gamma_1 + \gamma_1$. Repeat the same argument to get $\gamma_1 = \epsilon$ or $\gamma_1 = \gamma_2 + \gamma_2$. After finitely many steps we are bound to get $\beta = \epsilon \cdot 2k$ where $2k = k_1$ and $\epsilon_1 < \epsilon \Rightarrow \epsilon_1 \cdot \omega \leq \epsilon$ and of course $\epsilon_1 < \epsilon \Rightarrow \epsilon \not\to \epsilon_1$.
Let $\varphi(x, y, \bar{P})$ and $<^*$ be as usual and $\delta := \mathrm{cf}(\beta) = \mathrm{cf}(\epsilon)$. Let $\alpha = \beta + \epsilon^*$ if $\epsilon^* < \epsilon$ then $\epsilon^* + \beta = \beta$ and $\alpha$ doesn't witness weirdness, therefore $\epsilon^* \geq \epsilon$.
Proceed as before: choose $\{x_i^0\}_{i<\delta}, \{x_i^1\}_{i<\delta}, \ldots, \{x_i^k\}_{i<\delta}$ with $\{x_i^l\}_{i<\delta} \subseteq [\epsilon \cdot l, \epsilon(l+1))$, homogeneous, unbounded and $<^*$ incresing.

By the composition theorem it will follow that $\mathrm{otp}([\epsilon \cdot l, \epsilon(l+1)), <^*) \geq \epsilon$ and by homogeneity we will have, for $0 < i, j < \omega$ and $l \leq k$, $x_i^l <^* x_j^{l+1}$. It follows that $\mathrm{otp}(\alpha, <^*) \geq (\epsilon \cdot k) + 1 = \beta + 1$ and a contradiction.

♡

**Theorem 4.7.** *Well ordering of ordinals are obtained only by the following process: let $\langle P_0, P_1, \ldots, P_{n-1} \rangle$ be a partition of $\alpha$ and*

$$i <^* j \iff \big[(\exists k < n)(i \in P_k \,\&\, j \in P_k \,\&\, i < j)\big] \vee \big[i \in P_{k_1} \,\&\, j \in P_{k_2} \,\&\, k_1 < k_2\big].$$

♡

**Proposition 4.8.** *Log(C) is well defined.*

**Proof.** Let $(C, <^*)$ be a scattered chain and let $(\alpha, <)$ and $(\beta, <)$ be results of a definable well orderings of $(C, <^*)$ where in addition (by 4.2) there is $\psi(x, y, \bar{Q})$ that defines $C$ in $\alpha$. So $\alpha \to \beta$ and by 4.5 and 4.6 $\alpha < \omega^\omega \iff \beta < \omega^\omega$ and $\alpha \in [\omega^k, \omega^{k+1}) \iff \beta \in [\omega^k, \omega^{k+1})$.

♡

## 5. $(\omega^\omega, <)$ and longer chains

The following lemma is a part of Theorem 3.5(B) in [Sh]:

**Lemma 5.1.** *Let $I$ be a well ordered chain of order type $\geq \omega^k$. Let $f\colon I^2 \to \{t_0, t_1 \ldots, t_{l-1}\}$ be an additive colouring and assume that for $\alpha < \beta \in I$, $f(\alpha, \beta)$ depends only on the order type in $I$ of the segment $[\alpha, \beta)$.*
*Then there is $i < l$ such that for some $p \leq l$, for every $r \geq p$, if $\mathrm{otp}\big([\alpha, \beta)\big) = \omega^r$ then $f(\alpha, \beta) = t_i$. Moreover, $t_i + t_i = t_i$.*

**Proof.** To avoid triviality assume $k > l$. For $\alpha < \beta$ in $I$ with $\mathrm{otp}\big([\alpha, \beta)\big) = \delta$, denote $f(\alpha, \beta)$ by $t(\delta)$ (makes sense by the assumptions).
By the pigeon-hole principle there are $1 \leq p \leq l$, $s > p$ and some $t_i$ with $t(\omega^p) = t(\omega^s) = t_i$. Now $\omega^{p+2} = \sum_{i<\omega} (\omega^{p+1} + \omega^p)$ and by the additivity of $f$:

$$t\big(\omega^{p+2}\big) = t\big(\sum_{i<\omega} (\omega^{p+1} + \omega^p)\big) = \sum_{i<\omega} t\big(\omega^{p+1} + \omega^p\big) = \sum_{i<\omega} \big(t(\omega^{p+1}) + t(\omega^p)\big) = \sum_{i<\omega} \big(t(\omega^{p+1}) + t(\omega^s)\big) =$$

$$\sum_{i<\omega} t\big(\omega^{p+1} + \omega^s\big) = \sum_{i<\omega} t\big(\omega^s\big) = \sum_{i<\omega} t\big(\omega^p\big) = t\big(\sum_{i<\omega} \omega^p\big) = t\big(\omega^{p+1}\big).$$

Hence

$$t\big(\omega^{p+2}\big) = t\big(\omega^{p+1}\big).$$

Using this and as $\omega^{p+3} = \sum_{i<\omega} (\omega^{p+2} + \omega^{p+1})$ we have

$$t\big(\omega^{p+3}\big) = t\big(\sum_{i<\omega} (\omega^{p+2} + \omega^{p+1})\big) = \sum_{i<\omega} t\big(\omega^{p+2} + \omega^{p+1}\big) = \sum_{i<\omega} \big(t(\omega^{p+2}) + t(\omega^{p+1})\big) =$$

$$\sum_{i<\omega} \big(t(\omega^{p+1}) + t(\omega^{p+1})\big) = \sum_{i<\omega} t\big(\omega^{p+1}\big) = t\big(\sum_{i<\omega} \omega^{p+1}\big) = t\big(\omega^{p+2}\big).$$

Hence
$$t(\omega^{p+3}) = t(\omega^{p+2}).$$

So for every $j > 0$, $t(\omega^{p+1}) = t(\omega^{p+j})$ and in particular $t(\omega^{p+1}) = t(\omega^{s}) = t(\omega^{p}) = t_i$.
This proves the first part of the lemma. As for the moreover clause, since $\omega^{p+1} = \omega^p + \omega^{p+1}$ we have
$$t_i = t(\omega^{p+1}) = t(\omega^p + \omega^{p+1}) = t(\omega^p) + t(\omega^{p+1}) = t_i + t_i.$$

♡

**Proposition 5.2.** *The formula $\varphi(X, Y)$ saying "if $Y$ is without a last element then $X \subseteq Y$ is an $\omega$-sequence unbounded in $Y$ (and if not then $X = \emptyset$)" can not be uniformized in $(\omega^\omega, <)$.*
*Moreover, if $\psi_m(X, Y, \bar{P}_m)$ uniformizes $\varphi$ on $\omega^m$ then one of the sets $\{\mathrm{dp}(\psi_m) : m < \omega\}$ or $\{\lg(\bar{P}_m) : m < \omega\}$ is unbounded.*

**Proof.** Suppose the second statement fails, then:
(†) there is a formula $\psi(X, Y, \bar{Z})$ such that for an unbounded set $I \subseteq \omega$, for every $m \in I$ there is $\bar{P}_m \subseteq \omega^m$ such that $\psi(X, Y, \bar{P}_m)$ uniformizes $\varphi$ on $\omega^m$.
Let $\bar{P}_m = \bar{P}$ let $n = \mathrm{dp}(\psi) + 1$ and $M := |\{\mathrm{Th}^n(C; X, Y, \bar{Z}) : C \text{ a chain }, X, Y, \bar{Z} \subseteq C, \lg(\bar{Z}) = \lg(\bar{P})\}|$.
Let $m \in I$ be large enough ($m > 2M + 3$ will do), and let's show that $\psi$ doesn't work for $\omega^m$ and a subset $Y_1$ that will be defined now.
If $\alpha < \omega^m$ then $\alpha = \omega^{m-1}k_{m-1} + \omega^{m-2}k_{m-2} + \ldots + \omega k_1 + k_0$. Let $k(\alpha) := \min\{i : k_i \neq 0\}$ and let $A_k := \{\alpha < \omega^m : k(\alpha) = k\}$. Note that $\mathrm{otp}(A_k) = \omega^{m-k}$.
For $k \in \{1, 2, \ldots, m-1\}$ we will choose $Y_k \subseteq A_k$ with $\mathrm{otp}(Y_k) = \mathrm{otp}(A_k) = \omega^{m-k}$ such that for $\alpha < \beta$ in $Y_k$:
$$(*) \quad \mathrm{Th}^n(\omega^m; \bar{P}, Y_k) \restriction_{[\alpha,\beta)} \text{ depends only on } \mathrm{otp}([\alpha, \beta) \cap Y_k)$$

we will start with $k = m - 1$ and proceed by inverse induction:
Let $A_{m-1} = \langle \alpha_j : j < \omega \rangle$. Let for $l < p < \omega$, $h(l, p) := \mathrm{Th}^n(\omega^m; \bar{P}, \alpha_l) \restriction_{[\alpha_l,\alpha_p)}$. Let $J \subseteq \omega$ be homogeneous with respect to this colouring namely, for some fixed theory $t_{m-1}$, for every $l < p$ in $J$,
$$\mathrm{Th}^n(\omega^m; \bar{P}, \alpha_l) \restriction_{[\alpha_l,\alpha_p)} = t_{m-1}.$$

