On Distinguishing Quotients of Symmetric Groups

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Abstract

A study is carried out of the elementary theory of quotients of symmetric groups in a similar spirit to [10]. Apart from the trivial and alternating subgroups, the normal subgroups of the full symmetric group $S(\mu)$ on an infinite cardinal μ are all of the form $S_\kappa(\mu)$ = the subgroup consisting of elements whose support has cardinality $< \kappa$, for some $\kappa \leq \mu^+$. A many-sorted structure $\mathcal{M}_{\kappa\lambda\mu}$ is defined which, it is shown, encapsulates the first order properties of the group $S_\lambda(\mu)/S_\kappa(\mu)$. Specifically, these two structures are (uniformly) bi-interpretable, where the interpretation of $\mathcal{M}_{\kappa\lambda\mu}$ in $S_\lambda(\mu)/S_\kappa(\mu)$ is in the usual sense, but in the other direction is in a weaker sense, which is nevertheless sufficient to transfer elementary equivalence. By considering separately the cases $cf(\kappa) > 2^{\aleph_0}, cf(\kappa) \leq 2^{\aleph_0} < \kappa, \aleph_0 < \kappa < 2^{\aleph_0}$, and $\kappa = \aleph_0$, we make a further analysis of the first order theory of $S_\lambda(\mu)/S_\kappa(\mu)$, introducing many-sorted second order structures $\mathcal{N}^2_{\kappa\lambda\mu}$, all of whose sorts have cardinality at most $2^{\aleph_0}$, and in terms of which we can completely characterize the elementary theory of the groups $S_\lambda(\mu)/S_\kappa(\mu)$.

1 Introduction

In [5], [6], [10], and [11] a study was made of the elementary theory of infinite symmetric groups, and a number of natural questions arising were answered. In this paper we examine the quotients of normal subgroups of infinite symmetric groups in the light of similar questions. Now the normal subgroups of infinite symmetric groups are easily describable in terms of the cardinalities of support sets. More exactly, the *support* of $g \in \mathrm{Sym}(\Omega)$ is the set of elements of Ω moved by g. The non-trivial normal subgroups of $\mathrm{Sym}(\mu)$ where μ is an infinite cardinal are then of the form $S_\kappa(\mu) = \{g \in \mathrm{Sym}(\mu) : |supp\ g| < \kappa\}$ for some cardinal κ,

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and the alternating group $A(\mu)$, (see [9] for example), and the objects of study here are the factors $S_\lambda(\mu)/S_\kappa(\mu)$ for $\kappa < \lambda$.

The problem of which of these groups are isomorphic is mentioned in [9], but we concentrate exclusively here on the situation with regard to elementary equivalence. We shall find that many of the ideas from [10] carry through, though with more complicated proofs.

Interpretability results about the groups $S_\lambda(\mu)/S_\kappa(\mu)$ also provide information about their outer automorphisms, as was explored for instance for $\mathrm{Sym}(\omega)/S_\omega(\omega)$ in [1], and in a related context in [3]. A survey of this aspect is given in [12]. The result proved in [1] is that the outer automorphism group of $\mathrm{Sym}(\omega)/S_\omega(\omega)$ is infinite cyclic, with a typical outer automorphism being induced by the map $n \mapsto n+1$. The method used there incorporates a second order interpretation of the relevant ring of sets in the quotient group. One of things we are able to show here is that this can actually be done in a first order fashion. The existence of this interpretation is also applied in [12] to show that the outer automorphism group of $\mathrm{Sym}(\mu)/S_\omega(\mu)$ is infinite cyclic for any μ, extending the result from [1]. What the outer automorphism group of $S_\lambda(\mu)/S_\kappa(\mu)$ is in general is still open—it seems conceivable that it is trivial whenever $\kappa > \aleph_0$.

The first order interpretation of the ring of sets in the quotient group was carried out originally by Rubin in [7] (see also [8]) by a different method. Two of his main results are [7] Theorems 4.2, 4.3 which state:

$$(1) \qquad \{(S_\lambda(\mu)/S_\kappa(\mu), \mathbf{B}_{\kappa\lambda\mu}; \ldots) : \kappa < \lambda \leq \mu^+\}$$

is interpretable in

$$(2) \qquad \{(S_\lambda(\mu)/S_\kappa(\mu), \circ) : \kappa < \lambda \leq \mu^+\}$$

where if $\mathcal{P}_\lambda(\mu), \mathcal{P}_\kappa(\mu)$ are the rings of subsets of μ of cardinality $< \lambda, < \kappa$ respectively, then $\mathbf{B}_{\kappa\lambda\mu}$ is the boolean algebra generated by $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ and $\ldots$ signifies the 'natural relations and functions',

(2)

$$(2)\{(S(\mu)/S_\kappa(\mu) : \kappa \leq \mu, cf(\kappa) > 2^{\aleph_0}\}$$

is bi-interpretable (in a suitable sense) with

$$\{\{(2^{\aleph_0}, [\kappa, \mu], \mathcal{P}_{(2^{\aleph_0})^+}((2^{\aleph_0} \cup [\kappa, \mu])^2); <, E)\} : \kappa \leq \mu, cf(\kappa) > 2^{\aleph_0}\}$$

where $[\kappa, \mu] = \{\nu : \nu \text{ is a cardinal and } \kappa \leq \nu \leq \mu\}$, and $(a, b, R) \in E \Leftrightarrow (a, b) \in R \in \mathcal{P}_{(2^{\aleph_0})^+}((2^{\aleph_0} \cup [\kappa, \mu])^2)$.

Our corresponding results are Theorems 2.6, 4.3, and Corollaries 3.9, 4.4. Combining Rubin's result with (1) and (2) with [10],[11]gives a full classification of the elementary types of the groups in the class $\{(S(\mu)/S_\kappa(\mu) : \kappa \leq \mu, cf(\kappa) > 2^{\aleph_0}\}$, that is two cardinls. The case $cf(\kappa) \leq 2^{\aleph_0}$ which Rubin gives as an open question is treated in our final section.

The notation used is fairly standard. We use κ, λ, μ, and ν to stand for cardinals (usually infinite), and $|X|$ for the cardinality of the set X. If Ω is any set we write $\mathrm{Sym}(\Omega)$ for the group of all permutations of Ω (1–1 maps from Ω onto itself), with permutations acting on the right, and we write $S(\mu)$ for $\mathrm{Sym}(\mu)$ for any cardinal μ. For $g \in \mathrm{Sym}(\Omega)$ we let *supp* g be the support of g. If we are working in $S_\lambda(\mu)/S_\kappa(\mu)$ (where $S_\lambda(\mu), S_\kappa(\mu)$ are as introduced above) then we refer to sets of cardinality less than κ as *small*. We use overlines such as $\overline{x}$ to stand for finite sequences ('tuples') $(x_1, x_2, \ldots, x_n)$. By a *permutation representation* or *action* of a group G we understand a homomorphism θ from G into $\mathrm{Sym}(X)$ for some set X. The representation is *faithful* if θ is 1–1, it is *transitive* if for any $x, y \in X$ there is $g \in G$ such that $x(g\theta) = y$, and it is *trivial* if its image is the trivial group.

If X is a subset (or sequence of elements) of a group G, we let $\langle X \rangle$ denote the subgroup generated by X. If $g, h \in G$ we write g^h for the conjugate $h^{-1}gh$ of g by h. If $\overline{g}$ is a sequence of members of G and $h \in G$, we write $\overline{g}^h$ for the sequence whose ith entry is g_i^h, and if $\overline{g}, \overline{h}$ are sequences of members of G of the same length, we let $\overline{g} * \overline{h}$ be the sequence whose ith entry is $g_i h_i$. If $\overline{g}_1^h = \overline{g}_2$ for some $h, \overline{g}_1$ and $\overline{g}_2$ are said to be *conjugate*. If $N \leq G$ and $\overline{f} = (f_1, \ldots, f_n) \in G^n$ we let $N.\overline{f} = (Nf_1, \ldots, Nf_n)$.

We write $\mathcal{P}(X)$ for the power set of the set X, and $\mathcal{P}_\kappa(X)$ for the set of subsets of X of cardinality less than κ. Then $\mathcal{P}(X)$ is a boolean algebra, and each $\mathcal{P}_\kappa(X)$ for κ infinite is a ring of sets. Moreover, if $\aleph_0 \leq \kappa < \lambda \leq |X|^+, \mathcal{P}_\kappa(X)$ is an ideal of $\mathcal{P}_\lambda(X)$, so we may study the quotient ring $\mathcal{P}_\lambda(X)/\mathcal{P}_\kappa(X)$, which is a boolean algebra just in the case where $\lambda = |X|^+$ (that is, where $\mathcal{P}_\lambda(X) = \mathcal{P}(X)$).

In the remainder of this introductory section we give an outline of the main arguments of the paper.

Our analysis of the quotient groups $S_\lambda(\mu)/S_\kappa(\mu)$ is carried out using certain many sorted structures $\mathcal{M}_{\kappa\lambda\mu}$ and $\mathcal{N}^2_{\kappa\lambda\mu}$. (There is also a simpler version $\mathcal{M}^*_{\kappa\lambda\mu}$ of $\mathcal{M}_{\kappa\lambda\mu}$ applicable just in the case $cf(\kappa) > 2^{\aleph_0}$.) These structures are devised with the object of describing the permutation action of tuples of elements of $S_\lambda(\mu)$, modulo small sets. The essential properties of such an n-tuple $\overline{g} = (g_1, g_2, \ldots, g_n)$ are described by its action on the orbits of the subgroup $\langle \overline{g} \rangle$. In fact, if $\overline{g}_1$ and $\overline{g}_2$ are n-tuples of elements of $S_\lambda(\mu)$ then $\overline{g}_1$ and $\overline{g}_2$ are conjugate if and only if the orbits of $\langle \overline{g}_1 \rangle$ and $\langle \overline{g}_2 \rangle$ can be put into 1–1 correspondence in such a way that the action of $\overline{g}_1$ on each orbit of $\langle \overline{g}_1 \rangle$ is isomorphic to that of $\overline{g}_2$ on the corresponding orbit of $\langle \overline{g}_2 \rangle$. Similar remarks apply in the quotient group, except that we have to allow fewer than κ 'mistakes' (by passing to equivalence classes of a suitable equivalence relation).

These considerations lead us to observe that what should represent $\overline{g}$ in $\mathcal{M}_{\kappa\lambda\mu}$ is a list of how many $\langle \overline{g} \rangle$-orbits there are of the various possible isomorphism types, where by 'isomorphic' here we mean 'under the action of $\overline{g}$'. Included among the sorts of $\mathcal{M}_{\kappa\lambda\mu}$ are therefore, for each positive integer n, the family IS_n of isomorphism types of pairs $(A, \overline{f})$, where $\overline{f}$ is an n-tuple of

permutations of A acting transitively on A. We keep track of the 'list' of how many orbits there are of the various types by means of a function h from IS_n to cardinals, and the family of all these forms a further collection of sorts F_n. In F_n we have to identify two functions under an equivalence relation $\mathcal{E}_n$ if they arise from members of $S_\lambda(\mu)$ lying in the same coset of $S_\kappa(\mu)$.

Already it is clear that $\mathcal{M}_{\kappa\lambda\mu}$ will have second order features (not surprisingly, since elements of $S_\lambda(\mu)$ are *subsets* of μ^2), but it is still officially construed at this stage as a first order structure. The main reason for this is that at present we cannot identify the elements of F_n as functions from IS_n to $Card$ (= the set of cardinals $< \lambda$), as we would like, because, as just remarked, the members of F_n are $\mathcal{E}_n$-classes, and $\mathcal{E}_n$ is not in general compatible with application. This point is responsible for many of the complications in the paper. In the special case $cf(\kappa) > 2^{\aleph_0}$, we *can* so identify them, and the analysis is considerably simplified. If we do not assume $cf(\kappa) > 2^{\aleph_0}$, then the best we can do to point the connection between IS_n and F_n is to consider an 'application' function App_n which acts on $F_n \times IS_n$ and gives values in $Card^- = \{\nu \in Card : \nu = 0 \vee \kappa \leq \nu < \lambda\}$. This then *will* be compatible with $\mathcal{E}_n$, which is why all values $< \kappa$ are replaced by 0. The final sort in $\mathcal{M}_{\kappa\lambda\mu}$ is therefore $Card^-$, and various relations and functions are included in its signature to express which of its properties mirror the first order properties of $S_\lambda(\mu)/S_\kappa(\mu)$. The most important of these is App_n, but we also need relations Eq and $Prod$ corresponding to 'equality' and 'product in the group', and 'projections' $Proj_n$ to handle existential quantification. Here $Eq \subseteq F_2$, $Prod \subseteq F_3$, and $Proj_n$ is a function from F_{n+1} to F_n. Corresponding relations Eq^1, $Prod^1$, and $Proj^1_n$ are defined on the IS_n, which in 'nice' cases are sufficient to express Eq, $Prod$, and $Proj_n$.

The minimum goal in defining the structures $\mathcal{M}_{\kappa\lambda\mu}$ is that $S_{\lambda_1}(\mu_1)/S_{\kappa_1}(\mu_1)$ and $S_{\lambda_2}(\mu_2)/S_{\kappa_2}(\mu_2)$ should be elementarily equivalent if and only if $\mathcal{M}_{\kappa_1\lambda_1\mu_1}$ and $\mathcal{M}_{\kappa_2\lambda_2\mu_2}$ are (Corollary 4.4), and in a sense this 'solves the problem' of which of the quotient groups are elementarily equivalent. More precise information is however avalaible. In particular, $\mathcal{M}_{\kappa\lambda\mu}$ is 'explicitly interpretable' in $G = S_\lambda(\mu)/S_\kappa(\mu)$; this is 'interpretability' in the usual sense, meaning that each sort and relation and function of $\mathcal{M}_{\kappa\lambda\mu}$ can be represented by a definable (without parameters, in fact) relation on some power of G. In the other direction we cannot hope for explicit interpretability, as one sees just by looking at the cardinalities of the structures; a weaker property which we call 'semi-interpretability' (Definition 2.5) is established here, which is still strong enough to transfer elementary equivalence. The fact that $S_\lambda(\mu)/S_\kappa(\mu)$ is semi-interpretable in $\mathcal{M}_{\kappa\lambda\mu}$ is shown in Theorem 2.6, and essentially involves making precise the discussion in the previous paragraph. It goes by induction on formulae of the language of group theory. For the basis cases we use Eq and $Prod$, and for the key induction step (existential quantification), $Proj_n$.

The method for interpreting $\mathcal{M}_{\kappa\lambda\mu}$ in $G = S_\lambda(\mu)/S_\kappa(\mu)$ is described in Sections 3 and 4. In Section 3 we show how the quotient ring of sets $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ can be interpreted. The ideas behind McKenzie's corresponding calculations for

the symmetric group [5] are followed, but with considerably greater complications. The key point is to express disjointness of supports ('almost disjointness' actually, meaning that they intersect in a small set). Now clearly, if two permutations have almost disjoint supports, then they commute in G. The converse is very far from true, but we follow this as a first idea, and study the configurations of certain commuting elements in sufficient details to express disjointness. Specifically we consider sequences $\overline{g}$ of length 60 which satisfy the diagram (which we write alt_5) of $A(5)$, the alternating group on 5 symbols, in some fixed enumeration. This group is chosen because it is simple, and its outer automorphisms and transitive permutation representations are easy to describe. Now apart from a small set, any 60-tuple satisfying alt_5 is determined up to conjugacy by how many orbits it has of the (finitely many) possible transitive permutation representations. Indeed this is precisely the information given by the element of F_{60} corresponding to such a tuple. By means of a (rather technical) analysis of how these interact we can derive a formula which holds for two elements satisfying alt_5 if and only if (they have a special form and) their supports are almost disjoint. Using this we find another formula which says that two involutions have almost disjoint supports, and elements of $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ are then represented by (cosets of) involutions of G. This gives the interpretation of the quotient ring of sets in G, and that of the action of G on $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ follows easily. Moreover, all the other items of the signature of $\mathcal{M}_{\kappa\lambda\mu}$ can be interpreted without much further difficulty, though there are some slight complications in special cases, such as $\lambda = \mu^+$ or $\kappa = \aleph_0$. It is important that we can distinguish each special case by a first order formula. For instance, the structures in which $\lambda = \mu^+$ may be singled out by a formula saying that there is a group element such that the only element disjoint from it is the identity (an element which moves every element of μ for instance).

Although we generally expect $\mathcal{M}_{\kappa\lambda\mu}$ to have much smaller cardinality than $S_\lambda(\mu)/S_\kappa(\mu)$, and it expresses the structure of the group in a more compact form, the $\mathcal{M}_{\kappa\lambda\mu}$ still form a proper class, in view of the presence of the sort $Card^-$. In Sections 5 and 6 we introduce the structures $\mathcal{N}^2_{\kappa\lambda\mu}$, all of whose sorts have cardinality $\leq 2^{\aleph_0}$, in order to be able to reduce the problem about elementary equivalence of the groups to questions about ordinals of cardinality $\leq 2^{\aleph_0}$. In addition, the fact that $\mathcal{M}_{\kappa\lambda\mu}$ is a 'second order structure in disguise' is brought more out into the open, since $\mathcal{N}^2_{\kappa\lambda\mu}$ genuinely *is* second order (hence the superscript 2). The language used to describe $\mathcal{N}^2_{\kappa\lambda\mu}$ has first order variables ranging over each of its sorts, and for each n-tuple of sorts, n-ary relations whose ith place lies in the ith sort in the list. See Definitions 5.4 and 6.2. In some cases we have to restrict the cardinality of the relations over which the second order variables range.

Looking first at the more straightforward case, to indicate the main ideas, suppose that $cf(\kappa) > 2^{\aleph_0}$. We show that now App_n can genuinely be construed as 'application', so that we may fully describe F_n in terms of IS_n and $Card^-$.

What therefore controls the structure $\mathcal{M}_{\kappa\lambda\mu}$ is $Card^-$, and more specifically its order-type $\alpha = \alpha(\kappa, \lambda, \mu)$. The crucial ordinals needed to describe the elementary theory of $\mathcal{M}_{\kappa\lambda\mu}$ are found by writing α in 'base Ω Canotor normal form' where $\Omega = (2^{\aleph_0})^+$, and the countable list of ordinals $\alpha_{[n]}$ (the Cantor coefficients) and certain cofinalities $\alpha^{[n]}$ are what replace $Card^-$ in $\mathcal{N}^2_{\kappa\lambda\mu}$. Theorem 5.5 asserts that $\mathcal{N}^2_{\kappa\lambda\mu}$ is (explicitly) interpretable in a reduct $\mathcal{M}^*_{\kappa\lambda\mu}$ of $\mathcal{M}_{\kappa\lambda\mu}$ (and hence in $\mathcal{M}_{\kappa\lambda\mu}$). Non-empty subsets of $Card^-$ of cardinality $< \Omega$ may be encoded by members of F_2 (we have to use F_2 rather than F_1 since $|IS_1| = \aleph_0$ but $|IS_2| = 2^{\aleph_0}$), and it is not hard to express all the individual terms of the base Ω Cantor normal form for α. To express facts about cofinalities we have to quantify over binary relations on $Card^-$ of cardinality $< \Omega$, which may be encoded using members of F^2_2. To express the full second order logic described above we use longer tuples from possibly higher F_ns.

The transfer of properties from $\mathcal{N}^2_{\kappa\lambda\mu}$ to $\mathcal{M}_{\kappa\lambda\mu}$ ($\mathcal{M}^*_{\kappa\lambda\mu}$ actually suffices in this case) is not even by a semi-interpretation. Theorem 5.9 shows directly how to express an arbitrary formula of the first order language of $\mathcal{M}^*_{\kappa\lambda\mu}$ by a second order formula of the language of $\mathcal{N}^2_{\kappa\lambda\mu}$. Parameters are transferred using 'k-representations', where this means that a tuple of elements of $\mathcal{M}^*_{\kappa\lambda\mu}$ (of possibly varying sorts) is represented by a (longer) tuple of elements of $\mathcal{N}^2_{\kappa\lambda\mu}$ including partial maps from IS_2 to IS_2 encoding $\alpha_{[0]}, \ldots, \alpha_{[k-1]}$ and $\alpha^{[0]}, \ldots, \alpha^{[k-1]}$.

