On the existence of universal models

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Abstract

Suppose that $\lambda = \lambda^{<\lambda} \geq \aleph_0$, and we are considering a theory T. We give a criterion on T which is sufficient for the consistent existence of λ^{++} universal models of T of size λ^+ for models of T of size $\leq \lambda^+$, and is meaningful when $2^{\lambda^+} > \lambda^{++}$. In fact, we work more generally with abstract elementary classes. The criterion for the consistent existence of universals applies to various well known theories, such as triangle-free graphs and simple theories.

Having in mind possible applications in analysis, we further observe that for such λ, for any fixed $\mu > \lambda^+$ regular with $\mu = \mu^{\lambda^+}$, it is consistent that $2^\lambda = \mu$ and there is no normed vector space over $\mathbb{Q}$ of size $< \mu$ which is universal for normed vector spaces over $\mathbb{Q}$ of dimension λ^+ under the notion of embedding h which specifies (a, b) such that $||h(x)||/||x|| \in (a, b)$ for all x.

[1]

0 Introduction.

We study the existence of universal models for certain natural theories, which are not necessarily first order. This paper is self-contained, and it continues Saharon Shelah's [Sh 457] and [Sh 500]. An example of a theory to which our results can be applied is the theory of triangle-free graphs, or any simple theory (in the sense of [Sh 93]). For T a theory with a fixed notion of an embedding between its models, we say that a model M^* of T is universal for models of T (of size λ) if every model M of T of size λ, embeds into M^*. We similarly define when a family of models is jointly universal for models of size λ. More generally, we consider universals in an abstract elementary class, see Definition 1.9.

Two well known theorems on the existence of universal models for first order theories T (see [ChKe]) are

1. Under GCH, there is a universal model of T of cardinality λ for every $\lambda > |T|$.
2. If $2^{<\lambda} = \lambda > |T|$, then there is a universal model of T of cardinality λ.

Without the above assumptions, it tends to be hard for a first order theory to have a universal model, see [Sh 457] for a discussion and further references.

Although the problem of the existence of universal models for first-order theories (i.e. elementary classes of all models of such a theory) is the one which has been studied most extensively, there are of course many natural theories which are not first order. To approach such questions, we view the problem from the point of view of abstract elementary classes, which were introduced in [Sh 88] (in §1 we recall the definitions), and in a more

[1]In the list of publications of S. Shelah, this is publication number 614. Both authors thank the United States-Israel Binational Science Foundation for a partial support and various readers of the manuscript. for their helpful comments.

AMS Subject Classification: 03E35, 03C55.

Keywords: universal models, approximation families, consistency results.

specialized form earlier by Bjarni Jónsson, see [ChKe]. Such classes will be throughout denoted by $\mathcal{K}$, and if λ is a cardinal, the family of elements of $\mathcal{K}$ which have size λ will be denoted by $\mathcal{K}_\lambda$.

In [Sh 457] S. Shelah introduced the notion of an approximation family and studied abstract elementary classes with a "simple" (here called "workable", to differentiate them from simple theories in the sense of [Sh 93]) λ-approximation family. One of the results mentioned in [Sh 457] is that for λ an uncountable cardinal satisfying $\lambda = \lambda^{<\lambda}$, it is consistent that every abstract elementary class $\mathcal{K}$ which has a workable λ-approximation family, has an element of size λ^{++} which is universal for the elements if $\mathcal{K}$ which have size λ^+, i.e. $\mathcal{K}_{\lambda^+}$. Although the main idea of the proof there was correct, there were many incorrect details and omissions that made the proof and theorem incorrect as stated. In this paper we give a somewhat different proof of this result, and we also deal with $\lambda = \aleph_0$. Our results give a precise criterion for a class to be amenable to the theorem about consistency of the existence of a small family of models in $\mathcal{K}_{\lambda^+}$ that are universal for $\mathcal{K}_{\lambda^+}$. Among the classes which satisfy this criterion are the class of triangle-free graphs under embeddings (as shown in [Sh 457]) or in fact the elementary class of models of any simple theory, as shown in [Sh 500].

A complete definition of a λ-approximation family $K_{\rm ap}$ is given in §1, but let us try to give an intuitive idea here. The easiest way to look at this is to say that $K_{\rm ap}$ is a forcing notion whose generic gives an element of $\mathcal{K}_{\lambda^+}$. A natural example is to take a theory T, consider the class of all its models N of size λ^+ (with universe a subset of the ordinals $< \lambda^+$), and define $K_{\rm ap}$ as the set of all M of size $< \lambda$ which are an elementary submodel to some such N, the order being $\prec$. So, for example, the union of an elementary chain of elements in $K_{\rm ap}$ is an element of $\mathcal{K}$.