By the composition theorem, for every $l < p$ in $J$,
$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_{m-1}) \restriction_{[\alpha_l,\alpha_p)} = t_{m-1} \cdot |Y_{m-1} \cap [\alpha_l, \alpha_p)|$$

and this proves $(*)$ for $Y_{m-1}$.

Rename $Y_{m-1}$ by $\langle \alpha_i : i < \omega \rangle$. In each segment $[\alpha_i, \alpha_{i+1})$ choose $\langle \beta^i_l : 0 < l < \omega \rangle \subseteq A_{m-2}$ increasing and cofinal such that for every $l < p < \omega$ the theory $\mathrm{Th}^n(\omega^m; \bar{P}, \beta^i_l) \restriction_{[\beta^i_l,\beta^i_p)}$ is constant.
Returning to $Y_{m-1}$, for $i < j < \omega$ let
$$h_1(i, j) := \langle \mathrm{Th}^n(\omega^m; \bar{P}) \restriction_{[\alpha_i,\beta^{j-1}_1)}, \mathrm{Th}^n(\omega^m; \bar{P}, \beta^{j-1}_1) \restriction_{[\beta^{j-1}_1,\beta^{j-1}_2)} \rangle$$

w.l.o.g. (by thinning out and re-renaming and noting that we don't harm $(*)$) $Y_{m-1}$ is homogeneous with respect to this colouring.

Hence, for some theories $t^*$ and $t_{m-2}$, for every $i<j<\omega$ we have

$$h_1(i,j)=\langle t^*,t_{m-2}\rangle$$

Let $Y_{m-2}:=\langle \beta^i_l:0<l<\omega, i<\omega\rangle$, clearly $\mathrm{otp}(Y_{m-2})=\omega^2$. Let's check $(*)$ for $Y_{m-2}$:
Firstly, note that for $l<p<\omega$,

$$\mathrm{Th}^n(\omega^m;\bar P,Y_{m-2})\restriction_{[\beta^i_l,\beta^i_p)}=t_{m-2}\cdot(p-l).$$

Secondly, for $i<j<\omega$ $\quad\mathrm{Th}^n(\omega^m;\bar P,Y_{m-2})\restriction_{[\beta^i_l,\beta^j_p)}=$

$$\mathrm{Th}^n(\omega^m;\bar P,Y_{m-2})\restriction_{[\beta^i_l,\alpha_{i+1})}+\mathrm{Th}^n(\omega^m;\bar P,Y_{m-2})\restriction_{[\alpha_{i+1},\alpha_{i+2})}+\dots$$

$$+\mathrm{Th}^n(\omega^m;\bar P,Y_{m-2})\restriction_{[\alpha_{j-1},\alpha_j)}+\mathrm{Th}^n(\omega^m;\bar P,Y_{m-2})\restriction_{[\alpha_j,\beta^j_p)}$$

where the first theory is equal to $t_{m-2}\cdot\omega$, the last theory is $t^*+t_{m-2}\cdot(p-l)$, and the middle theories are $t^*+t_{m-2}\cdot\omega$. These observations prove $(*)$ for $Y_{m-2}$.

For defining $Y_{m-3}$ let's restrict ourselves to a segment $[\alpha_i,\alpha_{i+1})$ where $\alpha_i,\alpha_{i+1}\in Y_{m-1}$. In this segment we have defined $\langle\beta^i_l:0<l<\omega\rangle\subseteq Y_{m-2}$. Now choose in each $[\beta^i_l,\beta^i_{l+1})$ an increasing cofinal sequence $\langle\gamma^{i,l}_j:0<j<\omega\rangle$ such that for $j<p<\omega$, $\mathrm{Th}^n(\omega^m;\bar P,\gamma^{i,l}_j)\restriction_{[\gamma^{i,l}_j,\gamma^{i,l}_p)}$ is constant.
For $0<l<p<\omega$ let

$$h^i_1(l,p):=\big\langle\mathrm{Th}^n(\omega^m;\bar P)\restriction_{[\beta^i_l,\gamma^{i,p-1}_1)},\ \mathrm{Th}^n(\omega^m;\bar P,\gamma^{i,p-1}_1)\restriction_{[\gamma^{i,p-1}_1,\gamma^{i,p-1}_2)}\ \big\rangle$$

and again w.l.o.g we may assume that $\langle\beta^i_l:0<l<\omega\rangle$ is homogeneous with respect to $h^i_1$.
Next, for $i<j<\omega$ define

$$h_2(i,j):=\big\langle\mathrm{Th}^n(\omega^m;\bar P)\restriction_{[\alpha_i,\gamma^{j-1,1}_1)},\ \mathrm{Th}^n(\omega^m;\bar P,\gamma^{j-1,1}_1)\restriction_{[\gamma^{j-1,1}_1,\gamma^{j-1,1}_2)}\ \big\rangle$$

by thinning out and renaming we may assume that $Y_{m-1}$ is homogeneous with respect to $h_2$, now $Y_{m-2}$ is also thinned out but each new $\langle\beta^i_l:0<l<\omega\rangle$ which is some old $\langle\beta^{i^*}_l:0<l<\omega\rangle$ is still homogeneous.
As a result we will have, for some theories $t^{**},t^{***},t_{m-3}$:

$$(\forall i<j<\omega)(\forall 0<l<p<\omega)\big[h^i_1(l,p)=\langle t^{**},t_{m-3}\rangle\ \&\ h_2(i,j)=\langle t^{***},t_{m-3}\rangle\big].$$

Let $Y_{m-3}:=\{\gamma^{i,l}_j:i<\omega,0<l<\omega,0<j<\omega\}$, as before $(*)$ holds by noting that if for example $i_1<i_2<\omega$ and $1<l_2$ then

$$\mathrm{Th}^n(\omega^m;\bar P,\gamma^{i_1,l_1}_{j_1})\restriction_{[\gamma^{i_1,l_1}_{j_1},\gamma^{i_2,l_2}_{j_2})}=t_{m-3}\cdot\omega+(t^{**}+t_{m-3}\cdot\omega)\cdot\omega+[t^{***}+(t^{**}+t_{m-3}\cdot\omega)\cdot\omega]\cdot(i_2-i_1-1)+$$

$$t^{***}+t_{m-3}\cdot\omega+(t^{**}+t_{m-3}\cdot\omega)(l_2-1)+t^{**}+t_{m-3}\cdot(j_2-1)$$

and similarly for the other possibilities.

$Y_{m-4},Y_{m-5},\dots,Y_1$ are defined by using the same prescription i.e. $Y_{m-l}$ is defined by taking a homogenous sequence between two successive elements of $Y_{m-l-1}$ then homogenous sequences between two successive elements of $Y_{m-l-2}$ by using colouring of the form $h_1, h_2, \ldots$. The thinning out and w.l.o.g's for already defined $Y_{m-k}$'s are not necessary but they ease notations considerably.