If $cf(\kappa) \leq 2^{\aleph_0}$, we can additionally interpret in $\mathcal{M}_{\kappa\lambda\mu}$ the base Ω Cantor normal form coefficients and cofinalities of the least ordinal α^* such that $(\exists\gamma)(\beta = \gamma + \alpha^*)$ where $\kappa = \aleph_\beta$, so this information needs to be added to $\mathcal{N}^2_{\kappa\lambda\mu}$, which now includes $\alpha^*_{[0]}, \ldots, \alpha^*_{[k-1]}$ and $\alpha^{*[0]}, \ldots, \alpha^{*[k-1]}$ (and also $cf(\kappa)$) as additional sorts. As remarked above, since we do not now automatically know that $\lambda > 2^{\aleph_0}$, we have to restrict the second order variables of $\mathcal{N}^2_{\kappa\lambda\mu}$ to range over relations of cardinality $< \lambda$. There are some additional complications in the cases $\kappa \leq 2^{\aleph_0}$ and $\kappa = \aleph_0$, though in all cases the outline described in the previous two paragraphs provides the basis of our analysis. Since the precise definition of $\mathcal{N}^2_{\kappa\lambda\mu}$ depends on which of these cases applies, it is important that they can all be distinguished by elementary formulae.

In summary the main conclusions are as follows. There are first order formulae of the language of group theory distinguishing those $S_\lambda(\mu)/S_\kappa(\mu)$ for which $\lambda \leq \mu$ or $\lambda = \mu^+$, and also the cases $cf(\kappa) > 2^{\aleph_0}$, $cf(\kappa) \leq 2^{\aleph_0} < \kappa$, $\aleph_0 < \kappa \leq 2^{\aleph_0}$, and $\kappa = \aleph_0$. In the case $\lambda \leq \mu$ and $cf(\kappa) > 2^{\aleph_0}$ the following holds:

for any given ordinals $\alpha_l, \alpha^l < \Omega$ there is a first order theory T in the language of group theory such that

> if $\kappa = \aleph_\beta, \lambda = \aleph_\gamma$, $\beta + \alpha = \gamma$, and $\alpha_{[n]} = \alpha_n, \alpha^{[n]} = \alpha^n$ for each n,
> then the first order theory of the group $S_\lambda(\mu)/S_\kappa(\mu)$ is equal to T,

with similar statements in the other cases (including reference to the $\alpha^*_{[n]}, \alpha^{*[n]}$

and so on corresponding to the exact definition of $\mathcal{N}^2_{\kappa\lambda\mu}$).

2 The Basic Machinery

Since we are aiming at a two-way interpretation, where the technically most involved step is the representation of many notions inside the quotient group $S_\lambda(\mu)/S_\kappa(\mu)$, we describe in this section the structure whose bi-interpretability with this group is to be shown. In one direction this is interpretability in the usual sense (called 'explicit interpretability' in [10]), but in the other only what we may term 'semi-interpretability',—which is still sufficient for the transfer of elementary properties. We suppose that $\aleph_0 \leq \kappa < \lambda \leq \mu^+$. The interpretation is most straightforward when $cf(\kappa) > 2^{\aleph_0}$, but we can handle the general case at the expense of some additional work. In the main presentation we assume $\kappa > \aleph_0$, indicate how the argument simplifies when $cf(\kappa) > 2^{\aleph_0}$, and what extra is required when $\kappa = \aleph_0$. We remark that in [7] Rubin showed how to interpret the quotient ring $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ in the group, which is also one of our main goals, though his methods were very different from those we use.

Definition 2.1 *(i) For a finite sequence $\overline{f} = (f_1, f_2, \ldots, f_n)$ of members of $S_\lambda(\mu)$ we let supp $\overline{f} = \bigcup_{i=1}^n$ supp f_i.*

(ii) For a positive integer n let IS_n be the family of isomorphism classes of pairs $(A, \overline{g})$ where $\overline{g} \in (\mathrm{Sym}(A))^n$ and $\langle \overline{g} \rangle$ acts transitively on A (and if $\lambda \leq \mu$, then not every g_i is equal to the identity).

(iii) $Card = \{\nu : \nu$ a cardinal such that $\nu < \lambda\}$.

(iv) $Card^- = \{0\} \cup \{\nu \in Card : \kappa \leq \nu\}$.

(v) If $\overline{f} \in (S_\lambda(\mu))^n$ let $\chi = Ch_{\overline{f}}$ be the function from IS_n to $Card$ given by $Ch_{\overline{f}}((A, \overline{g})_{\cong}) =$ the number of orbits B of $\langle \overline{f} \rangle$ such that $(B, \overline{f}) \cong (A, \overline{g})$.

(vi) $Ch_n = \{Ch_{\overline{f}} : \overline{f} \in (S_\lambda(\mu))^n\}$.

(vii) For cardinals $\kappa_1 \leq \kappa_2$ we define $\kappa_2 - \kappa_1$ to be the least cardinal κ_3 such that $\kappa_1 + \kappa_3 = \kappa_2$, and we let $|\kappa_1 - \kappa_2| = |\kappa_2 - \kappa_1| = \kappa_2 - \kappa_1$.

(viii) We define an equivalence relation $\mathcal{E}_n$ on Ch_n by letting $\chi_1 \mathcal{E}_n \chi_2$ if $\sum\{|\chi_1(t) - \chi_2(t)| : t \in IS_n\} < \kappa$.

(ix) For each $n \geq 1$ let F_n be the set of functions $h : IS_n \to Card$ such that

$$\sum\{h(t) : t \in IS_n\} < \lambda, \text{ and if } \lambda = \mu^+, \sum\{h(t) : t \in IS_n\} = \mu,$$

modulo $\mathcal{E}_n$.

Remark 2.2 *In 'nice' cases $cf(\kappa) > 2^{\aleph_0}$ we may replace $Card$ by $Card^-$, and then the definition of $Ch_{\overline{f}}$ is modified by replacing all values less than κ by 0. Corresponding to this, for $\chi \in Ch_n$ we let $\chi^-(t) = \chi(t)$ if $\chi(t) \geq \kappa$ and $\chi^-(t) = 0$ otherwise. We consider this case further in section 5. If $\kappa = \aleph_0$, the definition*

of $\mathcal{E}_n$ is modified; here we let $\chi_1 \mathcal{E}_n \chi_2$ if $\sum\{|A_t|.|\chi_1(t) - \chi_2(t)| : t \in IS_n\} < \aleph_0$ where $t = (A_t, \overline{g})_{\cong}$ (and this says that $\chi_1(t) = \chi_2(t)$ whenever A_t is infinite, and $\{t : \chi_1(t) \neq \chi_2(t)\}$ is finite).

Lemma 2.3 *(i) For any $\overline{f} \in (S_\lambda(\mu))^n, \sum\{Ch_{\overline{f}}(t) : t \in IS_n\} < \lambda$.*

(ii) For any function h from IS_n to Card such that $\sum\{h(t) : t \in IS_n\} < \lambda$, and if $\lambda = \mu^+, \sum\{h(t) : t \in IS_n\} = \mu$, there is $\overline{f} \in (S_\lambda(\mu))^n$ with $Ch_{\overline{f}} = h$.

(iii) For $\overline{f}, \overline{g} \in (S_\lambda(\mu))^n, Ch_{\overline{f}} \mathcal{E}_n Ch_{\overline{g}}$ if and only if $S_\kappa(\mu).\overline{f}$ and $S_\kappa(\mu).\overline{g}$ are conjugate.

Proof (i) For each $t \in IS_n, Ch_{\overline{f}}(t)$ is the number of orbits of $\langle \overline{f} \rangle$ on μ of that isomorphism type. Hence if $\lambda \leq \mu, \sum_t Ch_{\overline{f}}(t)$ is equal to the number of non-trivial orbits of $\langle \overline{f} \rangle$ on μ, which has cardinality at most $|supp\ \overline{f}|$. But $supp\ \overline{f}$ has cardinality less than λ. If $\lambda = \mu^+, \sum_t Ch_{\overline{f}}(t)$ equals the number of orbits of $\langle \overline{f} \rangle$ on μ, which is $\leq \mu < \lambda$. (If $\kappa = \aleph_0$, instead we have $\sum\{|A_t|.Ch_{\overline{f}}(t) : t \in IS_n\} < \lambda$.)

(ii) First suppose $\lambda \leq \mu$. For each $t \in IS_n$ choose a representative $(A_t, \overline{g}_t)$ of that isomorphism type. Identify $\bigcup\{A_t \times h(t) : t \in IS_n\}$ with a subset A of μ, and let $\overline{f}$ act on $A_t \times h(t)$ as $\overline{g}_t$ does and fix all points outside A. Then for each $t, \langle \overline{f} \rangle$ has precisely $h(t)$ orbits of type t. If $\lambda = \mu^+$ and $\sum\{h(t) : t \in IS_n\} = \mu$, where now the trivial isomorphism type is allowed, we may identify $\bigcup\{A_t \times h(t) : t \in IS_n\}$ with the whole of μ. (If $\kappa = \aleph_0, h(t)$ is replaced by $|A_t|.h(t)$.)

(iii) Altering a member of $(S_\lambda(\mu))^n$ on a set of cardinality $< \kappa$ does not change its Ch_n-value (*modulo* $\mathcal{E}_n$), so if $S_\kappa(\mu).\overline{f}^h = S_\kappa(\mu).\overline{g}$ it follows that $Ch_{\overline{f}^h}\ \mathcal{E}_n\ Ch_{\overline{g}}$. But h furnishes an isomorphism of $(A, \overline{f})$ to $(Ah, \overline{f}^h)$ for each orbit A of $\langle \overline{f} \rangle$, and so $Ch_{\overline{f}} = Ch_{\overline{f}^h}\ \mathcal{E}_n\ Ch_{\overline{g}}$.

Conversely, if $Ch_{\overline{f}}\ \mathcal{E}_n\ Ch_{\overline{g}}$, by altering $\overline{f}$ on a set of cardinality less that κ we may suppose that $Ch_{\overline{f}} = Ch_{\overline{g}}$. There is therefore a 1–1 correspondence between the orbits of $\langle \overline{f} \rangle$ and $\langle \overline{g} \rangle$ which preserves the isomorphism type in IS_n, and which maps singleton orbits to singleton orbits. Moreover this may be chosen having support of size $< \lambda$ (since $|supp\ \overline{f}|, |supp\ \overline{g}| < \lambda$). This gives rise to the desired conjugacy h. $\square$

Definition 2.4 *Given the cardinals κ, λ, μ we form a many-sorted structure $\mathcal{M} = \mathcal{M}_{\kappa\lambda\mu}$ with sorts grouped as follows:*

sorts 1: a sort IS_n for each $n \geq 1$ (having cardinality $2^{\aleph_0}$ for $n \geq 2, IS_1$ of cardinality $\aleph_0$),

sort 2: $Card^-$, (in which 0 and κ, as the first two elements, are definable, so do not need to be explicitly named),

sorts 3: a sort F_n for each $n \geq 1$.

The signature taken is as follows:
unary relations Eq^1 *on* IS_2 *and* $Prod^1$ *on* IS_3 *given by*

$$Eq^1 = \{t \in IS_2 : t = ((A, g_1, g_2))_{\cong} \to g_1 = g_2\},$$
$$Prod^1 = \{t \in IS_3 : t = ((A, g_1, g_2, g_3))_{\cong} \to g_1 g_2 = g_3\},$$

for each n *a binary relation* $Proj^1_n \subseteq IS_{n+1} \times IS_n$ *given by*

$$Proj^1_n = \{(t_1, t_2) \in IS_{n+1} \times IS_n : \exists A_1 \exists A_2 \exists g_1 \exists g_2 \ldots \exists g_{n+1}(A_1 \supseteq A_2$$
$$\wedge\, t_1 = ((A_1, g_1, \ldots, g_{n+1}))_{\cong} \wedge\, t_2 = ((A_2, g_1|A_2, \ldots, g_n|A_2))_{\cong})\},$$

$<$ *on sort* 2, *the usual ordering of cardinals,*
for each n *a function* App_n *from* $F_n \times IS_n$ *to* $Card^-$ *given by* $App_n(x, y) = \nu$ *provided that for some* h *with* $(h)_{\mathcal{E}_n} = x, h(y) = \nu$ *(noting that the value of* $h(y)$ *is well-defined for* $\nu \geq \kappa$, *and for* $\nu < \kappa$, *all values are replaced by* 0,*—see the definition of* χ^- *above),*
unary predicates Eq *on sort* F_2 *and* $Prod$ *on sort* F_3 *given by* $Eq(h), Prod(h)$ *hold if*

$$\textstyle\sum\{h(t) : t \in IS_2 \wedge (t = ((A, g_1, g_2))_{\cong} \to g_1 \neq g_2)\} = 0,$$
$$\textstyle\sum\{h(t) : t \in IS_3 \wedge (t = ((A, g_1, g_2, g_3))_{\cong} \to g_1 g_2 \neq g_3)\} = 0,$$

respectively, (where as the sorts 3 *consist of functions modulo* $\mathcal{E}_n$, *saying that these sums are zero means in effect that they are* $< \kappa$*),*
and functions $Proj_n$ *from sort* F_{n+1} *to sort* F_n *such that if* $h : IS_{n+1} \to Card$ *then* $Proj_n(h) : IS_n \to Card$ *is given by* $Proj_n(h)(t) = \sum\{|B_{t't}|.h(t') : t' \in IS_{n+1}\}$ *where for each* $t' = ((A, \overline{g}))_{\cong} \in IS_{n+1}$, *and* $\overline{g}$ *of length* $n+1, B_{t't}$ *is the set of all orbits of* $\langle g_1, g_2, \ldots, g_n\rangle$ *on* A *on which* $(g_1, g_2, \ldots, g_n)$ *has isomorphism type* t. *(Note that* $|B_{t't}|$ *is independent of the particular choice of* $(A, \overline{g})$ *corresponding to* t'. *Note also that strictly speaking here and in the definition of* Eq, $Prod$, *we should work with the* $\mathcal{E}_n$*-classes determined by* $h, Proj_n(h)$.*)*

We include $Proj_n$ in order to handle existential quantification in the forthcoming induction (Theorem 2.6). The definitions of Eq and $Prod$ apply just in the case $\kappa > \aleph_0$, and are intended to express equality and products in $S_\lambda(\mu)$ up to fewer than κ mistakes. For $\kappa = \aleph_0$, instead of summing the relevant $h(t)$ we sum $|A'_t|.h(t)$ where $t = ((A_t, f_1, f_2))_{\cong}$ or $((A_t, f_1, f_2, f_3))_{\cong}$ and $A'_t = \{\alpha \in A_t : \alpha f_1 \neq \alpha f_2\}$ or $\{\alpha \in A_t : \alpha f_1 f_2 \neq \alpha f_3\}$ respectively.

In the general case the inclusion of the sorts IS_n and $Card^-$ is unnecessary, at any rate as far as the proof of Theorem 2.6 is concerned. On the other hand in all cases they can be naturally represented within $S_\lambda(\mu)/S_\kappa(\mu)$, and in the special case $cf(\kappa) > 2^{\aleph_0}$ the 'application' functions App_n genuinely identify the members of F_n as functions from IS_n to $Card^-$ (since here the equivalence relation $\mathcal{E}_n$ can be dispensed with), meaning that $App'_n : F_n \to (Card^-)^{IS_n}$ given by $App'_n(h)(t) = App_n(h, t)$ is 1–1. In section 5 we shall also see that

$Proj_n$ is definable from $Proj^1_n$ and App_n in this case, and similarly for Eq and $Prod$, easing the analysis of the $\mathcal{M}_{\kappa\lambda\mu}$.

The sense in which we can show that $S_\lambda(\mu)/S_\kappa(\mu)$ is interpretable in $\mathcal{M}_{\kappa\lambda\mu}$ is weaker than the usual one and is given in the following definition.

Definition 2.5 *For structures $\mathcal{M}$ and $\mathcal{N}$ we say that $\mathcal{M}$ is* semi-interpretable *in $\mathcal{N}$ if there is a recursive function F from formulae of the language of $\mathcal{M}$ to formulae of the language of $\mathcal{N}$ and there are functions $f_n : \mathcal{M}^n \to \mathcal{N}$ such that for all $\overline{a} \in \mathcal{M}^n$ and $\varphi(\overline{x})$ with n free variables, $\mathcal{M} \models \varphi[\overline{a}] \Leftrightarrow \mathcal{N} \models F(\varphi)[f_n(\overline{a})]$. If the same F serves over a class of pairs of structures then we say that the first of each pair is* uniformly semi-interpretable *in the second.*

Theorem 2.6 *For every first order formula $\varphi(x_0, \ldots, x_{n-1})$ of the theory of groups there is an effectively determined first order formula $\psi(y)$ of the language of $\mathcal{M}$ such that for all κ, λ, μ, and for every $\overline{f} \in (S_\lambda(\mu))^n$,*

$$S_\lambda(\mu)/S_\kappa(\mu) \models \varphi[S_\kappa(\mu).\overline{f}] \Leftrightarrow \mathcal{M}_{\kappa\lambda\mu} \models \psi[(Ch_{\overline{f}})_{\mathcal{E}_n}].$$

Proof We construct ψ by induction. First suppose that φ is atomic. It suffices to consider formulae of the form $x_0 = x_1$ and $x_0x_1 = x_2$ for variables x_0, x_1, x_2. If $\varphi(x_0, x_1)$ is $x_0 = x_1$ we take for $\psi(y)$ the formula $Eq(y)$. Then

$S_\lambda(\mu)/S_\kappa(\mu) \models S_\kappa(\mu).f_1 = S_\kappa(\mu).f_2$

$\Leftrightarrow$ $|\{\alpha : \alpha f_1 \neq \alpha f_2\}| < \kappa$

$\Leftrightarrow$ the union of the orbits of $\langle f_1, f_2\rangle$ on which the actions of f_1 and f_2 are distinct has cardinality $< \kappa$ (since $\kappa > \aleph_0$)

$\Leftrightarrow$ $\sum\{|A_t|.Ch_{\overline{f}}(t) : t \in IS_2 \wedge (t = ((A_t, g_1, g_2))_{\cong} \to g_1 \neq g_2)\} < \kappa$

$\Leftrightarrow$ $\mathcal{M} \models Eq((Ch_{\overline{f}})_{\mathcal{E}_n})$ (using $\kappa > \aleph_0$ again)

$\Leftrightarrow$ $\mathcal{M} \models \psi[(Ch_{\overline{f}})_{\mathcal{E}_n}]$,

and similarly for the formula $x_0x_1 = x_2$ (using $Prod$).

The propositional induction steps are straightforward.

Finally suppose that $\varphi(\overline{x})$ is $\exists y\varphi_1(\overline{x}, y)$.

It is easily checked that for any $\overline{f} = (f_1, f_2, \ldots, f_n)$ and f_{n+1} in $S_\lambda(\mu)$,

$$Proj_n(Ch_{\overline{f}, f_{n+1}}) = Ch_{\overline{f}}.$$

¿From this it follows that if $h \in F_{n+1}$, then $Proj_n(h) = Ch_{\overline{f}}$ if and only if $h = Ch_{\overline{f}, f_{n+1}}$ for some f_{n+1}. Continuing the proof we deduce that

$S_\lambda(\mu)/S_\kappa(\mu) \models \exists x\varphi_1(S_\kappa(\mu).\overline{f}, x)$

$$\begin{aligned}
&\Leftrightarrow \quad \text{for some } g \in S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models \varphi_1[S_\kappa(\mu).\overline{f}, S_\kappa(\mu).g] \\
&\Leftrightarrow \quad \text{for some } g \in S_\lambda(\mu), \mathcal{M} \models \psi_1[(Ch_{\overline{f},g})_{\varepsilon_{n+1}}] \\
&\Leftrightarrow \quad \mathcal{M} \models \exists x(\psi_1(x) \wedge Proj_n(x) = (Ch_{\overline{f}})_{\varepsilon_n}),
\end{aligned}$$

where ψ_1 is a formula corresponding to φ_1 as given by the induction hypothesis, and so we take for $\psi(\overline{x})$ the formula $\exists y(\psi_1(y) \wedge Proj_n(y) = \overline{x})$. $\square$

3 Interpreting $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ in $S_\lambda(\mu)/S_\kappa(\mu)$

In this section we show how it is possible to interpret many 'set-theoretical' properties inside $S_\lambda(\mu)/S_\kappa(\mu)$, by representing subsets of μ via supports of suitably chosen elements (always up to fewer than 'κ mistakes'), and consequently to interpret the ring $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$. The key idea is to use sequences whose entries are transitive representations of a specific finite *non-abelian* group to represent the subsets, which enables us to capture disjointness of their supports via a commutativity condition. We introduce the necessary formulae one by one, and outline why they represent what is required.

Let G be a fixed finite group of order n, and let $\overline{G} = (a_1, a_2, \ldots, a_n)$ be a fixed enumeration of G for which $a_1 = id$, the identity. In what follows we shall in fact just use $G = A(5)$, the alternating group on $\{0,1,2,3,4\}$. This is for three reasons: it is the smallest non-abelian simple group; its transitive permutation representations are easy to describe; and (a small point needed in the proof) its outer automorphism group is also well known (and is just $S(5)$).

Let $diag(\overline{G}, \overline{x})$ be the conjunction over all i, j, k between 1 and n for which $a_i a_j = a_k$ of the formulae $x_i x_j = x_k$. This is intended to say that $(x_1, x_2, \ldots, x_n)$ is a 'copy' of G (in the specified enumeration), but actually just says that it is a homomorphic image. We write $diag(\overline{A(5)}, \overline{x})$ as $alt_5(\overline{x})$.

Lemma 3.1 *Suppose $\overline{f} \in (S_\lambda(\mu))^n$ is such that $S_\lambda(\mu)/S_\kappa(\mu) \models diag(\overline{G}, S_\kappa(\mu).\overline{f})$. Then there is a small union X of $\langle \overline{f} \rangle$-orbits such that if $\alpha \in \mu - X$, and $a_i a_j = a_k$, then $\alpha f_i f_j = \alpha f_k$.*

Proof Let $X_{ijk} = \{\alpha \in \mu : \alpha f_i f_j \neq \alpha f_k\}$ and let $X = \bigcup\{X_{ijk} : a_i a_j = a_k\}$. By definition of *diag*, and of $S_\lambda(\mu)/S_\kappa(\mu)$, $|X| < \kappa$. So it suffices to observe that $\mu - X$ is closed under the action of $\langle \overline{f} \rangle$. Let $\alpha \notin X$ and $1 \leq r \leq n$. Suppose that i, j, k are arbitrary subject to $a_i a_j = a_k$. Then there are s, t such that $a_r a_i = a_s$ and $a_r a_k = a_t$. We find that $a_s a_j = a_r a_i a_j = a_r a_k = a_t$. Since $\alpha \notin X, \alpha f_r f_i f_j = \alpha f_s f_j = \alpha f_t = \alpha f_r f_k$. Thus $\alpha f_r \notin X_{ijk}$ and so $\alpha f_r \notin X$ as required. $\square$

Lemma 3.2 *For any $\overline{f}, S_\lambda(\mu)/S_\kappa(\mu) \models alt_5(S_\kappa(\mu).\overline{f})$ if and only if there is a small union X of orbits of $\langle \overline{f} \rangle$ on μ such that for every orbit Y of $\langle \overline{f} \rangle$ on $\mu - X$, the action of $\langle \overline{f} \rangle$ on Y is isomorphic to some action of $A(5)$ (so that*

$|Y| = 1, 5, 6, 10, 12, 15, 20, 30$, *or* 60, *and then we say that* $\overline{f}$ *acts as* $A(5)$ *with this degree).*

Proof This is immediate from Lemma 3.1 on remarking that for orbits Y outside X, the action of $\langle \overline{f} \rangle$ on Y precisely corresponds to some transitive action of $A(5)$. The fact that the possible values of $|Y|$ are as stated follows from the fact that any transitive action of $A(5)$ is isomorphic to its action on a coset space $[A(5) : H]$ for some subgroup H of $A(5)$, and the possible orders of subgroups of $A(5)$ are $1, 2, 3, 4, 5, 6, 10, 12$, and 60. $\square$

With this lemma in mind we may define for any $\overline{f} \in S_\lambda(\mu)$ of length 60 the cardinals $\nu_m(\overline{f})$ for $m \in \{1, 5, 6, 10, 12, 15, 20, 30, 60\}$ by $\nu_m(\overline{f}) =$ the number of $\langle \overline{f} \rangle$-orbits on μ on which $\overline{f}$ acts as $A(5)$ with degree m. The significant values (that is, those which are preserved under passing to the coset $S_\kappa(\mu).\overline{f}$) are those $\nu_m(\overline{f})$ which are $\geq \kappa$, and these provide a 'profile' of $\overline{f}$ characterizing it up to conjugacy.

To make further progress we need to analyse with some care some properties of the possible faithful transitive actions of $A(5)$, which we do in the next three lemmas.

Lemma 3.3 *Suppose that* H *and* K *are proper subgroups of* $A(5)$*. Then for some* $a \in A(5), |H \cap a^{-1}Ka| \leq 3$*. Moreover, if there is* a *such that* $|H \cap a^{-1}Ka| = 3$ *but no* b *such that* $|H \cap b^{-1}Kb| < 3$*, then* $|H| = |K| = 12$*.*

Proof As $A(5)$ is simple, $|H|, |K| \leq 12$. If H or K has order ≤ 3 we just let $a = id$. Assuming without loss of generality that $|H| \geq |K|$ we are left with the following possibilities for $(|H|, |K|)$:

$(12, 12), (12, 10), (12, 6), (12, 5), (12, 4), (10, 10), (10, 6), (10, 5), (10, 4)$, $(6, 6), (6, 5), (6, 4), (5, 5), (5, 4), (4, 4)$.

The subgroups of $A(5)$ of orders $12, 10, 6, 5, 4$ are determined uniquely up to conjugacy in $S(5)$ (as is easy to check) and so by replacing by a conjugate by a member of $A(5)$ may be taken to lie in the following list:

$12 : A(4)$ (regarded as the stabilizer of 4 in $A(5)$),
$10 : \langle (01234), (14)(23) \rangle, \langle (01243), (13)(24) \rangle$,
$6 : \quad \langle (012), (01)(34) \rangle$,
$5 : \quad \langle (01234) \rangle, \langle (01243) \rangle$,
$4 : \quad \langle (01)(23), (02)(13) \rangle$.

The following cases can be at once ruled out as $|H \cap K| \leq 3$ is already true: $(12, 10), (12, 6), (12, 5), (10, 6), (10, 4), (6, 5), (6, 4), (5, 4)$. In all the remaining cases, which are $(12, 12), (12, 4), (10, 10), (10, 5), (6, 6), (5, 5)$, and $(4, 4)$, the conjugator (234) will serve as a, as is easy to check.

Now for the final part, suppose that $|H \cap a^{-1}Ka| = 3$ for some a and that $|H \cap b^{-1}Kb| \geq 3$ for all b. Then $|H|$ and $|K|$ are multiples of 3. If H or K has order 12 or 6, we take it as above, and if 3 we take it as $\langle (012) \rangle$. In all cases

except for $|H| = |K| = 12$ we find that for $b = (243), |H \cap b^{-1}Kb| = 1$ or 2, and we conclude that H and K must both have order 12. □

Lemma 3.4 *Let D be the diagonal subgroup $\{(a_i, a_i) : 1 \leq i \leq 60\}$ of $A(5) \times A(5)$. Then for any subgroup H of $A(5) \times A(5)$ of order 12 or 36, there is a such that $|a^{-1}Ha \cap D| \neq 3$.*

Proof Suppose otherwise. Thus $|H| = 12$ or 36, and for every $a \in A(5) \times A(5), |a^{-1}Ha \cap D| = 3$. In particular $|H \cap D| = 3$ so we suppose that $H \cap D = \langle((012), (012))\rangle$. Let $a = ((13)(24), id)$. Since $|a^{-1}Ha \cap D| = 3$ there are $i < j < k$ such that $((ijk), (ijk)) \in a^{-1}Ha$. Also $((034), (012)) \in a^{-1}Ha$. If $|\{i, j, k\} \cap \{0, 3, 4\}| = 1$ then $\langle(ijk), (034)\rangle$ contains an element of order 5, contrary to $|a^{-1}Ha| = 12$ or 36. Hence $|\{i, j, k\} \cap \{0, 3, 4\}| = 2$ or 3. Similarly $|\{i, j, k\} \cap \{0, 1, 2\}| = 2$ or 3. Therefore $i = 0$ and $j = 1$ or $2, k = 3$ or 4.
Case 1: $(ijk) = (013)$. Then $((012), (012)), ((031), (013)) \in H$. But these two elements generate a group of order 144 ($A(4) \times A(4)$ in fact).
Case 2: $(ijk) = (014)$. Therefore $((012), (012)), ((032), (014)) \in H$. Since H has no element of order $5, H \leq A(4) \times A(\{0, 1, 2, 4\})$. Let $b = ((014), id)$. Then $b^{-1}Hb \leq A(\{1, 4, 2, 3\}) \times A(\{0, 1, 2, 4\})$. If $((i'j'k'), (i'j'k'))$ lies in $b^{-1}Hb$ with $i' < j' < k'$ then $\{i', j', k'\} \subseteq \{1, 4, 2, 3\} \cap \{0, 1, 2, 4\}$, so $(i'j'k') = (124)$. Then $((021), (124)) \in H$, so that $(id, (02)(14)) \in H$, from which it follows that $|H| \neq 12, 36$.
Case 3: $(ijk) = (023)$. Then $((012), (012)), ((041), (023)) \in H$, and we argue as in Case 2, with $b = ((013), id)$. This time we find that $((021), (123)) \in H$, so that $(id, (02)(13)) \in H$, and $|H| \neq 12, 36$.
Case 4: $(ijk) = (024)$. Then $((012), (012)), ((042), (024)) \in H$ so $((01)(24), (014)) \in H$, and $|H| = 144$ as in Case 1. □

Lemma 3.5 *Suppose that $\overline{f}, \overline{g}$ are subgroups of $\mathrm{Sym}(X)$ isomorphic to $A(5)$ (in the specified listings) which centralize each other, and such that $\langle \overline{f}, \overline{g} \rangle$ is transitive on X. Then $\overline{f} * \overline{g}$ has an orbit of length at least 20. Moreover, if $\overline{f} * \overline{g}$ has an orbit of length 20 then it also has an orbit of some other length greater than 1.*

Proof Let $\mathcal{X} = \{X_i : i < m\}$ and $\mathcal{Y} = \{Y_j : j < n\}$ be the families of orbits of $\overline{f}$ and $\overline{g}$ respectively. Then as $\overline{f}$ and $\overline{g}$ commute, $\overline{f}$ and $\overline{g}$ each preserve $\mathcal{X}$ and $\mathcal{Y}$ (setwise), and hence also $\mathcal{Z} = \{X_i \cap Y_j : i < m, j < n\}$. Moreover by transitivity of $\langle \overline{f}, \overline{g} \rangle$ on X the actions of $\overline{f}$ on its orbits are all isomorphic, as are the actions of $\overline{g}$ on its orbits. Since $\overline{f}, \overline{g}$ are isomorphic to $A(5)$, these orbits are all non-trivial, and since $A(5)$ is simple, they all have at least 5 members.
Case 1: $m = n = 1$. Thus $\overline{f}$ and $\overline{g}$ are both transitive.

In this situation it is standard that $\overline{f}$ and $\overline{g}$ both act regularly (see [13, Theorem 3.2.9]). For suppose that $xf_i = x$. Then for each $j, (xg_j)f_i = xf_ig_j = xg_j$ and as $\overline{g}$ is transitive, $f_i = id$. Similarly $\overline{g}$ is regular. By suitably labelling

the elements of X we may suppose that $X = A(5)$ and $\overline{f}$ is the right regular action, in other words $(a_i)f_j = a_ia_j$ for each i and j.

Now we appeal essentially to the fact that the centralizer of the right regular action is the left regular action (see [13, Theorem 3.2.10]). Let $a_1g_i = a_r$. Then $a_jg_i = a_1f_jg_i = a_1g_if_j = a_rf_j = a_ra_j$. Hence g_i is multiplication on the left by a_r. Let us write a_r as $a_i\theta$. Thus θ is 1–1 since if $a_i\theta = a_j\theta, a_1g_i = a_1g_j$ and $i = j$ (by regularity). So also θ is onto. Moreover it is an anti-homomorphism, since $(a_ia_{i'})\theta = a_1(g_ig_{i'}) = ((a_i\theta)a_1)g_{i'} = (a_{i'}\theta)(a_i\theta)a_1 = (a_{i'}\theta)(a_i\theta)$. Thus φ given by $a_i\varphi = (a_i^{-1})\theta$ is an automorphism of $A(5)$. So for some $s \in S(5), a_i\varphi = s^{-1}a_is$ for all i, so that $a_i(f_jg_j) = s^{-1}a_j^{-1}sa_ia_j$. Now the length of the orbit of f_jg_j containing a_i is equal to the index of its stabilizer in $A(5)$. But $s^{-1}a_j^{-1}sa_ia_j = a_i \Leftrightarrow a_j^{-1}sa_i = sa_ia_j^{-1} \Leftrightarrow a_j \in C_{A(5)}(sa_i) = A(5) \cap C_{S(5)}(sa_i)$. Now sa_i either ranges over $A(5)$ or over $S(5) - A(5)$. If $A(5)$ let $sa_i = (012)$ or (01234). Then $|C_{A(5)}(sa_i)| = 3$ or 5 and so there are orbits of lengths 20 and 12. If $S(5) - A(5)$ let $sa_i = (0123)$. Then $|C_{A(5)}(sa_i)| = 2$ and so there is an orbit of length 30.

Case 2: $m = 1 \ \wedge \ n > 1$ (or similarly $m > 1 \ \wedge \ n = 1$).

Then $\overline{f}$ is transitive, so by the same proof as above, $\overline{g}$ acts semiregularly (that is, only the identity has any fixed point). Hence $\overline{g}$ acts regularly on each orbit, and so each orbit has size 60. But then $|X| > 60$, contrary to $\overline{f}$ transitive on X.

Case 3: $m, n > 1$.

Since $\langle \overline{f}, \overline{g} \rangle$ is transitive, the actions of $\overline{f}$ on $\mathcal{Y}$ and $\overline{g}$ on $\mathcal{X}$ are both transitive, and hence faithful. Moreover $\langle \overline{f}, \overline{g} \rangle$ acts transitively on $\mathcal{Z} = \{X_i \cap Y_j : i < m, j < n\}$ (which in particular means that all $X_i \cap Y_j$ are non-empty of equal size).

We show that some orbit of $\overline{f} * \overline{g}$ in its action on $\mathcal{Z}$ has length ≥ 20, and it will follow that the same applies to its action on X. Now the length of the orbit containing $X_i \cap Y_j$ is equal to the index of its stabilizer, and as $(X_i \cap Y_j)f_kg_k = X_ig_k \cap Y_jf_k$, $(X_i \cap Y_j)f_kg_k = X_i \cap Y_j \Leftrightarrow X_ig_k = X_i \wedge Y_jf_k = Y_j$. Hence $\{a_k : (X_i \cap Y_j)f_kg_k = X_i \cap Y_j\} = \{a_k : X_ig_k = X_i\} \cap \{a_k : Y_jf_k = Y_j\}$ and all we have to do is to show that for some i, j, the right hand side has order ≤ 3. Let $H = \{a_k : X_0g_k = X_0\}$ and $K = \{a_k : Y_0f_k = Y_0\}$. Then the stabilizers of the other X_i and Y_j are just the conjugates of these. For instance $\{a_k : X_0g_ig_k = X_0g_i\} = \{a_k : a_ia_ka_i^{-1} \in H\} = a_i^{-1}Ha_i$ and $\{a_k : Y_0f_jf_k = Y_0f_j\} = a_j^{-1}Ka_j$.

For our choice we take $i = 0$ and select j by using Lemma 3.3.

Finally we have to show (still in Case 3) that not all orbits of $\overline{f} * \overline{g}$ can have length 20 or 1. Suppose otherwise. Since $m \geq 5$ and $\overline{f} * \overline{g}$ acts transitively on $\{X_i : i < m\}$, none of the orbits can have length 1. Applying the last clause of Lemma 3.3 we find that $|H| = |K| = 12$, and so $m = n = 5$. Therefore $|X| = 25|X_0 \cap Y_0|$ and since this is a multiple of 20, and $|X_0| = 5|X_0 \cap Y_0|$ is a factor of $60, |X_0 \cap Y_0| = 4$ or 12, so $|X| = 100$ or 300.

Pick $x \in X_0 \cap Y_0$ and let $L = \{(a_i, a_j) \in A(5) \times A(5) : xf_ig_j = x\}$. Since $A(5) \times A(5)$ acts transitively on X via $(\overline{f}, \overline{g}), |L| = 60^2/|X| = 12$ or 36. By Lemma 3.4 (and with D as there), there are i, j such that $|(a_i, a_j)^{-1}L(a_i, a_j) \cap$

$D| \neq 3$. Let $y = xf_ig_j$. Then $yf_kg_k = y \Leftrightarrow xf_if_kg_jg_k = xf_ig_j \Leftrightarrow xf_if_kf_i^{-1}g_jg_kg_j^{-1} = x \Leftrightarrow (a_ia_ka_i^{-1}, a_ja_ka_j^{-1}) \in L \Leftrightarrow (a_k, a_k) \in (a_i, a_j)^{-1}L(a_i, a_j)$. Hence $|\{a_k : yf_kg_k = y\}| = |(a_i, a_j)^{-1}L(a_i, a_j) \cap D| \neq 3$, and so the orbit of y under the action of $\overline{f} * \overline{g}$ does not have length 20 after all. $\square$

We now move towards the construction of a formula which is intended to say that $\overline{x}$ acts as $A(5)$ on all but a small set of its orbits, and that each such orbit has length 1 or 5. Actually we stop short of doing this (even though it can be done) and just find a formula restricting the range of representations possible—as this provides a quicker route to our goal. We require the following auxiliary formulae:

$comm_{m,n}(\overline{x}, \overline{y})$: $\bigwedge_{1 \leq i \leq m, 1 \leq j \leq n} x_iy_j = y_jx_i$,
where m and n are the lengths of $\overline{x}$ and $\overline{y}$. This asserts that each entry of $\overline{x}$ commutes with each entry of $\overline{y}$.

$conj_n(\overline{x}, \overline{y})$: $(\exists z)(\overline{x}^z = \overline{y})$.

In practice we omit the subscripts from $comm_{m,n}$ and $conj_n$ (and other similar formulae).

$indec(\overline{x})$: $alt_5(\overline{x}) \wedge (\forall \overline{y})(\forall \overline{z})(comm(\overline{y}, \overline{z}) \wedge alt_5(\overline{y}) \wedge alt_5(\overline{z}) \wedge \overline{x} = \overline{y} * \overline{z} \rightarrow (conj(\overline{x}, \overline{y}) \vee conj(\overline{x}, \overline{z}))$.

Lemma 3.6 *For any sequence $\overline{f}$ of elements of $S_\lambda(\mu)$ of length 60, $S_\lambda(\mu)/S_\kappa(\mu) \models indec(S_\kappa(\mu).\overline{f})$ if and only if $|\mu - supp\ \overline{f}| = \mu$, $\langle \overline{f} \rangle$ acts as $A(5)$ on all orbits outside a small subset of μ, $\nu_{30}(\overline{f}), \nu_{60}(\overline{f}) < \kappa$, and there is at most one $m \in \{5, 6, 10, 12, 15, 20\}$ for which $\nu_m(\overline{f}) \geq \kappa$.*

Proof We remark that we need to stipulate $|\mu - supp\ \overline{f}| = \mu$ in view of the possibility that $\lambda = \mu^+$. Let us say that $\overline{f} \notin S_\kappa(\mu)$ is *indecomposable* if $S_\lambda(\mu)/S_\kappa(\mu) \models indec(S_\kappa(\mu).\overline{f})$.

First observe that if $\nu_{m_1}(\overline{f}), \nu_{m_2}(\overline{f}) \geq \kappa$ where $1 < m_1 < m_2$ then we may write $\overline{f}$ as $\overline{g} * \overline{h}$ where $\overline{g}$ is the restriction of $\overline{f}$ to the union of its orbits of length m_1 (that is it agrees with $\overline{f}$ there and fixes all other points), and $\overline{h}$ is the restriction of $\overline{f}$ to the complement of the union of these orbits. Clearly $\overline{f}, \overline{g}$ commute, $S_\kappa(\mu).\overline{g}$ and $S_\kappa(\mu).\overline{h}$ satisfy $alt_5(\overline{x})$, and $\overline{f} = \overline{g} * \overline{h}$. But neither $S_\kappa(\mu).\overline{g}$ nor $S_\kappa(\mu).\overline{h}$ is conjugate to $S_\kappa(\mu).\overline{f}$.