We will show now that $\psi$ doesn't choose an unbounded $\omega$-sequence in $Y_1$ that is, for every $\omega$-sequence $X \subseteq Y_1$ there is an $\omega$-sequence $X' \subseteq Y_1$ such that $\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \mathrm{Th}^n(\omega^m; \bar{P}, Y_1, X')$.
By $(*)$, for $\alpha < \beta$ in $Y_1$ the additive colouring $f(\alpha, \beta) := \mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\alpha,\beta)}$ depends only on $\mathrm{otp}\big([\alpha, \beta) \cap Y_1\big)$ hence we can apply lemma 5.1 and conclude that for some $p \leq m/2$, for every $r \geq p$, $\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\alpha,\beta)}$ is equal to some fixed theory $t$ whenever $\mathrm{otp}\big([\alpha, \beta) \cap Y_1\big) = \omega^r$. (Remember that $f$ has at most $M$ possibilities and that $m > 2M$). Moreover, we know that $t + t = t$.
Assume now that for some $X \subseteq Y_1$, $\psi(X, Y_1, \bar{P})$ holds, so $X$ is a cofinal $\omega$-sequence. Let $X = \{\delta_i : i < \omega\}$. As $\mathrm{otp}(Y_1) = \omega^{m-1}$ for unboundedly many $i$'s we have $\mathrm{otp}\big([\delta_i, \delta_{i+1}) \cap Y_1\big) \geq \omega^{m-2} > \omega^p$.
Let $\beta_i := \mathrm{otp}\big([\delta_i, \delta_{i+1}) \cap Y_1\big)$ and denote by $t(\epsilon)$ the theory $\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\alpha,\beta)}$ when $\mathrm{otp}\big([\alpha, \beta) \cap Y_1\big) = \epsilon$ (by $(*)$ it doesn't matter which $\alpha$ and $\beta$ we use).
We are interested in $\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X)$ which is
$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \emptyset) \restriction_{[0,\delta_0)} + \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_0) \restriction_{[\delta_0,\delta_1)} + \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_1) \restriction_{[\delta_1,\delta_2)} + \ldots$.
As $\delta_i$ is the first element in $[\delta_i, \delta_{i+1}) \cap Y_1$, $\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i,\delta_{i+1})}$ is determined by $\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_i,\delta_{i+1})} = t(\beta_i)$ and abusing notations we will say

$$(**) \quad \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) \;\simeq\; t(\delta_0) + \sum_{i<\omega} t(\beta_i).$$

Let $i < \omega$ be such that $\beta_i \geq \omega^{m-2}$ and let $j > i$ be the first with $\beta_j \geq \omega^{m-2}$.
First case: $i = j + 1$.
Let $\beta_i = \mathrm{otp}\big([\delta_i, \delta_{i+1}) \cap Y_1\big) = \omega^{m-2} \cdot k_1 + \epsilon_1$ and $\beta_{i+1} = \mathrm{otp}\big([\delta_{i+1}, \delta_{i+2}) \cap Y_1\big) = \omega^{m-2} \cdot k_2 + \epsilon_2$ where $k_1, k_2 \geq 1$ and $\epsilon_1, \epsilon_2 < \omega^{m-2}$.
Define $\gamma :=$ the $\omega^{m-2} \cdot k_1 + \omega^{m-3} + \epsilon_1$'th successor of $\delta_i$ in $Y_1$. So $\delta_{i+1} < \gamma < \delta_{i+2}$ but $\mathrm{otp}\big([\delta_{i+1}, \delta_{i+2}) \cap Y_1\big) = \beta_{i+1}$ hence

$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\gamma,\delta_{i+2})} = \mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_{i+1},\delta_{i+2})} = t(\beta_{i+1})$$

hence

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \gamma) \restriction_{[\gamma,\delta_{i+2})} = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_{i+1}) \restriction_{[\delta_{i+1},\delta_{i+2})} .$$

On the other hand,

$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_i,\gamma)} = t(\omega^{m-2} \cdot k_1) + t(\omega^{m-3}) + t(\epsilon_1)$$

but $m - 3 \geq p$ hence $t(\omega^{m-3}) = t(\omega^{m-2}) = t$ moreover $t + t = t$ and it follows that

$$t(\omega^{m-2} \cdot k_1) + t(\omega^{m-3}) = t(\omega^{m-2}) \cdot k_1 + t(\omega^{m-3}) = t(\omega^{m-2}) \cdot (k_1 + 1) = t(\omega^{m-2}) \cdot (k_1) = t(\omega^{m-2} \cdot k_1)$$

hence

$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_i,\gamma)} = t(\omega^{m-2} \cdot k_1) + +t(\epsilon_1) = \mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_i,\delta_{i+1})} = t(\beta_{i+1})$$

hence

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i,\gamma)} = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i,\delta_{i+1})} .$$

Now all other relevant theories are left unchanged therefore, letting $X' := X \setminus \{\delta_{i+1}\} \cup \{\gamma\}$ we get $X \neq X'$ but

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \mathrm{Th}^n(\omega^m; \bar{P}, Y_1, X')$$

.

General case: $j = i + l$.
Look at $\delta_{i+1}, \delta_{i+2}, \ldots, \delta_{i+l-1}, \delta_{i+l} = \delta_j$. We'll define $\gamma_1, \gamma_2, \ldots, \gamma_l$ with $\delta_{i+k} < \gamma_k < \delta_{i+k+1}$ for $0 < k < l$ and $\gamma_l = \delta_{i+l} = \delta_j$. This will be done by 'shifting' the $\delta_{i+k}$'s by $\omega^{m-3}$ (remember that $\beta_{i+k} < \omega^{m-2}$ for $0 < k < l$).
Assume as before that $\beta_i = \mathrm{otp}\big([\delta_i, \delta_{i+1}) \cap Y_1\big) = \omega^{m-2} \cdot k_1 + \epsilon_1$ where $k_1 \geq 1$ and $\epsilon_1 < \omega^{m-2}$.
Define $\gamma_1 :=$ the $\omega^{m-2} \cdot k_1 + \omega^{m-3} + \epsilon_1$'th successor of $\delta_i$ in $Y_1$, $\gamma_2 :=$ the $\beta_{i+1}$'th successor of $\gamma_1$ in $Y_1$, $\gamma_3 :=$ the $\beta_{i+2}$'th successor of $\gamma_2$ in $Y_1$ and so on, $\gamma_l$ will clearly be equal to $\delta_j$.
As before we have for $1 < k \leq l$, (by preserving the order types)

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \gamma_k) \restriction_{[\gamma_k, \gamma_{k+1})} = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_{i+k}) \restriction_{[\delta_{i+k}, \delta_{i+k+1})} .$$

and (using $t + t = t$)

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i, \gamma_1)} = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i, \delta_{i+1})} .$$

Letting $X' := X \setminus \{\delta_{i+1}, \delta_{i+2}, \ldots, \delta_{j-1}\} \cup \{\gamma_1, \gamma_2, \ldots, \gamma_{l-1}\}$ we get $X \neq X'$ but

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \mathrm{Th}^n(\omega^m; \bar{P}, Y_1, X')$$

.

Since $\mathrm{dp}(\psi) = n - 1$, $X$ is not the unique $\omega$-sequence chosen by $\psi$ from $Y_1$. Therefore, $\psi$ does not uniformize $\varphi$ on $\omega^m$, a contradiction.
[complete, using composition theorem, for $\omega^\omega$]

♡

**Theorem 5.3.** *If $C$ has the uniformization property then $Log(C) < \omega$.*

♡

## 6. Very Tame Trees

**Proposition 6.1.** *If the ordinals $\alpha$ and $\beta$ have the uniformization property then so do $\alpha + \beta$ and $\alpha\beta$.*

**Proof.** $\alpha + \beta$ is similar to $\alpha + \alpha = \alpha \cdot 2$ and we leave it to the reader. We shall prove that $\alpha \cdot \beta$ has the uniformization property.
Let $\varphi(X, Y, \bar{Q})$ be p.u in $\alpha\beta$ with $\mathrm{dp}(\varphi) = n$ and $\lg(\bar{Q}) = l$. Let $\langle t_0, \ldots, t_{a-1} \rangle$ be an enumeration of the the theories in $\mathcal{T}_{n,l+2}$. For $i < a$ and $X, Y \subseteq \alpha\beta$ define $P_i(X, Y, \bar{Q}) \subseteq K := \{\alpha\gamma : \gamma < \beta\}$ by

$$P_i(X, Y, \bar{Q}) := \big\{\alpha\gamma : \mathrm{Th}^n(\alpha\beta; X, Y, \bar{Q}) \restriction_{[\alpha\gamma, \alpha\gamma+\alpha)} = t_i\big\}$$

it follows that, for every $X, Y \subseteq \alpha\beta$, $\bar{P} = \bar{P}(X, Y, \bar{Q}) = \langle P_0(X, Y, \bar{Q}), \ldots, P_{a-1}(X, Y, \bar{Q}) \rangle$ is a partition of $K$ that is definable from $X, Y, \bar{Q}$ and $K$.