Next suppose that $\nu_{30}(\overline{f}) \geq \kappa$, and let X be the union of all orbits of $\overline{f}$ of length 30 on which $\overline{f}$ acts as $A(5)$. We let $\overline{g}$ and $\overline{f}$ agree on $\mu - X$ and $\overline{h}$ fix $\mu - X$ pointwise. Let Y be a typical orbit of $\overline{f}$ contained in X, (and hence of length 30). Now if H and K are subgroups of $A(5)$ of orders 12 and 10, then $A(5)$ has a transitive action of degree 30 on $[A(5) : H] \times [A(5) : K]$, (since $|H \cap K| = 2$), which is therefore isomorphic to the action of $\langle \overline{f} \rangle$ on Y. So we may let $Y = \{\alpha_{(Ha_i, Ka_j)} : Ha_i \in [A(5) : H], Ka_j \in [A(5) : K]\}$ in such a way that for each $k, \alpha_{(Ha_i, Ka_j)}f_k = \alpha_{(Ha_ia_k, Ka_ja_k)}$. The point is that this expresses the action of $\overline{f}$ on Y as a commuting 'product' of actions having orbits of sizes 5 and 6. We let $\alpha_{(Ha_i, Ka_j)}g_k = \alpha_{(Ha_ia_k, Ka_j)}$ and $\alpha_{(Ha_i, Ka_j)}h_k = \alpha_{(Ha_i, Ka_ja_k)}$.

This therefore defines the actions of $\overline{g}$ and $\overline{h}$ on the orbits of $\overline{f}$ having length 30. It is clear that neither $S_\kappa(\mu).\overline{g}$ nor $S_\kappa(\mu).\overline{h}$ can be conjugate to $S_\kappa(\mu).\overline{f}$, since $\nu_{30}(\overline{g}), \nu_{30}(\overline{h}) < \kappa$. But $S_\kappa(\mu).\overline{g}$ and $S_\kappa(\mu).\overline{h}$ fulfil the other requirements on $\overline{y}$ and $\overline{z}$ in *indec*, and so we conclude that $S_\kappa(\mu).\overline{f}$ cannot satisfy *indec*.

If $\nu_{60}(\overline{f}) \geq \kappa$, a similar argument applies, but this time taking $|H| = 12$ and $|K| = 5$.

Now suppose that $|\mu - supp\ \overline{f}| < \mu$. Let X be a union of orbits of $\langle \overline{f} \rangle$ such that $|X| = |\mu - X| = \mu$, and let $\overline{g}$ and $\overline{h}$ be the restrictions of $\overline{f}$ to X and $\mu - X$ respectively. Then $S_\kappa(\mu).\overline{g}$ and $S_\kappa(\mu).\overline{h}$ provide witnesses for y and z violating $indec(S_\kappa(\mu).\overline{f})$.

Conversely, suppose that $|\mu - supp\ \overline{f}| = \mu$ and for some $m \in \{5, 6, 10, 12, 15, 20\}$, the union X of the orbits of $\langle \overline{f} \rangle$ of length m on which $\overline{f}$ acts as $A(5)$ has cardinality $\geq \kappa$, and that $\overline{f}$ fixes all but a small subset Y of $\mu - X$. We verify $indec(S_\kappa(\mu).\overline{f})$. Suppose $S_\kappa(\mu).\overline{g}$ and $S_\kappa(\mu).\overline{h}$ are witnesses for $\overline{y}$ and $\overline{z}$ in *indec*. If $|supp\ \overline{g} \cap supp\ \overline{h}| < \kappa$ then $\overline{g}$ and $\overline{h}$ are restrictions of $\overline{f}$ (meaning that apart from a small set, their supports are contained in $supp\ \overline{f}$, and on their supports they agree with $\overline{f}$), and so, as $|supp\ \overline{g}| + |supp\ \overline{h}| = |supp\ \overline{f}|$, either $|supp\ \overline{g}| = |supp\ \overline{f}|$ or $|supp\ \overline{h}| = |supp\ \overline{f}|$, so that one of $S_\kappa(\mu).\overline{g}, S_\kappa(\mu).\overline{h}$ is conjugate to $S_\kappa(\mu).\overline{f}$.

So we suppose that $|supp\ \overline{g} \cap supp\ \overline{h}| \geq \kappa$ and aim for a contradiction. Since $\overline{g}$ and $\overline{h}$ commute *mod* $S_\kappa(\mu)$, by increasing Y if necessary we may assume they commute outside Y. Let Z be a typical orbit of $\langle \overline{g}, \overline{h} \rangle$ on $(supp\ \overline{g} \cap supp\ \overline{h}) - Y$. Then the restrictions of $\overline{g}$ and $\overline{h}$ to Z fulfil the hypotheses of Lemma 3.5, and so $\overline{g} * \overline{h}$ either has an orbit on Z of length greater than 20, or orbits there of length 20 and some other length greater than 1. Since this applies to all possible choices of Z, either there are $\geq \kappa$ Zs for which there is an orbit of length greater than 20, or there are $\geq \kappa$ Zs containing an orbit of length 20, and of some other length greater than 1. But each of these is contrary to the hypothesis on $\overline{f}$. $\square$

We are now able to express disjointness of certain sequences, which is the key to recovering the appropriate ring of sets inside $S_\lambda(\mu)/S_\kappa(\mu)$. From this we shall be able to express disjointness of involutions (meaning disjointness of their supports), which are actually the elements we shall use to represent sets, and of more general sequences. But the first approximation uses elements satisfying *indec* and acting in the same way. Let us say that two such elements $S_\kappa(\mu).\overline{f}$ and $S_\kappa(\mu).\overline{g}$ have the *same action* if $\nu_m(\overline{f}) \geq \kappa$ and $\nu_m(\overline{g}) \geq \kappa$ for the same $m > 1$.

$disj_1(\overline{x}, \overline{y}) : indec(\overline{x}) \wedge indec(\overline{y}) \wedge comm(\overline{x}, \overline{y}) \wedge indec(\overline{x} * \overline{y})$.

Lemma 3.7 *For any sequences $\overline{f}$ and $\overline{g}$ of elements of $S_\lambda(\mu) - S_\kappa(\mu)$ of length 60, $S_\lambda(\mu)/S_\kappa(\mu) \models disj_1(S_\kappa(\mu).\overline{f}, S_\kappa(\mu).\overline{g})$ if and only if $\overline{f}$ and $\overline{g}$ are indecomposable with the same action, $|\mu - (supp\ \overline{f} \cup supp\ \overline{g})| = \mu$, and $|supp\ \overline{f} \cap supp\ \overline{g}| < \kappa$.*

Proof It is clear that if two indecomposable sequences in $S_\lambda(\mu)/S_\kappa(\mu)$ have the same action, and almost disjoint supports (meaning that the intersection of their supports has cardinality less than κ), then they commute, and their pointwise product also is indecomposable (provided that the union of their supports does not have small complement). Conversely suppose that the given conditions apply. Then as in the previous proof, if the supports of $\overline{f}$ and $\overline{g}$ are not almost disjoint, then indecomposability of $\overline{f} * \overline{g}$ is violated. It also follows that $\overline{f}, \overline{g}$, and $\overline{f} * \overline{g}$ must all have the same action. □

It is now possible to find formulae expressing the following concepts inside $S_\lambda(\mu)/S_\kappa(\mu)$:

membership in $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$,

the boolean operations on $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$,

the action of $S_\lambda(\mu)/S_\kappa(\mu)$ on $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$.

First we represent members of $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ by involutions, and let $set(x)$ be the formula $x^2 = 1$, (where for present purposes it is easier to count the identity as an 'involution'). The idea is that each involution will encode its support (so for example the identity represents the empty set). Of course this only makes any sense if we can tell when two involutions encode the same set.

Now let i be such that a_i has order 2 in $A(5)$. Then for $g \in S_\lambda(\mu)$ with $|\mu - supp\ g| = \mu$, $S_\kappa(\mu).g$ has order 2 if and only if there is some indecomposable $\overline{g}$ with $\nu_5(\overline{g}) \geq \kappa$ such that $S_\kappa(\mu).g = S_\kappa(\mu).g_i$.

$disj'(x,y) : set(x) \wedge set(y) \wedge \exists \overline{z} \exists \overline{t}(z_i = x \wedge t_i = y \wedge disj_1(\overline{z}, \overline{t}))$.

$disj(x,y) : \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists y_1 \exists y_2 \exists y_3 \exists y_4 (x = x_1x_2x_3x_4 \wedge y = y_1y_2y_3y_4 \wedge \bigwedge_{1 \leq i,j \leq 4} disj'(x_i, y_j))$.

Here the idea is that $disj'$ should express disjointness of (sets encoded by) involutions, and $disj$ should express disjointness of (the supports of) arbitrary permutations. Because of the possibility that $\lambda = \mu^+$ we use products of four elements rather than just two, since we need to be able to express an arbitrary group element in terms of involutions the complements of whose supports have cardinality μ.

$subset(x,y) : set(x) \wedge set(y) \wedge \forall z(disj(y,z) \to disj(x,z))$,

$sameset(x,y) : set(x) \wedge set(y) \wedge \forall z(disj(y,z) \leftrightarrow disj(x,z))$,

$union(x,y,z) : set(x) \wedge set(y) \wedge set(z) \wedge \forall t(subset(x,t) \wedge subset(y,t) \leftrightarrow subset(z,t))$,

$intersect(x,y,z) : set(x) \wedge set(y) \wedge set(z) \wedge \forall t(subset(t,x) \wedge subset(t,y) \leftrightarrow subset(t,z))$,

$union_n(\overline{x}, y) : (\forall z)(disj(z,y) \leftrightarrow \bigwedge_{i=1}^n disj(z, x_i))$,

$map(x,y,z) : set(x) \wedge set(y) \wedge sameset(z^{-1}xz, y)$,

The following result sums up what these formulae express.

Lemma 3.8 *(i) For any $f, g \in S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models disj'(S_\kappa(\mu).f, S_\kappa(\mu).g)$ if and only if $S_\kappa(\mu).f$ and $S_\kappa(\mu).g$ are involutions such that $|supp\ f \cap supp\ g| < \kappa$ and $|\mu - (supp\ f \cup supp\ g)| = \mu$.*

(ii) For any $f, g \in S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models disj(S_\kappa(\mu).f, S_\kappa(\mu).g)$ *if and only if* $|supp\ f \cap supp\ g| < \kappa$.

(iii) For any $f, g \in S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models subset(S_\kappa(\mu).f, S_\kappa(\mu).g)$ *if and only if* $S_\kappa(\mu).f$ *and* $S_\kappa(\mu).g$ *are involutions such that* $|supp\ f - supp\ g| < \kappa$.

(iv) For any $f, g \in S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models sameset(S_\kappa(\mu).f, S_\kappa(\mu).g)$ *if and only if* $S_\kappa(\mu).f$ *and* $S_\kappa(\mu).g$ *are involutions such that* $|supp\ f - supp\ g|$, $|supp\ g - supp\ f| < \kappa$.

(v) For any $f, g, h \in S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models union(S_\kappa(\mu).f, S_\kappa(\mu).g, S_\kappa(\mu).h)$ *if and only if* $S_\kappa(\mu).f, S_\kappa(\mu).g$, *and* $S_\kappa(\mu).g$ *are involutions such that* $supp\ f \cup supp\ g$ *and* $supp\ h$ *differ by a set of cardinality* $< \kappa$.

(vi) Similarly for intersections.

(vii) For any $\overline{f}, g \in S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models union_n(S_\kappa(\mu).\overline{f}, S_\kappa(\mu).g)$ *if and only if* $\bigcup_{i=1}^n supp(f_i)$ *and* $supp\ g$ *differ by a set of cardinality* $< \kappa$.

(viii) For any $f, g, h \in S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models map(S_\kappa(\mu).f, S_\kappa(\mu).g, S_\kappa(\mu).h)$ *if and only if* $S_\kappa(\mu).f$ *and* $S_\kappa(\mu).g$ *are involutions and* $(supp\ f)h$ *and* $supp\ g$ *differ by a set of cardinality* $< \kappa$.

Proof (ii) follows from the fact that any permutation may be written as a product of two involutions, and any involution may be written as a product of two involutions the complement of whose support has cardinality μ. The rest of the proof is straightforward. □

Corollary 3.9 *The ring of sets* $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ *and the natural action of* $S_\lambda(\mu)/S_\kappa(\mu)$ *on this ring are interpretable inside the group* $S_\lambda(\mu)/S_\kappa(\mu)$.

This result is due to Rubin [7] Theorem 4.3, but using different methods. An alternative route to the same conclusion, avoiding so much detail on permutation representations, starts by interpreting $\mathcal{P}_\lambda(\mu)/\mathcal{P}_\kappa(\mu)$ in $S_\lambda(\mu)/S_\kappa(\mu)$ using parameters $\overline{f^*}, \overline{f^{**}}$. The first of these acts as $A(5)$ with orbits of degree 5 and 1 only, and with the aid of the second, disjointness can be expressed more rapidly. The parameters are then eliminated at a later stage.

4 Interpreting $\mathcal{M}_{\kappa\lambda\mu}$ in $S_\lambda(\mu)/S_\kappa(\mu)$

This is carried out as follows:

members of IS_n are represented by 'pure' n-tuples, being those for which almost all orbits are isomorphic, modulo isomorphism of this action,

members of $Card^-$ are represented by group elements which encode sets, (that is, involutions), modulo the relation of having equal cardinality,

members of F_n are represented by n-tuples of group elements, modulo conjugacy.

In addition we have to show definability of the relations and functions in the signature.

First we show how to distinguish the case $\lambda = \mu^+$ (which has already required special treatment in the previous section). We use the formula

$$max : (\exists x)(\forall y)(disj(x,y) \to y = 1)$$

(expressing that λ has its *maximum* value)

Remark 4.1 *$\lambda = \mu^+$ if and only $S_\lambda(\mu)/S_\kappa(\mu) \models max$.*

To carry out the interpretation more formally we require the following formulae:
$disj_n(\overline{x},\overline{y}) : \bigwedge_{1\le i,j\le n} disj(x_i,y_j)$,
$restr_n(\overline{x},\overline{y}) : \exists \overline{z}(disj_n(\overline{x},\overline{z}) \wedge \overline{x} * \overline{z} = \overline{y})$,
$\overline{x} = 1 : \bigwedge_{i=1}^n x_i = 1$,
$compat_n(\overline{x},\overline{y}) : \exists \overline{z} \exists t(\overline{z} \ne 1 \wedge restr_n(\overline{z},\overline{x}) \wedge restr_n(\overline{z}^t,\overline{y}))$,
$pure_n(\overline{x}) : \forall \overline{y} \forall \overline{z}(\overline{y} \ne 1 \wedge \overline{z} \ne 1 \wedge restr_n(\overline{y},\overline{x}) \wedge restr_n(\overline{z},\overline{x})$
$\to compat_n(\overline{y},\overline{z})) \wedge (\neg max \to \overline{x} \ne 1)$,
$iso_n(\overline{x},\overline{y}) : pure_n(\overline{x}) \wedge pure_n(\overline{y}) \wedge (compat_n(\overline{x},\overline{y}) \vee \overline{x} = \overline{y} = 1)$.

Lemma 4.2 *(i) For any finite sequence $\overline{f}$ of members of $S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models pure_n(S_\kappa(\mu).\overline{f})$ if and only if the non-trivial actions of $\overline{f}$ on all but a small union of the orbits of $\langle \overline{f} \rangle$ are isomorphic or, if $\lambda = \mu^+$, almost all orbits have size 1..*

(ii) For any sequences $\overline{f},\overline{g}$ in $S_\lambda(\mu), S_\lambda(\mu)/S_\kappa(\mu) \models iso_n(S_\kappa(\mu).\overline{f}, S_\kappa(\mu).\overline{g})$ if and only if the actions of $\overline{f}$ and $\overline{g}$ on all but a small union of orbits of $\langle \overline{f} \rangle, \langle \overline{g} \rangle$ have the same isomorphism type in IS_n.

Note that it is not enough to talk of the actions of $\langle \overline{f} \rangle$ on its orbits; we need to distinguish the generating tuple $\overline{f}$ in order to capture IS_n. Observe that the final parts of the formulae $pure_n$ and iso_n cover the case $\lambda = \mu^+$, and correspond to the remark in parentheses in Definition 2.1(ii). Similar remarks apply to the treatment of F_n.

As mentioned above, for sort 2 we just use involutions, this time modulo the equivalence relation given by
$samecard(x,y) : set(x) \wedge set(y) \wedge \exists x_1 \exists x_2 \exists y_1 \exists y_2(disj(x_1,x_2) \wedge disj(y_1,y_2) \wedge$
$x = x_1x_2 \wedge y = y_1y_2 \wedge conj(x_1,y_1) \wedge conj(x_2,y_2))$.

This is slightly more complicated than the expected '$set(x) \wedge set(y) \wedge conj(x,y)$' in view of the case $\lambda = \mu^+$. And the sorts 3 have already been remarked on.

It remains to show that the relations and functions of $\mathcal{M}_{\kappa\lambda\mu}$ are definable.

First the ordering $\le$ (and hence $<$) on $Card$ is definable by
$lesseq(x,y) : set(x) \wedge set(y) \wedge (\exists z)(subset(z,y) \wedge samecard(x,z))$.

To define $Eq^1, Eq, Prod^1$ and $Prod$ we use
$eq^1(x_1,x_2) : pure_2(x_1,x_2) \wedge x_1 = x_2, eq(x_1,x_2) : x_1 = x_2$,
$prod^1(x_1,x_2,x_3) : pure_3(x_1,x_2,x_3) \wedge x_1x_2 = x_3$
and
$prod(x_1,x_2,x_3) : x_1x_2 = x_3$

respectively. Note that there is a slight difference between Eq^1 and Eq (and between $Prod^1$ and $Prod$), since in the former case (x_1, x_2) is meant to represent a member of IS_2, but in the latter, of F_2.

We may define $Proj^1_n$ and $Proj_n$ by

$proj^1_n((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_n)) : pure_{n+1}(x_1, \ldots, x_{n+1}) \wedge pure_n(y_1, \ldots, y_n) \wedge$
$iso_n(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and

$proj_n((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_n)) : conj_n((x_1, \ldots, x_n), (y_1, \ldots, y_n))$,

and App_n by

$app_n(\overline{x}, \overline{y}, z) : pure_n(\overline{y}) \wedge [((\exists \overline{t})(pure_n(\overline{t}) \wedge compat_n(\overline{y}, \overline{t}) \wedge restr_n(\overline{t}, \overline{x})$
$\wedge (\forall \overline{u})(restr_n(\overline{t}, \overline{u}) \wedge restr_n(\overline{u}, \overline{x}) \wedge pure_n(\overline{u}) \rightarrow \overline{t} = \overline{u}) \wedge (\exists v)(union_n(\overline{t}, v)$
$\wedge samecard(v, z))) \vee ((\forall \overline{t})(compat_n(\overline{y}, \overline{t}) \rightarrow \neg restr_n(\overline{t}, \overline{x})) \wedge z = 1)]$,

which we may paraphrase as 'either there is a maximal pure restriction $\overline{t}$ of $\overline{x}$ compatible with $\overline{y}$ and of cardinality (coded by) z, or $\overline{x}$ has no restriction compatible with $\overline{y}$ and $z = 1$ (that is, codes 0)'. If $\lambda = \mu^+$, app_n is modified to cover the case $\overline{y} = 1$, and if $\kappa = \aleph_0$ we have to count orbits rather than their union, and the statement about v is modified to express 'there is a set having the same cardinality as z which intersects each orbit of $\overline{t}$ and is minimal subject to this'. To justify this we further note that the case $\kappa = \aleph_0$ can be distinguished by the sentence

$$(\exists x)(\forall y)(restr_1(y, x) \rightarrow (y = 1 \vee y = x)).$$

We have proved the following:

Theorem 4.3 *$\mathcal{M}_{\kappa\lambda\mu}$ is interpretable in the group $S_\lambda(\mu)/S_\kappa(\mu)$.*

We remark that 'interpretability' here is taken in the usual sense (called 'explicit interpretability' in [10]). This means for instance that, rather than just transferring the first order properties, we are able to deduce that whenever $S_{\lambda_1}(\mu_1)/S_{\kappa_1}(\mu_1) \cong S_{\lambda_2}(\mu_2)/S_{\kappa_2}(\mu_2)$ then $\mathcal{M}_{\kappa_1\lambda_1\mu_1} \cong \mathcal{M}_{\kappa_2\lambda_2\mu_2}$, and hence to try to distinguish the groups $S_\lambda(\mu)/S_\kappa(\mu)$ up to isomorphism as well as up to elementary equivalence. But for us here the following is the point.

Corollary 4.4 *$\mathcal{M}_{\kappa_1\lambda_1\mu_1} \equiv \mathcal{M}_{\kappa_2\lambda_2\mu_2}$ if and only if $S_{\lambda_1}(\mu_1)/S_{\kappa_1}(\mu_1) \equiv S_{\lambda_2}(\mu_2)/S_{\kappa_2}(\mu_2)$.*

Proof This follows from Theorems 2.6 and 4.3. □

In the next sections we give more details about the circumstances under which $\mathcal{M}_{\kappa_1\lambda_1\mu_1} \equiv \mathcal{M}_{\kappa_2\lambda_2\mu_2}$.

5 Refinements and the case $cf(\kappa) > 2^{\aleph_0}$

We now make some remarks about distinguishing the elementary theories of $S_\lambda(\mu)/S_\kappa(\mu)$ for different values of κ, λ, μ, which by Corollary 4.4 is equivalent

to distinguishing the $\mathcal{M}_{\kappa\lambda\mu}$. In the first place, according to Remark 4.1, the case $\lambda = \mu^+$ can be singled out in $S_\lambda(\mu)/S_\kappa(\mu)$ by means of the sentence *max* of the language of group theory, and hence also by a suitable sentence in $\mathcal{M}_{\kappa\lambda\mu}$. So we may treat the cases $\lambda \leq \mu, \lambda = \mu^+$ separately. Now when $\lambda \leq \mu$ the cardinal μ actually plays no part at all in the structure $\mathcal{M}_{\kappa\lambda\mu}$, so we at once see that for fixed $\kappa \leq \lambda$, all the $\mathcal{M}_{\kappa\lambda\mu}$ with $\mu \geq \lambda$ are elementarily equivalent. More is even true at this stage, since many of the $\mathcal{M}_{\kappa\lambda\mu}$ are in fact isomorphic. For instance if $cf(\kappa_1), cf(\kappa_2) \geq (2^{\aleph_0})^+$ then $\mathcal{M}_{\kappa_1\kappa_1^+\mu_1} \cong \mathcal{M}_{\kappa_2\kappa_2^+\mu_2}$ (for $\mu_1 \geq \kappa_1^+, \mu_2 \geq \kappa_2^+$) since in this case $Card^- = \{0, \kappa_1\}, \{0, \kappa_2\}$ respectively, and similarly $\mathcal{M}_{\kappa_1\kappa_1^{++}\mu_1} \cong \mathcal{M}_{\kappa_2\kappa_2^{++}\mu_2}$ etc.

We know of course that $Th(\mathcal{M}_{\kappa\lambda\mu})$ can only take at most $2^{\aleph_0}$ values, and so there will be many pairs of distinct triples giving elementarily equivalent models. In [11] this was however illustrated more explicitly, and we carry out a similar analysis here. There a characterization of elementary equivalence was provided based on the second order theory of certain many-sorted ordinal structures, whose sorts all had cardinality $\leq 2^{\aleph_0}$, and we give a parallel treatment. While doing so we give a few more details about the material from [11] (which in its turn is related to [4]). First we show how a suitable second order logic can be represented in the structures $\mathcal{M}_{\kappa\lambda\mu}$. Small modifications are made in the case $\kappa = \aleph_0$ (distinguishable in the language of group theory), which we do not spell out explicitly.

To represent subsets of IS_n in $\mathcal{M}_{\kappa\lambda\mu}$ is rather straightforward, but subsets of $IS_m \times IS_n$ are harder to deal with. We use 'products' (similar to the method of section 3) to help us to do this. We say that $t \in IS_{m+n}$ is a *product* of $t_1 \in IS_m$ and $t_2 \in IS_n$ if t has the form $((A, \overline{f}))_\cong$ where $A = \{\alpha_{xy} : x \in X, y \in Y\}$, the $\langle f_1, \ldots, f_m\rangle$-orbits of A are $\{\alpha_{xy} : y \in Y\}$ for $x \in X$, all of type t_1, the $\langle f_{m+1}, \ldots, f_{m+n}\rangle$-orbits of A are $\{\alpha_{xy} : x \in X\}$ for $y \in Y$, all of type t_2, and the actions of f_i and f_j on A for $1 \leq i \leq m < j \leq m+n$ commute. We say that $h \in F_{m+n}$ is a *product* if whenever $h = (h')_{\mathcal{E}_{m+n}}$, $\sum\{h'(t) : t \in IS_{m+n} \wedge t$ not a product$\} < \kappa$.

The idea here is that if $\overline{f}$ acts as a product on almost all of its orbits, then we can uniquely recover its actions on the first m and last n co-ordinates, so that products provide a way of encoding sets of ordered pairs. As illustrated in section 3 however, the actions of tuples may commute without their being a product, and so the natural condition to try to capture expressibility as a product, namely commutativity, does not work. This time however this does not matter; the point being that when two actions commute, and together generate a transitive action, the projections onto the two sets of co-ordinates are uniquely determined. Let us therefore say that $h \in F_{m+n}$ is a *product* if whenever $h = (h')_{\mathcal{E}_{m+n}}$,

$$\sum\{h'(t) : t \in IS_{m+n} \wedge t = ((A, (\overline{f}_1, \overline{f}_2)))_\cong \to \neg comm_{m,n}(\overline{f}_1, \overline{f}_2)\} < \kappa.$$

We now represent subsets of IS_n of cardinality $< \lambda$ by $h \in F_n$ such that

$(\forall t \in IS_n)h(t) \leq \kappa$, and subsets of $IS_m \times IS_n$ of cardinality $< \lambda$ by $k \in F_{m+n}$ which are products and such that $(\forall t \in IS_{m+n})k(t) \leq \kappa$. The subset of IS_n encoded by h is then $\{t : h(t) = \kappa\}$ and the subset of $IS_m \times IS_n$ encoded by k is $\{(t_1, t_2) \in IS_m \times IS_n : (\exists t \in IS_{m+n})(k(t) = \kappa$ and t_1, t_2 are the members of IS_m, IS_n determined on co-ordinates 1 to m and $m+1$ to $m+n$ respectively)$\}$. (The definition of 'product' ensures that these are uniquely determined from t, since if the actions of $\overline{f}_1$ and $\overline{f}_2$ on the $\overline{f}$-orbit A commute then they each preserve the set of orbits of the other, and all actions of $\overline{f}_1$ on its orbits are isomorphic, and similary for $\overline{f}_2$.)

Lemma 5.1 *There are formulae of the language of $\mathcal{M}_{\kappa\lambda\mu}$ expressing the following:*

(i) h encodes a subset of IS_n,
(ii) k encodes a subset of $IS_m \times IS_n$,
(iii) $mem_n(t, h) : t$ lies in the set encoded by h,
$mem_{m,n}(t_1, t_2, k) : (t_1, t_2)$ lies in the set encoded by k,
(iv) $equal_n(h, h') : h, h'$ encode the same subset of IS_n,
$equal_{m,n}(k, k') : k, k'$ encode the same subset of $IS_m \times IS_n$,
(v) $fun_{m,n}(k) : k$ encodes a function (from a subset of IS_m into IS_n),
(vi) one-one$fun_{m,n}(k) : k$ encodes a 1–1 function.

Proof (i) h encodes a subset of IS_n if and only if $(\forall t \in IS_n)App_n(h, t) \leq \kappa$.

(ii) By appeal to Theorem 2.6 we may express projections of $h \in F_{m+n}$ to co-ordinates 1 to m and $m+1$ to $m+n$, and then use the formula $comm_{m,n}$.

(iii) $mem_n(t, h)$ is taken as $App_n(h, t) = \kappa$.

For $mem_{m,n}(t_1, t_2, k)$ we take $(\exists t \in IS_{m+n})(App_{m+n}(h, t) = \kappa \wedge t_1, t_2$ are the projections of t onto co-ordinates 1 to m and $m+1$ to $m+n$ respectively). (The fact that we can express these more generalized projections here follows by appeal to Theorem 2.6, though they could also have been included in the signature of the $\mathcal{M}_{\kappa\lambda\mu}$ if desired.)

(iv), (v), and (vi) follow from (iii). $\square$

For the remainder of this section we specialize to the case $cf(\kappa) > 2^{\aleph_0}$, to avoid complications. We return to the general case in section 6. One of the benefits of assuming $cf(\kappa) > 2^{\aleph_0}$ is that we can dispense altogether with the equivalence relations $\mathcal{E}_n$. This is because any function from F_n to $Card$ is $\mathcal{E}_n$-equivalent to a unique function from F_n into $Card^-$ (obtained by replacing all values below κ by 0). Various other simplifications and interdefinabilities in this case are described in the following theorem.

Theorem 5.2 *Suppose that $\kappa < \lambda \leq \mu^+$ and $cf(\kappa) > 2^{\aleph_0}$. Then*

(i) for each $h \in F_n$ there is a unique $\mathcal{E}_n$-representative which is a function from IS_n to $Card^-$ (so that from now on in this section we dispense with $\mathcal{E}_n$ and regard F_n as a subset of $(Card^-)^{IS_n}$),

(ii) for each $n, Sum_n : F_n \to Card^-$ *given by* $Sum_n(h) = \sum\{h(t) : t \in IS_n\}$ *is definable in* $\mathcal{M}_{\kappa\lambda\mu}$,

(iii) $Eq, Prod,$ *and* $Proj_n$ *are all definable in*

$$\mathcal{M}^*_{\kappa\lambda\mu} = ((IS_n)_{n\geq 1}, Card^-, (F_n)_{n\geq 1}; Eq^1, Prod^1, (Proj^1_n)_{n\geq 1}, <, (App_n)_{n\geq 1}),$$

and conversely $Eq^1, Prod^1,$ *and* $Proj^1_n$ *are all definable in*

$$((IS_n)_{n\geq 1}, Card^-, (F_n)_{n\geq 1}; <, (App_n)_{n\geq 1}, Eq, Prod, (Proj_n)_{n\geq 1}).$$

Proof (ii) This is because $Sum_n(h)$ may also be written as $\sup\{h(t) : t \in IS_n\}$ (since $\kappa > 2^{\aleph_0}$), so that

$$Sum_n(h) = \alpha \Leftrightarrow (\forall t) App_n(h,t) \leq \alpha \ \wedge \ (\forall \beta < \alpha)(\exists t)(\beta < App_n(h,t)).$$

(iii) $Eq = \{h \in F_2 : (\forall t \in IS_2)(App_2(h,t) \neq 0 \to Eq^1(t))\}$,
$Prod = \{h \in F_3 : (\forall t \in IS_3)(App_3(h,t) \neq 0 \to Prod^1(t))\}$.
For $Proj_n$ we remark that $B_{t't} \neq \emptyset \Leftrightarrow Proj^1_n(t',t)$, and so

$$Proj_n(h)(t) = \sum\{h(t') : Proj^1_n(t',t)\} = \sup\{h(t') : Proj^1_n(t',t)\}.$$

As in (ii) we see that

$$\begin{array}{l} Proj_n(h)(t) = \alpha \Leftrightarrow (\forall t')(Proj^1_n(t',t) \to App_{n+1}(h,t') \leq \alpha) \\ \wedge\, (\forall \beta < \alpha)(\exists t' \in IS_{n+1})(Proj^1_n(t',t) \ \wedge \ \beta < App_{n+1}(h,t')). \end{array}$$

Conversely we have

$$\begin{array}{c} Eq^1 = \{t \in IS_2 : (\exists h \in F_2)(Eq(h) \ \wedge \ App_2(h,t) \neq 0)\}, \\ Prod^1 = \{t \in IS_3 : (\exists h \in F_3)(Prod(h) \ \wedge \ App_3(h,t) \neq 0)\}, \end{array}$$

and

$$Proj^1_n(t_1,t_2) \Leftrightarrow (\forall h \in F_{n+1})(App_{n+1}(h,t_1) \neq 0 \to Proj_n(h)(t_2) \neq 0). \quad \square$$

This theorem tells us that when $cf(\kappa) > 2^{\aleph_0}$ it suffices to consider the structures $\mathcal{M}^*_{\kappa\lambda\mu}$, and here, since the members of F_n are now viewed as functions from IS_n to $Card^-$, this amounts to a version of second order logic on the sorts 1 and 2, together with $Eq^1, Prod^1, Proj^1_n$, and $<$. The sorts IS_n are independent of κ, λ, μ, and so the main point is to analyse $Card^-$. We give an analysis of this situation similar to that in [11] which involves defining suitable 'small' ordinals (meaning of cardinality $\leq 2^{\aleph_0}$), sufficient to capture the elementary theory.

In what follows we extend the definition of 'cofinality' to zero or successor ordinals by letting $cf(0) = 0$ and $cf(\alpha + 1) = 1$. Let $\Omega = (2^{\aleph_0})^+$. Then any ordinal α may be written uniquely in the form

$$\alpha = \Omega^\omega.\alpha_\omega + \ldots + \Omega^n.\alpha_{[n]} + \ldots + \Omega.\alpha_{[1]} + \alpha_{[0]}$$

where Ω^ω is the ordinal power, $\alpha_{[n]} < \Omega$ for $n \in \omega$, and $\{n : \alpha_{[n]} \neq 0\}$ is finite. We write $\alpha[n] = \Omega^\omega.\alpha_\omega + \ldots + \Omega^{n+1}.\alpha_{[n+1]}$ and let

$$\alpha^{[n]} = \begin{cases} 1 + cf(\alpha[n]) & \text{if } cf(\alpha[n]) < \Omega, \\ 0 & \text{otherwise} \end{cases}$$

For ordinals α, β, and $k \in \omega$, let $\alpha \sim_k \beta$ if $\alpha_{[l]} = \beta_{[l]}$ and $\alpha^{[l]} = \beta^{[l]}$ for all $l \leq k$, and $\alpha \sim \beta$ if $\alpha \sim_k \beta$ for all k. For a set of ordinals $\{\alpha\} \cup A$ let $\gamma(\alpha, A)$ be the order-type of $\{\beta \leq \alpha : (\forall \gamma \in A)(\gamma < \alpha \to \gamma < \beta)\}$. Note that this set is the final segment of $\alpha \cup \{\alpha\}$ consisting of all (strict) upper bounds of $A \cap \alpha$.

The following lemma is stated in [10] and a proof outlined. The result is also related to [4]. We give fuller details here for the reader's benefit.

Lemma 5.3 *(i) $\sim_k$ is an equivalence relation. For each α there is $\beta < \Omega^{k+2}$ with $\alpha \sim_k \beta$; and if $\alpha \sim_k \beta$ then $\alpha < \Omega^{k+1}$ if and only if $\beta < \Omega^{k+1}$, and each of these implies $\alpha = \beta$.*

(ii) If $\alpha \geq \Omega^{k+1}$ then for any $\beta, \alpha \sim_k \beta + \alpha$.

(iii) If $\alpha_\gamma \sim_k \beta_\gamma$ for each $\gamma < \delta$, then $\sum_{\gamma<\delta} \alpha_\gamma \sim_k \sum_{\gamma<\delta} \beta_\gamma$.

(iv) If $\alpha \sim_{k+1} \beta$, and $A \subseteq \alpha$ with $|A| < \Omega$, there is an order-preserving map $F : A \to \beta$ such that for each $a \in A \cup \{\alpha\}, \gamma(a, A) \sim_k \gamma(F(a), F(A))$ (where $F(\alpha)$ is taken to equal β).

Proof (i) Let $\beta = \Omega^{k+1}.\beta_{[k+1]} + \Omega^k.\alpha_{[k]} + \ldots + \Omega.\alpha_{[1]} + \alpha_{[0]}$ where $\beta_{[k+1]}$ is given as follows:

$$\beta_{[k+1]} = \begin{cases} \alpha_{[n]} & \text{if } \alpha_{[n]} \neq 0 \text{ for some least } n > k, \\ \omega & \text{if } \alpha_{[n]} = 0 \text{ for all } n > k \text{ and } \alpha_\omega \text{ is a successor}, \\ cf(\alpha_\omega) & \text{if } \alpha_{[n]} = 0 \text{ for all } n > k, \alpha_\omega \text{ a limit ordinal and } cf(\alpha_\omega) < \Omega \\ 0 & \text{otherwise} \end{cases}$$

Then $\alpha_{[l]} = \beta_{[l]}$ for $l \leq k$ is immediate. If $\alpha < \Omega^{k+1}$ the final clause applies, so $\beta = \alpha < \Omega^{k+1}$. Also if $\beta < \Omega^{k+1}$ then $\alpha < \Omega^{k+1}$ so the last part also follows.

Now suppose the first clause applies. Then if $l \leq k$,

$$\alpha[l] = \Omega^\omega.\alpha_\omega + \ldots + \Omega^n.\alpha_{[n]} + \Omega^k.\alpha_{[k]} + \ldots + \Omega^{l+1}.\alpha_{[l+1]}$$

$$\text{and } \beta[l] = \Omega^{k+1}.\alpha_{[n]} + \Omega^k.\alpha_{[k]} + \ldots + \Omega^{l+1}.\alpha_{[l+1]}$$

which have equal cofinalities as $\alpha_{[n]} \neq 0$. If the second or third clause applies, then

$$\alpha[l] = \Omega^\omega.\alpha_\omega + \Omega^k.\alpha_{[k]} + \ldots + \Omega^{l+1}.\alpha_{[l+1]}$$

$$\text{and } \beta[l] = \Omega^{k+1}.\beta_{[k+1]} + \Omega^k.\alpha_{[k]} + \ldots + \Omega^{l+1}.\alpha_{[l+1]}.$$

If $\alpha^{[l]} \neq \beta^{[l]}$ then $\alpha_{[k]} = \ldots = \alpha_{[l+1]} = 0$ so $\alpha[l] = \Omega^{\omega}.\alpha_{\omega}$ and $\beta[l] = \Omega^{k+1}.\beta_{[k+1]}$. But if clause 2 applies, $cf(\alpha[l]) = \omega = cf(\beta[l])$, and if clause 3 applies, $cf(\alpha[l]) = cf(\alpha_{\omega}) = \beta_{[k+1]} = cf(\beta[l])$ after all.

(ii) As $\alpha \geq \Omega^{k+1}, \alpha_{\omega} \neq 0$, or $\alpha_{[n]} \neq 0$ for some $n > k$. Write $\alpha = \Omega^{N}.\alpha_{[N]} + \ldots + \alpha_{[0]}$ where $N > k, \alpha_{[N]} \neq 0$ (and where $N = \omega, \alpha_{[N]} = \alpha_{\omega}$ is allowed). Writing β in a similar way, if $n < N, \Omega^{n}.\beta_{[n]} + \Omega^{N}.\alpha_{[N]} = \Omega^{N}.\alpha_{[N]}$, and so $\beta + \alpha = \Omega^{\omega}.\beta_{\omega} + \ldots + \Omega^{N}.(\beta_{[N]} + \alpha_{[N]}) + \Omega^{N-1}.\alpha_{[N-1]} + \ldots + \alpha_{[0]}$. For $l \leq k$ we have

$$\alpha[l] = \Omega^{N}.\alpha_{[N]} + \ldots + \Omega^{l+1}.\alpha_{[l+1]}$$

$$\text{and } (\beta + \alpha)[l] = \Omega^{\omega}.\beta_{\omega} + \ldots + \Omega^{N}.(\beta_{[N]} + \alpha_{[N]}) + \ldots + \Omega^{l+1}.\alpha_{[l+1]}.$$

The only way in which $cf(\alpha[l])$ can be unequal to $cf((\beta + \alpha)[l])$ is for $\alpha[l] = \Omega^{N}.\alpha_{[N]} \neq \Omega^{\omega}.\beta_{\omega} + \ldots + \Omega^{N}.(\beta_{[N]} + \alpha_{[N]}) = (\beta + \alpha)[l]$. But if $\alpha_{[N]}$ is a limit ordinal, $cf(\Omega^{N}.\alpha_{[N]}) = cf(\alpha_{[N]}) = cf(\Omega^{N}.(\beta_{[N]} + \alpha_{[N]}))$, and if it is a successor, both cofinalities are equal to $cf(\Omega^{N})$, so we deduce that $\alpha^{[l]} = \beta^{[l]}$, and hence that $\alpha \sim_{k} \beta$.

(iii) If $\delta = 0$ or 1 the result is immediate. Next suppose $\delta = 2$. If $\alpha_1 \geq \Omega^{k+1}$ then by (i) also $\beta_1 \geq \Omega^{k+1}$ (and vice versa), so by (ii) $\alpha_0 + \alpha_1 \sim_{k} \alpha_1 \sim_{k} \beta_1 \sim_{k} \beta_0 + \beta_1$. Otherwise if m is greatest such that $\alpha_{1[m]} \neq 0$ then $m \leq k$ and also m is the greatest such that $\beta_{1[m]} \neq 0$, and $\alpha_0 + \alpha_1 =$

$$\Omega^{\omega}.\alpha_{0\omega} + \ldots + \Omega^{k+1}.\alpha_{0[k+1]} + \ldots + \Omega^{m}.(\alpha_{0[m]} + \alpha_{1[m]}) + \Omega^{m-1}.\alpha_{1[m-1]} + \ldots + \alpha_{1[0]},$$

and $\beta_0 + \beta_1 =$

$$\Omega^{\omega}.\beta_{0\omega} + \ldots + \Omega^{k+1}.\beta_{0[k+1]} + \ldots + \Omega^{m}.(\beta_{0[m]} + \beta_{1[m]}) + \Omega^{m-1}.\beta_{1[m-1]} + \ldots + \beta_{1[0]}.$$

¿From $\alpha_i^{[l]} = \beta_i^{[l]}$, for $i = 0, 1, l \leq k$ it follows that $(\alpha_0 + \alpha_1)^{[l]} = (\beta_0 + \beta_1)^{[l]}$.