$\alpha \cdot \beta = \sum_{\gamma<\beta}[\alpha\gamma, \alpha\gamma+\alpha)$ and by theorem 3.8 there is $m = m(n,l)$ such that $\mathrm{Th}^n(K;\bar{P}(X,Y,\bar{Q}))$ determines $\mathrm{Th}^n(\alpha\beta;X,Y,\bar{Q})$.
Let $\mathcal{R} = \{r_0,\ldots,r_{c-1}\}$ be the set of theories that satisfy, for every $X,Y \subseteq \alpha\beta$:

$$\mathrm{Th}^n(K;\bar{P}(X,Y,\bar{Q})) \in \mathcal{R} \Rightarrow \alpha\beta \models \varphi(X,Y,\bar{Q}).$$

Now let $\langle s_0,\ldots,s_{b-1}\rangle$ be an enumeration of the the theories in $\mathcal{T}_{n+1,l+1}$. For $i < b$ and $Y \subseteq \alpha\beta$ define $R^0_i(Y,\bar{Q}) \subseteq K$ by

$$R^0_i(Y,\bar{Q}) := \{\alpha\gamma : \mathrm{Th}^{n+1}(\alpha\beta;Y,\bar{Q}) \restriction_{[\alpha\gamma,\alpha\gamma+\alpha)} = s_i\}$$

as before, for every $Y \subseteq \alpha\beta$, $\bar{R}^0 = \bar{R}^0(Y,\bar{Q}) = \langle R^0_0(Y,\bar{Q}),\ldots,R^0_{b-1}(Y,\bar{Q})\rangle$ is a partition of $K$ that is definable from $Y,\bar{Q}$ and $K$.
Now let $\bar{R}^1 = \langle R^1_0,\ldots,R^1_{a-1}\rangle$ be any partition of $K$. We will say that $\bar{R}^0(Y,\bar{Q})$ and $\bar{R}^1$ are <u>coherent</u> if
(1) $\alpha\gamma \in (R^0_i \cap R^1_j)$ implies that for every chain $C$, $B \subseteq C$ and $\bar{D} \subseteq C$ of length $l$:
if $\mathrm{Th}^{n+1}(C;B,\bar{D}) = s_i$ then $(\exists A \subseteq C)[\mathrm{Th}^n(C;A,B,\bar{D}) = t_j]$,
(2) $\mathrm{Th}^n(K;\bar{R}^1) \in \mathcal{R}$.
Since $a,b$ and $c$ are finite, there is a formula $\theta_1(\bar{U},\bar{W})$ (with $\lg(\bar{U}) = b$ and $\lg(\bar{W}) = a$) such that for any $\bar{R}^0,\bar{R}^1 \subseteq K$,
$K \models \theta_1(\bar{R}^0,\bar{R}^1)$ iff $\bar{R}^0$ and $\bar{R}^1$ are coherent partitions of $K$.
Moreover, as $K \cong \beta$ and $\beta$ has the uniformization property, there exists $\bar{S} \subseteq K$ and a formula $\theta_2(\bar{U},\bar{W},\bar{S})$ such that for every $\bar{R}^0 \subseteq K$
if $(\exists\bar{W})\theta_1(\bar{R}^0,\bar{W})$ then $(\exists!\bar{W})[\theta_2(\bar{R}^0,\bar{W},\bar{S})\,\&\,\theta_1(\bar{R}^0,\bar{W})]$. Let $\theta(\bar{U},\bar{W},\bar{S}) := \theta_1 \wedge \theta_2$.
Now let $Y \subseteq \alpha\beta$, let $\bar{R}^0 = \bar{R}^0(Y,\bar{Q})$ and suppose that $\bar{R}^0$ and some $\bar{R}^1$ are coherent partitions of $K$. When $\alpha\gamma \in (R^0_i \cap R^1_j)$, we know by the first clause in the definition of coherence that
$(\exists X \subseteq \alpha\beta)[\mathrm{Th}^n(\alpha\beta;X,Y,\bar{Q}) \restriction_{[\alpha\gamma,\alpha\gamma+\alpha)} = t_j]$.
Now as $[\alpha\gamma,\alpha\gamma+\alpha) \cong \alpha$ and $\alpha$ has the uniformization property, there is $\bar{T}_\gamma \subseteq [\alpha\gamma,\alpha\gamma+\alpha)$ and a formula $\psi^\gamma_j(X,Y,\bar{T}_\gamma)$ (of depth $k(n,l)$ that depends only on $n$ and $l$) that uniformizes the formula that says "$\mathrm{Th}^n(\alpha\beta;X,Y,\bar{Q}) \restriction_{[\alpha\gamma,\alpha\gamma+\alpha)} = t_j$".
It follows that when $\psi^\gamma_j(X,Y,\bar{T}_\gamma)$ holds, $X \cap [\alpha\gamma,\alpha\gamma+\alpha)$ is unique.
W.l.o.g all $\bar{T}_\gamma$ have the same length and (by taking prudent disjunctions) $\psi^\gamma_j(X,Y,\bar{T}_\gamma) = \psi_j(X,Y,\bar{T}_\gamma)$ and let $\bar{T} = \cup_{\gamma<\beta}\bar{T}_\gamma$ (the union is disjoint). We are ready to define $U(X,Y,\bar{Q},\bar{T},\bar{S})$ that uniformizes $\varphi(X,Y,\bar{Q})$:
$U(X,Y,\bar{Q},\bar{T},\bar{S})$ says: "for every partition $\bar{R}^0$ of $K$ that is equal to [the definable] $\bar{R}^0(Y,\bar{Q})$ every $\bar{R}^1$ that is a [in fact the only] partition that satisfies $\theta(\bar{R}^0,\bar{R}^1,\bar{S})$, *if* $\alpha\gamma \in R^1_j$ and $D = [\alpha\gamma,\alpha\gamma+\alpha)$ [$\alpha\gamma$ and $\alpha\gamma+\alpha$ are two successive elements of $K$] *then* $D \models \psi_j(X\cap D, Y\cap D, \bar{Q}\cap D, \bar{T}\cap D)$".
Check that $U(X,Y,\bar{Q},\bar{T},\bar{S})$ does the job: clause (1) in the definition of coherence and the $\psi_j$'s guarantee that $X$ is unique, clause (2) guarantees that $U(X,Y,\bar{Q},\bar{T},\bar{S}) \Rightarrow \varphi(X,Y,\bar{Q})$.

♡

**Fact 6.2.** *Every finite chain has the uniformization property.*

♡

**Theorem 6.3.** *$(\omega,<)$ has the uniformization property.*

**Corollary 6.4.** *An ordinal $\alpha$ has the uniformization property iff $\alpha < \omega^\omega$.*

**Definition 6.5.** $(T, \lhd)$ is *very tame* if
1) $T$ is tame
2) $Sup\{\text{Log}(B) : B \subseteq T,\ B \text{ a branch}\} < \omega$

**Lemma 6.6.** *If $(T, \lhd)$ is not very tame then $(T, \lhd)$ does'nt have the uniformization property.*

**Proof.** If $T$ is not tame then by theorem 2.7 it doesn't have even a definable choice function. If $T$ is tame then either there is a a branch $B \subseteq T$ with $\text{Log}(B) = \infty$ or it has branches of unbounded Log. By 3.14(3) and 5.2 and using the definable well ordering of $T$, there is a formula $\varphi(X, Y, Z)$ that can't be uniformized.

♡

**Theorem 6.7.** *$(T, \lhd)$ has the uniformization property iff $(T, \lhd)$ is very tame.*

**Proof.** Assume $T$ is $(l^*, n^*, k^*)$ very tame and let $\varphi(X, Y, \bar{Q})$ be p.u in $T$ with $\text{dp}(\varphi) = n$ and $\lg(\bar{Q}) = l$.
As $T$ is $(n^*, k^*)$ tame it can be well ordered $T$ in the following way [the full construction is given in theorem A.2 in the appendix]: partition $T$ into a disjoint union of sub-branches, indexed by the nodes of a well founded tree $\Gamma$ and reduce the problem of a well ordering of $T$ to a problem of a well ordering of $\Gamma$. At the first step we pick a branch of $T$, call it $A_{\langle\rangle}$ and represent $T$ as $A_{\langle\rangle} \cup \bigoplus_{\eta \in \langle\rangle^+} T_\eta$ (where for $\tau \in \Gamma$, $\tau^+$ is the set $\{\nu : \nu$ an immediate successor of $\tau$ in $\Gamma\}$ ). At the second step we pick a branch $A_\eta$ in each $T_\eta$ and represent $T_\eta$ as $A_\eta \cup \bigoplus_{\nu \in \eta^+} T_\nu$. By tameness we finish after $\omega$ steps getting $T = \cup_{\eta \in \Gamma} A_\eta$ and the well ordering of $T$ is induced by the lexicographical well ordering of $\Gamma$ and the well ordering of each $A_\eta$ (which is scattered of Hdeg$\leq k^*$). We can choose a sequence of parameters $\bar{K}_0$ (with length depending on $n^*$ and $k^*$ only) and a set of representatives $K = \{u_\eta \in A_\eta : \eta \in \Gamma\}$ and using $\bar{K}_0$ we can define a binary relation $<^*$ on $K$ where $u_\eta <^* u_\nu$ will hold exactly when $\eta \lhd \nu$ in $\Gamma$, thus we can define the structure of $\Gamma$ in $T$. The sequence $\bar{K}_0$ will also enable us to define $T_\eta$ and $A_\eta$ from the representative $u_\eta$ and define a well ordering of each $A_\eta$.
Consequently, the order between two nodes $x, y \in T$ will be determined by the well order of the $A_\eta$'s (if they belong to the same $A_\eta$) or the well ordering of $\Gamma$ (if they belong to different $A_\eta$'s). The well ordering of the sets $\eta^+$ for $\eta \in \Gamma$ (hence the lexicographical well ordering of the well founded tree $\Gamma$) will be again defined using $\bar{K}_0$.
What we'll do here in order to uniformize $\varphi(X, Y, \bar{Q})$ is the following: given $Y \subseteq T$ we will use the decomposition $T = \cup_{\eta \in \Gamma} A_\eta$ and the fact that each $A_\eta$ is a scattered chain with $\text{Log}(A_\eta) < l^*$, (hence satisfies the uniformization property), to define a unique $X_\eta \subseteq A_\eta$. This will be done in such a way that when we glue the parts letting $X^* = \cup_{\eta \in \Gamma} X_\eta$ we will still get $T \models \varphi(X, Y, \bar{Q})$.
We will use the set of representatives $K$ and the fact that $A_\eta$ and $T_\eta$ are defined from $u_\eta$ but we won't always mention $\bar{K}_0$. We will also rely on the fact that $\Gamma$ is well founded (in fact, we only need to know that $\Gamma$ does not have a branch of order type $\geq \omega + 1$).
So let $Y \subseteq T$ and we want to define some $X^* = X^*(Y, \bar{Q}) \subseteq T$. The proof will go as follows: for each $\eta \in \Gamma$ we will define partitions $\bar{P}^1(Y, \bar{Q})_\eta$ and $\bar{P}^2(Y, \bar{Q})_\eta$ of $K_{\eta^+} := \{u_\nu : \nu \in \eta^+\}$ then, using the composition theorem 3.14 and similarly to the proof of proposition 6.1, we will define a notion of coherence and let $\bar{R}^1(Y, \bar{Q})_\eta$ and $\bar{R}^2(Y, \bar{Q})_\eta$ be a pair that is coherent with $\bar{P}^1(Y, \bar{Q})_\eta$ and $\bar{P}^2(Y, \bar{Q})_\eta$. The union $\bar{R}^1(Y, \bar{Q}) = \cup_{\eta \in \Gamma} \bar{R}^1(Y, \bar{Q})_\eta$ is a partition of $K$ and $\text{Th}^n(A_\eta; X_\eta, Y \cap A_\eta, \bar{Q} \cap A_\eta)$ will be determined by the unique member of $\bar{R}^1(Y, \bar{Q})$ to which $u_\eta$ belongs. Moreover, we will be able to choose $X_\eta$ uniquely and by coherence $X^* = \cup_{\eta \in \Gamma} X_\eta$ will satisfy $\varphi(X, Y, \bar{Q})$.

→ [3.12.]

To get started let $T = A_{\langle\rangle} \cup \bigoplus_{\eta\in\langle\rangle^+} T_\eta$. Now as in definition 3.12 $K_{\langle\rangle^+}$ has a natural structure of a chain with Log($K_{\langle\rangle^+}$)=Log($A_{\langle\rangle}$) $< l^*$ and by theorem 3.14(2) there is some $m = m(n,l)$ such that when $X \subseteq T$ is given, from $\mathrm{Th}^m(A_{\langle\rangle}; X, Y, \bar{Q})$ and $\langle \mathrm{Th}^m(T_\eta; X, Y, \bar{Q}) : \eta \in \langle\rangle^+\rangle$ we can compute $\mathrm{Th}^n(T; X, Y, \bar{Q})$.

Let $\langle s_0, \ldots, s_{b-1}\rangle$ be an enumeration of the the theories in $\mathcal{T}_{n+1,l+1}$.

Define $\bar{P}^1(Y,\bar{Q})_{\langle\rangle} = \langle P^1_0(Y,\bar{Q})_{\langle\rangle}, \ldots, P^1_{b-1}(Y,\bar{Q})_{\langle\rangle}\rangle$ a partition of $K_{\langle\rangle^+}$ by

$$\eta \in \mathcal{P}^1_i(Y,\bar{Q})_{\langle\rangle} \iff \mathrm{Th}^{n+1}(T_\eta; Y, \bar{Q}) = s_i$$

By the previous remarks $\bar{P}^1(Y,\bar{Q})_{\langle\rangle}$ is definable from $u_{\langle\rangle}, K, Y, \bar{Q}$ (and $\bar{K}_0$).

Define $\bar{P}^2(Y,\bar{Q})_{\langle\rangle} = \langle P^2_0(Y,\bar{Q})_{\langle\rangle}, \ldots, P^2_{b-1}(Y,\bar{Q})_{\langle\rangle}\rangle$ a partition of $K_{\langle\rangle^+}$ by

$$\eta \in \mathcal{P}^1_i(Y,\bar{Q})_{\langle\rangle} \iff \mathrm{Th}^{n+1}(A_\eta; Y, \bar{Q}) = s_i$$

Again, $\bar{P}^2(Y,\bar{Q})_{\langle\rangle}$ is definable from $u_{\langle\rangle}, K, Y, \bar{Q}$ and $\bar{K}_0$.

Let $\langle t_0, \ldots, t_{a-1}\rangle$ be an enumeration of the the theories in $\mathcal{T}_{n,l+2}$.

A partition of $K_{\langle\rangle^+}$, $\bar{R}^1 = \langle R^1_0, \ldots, R^1_{a-1}\rangle$ is coherent with $\bar{P}^1(Y,\bar{Q})_{\langle\rangle}$ if $P^1_i(Y,\bar{Q})_{\langle\rangle} \cap R^1_j \neq \emptyset$ implies "for every tree $S$ and $B, \bar{C} \subseteq S$ with $\lg(\bar{C}) = l$, if $\mathrm{Th}^{n+1}(S; B, \bar{C}) = s_i$ then there is $A \subseteq S$ such that $\mathrm{Th}^n(S; A, B, \bar{C}) = t_j$".

Similarly a partition of $K_{\langle\rangle^+}$, $\bar{R}^2 = \langle R^2_0, \ldots, R^2_{a-1}\rangle$ is coherent with $\bar{P}^2(Y,\bar{Q})_{\langle\rangle}$ if $P^2_i(Y,\bar{Q})_{\langle\rangle} \cap R^2_j \neq \emptyset$ implies

"for every chain $S$ and $B, \bar{C} \subseteq S$ with $\lg(\bar{C}) = l$, if $\mathrm{Th}^{n+1}(S; B, \bar{C}) = s_i$ then there is $A \subseteq S$ such that $\mathrm{Th}^n(S; A, B, \bar{C}) = t_j$".