We now prove the general case by transfinite induction. The successor case follows easily from the case $\delta = 2$. Suppose therefore that δ is a limit ordinal. Since $\alpha_{\gamma} \sim_{k} \beta_{\gamma}$ for $\gamma < \delta, \alpha_{\gamma} = 0 \Leftrightarrow \beta_{\gamma} = 0$, so we ignore any zero terms. Thus $cf(\sum_{\gamma<\delta} \alpha_{\gamma}) = cf(\sum_{\gamma<\delta} \beta_{\gamma})$ $(= cf(\delta))$. Also by (i), $\alpha_{\gamma} \geq \Omega^{k+1} \Leftrightarrow \beta_{\gamma} \geq \Omega^{k+1}$, and so $\{\gamma < \delta : \alpha_{\gamma} \geq \Omega^{k+1}\}$ is unbounded $\Leftrightarrow \{\gamma < \delta : \beta_{\gamma} \geq \Omega^{k+1}\}$ is unbounded. If each of these is unbounded, $\sum_{\gamma<\delta} \alpha_{\gamma} = \Omega^{k+1}.\alpha^{*}, \sum_{\gamma<\delta} \beta_{\gamma} = \Omega^{k+1}.\beta^{*}$ for some α^{*}, β^{*}. Otherwise for some $\gamma_0 < \delta, (\forall \gamma \geq \gamma_0)(\alpha_{\gamma}, \beta_{\gamma} < \Omega^{k+1})$ and as $\alpha_{\gamma} \sim_{k} \beta_{\gamma}$, by (i) $(\forall \gamma \geq \gamma_0)(\alpha_{\gamma} = \beta_{\gamma})$. In each case it follows that $\sum_{\gamma<\delta} \alpha_{\gamma} \sim_{k} \sum_{\gamma<\delta} \beta_{\gamma}$.

(iv) Given $\alpha \sim_{k+1} \beta$ and $A \subseteq \alpha, |A| < \Omega$ we write $\alpha = \alpha' + \xi, \beta = \beta' + \xi$ where α', β' are divisible by Ω^{k+2}, and $cf(\alpha') = cf(\beta')$ or $cf(\alpha'), cf(\beta') \geq \Omega$. First suppose $\xi = 0$.

If $cf(\alpha), cf(\beta) \geq \Omega$ we define $F : A \cup \{\alpha\} \to \beta \cup \{\beta\}$ by $F(\alpha) = \beta$ and otherwise inductively so that for each $a \in A, \gamma(a, A) \sim_{k} \gamma(F(a), F(A))$ and $\gamma(F(a), F(A)) < \Omega^{k+2}$. Suppose that $F(a')$ has been defined for $a' < a$ having these properties. Then $\gamma(a, A)$ is known and $F(a)$ has to be chosen. This is

possible by (i), and as $|A| < \Omega$ and Ω is regular, $F(a) < \Omega^{k+2} \leq \beta$. Moreover $\gamma(\alpha, A) \sim_k \gamma(\beta, F(A))$ is clear (since each of these order-types is cofinal with a positive multiple of Ω^{k+2}).

Next if $cf(\alpha) = cf(\beta) < \Omega$ we may write $\alpha = \sum_{\gamma<\lambda} \alpha_\gamma, \beta = \sum_{\gamma<\lambda} \beta_\gamma$ where each $\alpha_\gamma, \beta_\gamma$ has cofinality $\geq \Omega$ and is divisible by Ω^{k+2}. By the first case we define $F : A \cap \{\xi : \sum_{\delta<\gamma} \alpha_\delta \leq \xi < \sum_{\delta\leq\gamma} \alpha_\delta\} \to \{\xi : \sum_{\delta<\gamma} \beta_\delta \leq \xi < \sum_{\delta\leq\gamma} \beta_\delta\}$ for each $\gamma < \lambda$ and put the pieces together.

Finally for the case $\xi \neq 0$ we define $F : A \cap \alpha' \to \beta'$ as above and let $F(\alpha' + \gamma) = \beta' + \gamma$ for $\gamma < \xi$ and $\alpha' + \gamma \in A$. $\square$

Now we can prove the required bi-interpretability result. First we define the relevant structures.

Definition 5.4 *If $\alpha = \alpha(\kappa, \lambda, \mu)$ is the order-type of $Card^-$ in $\mathcal{M}_{\kappa\lambda\mu}$, we let*

$$\mathcal{N}^2_{\kappa\lambda\mu} = ((IS_n)_{n\geq 1}, (\alpha_{[n]})_{n\geq 0}, (\alpha^{[n]})_{n\geq 0}; Eq^1, Prod^1, (Proj^1_n)_{n\geq 1}, (<_n)_{n\geq 0}, (<^n)_{n\geq 0})$$

be the structure whose sorts are viewed as being pairwise disjoint (and all but finitely many $\alpha_{[n]}$ are empty), and $<_n, <^n$ are the usual (well-) orderings on $\alpha_{[n]}, \alpha^{[n]}$. The superscript 2 indicates that $\mathcal{N}^2_{\kappa\lambda\mu}$ is viewed as a second order structure in a very strong sense. This means that the language used to describe it, as well as including first order variables corresponding to each sort, also contains, for each tuple of sorts, variables ranging over relations whose ith entry lies in the ith sort of the tuple for each i. (Alternatively we can introduce sorts corresponding to each such tuple, adjoin all the natural relations, and work in first order logic).

In one direction the interpretability is 'explicit'.

Theorem 5.5 *If $cf(\kappa) > 2^{\aleph_0}$ then $\mathcal{N}^2_{\kappa\lambda\mu}$ is interpretable in $\mathcal{M}^*_{\kappa\lambda\mu}$.*

Proof The main point is to show how each $\alpha_{[n]}, \alpha^{[n]}$ may be represented in $\mathcal{M}^*_{\kappa\lambda\mu}$. Then we sketch how second order variables as described above are 'simulated' within the first order language of $\mathcal{M}^*_{\kappa\lambda\mu}$.

First we represent non-empty subsets of $Card^-$ of cardinality $< \Omega$ using members of F_2, the idea being that $h \in F_2$ represents its range. Since $2^{\aleph_0} < cf(\kappa) \leq \kappa < \lambda$, all such subsets of $Card^-$ can be represented. The following can then be expressed:

the set encoded by h_1 is a subset of that encoded by h_2:
$incl(h_1, h_2) : (\forall t_1)(\exists t_2)(h_1(t_1) = h_2(t_2))$,

(where as App_n is just 'application' we write $h_1(t_1)$ instead of $App_2(h_1, t_1)$ etc).

h_1 and h_2 encode the same set: $incl(h_1, h_2) \wedge incl(h_2, h_1)$,

h encodes a final segment of α:
$final(h) : (\forall t_1)(\forall \beta)(\exists t_2)(h(t_1) \leq \beta \to h(t_2) = \beta)$,

α is divisible by Ω $(\equiv \alpha_{[0]} = 0)$,

$div(\alpha,\Omega) : (\forall h)\neg final(h)$.

$\beta \in \alpha$ is divisible by Ω, Ω^{k+1}:

$div(\beta,\Omega) : (\forall h)(\forall \gamma<\beta)(\exists\delta)(\gamma \le \delta < \beta \wedge (\forall t)(h(t) \neq \delta))$,

$div(\beta,\Omega^{k+1}) : div(\beta,\Omega^k) \wedge (\forall h)(\forall\gamma<\beta)(\exists\delta)(\gamma \le \delta < \beta \wedge div(\delta,\Omega^k) \wedge (\forall t)(h(t)\neq\delta))$.

α is divisible by Ω^{k+1}:

$div(\alpha,\Omega^{k+1}) : div(\alpha,\Omega^k) \wedge (\forall h)(\forall\beta<\alpha)(\exists\gamma)(\beta \le \gamma < \alpha \wedge div(\gamma,\Omega^k) \wedge (\forall t)(h(t)\neq\gamma))$.

$\alpha_{[0]}$ is now represented by h such that

$$final(h) \wedge (\forall h')(incl(h,h') \wedge final(h') \to h = h'),$$

if such exists (and otherwise is 0). Similarly $\alpha_{[k]}$ is represented by h such that $(\forall t)(div(h(t),\Omega^k)) \wedge (\forall h')((\forall t)(div(h'(t),\Omega^k)) \wedge incl(h,h') \to h = h')$ if such exists (and otherwise is 0).

To encode facts about cofinalities we quantify over non-empty binary relations on $Card^-$ of cardinality $<\Omega$ using pairs (h_1,h_2) in F_2. Observe that if $\emptyset \neq R \subseteq (Card^-)^2, |R| < \Omega$, then for some $h_1,h_2 \in F_2, R = \{(h_1(t),h_2(t)) : t \in IS_2\}$. We can describe when R is an order-isomorphism thus:

$iso(h_1,h_2) : (\forall t_1)(\forall t_2)(h_1(t_1) \le h_1(t_2) \leftrightarrow h_2(t_1) \le h_2(t_2))$.

The set coded by h is then cofinal in α if

$cofinal(h) : (\forall\beta)(\exists t)(\beta \le h(t))$,

and h codes the cofinality of α, which is $<\Omega$ if

$cofinal(h) \wedge (\forall h')(cofinal(h') \to (\exists h_1)(\exists h_2)(iso(h_1,h_2) \wedge incl(h,h_1)$
$\wedge\, incl(h_1,h) \wedge incl(h_2,h'))$.

We may express $cf(\alpha) \ge \Omega$ by $(\forall h)\neg cofinal(h)$, and in a similar way for each $\beta \in \alpha$ we may express 'h codes a cofinal subset of β' and 'h codes $cf(\beta)$'. From this it should be clear that each $\alpha_{[n]}, \alpha^{[n]}$ can be represented (though presumably not uniformly).

Finally we show how to represent non-empty $n-$ary relations of cardinality $<\Omega$ on the sorts of the original structure according to sort provisos of the kind described above. Since the sorts of $\mathcal{N}^2_{\kappa\lambda\mu}$ all have cardinality $<\Omega$, this translates into *full* second order logic in this structure.

Without loss of generality consider a tuple of sorts in $\mathcal{M}^*_{\kappa\lambda\mu}$ of the form $(IS_{i_1},\dots,IS_{i_m},Card^-,\dots,Card^-)$ (where $Card^-$ occurs n times). We represent corresponding non-empty relations of cardinality $<\Omega$ by $m+n-$tuples of the form $(h_1,\dots,h_m,h'_1,\dots,h'_n)$ where $h_j \in F_{i_j+2}$ satisfies $fun_{2,i_j}(h_j)$ and $h'_j \in F_2$. Such an $m+n-$tuple represents

$$B = B(\overline{h}) = \{(H_1(t),\dots,H_m(t),h'_1(t),\dots,h'_n(t)) : t \in IS_2\}$$

where H_j is the function from IS_2 to IS_{i_j} determined by h_j.

Clearly $B(\overline{h})$ *is* a non-empty relation of the required kind of cardinality $<\Omega$, and conversely every such relation can be written as $B(\overline{h})$ for some $m+n$-tuple $\overline{h}$.

As in the proof of Lemma 5.1, $\bar{t} \in B(\bar{h})$ can be expressed in $\mathcal{M}^*_{\kappa\lambda\mu}$.

If R is a non-empty n-ary relation on $\mathcal{N}^2_{\kappa\lambda\mu}$ with specified sorts, then as each individual sort is definable as indicated above, R may be represented by a corresponding n-ary relation of the kind just discussed, in $\mathcal{M}^*_{\kappa\lambda\mu}$. $\square$

In the other direction we have a weaker notion than 'semi-interpretability', which is nevertheless sufficient to transfer elementary equivalence. The weakening just consists in having a whole family of representatives of a tuple rather than a single one. Let us say that for $k \in \mathbb{N}$ a *k-representation* of a tuple $(\bar{t}, \overline{\beta}, \overline{h})$ in $\mathcal{M}^*_{\kappa\lambda\mu}$ where each t_i lies in some $IS_j, \beta_i \in Card^-$, and each h_i lies in some F_j, is any tuple of the form $(A, <^*, \overline{g}, \bar{t}, \bar{b}, \overline{H})$ where

$<^*$ is a well-ordering of IS_2,

$A \subseteq IS_2, b_i \in A$,

if $h_i \in F_j$ then $H_i : IS_j \to A$,

$\overline{g} = (g_0, \dots, g_{k-1}, g^0, \dots, g^{k-1})$ where $g_i, g^i : A \cup \{\infty\} \to IS_2$,

for some order-preserving 1–1 map $\theta : A \to Card^-, \theta(b_i) = \beta_i$, $(\forall t \in IS_j)\theta(H_i(t)) = h_i(t)$, and for every $a \in A \cup \{\infty\}$ the order-type of $\{t \in IS_2 : t <^* g_j(a)\}$ equals $\gamma(\theta(a), \theta(A))_{[j]}$ and the order-type of $\{t \in IS_2 : t <^* g^j(a)\}$ equals $\gamma(\theta(a), \theta(A))^{[j]}$ (where we take $\theta(\infty) = \lambda$ ($> \beta$ for all $\beta \in Card^-$)).

We remark that all entries in this tuple except for the g_i, g^i lie in $\mathcal{N}^2_{\kappa\lambda\mu}$:

$$<^* \subseteq IS_2^2, A \subseteq IS_2, t_i \in IS_j, b_i \in IS_2, H_i \subseteq IS_j \times IS_2.$$

Moreover $g_i|A, g^i|A \subseteq IS_2^2$ lie in $\mathcal{N}^2_{\kappa\lambda\mu}$ so by making an easy modification to their 'official' definition, so do g_i, g^i. But θ does not (which is why it does not form part of the representation).

Lemma 5.6 *Any $(\bar{t}, \overline{\beta}, \overline{h})$ has a k-representation.*

Proof Let A' be the union of the set of entries of $\overline{\beta}$ and the ranges of the $h_i, A' \subseteq Card^-$. Then $|A'| \leq 2^{\aleph_0}$. Also $\gamma(a, A')_{[j]}, \gamma(a, A')^{[j]}$ each has order-type at most that of A'. We choose $A \subseteq IS_2$ of cardinality $|A'|$, a bijection $\theta : A \to A'$, and a well-ordering $<^*$ of IS_2 extending $\theta^{-1}(<)$. For $a \in A$ let $g_j(a)$ equal the $\gamma(\theta(a), \theta(A))_{[j]}$th element of IS_2 under $<^*$, $g_j(\infty) = \lambda$, and similarly for $g^j(a)$. Let $b_i = \theta^{-1}(\beta_i)$ and $H_i(t) = \theta^{-1}h_i(t)$ for each t. $\square$

Lemma 5.7 *In the language of $\mathcal{N}^2_{\kappa\lambda\mu}$ for each k there is a formula φ_k such that $\mathcal{N}^2_{\kappa\lambda\mu} \models \varphi_k[A, <^*, \overline{g}, b]$ if and only if*

$A \subseteq IS_2, <^*$ is a well-ordering of $IS_2, b \in IS_2$,

$\overline{g} = (g_0, \dots, g_{k-1}, g^0, \dots, g^{k-1})$ where the g_i, g^i are functions from $A \cup \{\infty\}$ into IS_2,

and if A and IS_2 are enumerated in $<^*$-increasing order as $\{a_\beta : \beta < \beta_0\}, \{b_\gamma : \gamma < \gamma_0\}$, and $b = b_{\gamma_1}$, and for each $\beta < \beta_0, \alpha_\beta$ is an ordinal for which $b_{(\alpha_\beta)_{[l]}} = g_l(a_\beta), b_{(\alpha_\beta)^{[l]}} = g^l(a_\beta)$ for $l < k$, then $\sum_{\beta<\beta_0} \alpha_\beta \sim_k \gamma_1$.

Proof The proof of this is obtained by formalizing a transfinite induction similar to that used in the proof of Lemma 5.3(iii). □

Lemma 5.8 *For each tuple of sorts and each k,*

(i) there is a formula rep_k of the language of $\mathcal{N}^2_{\kappa\lambda\mu}$ which holds in $\mathcal{N}^2_{\kappa\lambda\mu}$ for a tuple having the right sequence of sorts if and only if it is a k-representation (of some tuple),

(ii) there is a formula $isorep_k$ of the language of $\mathcal{N}^2_{\kappa\lambda\mu}$ which holds in $\mathcal{N}^2_{\kappa\lambda\mu}$ for a pair of tuples each having the right sequence of sorts if and only if there is some tuple of $\mathcal{M}^*_{\kappa\lambda\mu}$ of which they are both k-representations.

Proof (i) To tell whether a tuple is a k-representation we first verify lines 1 to 4 of the definition, which can all be expressed in the language of $\mathcal{N}^2_{\kappa\lambda\mu}$ (where we have second order logic). If they hold then the main point is to check whether θ can be defined to give the correct γ-values. For this we appeal to the previous lemma, and we also need to refer to the sorts $\alpha_{[j]}, \alpha^{[j]}$ for $j < k$ to ensure that the right $\gamma(\theta(\infty), \theta(A))_{[j]}, \gamma(\theta(\infty), \theta(A))^{[j]}$ values can be achieved. Then we may define $\overline{\beta}, \overline{h}$ by $\beta_i = \theta(b_i), h_i(t) = \theta(H_i(t))$.

(ii) Similar remarks apply except that we should now work with 'minimal' A, that is, those which are equal to the union of the $\{b_i\}$ and $range(H_i)$. □

Theorem 5.9 *For every (first order) formula $\varphi(x_0, \ldots, x_{n-1})$ of the language of $\mathcal{M}^*_{\kappa\lambda\mu}$ there is an effectively determined integer k and (second order) formula $\psi(y_0, \ldots, y_{2k+n+1})$ of the language of $\mathcal{N}^2_{\kappa\lambda\mu}$ such that for all κ, λ, μ with $cf(\kappa) > 2^{\aleph_0}$, for every $a_0, \ldots, a_{n-1}$ in $\mathcal{M}^*_{\kappa\lambda\mu}$ (having the correct sorts) and every k-representation $c_0, \ldots, c_{2k+n+1}$ of $\overline{a}$ in $\mathcal{N}^2_{\kappa\lambda\mu}$,*

$$\mathcal{M}^*_{\kappa\lambda\mu} \models \varphi[\overline{a}] \Leftrightarrow \mathcal{N}^2_{\kappa\lambda\mu} \models \psi[\overline{c}].$$

Proof We construct ψ by induction. The k is just the 'quantifier depth' of φ (for quantifications over F_j), as emerges from what follows.

First consider the case of atomic formulae, where we take $k = 0$. If $\varphi(x_0)$ is $Eq^1(x_0)$ we let $\psi(y_0)$ also be $Eq^1(y_2)$, (since sort IS_2 is the same in the two structures). Similarly for $Prod^1(x_0), Proj^1_n(x_0, x_1)$, and $x_0 = x_1$ where x_0, x_1 lie in the same IS_j. If $\varphi(x_0, x_1)$ is $x_0 = x_1$ or $x_0 < x_1$ where $x_0, x_1 \in Card^-$ we let $\psi(y_0, y_1, y_2, y_3)$ be $y_2 = y_3$ or $(y_2, y_3) \in y_1$ respectively. Consider $<$ for instance, and let $(A, <^*, b_0, b_1)$, a k-representation of (β_0, β_1), and θ, be given by the definition of what this means. Then $\theta(b_i) = \beta_i$, so $b_0 <^* b_1 \Leftrightarrow \beta_0 < \beta_1$ and $\mathcal{M}^*_{\kappa\lambda\mu} \models \varphi[\beta_0, \beta_1] \Leftrightarrow \beta_0 < \beta_1 \Leftrightarrow b_0 <^* b_1 \Leftrightarrow \mathcal{N}^2_{\kappa\lambda\mu} \models (b_0, b_1) \in <^* \Leftrightarrow \mathcal{N}^2_{\kappa\lambda\mu} \models \psi[A, <^*, b_0, b_1]$. For the remainder it suffices to consider $\varphi(x_0, x_1, x_2) \equiv x_1 = App_n(x_2, x_0)$ (since the other atomic formulae may be written in terms of this and the ones above). Here we let $\psi(y_0, y_1, y_2, y_3, y_4)$ be $(y_2, y_3) \in y_4$.