Finaly, a pair of partitions of $K_{\langle\rangle^+}$, $\langle \bar{R}^1, \bar{R}^2\rangle$ is <u>$t^*$-coherent</u> with the pair $\langle \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle}\rangle$ if

(1) $\bar{R}^1$ is coherent with $\bar{P}^1(Y,\bar{Q})_{\langle\rangle}$,

(2) $\bar{R}^2$ is coherent with $\bar{P}^2(Y,\bar{Q})_{\langle\rangle}$, and

(3) For every $X \subseteq T$, if $\mathrm{Th}^n(A_{\langle\rangle}; X, Y, \bar{Q}) = t^*$ and if for every $\eta \in \langle\rangle^+$ $\left[\mathrm{Th}^n(T_\eta; X, Y, \bar{Q}) = t_i \iff u_\eta \in R^1_i\right]$, then $T \models \varphi(X, Y, \bar{Q})$.

As $T \models (\exists X)\varphi(X, Y, \bar{Q})$ there are $t^*$ (that will be fixed from now on), $\bar{R}^1$ and $\bar{R}^2$ such that $\langle \bar{R}^1, \bar{R}^2\rangle$ is $t^*$-coherent with the pair $\langle \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle}\rangle$.

Moreover, "$\langle \bar{R}^1, \bar{R}^2\rangle$ is $t^*$-coherent with the pair $\langle \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle}\rangle$"
is determined by $\mathrm{Th}^k(K_{\langle\rangle^+}; \bar{R}^1, \bar{R}^2, \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle})$ where $k$ depends only on $n$ and $l$.

The first two clauses are clear (since $a$ and $b$ are finite) and for the third clause use theorem 3.14(2).

So the statement is expressed by a p.u formula $\psi^1(\bar{R}^1, \bar{R}^2, \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle})$ of depth $k$.

As by a previous remark $\mathrm{Log}(K_{\langle\rangle^+}) < l^*$ there is $\bar{S}_{\langle\rangle} \subseteq K_{\langle\rangle^+}$ and a formula $\psi_{\langle\rangle}(\bar{U}_1, \bar{U}_2, \bar{W}_1, \bar{W}_2, \bar{S}_{\langle\rangle})$ that uniformizes $\psi^1$.

To conclude the first step use $\mathrm{Log}(A_{\langle\rangle}) < l^*$ to define, by a formula $\theta_{\langle\rangle}(X, Y \cap A_{\langle\rangle}, \bar{Q} \cap A_{\langle\rangle}, \bar{O}_{\langle\rangle})$ and a sequence of parameters $\bar{O}_{\langle\rangle} \subseteq A_{\langle\rangle}$, a unique $X_{\langle\rangle} \subseteq A_{\langle\rangle}$ that will satisfy $\mathrm{Th}^n(A_\eta; X_{\langle\rangle}, Y, \bar{Q}) = t^*$.

The result of the first step is the following:

a) we have defined $X_{\langle\rangle} \subseteq A_{\langle\rangle}$ using $\bar{O}_{\langle\rangle} \subseteq A_{\langle\rangle}$ and $\theta_{\langle\rangle}$. $X_{\langle\rangle}$ is the intesection of the eventual $X^*$ with $A_{\langle\rangle}$.

b) we have chosen $\bar{R}^1_{\langle\rangle^+}, \bar{R}^2_{\langle\rangle^+} \subseteq K_{\langle\rangle^+}$ using $\psi$ and $\bar{S}_{\langle\rangle}$.

c) $\bar{R}^1_{\langle\rangle^+}$ and $\bar{R}^2_{\langle\rangle^+}$ tell us what are (for $\eta \in \langle\rangle^+$) the theories $\mathrm{Th}^n(T_\eta; X^*, Y, \bar{Q})$ and $\mathrm{Th}^n(A_\eta; X_\eta, Y, \bar{Q})$ respectively: if $u_\eta \in R^1_i$ then the eventual $X^* \cap T_\eta \subseteq T_\eta$ will satisfy $\mathrm{Th}^n(T_\eta; X^* \cap T_\eta, Y, \bar{Q}) = t_i$ and if $u_\eta \in R^2_j$ then then the soon to be defined $X_\eta \subseteq A_\eta$ will satisfy $\mathrm{Th}^n(A_\eta; X_\eta, Y, \bar{Q}) = t_j$.

We will proceed by induction on the level of $\eta$ in $\Gamma$ (remember, all the levels are $< \omega$) to define $\bar{S}_\eta, \bar{O}_\eta \subseteq A_\eta$ and $\bar{R}^1_{\eta^+}, \bar{R}^2_{\eta^+} \subseteq K_{\eta^+}$ and $X_\eta \subseteq T_\eta$.
The induction step:
We are at $\nu \in \Gamma$ where $\nu \in \eta^+$ and we want to define $\bar{S}_\nu, \bar{O}_\nu \subseteq A_\nu$, $\bar{R}^1_{\nu^+}, \bar{R}^2_{\nu^+} \subseteq K_{\nu^+}$ and $X_\nu \subseteq T_\nu$.
Now as $\bar{R}^1_{\eta^+}$ and $\bar{R}^2_{\eta^+}$ are defined, $u_\nu$ belongs to one member of $\bar{R}^1_{\eta^+}$ say the $i_1$'th and to one member of $\bar{R}^2_{\eta^+}$ say the $i_2$'th. This implies that there is some $X'_\nu \subseteq T_\nu$ such that $\mathrm{Th}^n(T_\nu; X'_\nu, Y, \bar{Q}) = t_{i_1}$ and $\mathrm{Th}^n(A_\nu; X'_\nu \cap A_\nu, Y, \bar{Q}) = t_{i_2}$.
Let $\bar{P}^1(Y,\bar{Q})_\nu$ and $\bar{P}^2(Y,\bar{Q})_\nu$ be partitions of $K_{\nu^+}$ that are defined as in the first step by saying, for $\tau \in \nu^+$, what are $\mathrm{Th}^{n+1}(T_\tau; Y, \bar{Q})$ and $\mathrm{Th}^{n+1}(A_\tau; Y, \bar{Q})$. $\langle \bar{R}^1_{\nu^+}, \bar{R}^2_{\nu^+} \rangle \subseteq K_{\nu^+}$ will be a pair that is <u>$t_{i_1}, t_{i_2}$-coherent</u> with $\langle \bar{P}^1(Y,\bar{Q})_\nu, \bar{P}^2(Y,\bar{Q})_\nu \rangle$ that is:
(1) $\bar{R}^1_{\nu^+}$ is coherent with $\bar{P}^1(Y,\bar{Q})_\nu$,
(2) $\bar{R}^2_{\nu^+}$ is coherent with $\bar{P}^2(Y,\bar{Q})_\nu$, and
(3) For every $X \subseteq T_\nu$ if $\mathrm{Th}^n(A_\nu; X, Y, \bar{Q}) = t_{i_2}$ and for every $\tau \in \nu^+$ $\left[\mathrm{Th}^n(T_\tau; X, Y, \bar{Q}) = t_i \iff u_\tau \in \text{the } i\text{'th member of } R^1_{\nu^+}\right]$, then $\mathrm{Th}^n(T_\nu; X, Y, \bar{Q}) = t_{i_1}$.
Using $\mathrm{Log}(K_{\nu^+}) < l^*$ choose $\bar{S}_{\nu^+} \subseteq K_{\nu^+}$ and $\psi_{i_1,i_2}(\bar{R}^1, \bar{R}^2, \bar{P}^1(Y,\bar{Q})_{\nu^+}, \bar{P}^2(Y,\bar{Q})_{\nu^+}, \bar{S}_{\nu^+})$ that uniformizes the formula that says "$\langle \bar{R}^1, \bar{R}^2 \rangle$ is $t_{i_1}, t_{i_2}$-coherent with $\langle \bar{P}^1(Y,\bar{Q})_\nu, \bar{P}^2(Y,\bar{Q})_\nu \rangle$". We may assume that $\psi_{i_1,i_2}$ depends only on $i_1$ and $i_2$ and that $\lg(\bar{S}_{\nu^+})$ is constant.
Use $\mathrm{Log}(A_\nu) < l^*$ to define, by a formula $\theta_{i_2}(X, Y \cap A_\nu, \bar{Q} \cap A_\nu, \bar{O}_\nu)$ and a sequence of parameters $\bar{O}_\nu \subseteq A_\nu$, a unique $X_\nu \subseteq A_\nu$ that will satisfy $\mathrm{Th}^n(A_\nu; X_\nu, Y, \bar{Q}) = t_{i_2}$. Again, we may assume that $\theta_{i_2}$ depends only on $i_2$ and that $\lg(\bar{O}_\nu)$ is constant.
So $\bar{S}_\nu$, $\bar{O}_\nu$, $\bar{R}^1_{\nu^+}$, $\bar{R}^2_{\nu^+}$ and $X_\nu$ are defined and we have concluded the inductive step. (Note that nothing will realy go wrong if $\nu$ doesn't have any successors in $\Gamma$).

Let $\bar{O} = \cup_{\eta\in\Gamma} \bar{O}_\eta$, $\bar{S} = \cup_{\eta\in\Gamma} \bar{S}_\eta$. The uniformizing formula $U(X, Y, \bar{Q}, \bar{O}, \bar{S}, K, \bar{K}_0)$ says:
"$X \cap A_{\langle\rangle}$ is defined as in the first step, and
for every pair of partitions $\langle \bar{P}^1, \bar{P}^2 \rangle$ of $K$ that agrees on each $K_{\eta^+}$ with [the definable]
$\langle \bar{P}^1_{\eta^+}(Y,\bar{Q}), \bar{P}^2_{\eta^+}(Y,\bar{Q}) \rangle$, (and agrees with $\langle \bar{P}^1_{\langle\rangle}, \bar{P}^2_{\langle\rangle} \rangle$ on $K_{\langle\rangle^+}$), and
for every $\langle \bar{R}^1, \bar{R}^2 \rangle$ that is a [in fact the only] pair of partitions that satisfies for every $u_\eta \in K$: if $u_\eta \in P^1_{i_1} \cap P^2_{i_2}$ then $\psi_{i_1,i_2}(\bar{R}^1 \cap K_{\eta^+}, \bar{R}^2 \cap K_{\eta^+}, \bar{P}^1 \cap K_{\eta^+}, \bar{P}^2 \cap K_{\eta^+}, \bar{S} \cap K_{\eta^+})$ holds, (and agrees with $\langle \bar{R}^1_{\langle\rangle}, \bar{R}^2_{\langle\rangle} \rangle$ on $K_{\langle\rangle^+}$),
for every $u_\eta \in K$ if $u_\eta \in R^2_i$ then $\theta_i(X \cap A_\eta, Y \cap A_\eta, \bar{Q} \cap A_\eta, \bar{O} \cap A_\eta)$ holds."

$U(X, Y, \bar{Q}, \bar{O}, \bar{S}, K, \bar{K}_0)$ does the job because it defines $X \cap A_\eta$ uniquely on each $A_\eta$ and because, (by the conditions of coherence) the union of the parts, $X$, satisfies $\varphi(X, Y, \bar{Q})$. Note also that $U$ does not depend on $Y$.

♡

## 7. Hopelessness of General Partial Orders

**Theorem 7.1.** *Every partial order $P$ can be embedded in a partial order $Q$ in which $P$ is first-order-definably well orderable.*

**Proof.**

♡

## *Appendix*

**Lemma A.1.** *Let $C$ be a scattered chain with* $\mathrm{Hdeg}(C) = n$. *Then there are* $\bar{P} \subseteq C$, $\lg(\bar{P}) = n-1$, *and a formula (depending on $n$ only)* $\varphi_n(x, y, \bar{P})$ *that defines a well ordering of $C$.*

**Proof.** By induction on $n = \mathrm{Hdeg}(C)$:
$n \leq 1$: $\mathrm{Hdeg}(C) \leq 1$ implies $(C, <_C)$ is well ordered or inversely well ordered. A well ordering of $C$ is easily definable from $<_C$.
$\mathrm{Hdeg}(C) = n + 1$: Suppose $C = \sum_{i \in I} C_i$ and each $C_i$ is of Hausdorff degree $n$. By the induction hypothesis there are a formula $\varphi_n(x, y, \bar{Z})$ and a sequence $\langle \bar{P}^i : i \in I \rangle$ with $\bar{P}^i \subseteq C_i$, $\bar{P}^i = \langle P^i_1, \ldots, P^i_{n-1} \rangle$ such that $\varphi_n(x, y, \bar{P}^i)$ defines a well ordering of $C_i$.
Let for $0 < k < n$, $P_k := \cup_{i \in I} P^i_k$ (we may assume that the union is disjoint) and $P_n := \cup\{C_i : i \text{ even}\}$.
We will define an equivalence relation $\sim$ by $x \sim y$ iff $\bigwedge_i (x \in C_i \Leftrightarrow y \in C_i)$.
$\sim$ and $[x]$, (the equivalence class of an element $x$), are easily definable from $P_n$ and $<_C$. We can also decide from $P_n$ if $I$ is well or inversely well ordered (by looking at subsets of $C$ consisted of nonequivalent elements) and define $<'$ to be $<$ if $I$ is well ordered and the inverse of $<$ if not. $\varphi_{n+1}(x, y, P_1, \ldots, P_n)$ will be defined by:

$$\varphi_{n+1}(x, y, \bar{P}) \Leftrightarrow \left[x \not\sim y \ \& \ x <' y\right] \vee \left[x \sim y \ \& \ \varphi_n(x, y, P_1 \cap [x], \ldots, P_{n-1} \cap [x])\right]$$

$\varphi_{n+1}(x, y, \bar{P})$ well orders $C$.

♡

**Theorem A.2.** *Let $T$ be a tame tree. If ${}^{\omega>}2$ is not embeddable in $T$ then there are $\bar{Q} \subseteq T$ and a monadic formula $\varphi(x, y, \bar{Q})$ that defines a well ordering of $T$.*

**Proof.** Assume $T$ is $(n^*, k^*)$ tame, recall definitions 4.1 and 4.2 and remember that for every $x \in T$, $rk(x)$ is well defined (i.e. $< \infty$). We will partition $T$ into a disjoint union of sub-branches, indexed by the nodes of a well founded tree $\Gamma$ and reduce the problem of a well ordering of $T$ to a problem of a well ordering of $\Gamma$.

Step 1. Define by induction on $\alpha$ a set $\Gamma_\alpha \subseteq {}^\alpha Ord$ (this is a our set of indices), for every $\eta \in \Gamma_\alpha$ define a tree $T_\eta \subseteq T$ and a branch $A_\eta \subseteq T_\eta$.
$\alpha = 0$ : $\Gamma_0$ is $\{\langle\rangle\}$, $T_{\langle\rangle}$ is $T$ and $A_{\langle\rangle}$ is a branch (i.e. a maximal linearily ordered subset) of $T$.
$\alpha = 1$ : Look at $(T \setminus A_{\langle\rangle})/\sim^1_{A_{\langle\rangle}}$, it's a disjoint union of trees and name it $\langle T_{\langle i \rangle} : i < i^* \rangle$, let $\Gamma_1 := \{\langle i \rangle : i < i^*\}$ and for every $\langle i \rangle \in \Gamma_1$ let $A_{\langle i \rangle}$ be a branch of $T_{\langle i \rangle}$.
$\alpha = \beta + 1$ : For $\eta \in \Gamma_\beta$ denote $(T_\eta \setminus A_\eta)/\sim^1_{A_\eta}$ by $\{T_{{}^\wedge \eta, i} : i < i_\eta\}$, let $\Gamma_\alpha = \{{}^\wedge \eta, i : \eta \in \Gamma_\beta,\ i < i_\eta\}$ and choose $A_{{}^\wedge \eta, i}$ to be a branch of $A_{{}^\wedge \eta, i}$.
$\alpha$ limit: Let $\Gamma_\alpha = \{\eta \in {}^\alpha Ord : \wedge_{\beta < \alpha} \eta \restriction_\beta \in \Gamma_\beta,\ \wedge_{\beta < \alpha} T_{\eta \restriction_\beta} \neq \emptyset\}$, let for $\eta \in \Gamma_\alpha$ $T_\eta = \cap_{\beta < \alpha} T_{\eta \restriction_\beta}$ and $A_\eta$ a branch of $T_\eta$. ($T_\eta$ may be empty).
Now, at some stage $\alpha \leq |T|^+$ we have $\Gamma_\alpha = \emptyset$ and let $\Gamma = \cup_{\beta < \alpha} \Gamma_\beta$. Clearly $\{A_\eta : \eta \in \Gamma\}$ is a partition of $T$ into disjoint sub-branches.
Notation: having two trees $T$ and $\Gamma$, to avoid confusion, we use $x, y, s, t$ for nodes of $T$ and $\eta, \nu, \sigma$ for nodes of $\Gamma$.