For the induction step the case of negation is immediate (we take the same k and the negation of the corresponding formula). For conjunction suppose that φ is $\varphi_1(x_0,..,x_{l-1},x_l,..,x_{m-1}) \wedge \varphi_2(x_0,\dots,x_{l-1},x_m,\dots,x_{n-1})$, where the x_i are distinct variables , and that $k_1,\psi_1(y'_0,..,y'_{2k_1+1},x'_0,.,x'_{m-1})$ corresponding to φ_1 and $k_2,\psi_2(y''_0,\dots,y''_{2k_2+1},x''_0,\dots,x''_{l-1},x''_m,\dots,x''_{n-1})$ corresponding to φ_2 have been chosen. Let $k = max(k_1,k_2)$ and $\psi(y_0,\dots,y_{2k+1},z_0,\dots,z_{l-1},z_l,\dots,z_{m-1},z_m,\dots,z_{n-1})$ be the formula

$$\psi_1(y_0,\dots,y_{2k_1+1},z_0,\dots,z_{m-1}) \wedge \psi_2(y_0,\dots,y_{2k_2+1},z_0,\dots,z_{l-1},z_m,\dots,z_{n-1}).$$

If $a_0,\dots,a_{n-1}$ in $\mathcal{M}^*_{\kappa\lambda\mu}$ have the correct sorts, and $(A,<^*,g_0,\dots,g_{k-1},g^0,\dots,g^{k-1},b_0,\dots,b_{n-1})$ is a k-representation in $\mathcal{N}^2_{\kappa\lambda\mu}$, then $(A,<^*,g_0,\dots,g_{k_1-1},g^0,\dots,g^{k_1-1},b_0,\dots,b_{m-1}),(A,<^*,g_0,\dots,g_{k_2-1},g^0,\dots,g^{k_2-1},b_0,\dots,b_{l-1},b_m,\dots,b_{n-1})$ are k_1-,k_2-representations of $(a_0,\dots,a_{m-1}),(a_0,\dots,a_{l-1},a_m,\dots,a_{n-1})$ respectively, and so the result goes through with this k.

Now consider the existential quantifier. Suppose $\varphi(x_0,\dots,x_{n-1})$ is $(\exists x_n)$ $\varphi'(x_0,\dots,x_{n-1},x_n)$, and that k',ψ' corresponding to φ' have been chosen.
Case 1: $x_n \in IS_j$. Let $k = k'$ and $\psi(y_0,\dots,y_{2k+n+1})$ be $(\exists y_{2k+n+2})\psi'(y_0,\dots,y_{2k+n+2})$ (where $y_{2k+n+2} \in IS_j$ too).

Suppose $a_0,\dots,a_{n-1} \in \mathcal{M}^*_{\kappa\lambda\mu}$ have the correct sorts, and $(A,<^*,\overline{g},\overline{c})$ is a k-representation of $\overline{a}$. Then for any $a_n \in IS_j$, $(A,<^*,\overline{g},\overline{c},a_n)$ is a k-representation of $(a_0,\dots,a_n)$. Hence

$$\begin{aligned}\mathcal{M}^*_{\kappa\lambda\mu} \models \varphi[a_0,\dots,a_{n-1}] &\\ \Leftrightarrow &\text{ for some } a_n \in IS_j, \mathcal{M}^*_{\kappa\lambda\mu} \models \varphi'[a_0,\dots,a_n]\\ \Leftrightarrow &\text{ for some } c_n \in IS_j, \mathcal{N}^2_{\kappa\lambda\mu} \models \psi'[A,<^*,\overline{g},c_0,\dots,c_n]\\ \Leftrightarrow &\ \mathcal{N}^2_{\kappa\lambda\mu} \models \psi[A,<^*,\overline{g},\overline{c}].\end{aligned}$$

Case 2: $x_n \in Card^-$. Any existential quantifiers over $Card^-$ may be eliminated in favour of quantifiers over IS_2 and F_2, since $(\exists x_n \in Card^-)\varphi'(x_0,\dots,x_n) \Leftrightarrow (\exists h \in F_2)(\exists t \in IS_2)\varphi'(x_0,\dots,x_{n-1},h(t))$.
Case 3: $x_n \in F_j$. Let $k = k'+1$ and $\psi(y_0,\dots,y_{2k+n+1})$ be the formula

$$\begin{gathered}(\exists z_0)\dots(\exists z_{2k+n+1})(\exists z)(rep_k(z_0,\dots,z_{2k+n+1},z) \wedge\\ isorep_k(y_0,\dots,y_{2k+n+1},z_0,\dots,z_{2k+n+1}) \wedge \psi''(z_0,\dots,z_{2k+n+1},z)),\end{gathered}$$

where $rep_k, isorep_k$ are the appropriate instances of the formulae provided by Lemma 5.8 (that is for the correct sequence of sorts), and $\psi''(z_0,\dots,z_{2k+n+1},z)$ is $\psi'(z'_0,\dots,z'_{2k+n-1},z)$ where $(z'_0,\dots,z'_{2k+n-1},z)$ is obtained from $(z_0,\dots,z_{2k+n+1},z)$ by deleting the two variables corresponding to g_k and g^k. For ease assume the variables in $\bigcup IS_i$ come first, then those in $Card^-$, then those in $\bigcup F_i$.

Let $(A,<^*,\overline{g},\overline{t},\overline{b},\overline{H})$ be a k-representation of $(\overline{t},\overline{\beta},\overline{h})$. Then

$\mathcal{M}^*_{\kappa\lambda\mu} \models \varphi[\bar{t},\bar{\beta},\bar{h}]$

$\Leftrightarrow$ for some $h \in F_j, \mathcal{M}^*_{\kappa\lambda\mu} \models \varphi'[\bar{t},\bar{\beta},\bar{h},h]$

$\Leftrightarrow$ for some $A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},H'$, where $H' : IS_j \to A'$,
$\mathcal{N}^2_{\kappa\lambda\mu} \models rep_k[A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},H'] \wedge$
$isorep_k[A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},A,<^*,\overline{g},\bar{t},\bar{b},\overline{H}] \wedge \psi''[A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},H']$

$\Leftrightarrow$ $\mathcal{N}^2_{\kappa\lambda\mu} \models \psi[A,<^*,\overline{g},\bar{t},\bar{b},\overline{H}]$.

The first and last steps are immediate. It is the intermediate equivalence which we have to justify.

Suppose then that $\mathcal{M}^*_{\kappa\lambda\mu} \models \varphi'[\bar{t},\bar{\beta},\bar{h},h]$, and let $(A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},H')$ be a k-representation of $(\bar{t},\bar{\beta},\bar{h},h)$ (which exists by Lemma 5.6). We get a corresponding k'-representation by omitting $g'_k, (g^k)'$, so by the induction hypothesis, $\mathcal{N}^2_{\kappa\lambda\mu} \models \psi''[A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},H']$. Also $\mathcal{N}^2_{\kappa\lambda\mu} \models rep_k[A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},H'] \wedge$ $isorep_k[A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},A,<^*,\overline{g},\bar{t},\bar{b},\overline{H}]$.

Conversely if $\mathcal{N}^2_{\kappa\lambda\mu} \models \psi[A,<^*,\overline{g},\bar{t},\bar{b},\overline{H}]$ there are $A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},H'$ which form a k-representation of $\bar{t},\bar{\beta},\bar{h},h$ for some h, and such that $\mathcal{N}^2_{\kappa\lambda\mu} \models rep_k[A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},H'] \wedge\ isorep_k[A',<',\overline{g'},\bar{t},\overline{b'},\overline{H'},A,<^*,\overline{g},\bar{t},\bar{b},\overline{H}]$. By Lemma 5.3(iv) there is some $H : IS_j \to A$ such that $(A,<^*,\overline{g},\bar{t},\bar{b},\overline{H},H)$ is a k'-representation of $(\bar{t},\bar{\beta},\bar{h},h)$. By the induction hypothesis again, $\mathcal{M}^*_{\kappa\lambda\mu} \models \varphi'[\bar{t},\bar{\beta},\bar{h},h]$ as required. $\square$

Corollary 5.10 *If $cf(\kappa_1), cf(\kappa_2) > 2^{\aleph_0}$ then the following are equivalent:*
(i) $\mathcal{M}_{\kappa_1\lambda_1\mu_1} \equiv \mathcal{M}_{\kappa_2\lambda_2\mu_2}$,
(ii) $\mathcal{N}^2_{\kappa_1\lambda_1\mu_1} \equiv \mathcal{N}^2_{\kappa_2\lambda_2\mu_2}$,
(iii) $S_{\lambda_1}(\mu_1)/S_{\kappa_1}(\mu_1) \equiv S_{\lambda_2}(\mu_2)/S_{\kappa_2}(\mu_2)$.

Proof This follows from Theorems 5.2, 5.5, and 5.9. $\square$

So in a certain sense, for cofinalities above $2^{\aleph_0}$, only a rather modest amount of information about the cardinals κ, λ, and μ is needed to distinguish the quotient groups, and in particular, whenever $\alpha(\kappa_1,\lambda_1,\mu_1) \sim \alpha(\kappa_2,\lambda_2,\mu_2)$ they are elementarily equivalent.

6 The case $cf(\kappa) \leq 2^{\aleph_0}$ and conclusions

In this section we begin by treating the rather more complicated case in which $cf(\kappa) \leq 2^{\aleph_0}$, and then summarize the conclusions in all cases. The first remark is that there is a first order sentence of the language of $\mathcal{M}_{\kappa\lambda\mu}$ which distinguishes this case, namely

$$(\exists h_1,h_2 \in F_2)(h_1 \neq h_2 \wedge (\forall t \in IS_2)(App_2(h_1,t) = App_2(h_2,t))).$$

So from now on we assume that $cf(\kappa) \leq 2^{\aleph_0}$.

We now describe the modification of $\mathcal{N}^2_{\kappa\lambda\mu}$ appropriate in this case, which varies slightly according as $\kappa \leq 2^{\aleph_0}$ or not, and $\kappa = \aleph_0$ or not (cases which we shall see below can be distinguished by formulae of the language of group theory). Let α^* be the least ordinal > 0 such that $(\exists\gamma)(\beta = \gamma + \alpha^*)$ where $\kappa = \aleph_\beta$. The definition of $\mathcal{N}^2_{\kappa\lambda\mu}$ is modified to include as additional sorts $cf(\kappa)$, and $\alpha^*_{[n]}, \alpha^{*[n]}$ for $n \geq 0$. Since α^* is by definition additively indecomposable, only at most one $\alpha^*_{[n]}$ can be non-zero, so the representation is somewhat redundant, and we have just $\omega+1$ possible cases. We also include (distinct) individual constants $c_0, c_\kappa \in IS_2$ in the structure. These may be chosen arbitrarily or, better, as *definable* elements (to ensure that the interpretation is without parameters).

Definition 6.1 *If $\alpha = \alpha(\kappa,\lambda,\mu)$ is the order-type of $Card^-$ in $\mathcal{M}_{\kappa\lambda\mu}$ and α^* is the least ordinal > 0 such that $(\exists\gamma)(\beta = \gamma + \alpha^*)$ where $\kappa = \aleph_\beta$, we let*

$$\mathcal{N}^2_{\kappa\lambda\mu} = ((IS_n)_{n\geq 1}, (\alpha_{[n]})_{n\geq 0}, (\alpha^{[n]})_{n\geq 0}, (\alpha^*_{[n]})_{n\geq 0}, (\alpha^{*[n]})_{n\geq 0}; Eq^1, Prod^1, \\ (Proj^1_n)_{n\geq 1}, (<_n)_{n\geq 0}, (<^n)_{n\geq 0}, (<^*_n)_{n\geq 0}, (<^{*n})_{n\geq 0}, c_0, c_\kappa, kap, fin)$$

be the structure whose sorts are viewed as being pairwise disjoint (and all but finitely many $\alpha_{[n]}$ and all but at most one $\alpha^*_{[n]}$ are empty), and $<_n, <^n, <^*_n, <^{*n}$ are the usual (well-) orderings on $\alpha_{[n]}, \alpha^{[n]}, \alpha^*_{[n]}, \alpha^{*[n]}$. As in Definition 5.4 the superscript 2 indicates that $\mathcal{N}^2_{\kappa\lambda\mu}$ is a second order structure, and the same restrictions are made as before on the second order variables which are allowed (where the new sorts are now allowed as entries in the tuples of sorts), except that since we no longer know for sure that $\lambda > 2^{\aleph_0}$, we have to restrict to quantification over relations of cardinality $< \lambda$. The constants c_0 and c_κ are distinct elements of IS_2, and kap and fin are unary relations on IS_2, kap picking out a subset of IS_2 of cardinality κ and fin the set of isomorphism types of finite sets, which are only included if $\kappa \leq 2^{\aleph_0}$, $\kappa = \aleph_0$ respectively.

The case $\kappa \leq 2^{\aleph_0}$ has to be treated separately because it is precisely here that Sum_n (summation of $h \in F_n$) cannot be identified with supremum. As we saw above, subsets of IS_2 can be represented in $\mathcal{M}_{\kappa\lambda\mu}$, and by various tricks (which we do not go into, but which are similar to ones described below for other purposes) one can express the property of having cardinality κ. In general there will be no *definable* such set however, so the interpretation of $\mathcal{N}^2_{\kappa\lambda\mu}$ in $\mathcal{M}_{\kappa\lambda\mu}$ in this case requires a parameter. If also $\kappa = \aleph_0$, we include a predicate fin picking out out the members of IS_2 corresponding to isomorphism types of finite sets. (This predicate *is* definable in $\mathcal{M}_{\kappa\lambda\mu}$.)

The fact that *kap* is not definable does not affect our main results however. We shall show that (in the relevant case), *kap* can be interpreted, and that we can express when the representations of members of our structure using two possible interpretations of *kap* represent the same object.

One main difference in this section is that we can no longer work with $\mathcal{M}^*_{\kappa\lambda\mu}$. Instead we refine the methods of section 5 to show how the second order logic

just mentioned can be represented in $\mathcal{M}_{\kappa\lambda\mu}$. We recall that in Lemma 5.1(ii) we saw how to say that two members of F_n or F_{m+n} encode the same subsets of IS_n or $IS_m \times IS_n$. In fact if $cf(\kappa) > 2^{\aleph_0}$ they encode the same set if and only if they are equal. But this is not true if $cf(\kappa) \leq 2^{\aleph_0}$ (as was essentially exploited above in devising a sentence to characterize this case). Life is easier if we use h which 'minimally encode' sets or relations. All this means is that the cumulative effect of values below κ is negligible, in other words $\sum\{h(t) : h(t) < \kappa\} < \kappa$, but we have to see how this can be formally expressed.

For $h_1, h_2 \in F_n$ we write $restr_n(h_1, h_2)$ for $\sum\{h_1(t) - h_2(t) : t \in IS_n\} < \kappa$. In $S_\lambda(\mu)/S_\kappa(\mu)$ this corresponds to a tuple representing h_1 being conjugate to a restriction of a tuple representing h_2 (expressed in section 4 by a corresponding formula $restr_n$), and so by Theorem 2.6 is first order expressible in the language of $\mathcal{M}_{\kappa\lambda\mu}$. Saying that h minimally encodes a set (or relation) then is expressed by $min(h)$:

$$(\forall h' \in F_n)((\forall t \in IS_n)(App_n(h,t) = \kappa \leftrightarrow App_n(h',t) = \kappa) \rightarrow restr_n(h,h')).$$

Now we show how to capture the behaviour of cardinals below κ in $\mathcal{M}_{\kappa\lambda\mu}$. Let us write $Card_{<\kappa}$ for $\{\nu \in Card : \nu < \kappa\}$. We can only hope to capture the 'tail' of $Card_{<\kappa}$. We encode (the tail of) a subset X of $Card_{<\kappa}$ by any $k \in F_2$ having X as range. (Of course subsets of $Card_{<\kappa}$ of cardinality $< \min(\Omega, \lambda)$ can be so encoded.) We can express 'k encodes some set' by $(\forall t \in IS_2)(App_2(k,t) = 0)$, and we say that such a k is *almost zero*. In the sense of the previous paragraph k encodes the empty subset of IS_2. As we wish to exclude 0 (that is, any k such that $\sum\{k(t) : t \in IS_2\} < \kappa$) we identify 0 as any $k \in F_2$ which *minimally* encodes the empty set.

In order to express when two almost zero members of F_2 encode the same subset of $Card_{<\kappa}$ it is easier to pass to those which are 'almost 1–1', meaning that

$$(\exists \nu < \kappa)(\forall t_1, t_2 \in IS_2)(k(t_1) = k(t_2) \geq \nu \rightarrow t_1 = t_2).$$

This requires a further technical trick.

Now if $h \in F_2$ minimally encodes a subset X of IS_2, and k_1, k_2 are almost zero, we can express 'k_1 encodes the restriction of k_2 to X' by the formula

$$restr_2(k_1,k_2) \wedge restr_2(k_1,h) \wedge (\forall k' \in F_2)(restr_2(k',k_2) \wedge restr_2(k',h) \rightarrow restr_2(k',k_1)).$$

If $f \in F_4$ and $one\text{-}one fun_{2,2}(f)$ we can express 'the function F coded by f carries k_1 to k_2' (meaning that F carries $\{t : k_1(t) > 0\}$ to $\{t : k_2(t) > 0\}$, and for each t with $k_1(t) > 0, k_2(F(t)) = k_1(t)$), via $S_\lambda(\mu)/S_\kappa(\mu)$ and Theorem 2.6 as follows:

'a tuple representing f has a restriction which projects to a conjugate of a tuple representing k_1 on co-ordinates 1,2, and to a conjugate of a tuple representing k_2 on co-ordinates 5,6'.

Using this we can now express 'k is almost 1–1' thus:

one-one(k) : k is almost zero and $\forall k_1 \forall k_2 \forall f(k_1, k_2$ non-zero restrictions of k to disjoint subsets of $IS_2 \wedge$ *one-onefun*$(f) \to \neg(f$ carries k_1 to $k_2))$.

For if k is not almost 1–1 there are cofinally many $\nu < \kappa$ such that $|k^{-1}(\nu)| \geq 2$ and we can find non-zero restrictions of k to disjoint subsets of IS_2 and a permutation taking one to the other.

The point of doing this is that we can now express 'almost zero k_1 and k_2 code the same (tail of a) subset of $Card_{<\kappa}$', and compare order-types of such subsets. For k_1 and k_2 code the same subset of $Card_{<\kappa}$ if and only if one can be carried to the other by a 1–1 function from a subset of IS_2 to IS_2.

We can now express $cf(\kappa) \leq 2^{\aleph_0} \wedge \kappa$ is a successor by

$$cf(\kappa) \leq 2^{\aleph_0} \text{ (already expressed) } \wedge (\forall k)(\textit{one-one}(k) \to k = 0),$$

if desired (though it corresponds to the special case $\alpha^* = 1$).

Now suppose that $cf(\kappa) \leq 2^{\aleph_0} \wedge \kappa$ is a limit. We wish to represent $cf(\kappa)$ and each $\alpha^*_{[n]}$ and $\alpha^{*[n]}$ in $\mathcal{M}_{\kappa\lambda\mu}$. We represent $cf(\kappa)$ by any k such that

one-one$(k) \wedge k \neq 0 \wedge (\forall k')($*one-one*$(k') \wedge k' \neq 0 \to (\exists g)(g$ a 1–1 map from a subset of IS_2 into $IS_2 \wedge (\forall t)(k(t) \leq k'(gt)))$.

For this we need to express $(\forall t)(k(t) \leq k'(gt)))$ in $\mathcal{M}_{\kappa\lambda\mu}$, and we use the same idea as above, going via $S_\lambda(\mu)/S_\kappa(\mu)$, and say that the projection to co-ordinates 1,2 of a tuple representing g has as a restriction a conjugate of k_1.

Now moving towards representing the $\alpha^*_{[n]}$ and $\alpha^{*[n]}$, we find a formula *subset*$^*(k_1, k_2)$ which expresses 'k_1, k_2 are almost 1–1, and the set encoded by k_1 is a subset of the set encoded by k_2' thus:

$$(\exists f)(\exists h)(\textit{one-onefunction}(f) \wedge h \text{ codes a subset of } IS_2 \wedge \\ k_1 \text{ is the restriction of } k_2 g \text{ to the set encoded by } h).$$

To represent $\alpha^*_{[n]}$ in $\mathcal{M}_{\kappa\lambda\mu}$, the main point is to find inductively a formula $div^*(h, \Omega^n)$ analogous to the *div* formulae considered earlier, expressing 'h encodes a function from IS_2 to $Card_{<\kappa}$ such that for every t, $h(t)$ is divisible by Ω^n'. For the basis case $div^*(h, \Omega^0)$ just says that h encodes a function from IS_2 to $Card_{<\kappa}$, in other words, h is 'almost zero'. We also need similar almost zero functions from IS_2^2 to $Card_{<\kappa}$.