Step 2. We want to show that $\Gamma_\omega = \emptyset$ hence $\Gamma$ is a well founded tree. Note that we made no restrictions on the choice of the $A_\eta$'s and we add one now in order to make the above statement true. Let ${}^\wedge \eta, i \in \Gamma$ define $A_{\eta, i}$ to be the sub-branch $\{t \in A_\eta : (\forall s \in A_{{}^\wedge \eta, i})[rk(t) \leq rk(s)]\}$ and $\gamma_{\eta, i}$ to be $rk(t)$ for some $t \in A_{\eta, i}$. By 5.5(1) and the inexistence of a stricly decreasing sequence of ordinals, $A_{\eta, i} \neq \emptyset$ and $\gamma_{\eta, i}$ is well defined. Note also that $s \in A_{{}^\wedge \eta, i} \Rightarrow rk(s) \leq \gamma_{\eta, i}$.

Proviso: For every $\eta \in \Gamma$ and $i < i_\eta$ the sub-branch $A_{\wedge\eta,i}$ contains every $s \in T_{\wedge\eta,i}$ with $rk(s) = \gamma_{\eta,i}$. Following this we claim: "$\Gamma$ does not contain an infinite, stricly increasing sequence". Otherwise let $\{\eta_i\}_{i<\omega}$ be one, and choose $s_n \in A_{\eta_n,\eta_{n+1}(n)}$ (so $s_n \in A_{\eta_n}$). Clearly $rk(s_n) \geq rk(s_{n+1})$ and by the proviso we get

$$rk(s_n) = rk(s_{n+1}) \Rightarrow rk(s_{n+1}) > rk(s_{n+2})$$

therefore $\{rk(s_n)\}_{n<\omega}$ contains an infinite, stricly decreasing sequence of ordinals which is absurd.

Step 3. Next we want to make "$x$ and $y$ belong to the same $A_\eta$" definable.

For each $\eta \in \Gamma$ choose $s_\eta \in A_\eta$, and let $Q \subseteq T$ be the set of representatives. Let $h{:}T \to \{d_0, \ldots, d_{n^*-1}\}$ be a colouring that satisfies: $h \restriction_{A_{\langle\rangle}} = d_0$ and for every $^\wedge\eta, i \in \Gamma$, $h \restriction_{A_{\wedge\eta,i}}$ is constant and, when $j < i$ and $s_{\wedge\eta,j} \sim^0_{A_\eta} s_{\wedge\eta,j}$ we have $h \restriction_{A_{\wedge\eta,i}} \neq h \restriction_{A_{\wedge\eta,j}}$. This can be done as $T$ is $(n^*, d^*)$ tame.

Using the parameters $D_0, \ldots, D_{n^*-1}$ ($x \in D_i$ iff $h(x) = d_i$), we can define $\vee_\eta x, y \in A_\eta$ by "$x, y$ are comparable and the sub-branch $[x, y]$ (or $[y, x]$) has a constant colour".

Step 4. As every $A_\eta$ has Hausdorff degree at most $k^*$, we can define a well ordering of it using parameters $P^\eta_1, \ldots, P^\eta_{k^*}$ and by taking $\bar{P}$ to be the (disjoint) union of the $\bar{P}^\eta$'s we can define a partial ordering on $T$ which well orders every $A_\eta$.

By our construction $\eta \lhd \nu$ if and only if there is an element in $A_\nu$ that 'breaks' $A_\eta$ i.e. is above a proper initial segment of $A_\eta$. (Caution, if $T$ does not have a root this may not be the case for $\langle\rangle$ and a $< n^*$ number of $\langle i\rangle$'s and we may need parameters for expressing that). Therefore, as by step 3 "being in the same $A_\eta$" is definable, we can define a partial order on the sub-branches $A_\eta$ (or the representatives $s_\eta$) by $\eta \lhd \nu \Rightarrow A_\eta \leq A_\nu$.

Next, note that "$\nu$ is an immediate successor of $\eta$ in $\Gamma$" is definable as a relation between $s_\nu$ and $s_\eta$ hence the set $A^+_\eta := A_\eta \cup \{s_{\wedge\eta,i}\}$ is definable from $s_\eta$. Now the order on $A_\eta$ induces an order on $\{s_{\wedge\eta,i}/\sim^0_{A_\eta}\}$ which is can be embedded in the complition of $A_\eta$ hence has Hdeg$\leq k^*$. Using additional parameters $Q^\eta_1, \ldots, Q^\eta_{k^*}$, we have a definable well ordering on $\{s_{\wedge\eta,i}/\sim^0_{A_\eta}\}$. As for the ordering on each $\sim^1_{A_\eta}$ equivalence class (finite with $\leq n^*$ elements), define it by their colours (i.e. the element with the smaller colour is the smaller according to the order).

Using $\bar{D}$, $\bar{P}$, $Q$ and $\bar{Q} = \cup_\eta \bar{Q}^\eta$ we can define a partial ordering which well orders each $A^+_\eta$ in such a way that every $x \in A_\eta$ is smaller then every $s_{\wedge\eta,i}$.

Summing up we can define (using the above parameters) a partial order on subsets of $T$ that well orders each $A_\eta$, orders sub-branches $A_\eta$, $A_\nu$ when the indices are comparable in $\Gamma$ and well orders all the "immediate successors" sub-branches of a sub-branch $A_\eta$.

Step 5. The well ordering of $T$ will be defined by $x < y \iff$

a) $x$ and $y$ belong to the same $A_\eta$ and $x < y$ by the well order on $A_\eta$; or

b) $x \in A_\eta$, $y \in A_\nu$ and $\eta \lhd \nu$; or

c) $x \in A_\eta$, $y \in A_\nu$, $\sigma = \eta \wedge \nu$ in $\Gamma$ (defined as a relation between sub-branches), $^\wedge\sigma, i \lhd \eta$, $^\wedge\sigma, j \lhd \nu$ and $s_{\wedge\sigma,i} < s_{\wedge\sigma,j}$ in the order of $A^+_\sigma$.

Note, that $<$ is a linear order on $T$ and every $A_\eta$ is a convex and well ordered sub-chain. Moreover $<$ is a linear order on $\Gamma$ and the order on the $s_\eta$'s is isomorphic to a lexicographic order on $\Gamma$.

Why is the above (which is clearly definable with our parameters) a well order? Because of the above note and because a lexicographic ordering of a well founded tree is a well order, provided that immediate successors are well ordered. In detail, assume $X = \{x_i\}_{i<\omega}$ is a stricly decreasing sequence of elements of $T$. Let $\eta_i$ be the unique node in $\Gamma$ such that $x_i \in A_{\eta_i}$ and by the above note w.l.o.g $i \neq j \Rightarrow \eta_i \neq \eta_j$. By the well foundedness of $\Gamma$ and clause (b) we may also assume w.l.o.g

that the $\eta_i$'s form an anti-chain in $\Gamma$. Look at $\nu_i := \eta_1 \wedge \eta_i$ which is constant for infinitely many $i$'s and w.l.o.g equals to $\nu$ for every $i$. Ask:
$(*)$ is there is an infinite $B \subseteq \omega$ such that $i, j \in B \Rightarrow x_i \sim^0_{A_\nu} x_j$ ?
If this occurs we have $\nu_1 \neq \nu$ with $\nu \lhd \nu_1$ such that for some infinite $B' \subseteq B \subseteq \omega$ we have $i \in B' \Rightarrow \nu_1 \lhd \eta_i$. (use the fact that $\sim^1_{A_\nu}$ is finite). W.l.o.g $B' = \omega$ and we may ask if $(*)$ holds for $\nu_1$. Eventually, since $\Gamma$ does not have an infinite branch, we will have a negative answer to $(*)$. We can conclude that w.l.o.g there is $\nu \in \Gamma$ such that $i \neq j \Rightarrow x_i \not\sim^0_{A_\nu} x_j$ i.e. the $x_i$'s "break" $A_\nu$ in "different places".
Define now $\nu_i$ to be the unique immediate successor of $\nu$ such that $\nu_i \lhd \eta_i$. The set $S = \{s_{\nu_i}\}_{i<\omega} \subseteq A^+_\nu$ is well ordered by the well ordering on $A^+_\nu$ and by clause (c) in the definition of $<$, $x_i > x_j \iff \nu_i > \nu_j$ so $S$ is an infinite stricly decreasing subset of $A^+_\nu$ – a contradiction.
This finishes the proof that there is a definable well order of $T$.

♡