Assuming inductively that $div^*(h, \Omega^n)$ has been found, we take for $div^*(h, \Omega^{n+1})$ the formula

$$div^*(h, \Omega^n) \wedge (\forall h')[(h' \text{ codes a function from } IS_2^2 \text{ to } Card_{<\kappa}) \wedge \\ (\forall t, t' \in IS_2)(h'(t, t') < h(t)) \to (\exists h'')(h'' \text{ codes a function from } IS_2 \text{ to} \\ Card_{<\kappa} \wedge (\forall t, t' \in IS_2)(h'(t, t') < h''(t) < h(t))].$$

We illustrated how to handle inequalities in this context above, so such a formula exists, and is clearly as required.

We can therefore represent each $\alpha^*_{[n]}$ in $\mathcal{M}_{\kappa\lambda\mu}$. Moreover, if $\alpha^*_{[n]} \neq 0$ for some n, $\alpha^* = \Omega^n . \alpha^*_{[n]}$, and all cofinalities are at once represented (equal to either 0

or $cf(\Omega^n.\alpha^*_{[n]}))$, and if $\alpha^*_{[n]} = 0$ for all n, $\alpha^* = \Omega^\omega.\alpha^*_\omega$, so the cofinalities are all equal to $cf(\kappa)$. Thus all the sorts of $\mathcal{N}^2_{\kappa\lambda\mu}$ are represented. The method for representing the second order logic on $\mathcal{N}^2_{\kappa\lambda\mu}$ described above is as in the proof of Theorem 5.5.

Next we show how to handle the case $\kappa \leq 2^{\aleph_0}$. Let us say that $h \in F_n$ *takes at most two values* if for some $h' : IS_2 \to Card_{<\kappa}$, $(h')_{\mathcal{E}_n} = h$ and $|range\ h'| \leq 2$. This notion is captured in $\mathcal{M}_{\kappa\lambda\mu}$ by the formula

$$(\forall X \subseteq IS_2)(\exists Y \subseteq X)(\text{all permutations of } X \text{ fixing } Y \text{ setwise also fix } h).$$

Observe that we need the $\forall\exists$ quantification because we can only quantify over subsets of IS_2 of cardinality $< \lambda$, and we have not insisted that $\lambda > 2^{\aleph_0}$.

We can now characterize $\kappa \leq 2^{\aleph_0}$ by means of the formula

$$(\exists h)(h \neq 0 \ \wedge\ h \text{ is almost zero } \wedge h \text{ takes at most two values}),$$

which justifies defining $\mathcal{N}^2_{\kappa\lambda\mu}$ by the cases $\kappa > 2^{\aleph_0}$ or $\kappa \leq 2^{\aleph_0}$. All the ingredients of this structure have been represented in $\mathcal{M}_{\kappa\lambda\mu}$ in the case $cf(\kappa) \leq 2^{\aleph_0} < \kappa$, and when $\kappa \leq 2^{\aleph_0}$ we interpret *kap* as a subset of IS_2 of cardinality κ. We remark that in this case, $|Card_{<\kappa}| \leq 2^{\aleph_0}$, and so this is an instance where the $\alpha^*_{[n]}$ and $\alpha^{[*n]}$ really are mostly redundant, since $\alpha^*_{[0]} = \alpha^*$, and all other $\alpha^*_{[n]}$ are zero. If $\kappa = \aleph_0$, we also have to represent fin, as mentioned earlier, and this is done as follows. Amplifying the remarks just before Theorem 4.3, let us say that an n-tuple $\overline{x} \in S_\lambda(\mu)/S_\kappa(\mu)$ is *irreducible* if $\overline{x} \neq 1$ and $\forall\overline{y}\forall\overline{z}(disj_n(\overline{y},\overline{z}) \wedge \overline{x} = \overline{y} * \overline{z} \to (\overline{y} = 1 \vee \overline{z} = 1))$. Then one easily checks that $S_\lambda(\mu)/S_\kappa(\mu) \models (\exists\overline{x})(\overline{x}$ irreducible$) \Leftrightarrow \kappa = \aleph_0$, and so, by Theorem 2.6, this can also be expressed in $\mathcal{M}_{\kappa\lambda\mu}$. Moreover, the same argument shows that irreducibility too can be expressed in $\mathcal{M}_{\kappa\lambda\mu}$, and we note that $t \in fin \Leftrightarrow (\forall h \in F_2)(h$ irreducible $\to App_2(H,t) = 0)$. For if $S_\kappa(\mu).\overline{g}$ is irreducible and $App_2(Ch_{\overline{g}},t) \neq 0$, where $t \in fin$, then $\langle\overline{g}\rangle$ must have infinitely many orbits of type t, so can be written as a non-trivial product of disjoint elements. On the other hand, if $t \notin fin$, then there is $S_\kappa(\mu).\overline{g} \neq 1$ such that $\langle\overline{g}\rangle$ has a single non-trivial orbit of type t.

This describes the essential steps in the proof of the following theorem.

Theorem 6.2 *For all κ, $\mathcal{N}^2_{\kappa\lambda\mu}$ is interpretable in $\mathcal{M}_{\kappa\lambda\mu}$. If $\kappa > 2^{\aleph_0}$ the interpretation is without parameters, and if $\kappa \leq 2^{\aleph_0}$ a parameter for kap is used.*

Proof The case $cf(\kappa) > 2^{\aleph_0}$ follows from Theorems 5.2 and 5.5, and the case $cf(\kappa) \leq 2^{\aleph_0}$ is covered by the above discussion. As remarked above, although the parameter *kap* is needed in the case $\kappa \leq 2^{\aleph_0}$, since 'having cardinality κ' is expressible, we can define when a subset of IS_n is a possible choice for its interpretation. □

To complete our analysis of the case $cf(\kappa) \leq 2^{\aleph_0}$ we show how $\mathcal{M}_{\kappa\lambda\mu}$ is (weakly) interpretable in $\mathcal{N}^2_{\kappa\lambda\mu}$ in this case in the sense of Theorem 5.9. This

will suffice to show that the structures $\mathcal{N}^2_{\kappa\lambda\mu}$ completely capture the first order theory of the groups $S_\lambda(\mu)/S_\kappa(\mu)$, which is our goal. Here we use a modification of the definition of a k-representation of a tuple $(\bar{t}, \overline{\beta}, \overline{h})$ in $\mathcal{M}_{\kappa\lambda\mu}$. Recall that without the assumption $cf(\kappa) > 2^{\aleph_0}$ we only know that $\overline{h}$ is a tuple of $\mathcal{E}_n$-classes of functions, which is one reason for the altered definition. Another point is that we need to capture the eventual behaviour of two well-order-types, namely $Card$ above *and* below κ. If $\kappa = \aleph_\beta$ and γ is least such that $\beta = \gamma + \alpha^*$, we let $Card^* = \{\nu \in Card : \nu = 0 \vee \aleph_\gamma \leq \nu < \lambda\}$. Then a k-representation of $(\bar{t}, \overline{\beta}, \overline{h})$ is defined to be any tuple of the form $(A, <^*, \overline{g}, \bar{t}, \bar{b}, \overline{H})$ such that

$<^*$ well-orders IS_2,

$A \subseteq IS_2, b_i \in A, c_0, c_\kappa \in A$,

if $h_i \in F_j$ then $H_i : IS_j \to A$,

$\overline{g}$ is a tuple of the form $(g_0, \ldots, g_{k-1}, g^0, \ldots, g^{k-1}, g^*_0, \ldots, g^*_{k-1}, g^{*0}, \ldots, g^{*\,k-1})$ where $g_i, g^i : A' \cup \{\infty\} \to IS_2, g^*_i, g^{*\,i} : A'' \to IS_2$, where $A' = \{a \in A : a \leq^* c_\kappa\}$ and $A'' = \{a \in A : c_\kappa < a\}$,

and for some 1–1 order-preserving map $\theta : A \to Card^*$,

θ takes c_0 to 0, c_κ to κ, and b_i to β_i for each i,

if $h_i \in F_j$ then for some $h'_i : IS_j \to Card^*, (h'_i)_{\mathcal{E}_j} = h_i$ and $(\forall t \in IS_j)\theta(H_i(t)) = h'_i(t)$,

the order-types of $\{t \in IS_2 : t <^* g_j(a)\}$ and $\{t \in IS_2 : t <^* g^j(a)\}$ are equal to $\gamma(\theta(a), \theta(A'))_{[j]}$ and $\gamma(\theta(a), \theta(A'))^{[j]}$ respectively, for each $a \in A'$,

and the order-types of $\{t \in IS_2 : t <^* g^*_j(a)\}$ and $\{t \in IS_2 : t <^* g^{*j}(a)\}$ are equal to $\gamma(\theta(a), \theta(A''))_{[j]}$ and $\gamma(\theta(a), \theta(A''))^{[j]}$ respectively, for each $a \in A''$.

Theorem 6.3 *For every (first order) formula $\varphi(x_0, \ldots, x_{n-1})$ of the language of $\mathcal{M}_{\kappa\lambda\mu}$ there is an effectively determined integer k and (second order) formula $\psi(y_0, \ldots, y_{4k+n+1})$ of the language of $\mathcal{N}^2_{\kappa\lambda\mu}$ such that for all κ, λ, μ with $cf(\kappa) \leq 2^{\aleph_0}$, for every $a_0, \ldots, a_{n-1}$ in $\mathcal{M}^*_{\kappa\lambda\mu}$ (having the correct sorts) and every k-representation $c_0, \ldots, c_{4k+n+1}$ of $\overline{a}$ in $\mathcal{N}^2_{\kappa\lambda\mu}$,*

$$\mathcal{M}^*_{\kappa\lambda\mu} \models \varphi[\overline{a}] \Leftrightarrow \mathcal{N}^2_{\kappa\lambda\mu} \models \psi[\overline{c}].$$

Proof We have to indicate the appropriate modifications in the proof of Theorem 5.9. We first remark on the analogues of Lemmas 5.7 and 5.8, which are required here too. Finding a formula to express the existence of a k-representation is much as before. Some modification is needed in Lemma 5.8, since we have to allow for the possibility that the H_i may be $\mathcal{E}_j$-equivalent, so that the lack of an order-isomorphism between the corresponding As need not determine whether or not the k-representations are isomorphic. This is handled using an additional existential quantifier.

Proceeding to the main proof, since we now have to work with $\mathcal{M}_{\kappa\lambda\mu}$ rather than $\mathcal{M}^*_{\kappa\lambda\mu}$, there are some extra atomic cases in the induction to consider. We concentrate on the formula $Eq(x_0)$, as this serves to illustrate the idea.

Since the structure $\mathcal{N}^2_{\kappa\lambda\mu}$, and the notion of '$k$-representation', is different in the cases $\kappa > 2^{\aleph_0}$ and $\kappa \leq 2^{\aleph_0}$, we treat the two separately, starting with the former, in which $cf(\kappa) \leq 2^{\aleph_0} < \kappa$. Let $(A, <^*, H)$ be a 0-representation of h. Then

$$\mathcal{M}_{\kappa\lambda\mu} \models Eq[h] \Leftrightarrow \sum\{h'(t) : t \in IS_2 \wedge (t = ((B, g_1, g_2))_{\cong} \to g_1 \neq g_2)\} < \kappa,$$

where $(h')_{\mathcal{E}_2} = h$ is as in the definition of k-representation, corresponding to H, and this is equivalent to

$$(\exists \nu < \kappa)(\forall t \in IS_2)(t = ((B, g_1, g_2))_{\cong} \to h'(t) \leq \nu).$$

For if $(\forall t \in IS_2)(t = ((B, g_1, g_2))_{\cong} \to h'(t) \leq \nu)$ then $\sum\{h'(t) : t \in IS_2 \wedge (t = ((B, g_1, g_2))_{\cong} \to g_1 \neq g_2)\} \leq 2^{\aleph_0}.\nu < \kappa$ as $2^{\aleph_0}, \nu < \kappa$. And if $\{h'(t) : t \in IS_2 \wedge (t = ((B, g_1, g_2))_{\cong} \to g_1 \neq g_2)\}$ is unbounded in $Card_{<\kappa}$, then $\sum\{h'(t) : t \in IS_2 \wedge (t = ((B, g_1, g_2))_{\cong} \to g_1 \neq g_2)\} \geq \sup\{h'(t) : t \in IS_2 \wedge (t = ((B, g_1, g_2))_{\cong} \to g_1 \neq g_2)\} \geq \kappa$.

Therefore

$$\begin{aligned} \mathcal{M}_{\kappa\lambda\mu} \models Eq[h] \quad &\Leftrightarrow \quad (\exists \nu < \kappa)(\forall t \in IS_2)(Eq^1(t) \to h'(t) \leq \nu) \\ &\Leftrightarrow \quad \mathcal{N}^2_{\kappa\lambda\mu} \models (\exists y \in IS_2)(\forall z \in IS_2)(y \in A \wedge y <^* \underline{c}_\kappa \wedge (Eq^1(z) \\ &\qquad\qquad \to H(t) \leq^* y)), \end{aligned}$$

and this provides the desired formula $\psi(y_0, y_1, y_2)$.

Now turning to the case where $\kappa \leq 2^{\aleph_0}$ we find that

$$\begin{aligned} \mathcal{M}_{\kappa\lambda\mu} \models Eq[h] \quad \Leftrightarrow \quad &(\exists \nu < \kappa)(\forall t \in IS_2)(t = ((B, g_1, g_2))_{\cong} \to h'(t) \leq \nu) \\ &\wedge |\{t \in IS_2 : h'(t) \neq 0\} < \kappa \\ &(\wedge(\forall t \in IS_2 - fin)(h'(t) = 0) \text{ when } \kappa = \aleph_0). \end{aligned}$$

The second clause can be expressed by using kap; one says that there is a 1–1 function from $\{t : H(t) \neq c_0\}$ into kap, and that no such function is onto. Similarly, when $\kappa = \aleph_0$, the final clause is expressed by $(\forall t \in IS_2 - fin)(H(t) \neq c_0)$.

In conclusion we note that although kap is used here, for any two possible choices for it, we can define when the representation of some object (for instance an ordinal) under the two values really represents the same object, and so the apparent arbitrariness is inessential. □

Further remarks

We first remark here that if κ is a successor cardinal, then the analysis at once becomes much easier. For if $cf(\kappa) > 2^{\aleph_0}$ then we may apply the results of section 5, and if $cf(\kappa) \leq 2^{\aleph_0}$ then $\alpha^* = 1$ and the extra sorts of the structure $\mathcal{N}^2_{\kappa\lambda\mu}$ play no essential part. Note however that although we can distinguish these two cases $(cf(\kappa) > 2^{\aleph_0}, cf(\kappa) \leq 2^{\aleph_0})$, we cannot distinguish when κ is a successor.

For as remarked at the beginning of section 5, if $cf(\kappa_1), cf(\kappa_2) \geq (2^{\aleph_0})^+$ then $\mathcal{M}_{\kappa_1\kappa_1^+\mu_1} \cong \mathcal{M}_{\kappa_2\kappa_2^+\mu_2}$, but $cf(\kappa) > 2^{\aleph_0}$ is compatible both with κ a successor and κ singular.

Arising out of this, we further note that in the general case, (if $\lambda > \kappa^+$), $\mathcal{M}_{\kappa\lambda\mu}$ and the disjoint sum of $\mathcal{M}_{\kappa^+\lambda\mu}$ and $\mathcal{M}_{\kappa\kappa^+\mu}$ are bi-interpretable, and so we can separate our problem into two parts, the first as in the previous paragraph, and the second of which is the true content of section 6.

Conclusions

In studying the elementary theory of the groups $G = S_\lambda(\mu)/S_\kappa(\mu)$ where $\aleph_0 \leq \kappa < \lambda \leq \mu^+$ we distinguish the following eight cases (by first order sentences of the language of group theory):

First we distinguish the cases $\lambda \leq \mu$ and $\lambda = \mu^+$. In each of these, the elementary theory of G is determined just by the values of κ and λ. Then we consider the cases

$$cf(\kappa) > 2^{\aleph_0},\ \ cf(\kappa) \leq 2^{\aleph_0} < \kappa,\ \ \aleph_0 < \kappa \leq 2^{\aleph_0},\ \ \text{and}\ \ \kappa = \aleph_0.$$

In each case we form a many-sorted second order structure $\mathcal{N}^2_{\kappa\lambda\mu}$ whose sorts all have cardinality $\leq 2^{\aleph_0}$, which captures the first order theory of G, meaning that

$$S_{\lambda_1}(\mu_1)/S_{\kappa_1}(\mu_1) \equiv S_{\lambda_2}(\mu_2)/S_{\kappa_2}(\mu_2) \Leftrightarrow \mathcal{N}^2_{\kappa_1\lambda_1\mu_1} \equiv \mathcal{N}^2_{\kappa_2\lambda_2\mu_2}.$$

For $cf(\kappa) > 2^{\aleph_0}$ we just require information about $Card^-$; in the other cases, information about the (large enough) cardinals below κ is also represented, and when $\kappa \leq 2^{\aleph_0}$ we also require extra unary predicate(s) on IS_2. We summarize this by the general form of Corollary 5.10:

Corollary 6.4 *If $\kappa_1 < \lambda_1 \leq \mu_1^+$ and $\kappa_2 < \lambda_2 \leq \mu_2^+$ then the following are equivalent:*

(i) $\mathcal{M}_{\kappa_1\lambda_1\mu_1} \equiv \mathcal{M}_{\kappa_2\lambda_2\mu_2}$,
(ii) $\mathcal{N}^2_{\kappa_1\lambda_1\mu_1} \equiv \mathcal{N}^2_{\kappa_2\lambda_2\mu_2}$,
(iii) $S_{\lambda_1}(\mu_1)/S_{\kappa_1}(\mu_1) \equiv S_{\lambda_2}(\mu_2)/S_{\kappa_2}(\mu_2)$.

For the case where $\lambda \leq \mu$ and $cf(\kappa) > 2^{\aleph_0}$ the following holds: For any given ordinals $\alpha_l, \alpha^l < \Omega$ there is a first order theory T in the language of group theory such that

if $2^{\aleph_0} < cf(\kappa) \leq \mu, \lambda \leq \mu, \kappa = \aleph_\beta, \lambda = \aleph_\gamma$, $\beta + \alpha = \gamma$, and $\alpha_{[n]} = \alpha_n, \alpha^{[n]} = \alpha^n$ for each n, then the first order theory of the group $S_\lambda(\mu)/S_\kappa(\mu)$ is equal to T,

with similar statements in the other cases (including reference to the $\alpha^*_{[n]}, \alpha^{*[n]}$ and kap, fin as appropriate).

Finally we remark on quotients by alternating and trivial groups. The class $\{S_\lambda(\mu)/A(\mu) : \aleph_0 \leq \lambda \leq \mu^+\}$ of quotients by alternating groups is definable

in the class of all quotients of symmetric groups, being precisely those with non-trivial centre. Moreover since the centre of $S_\lambda(\mu)/A(\mu)$ is just $S_\omega(\mu)/A(\mu)$, which has order 2, $S_\lambda(\mu)/S_\omega(\mu)$ can be easily interpreted in $S_\lambda(\mu)/A(\mu)$. It follows that if $S_{\lambda_1}(\mu_1)/A(\mu_1) \equiv S_{\lambda_2}(\mu_2)/A(\mu_2)$ then $S_{\lambda_1}(\mu_1)/S_\omega(\mu_1) \equiv S_{\lambda_2}(\mu_2)/S_\omega(\mu_2)$, but whether the converse is true is not at present clear, (though, as we have seen, the class $\{S_\lambda(\mu)/S_\omega(\mu) : \lambda, \mu\}$ *is* definable in $\{S_\lambda(\mu)/S_\kappa(\mu) : \kappa, \lambda, \mu\}$). The quotients by trivial groups are just the normal subgroups of $S_\lambda(\mu)$, which were studied in [10] and [11]. These may be distinguished from the other 'genuine' quotient groups we have studied (as in [12]) by means of the sentence

$$\exists x(x \neq 1 \wedge x^2 = 1 \wedge (\forall y)((xx^y)^2 = 1 \vee (xx^y)^3 = 1))$$

(which says that there is a transposition).

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