As we wish to use approximation families as forcing notions, we are led to discuss the closure and the chain condition. $K_{\rm ap}$ is said to be $(< \lambda)$-smooth, if every chain of length $< \lambda$ has a least upper bound. All λ-approximation families considered here satisfy this condition. There are indications that such an assumption is necessary for the universality results we wish to obtain, as if smoothness fails strongly there are no universals, see [GrSh 174].

As we intend to iterate with $(<\lambda)$-supports, our chain condition has to be a strong version of λ^+-cc, so to be preserved under such iterations. The one we use is $*^{\epsilon}_{\lambda}$ from [Sh 288], which is also the one used in [Sh 457]. This condition is a weakening of "stationary λ^+-cc". We recall the definition at the beginning of §2. The question now becomes which λ-approximation families yield such a chain condition. We call such approximation families workable. This notion is defined in §1. In [Sh 457] it is shown that triangle free graphs and the theory of an indexed family of independent equivalence equations have workable λ-approximation families.

In [Sh 500] and elsewhere, S. Shelah expresses the view that the existence of universal models has relevance to the general problem of classifying unstable theories. With this in mind, we can consider a theory as "simple" if it has a workable approximation family. In [Sh 93], another meaning of "simplicity" is considered: a theory is called simple if it does not have the tree property. In [Sh 500] it was shown that complete simple first order theories of size $<\lambda$ have workable approximation families in λ. This can be understood as showing that all simple theories behave "better" in the respect of universality than the linear orders do, as it is known by [KjSh 409] that when GCH fails, linear orders can have a universal in only a "few" cardinals. The hope of finding dividing lines via the existence of universal models is also realized for some non-simple theories, as it was shown by S. Shelah in [Sh 457] that some non-simple theories have workable approximation families, like the triangle-free graphs and the theory of an indexed family of independent equivalence relations, as simplest prototypes of non-simple theories. In [Sh 500], S. Shelah introduced a hierarchy NSOP_n for $3\leq n\leq\omega$ with the intention of encapturing by a formal notion the class of first order theories which behave "nicely" with respect to having universal models. Our research here continues [Sh 457].

We now give an idea of the proof of the positive consistency results. Details are explained in §1 and §2. The idea is that through a $(<\lambda)$-supports iteration of $(<\lambda)$-complete forcing we obtain the situation under which to every workable strong λ-approximation family K_{ap} there corresponds a tree of elements of K_{ap}. If K_{ap} approximates $\mathcal{K}$ and $\mathcal{K}$ is nice enough, then the models in this tree are organized so that the entire tree can be amalgamated to a model in $\mathcal{K}_{\lambda^+}$. Along the iteration we also make sure that every element of $\mathcal{K}_{\lambda^+}$ can be embedded into a model obtained as the union of one branch of such a tree. There are λ^{++} trees used for every approximation family, so the universal model obtained has size λ^{++}. Every individual forcing used in the iteration has $*^{\varepsilon}_{\lambda}$, but the proof of this for $\lambda > \aleph_0$ requires us to introduce an auxiliary step in the forcing.

In §3 we give a consistency result showing that with the same assumptions on λ^+ as above it is consistent that there is no universal normed vector space of size λ^+, even under a rather weak notion of embedding. We note that negative *consistency* results relevant to the universality problem tend to be much easier to obtain than the positive ones, especially as far as the first order theories are concerned.

We finish this introduction by giving more remarks on related results, and some conventions used throughout the paper.

The pcf theory of S. Shelah has proved to be a useful line of approach to the negative aspect of the problem of universality. This approach has been extensively applied by Menachem Kojman and S. Shelah (e.g. to linear orders [KjSh 409]), and later by each of them separately (M. Kojman on graphs [Kj], S. Shelah on Abelian groups [Sh 552] e.g). See [Sh 552] for the history and more references. One of the ideas involved is to use the existence of a club guessing sequence to prove that no universals exist. A related result of Mirna Džamonja in [Dž1] deals with uniform Eberlein compacta, and in [Dž2] she shows how the universality axioms presented in this paper can be applied to that class. Among the positive universality results, let us quote a paper by Rami Grossberg and S. Shelah [GrSh 174], in which it is shown that e.g. the class of locally finite groups has a universal model in any strong limit of cofinality $\aleph_0$ above a compact cardinal. This paper is also the first reference to the consideration of the universality spectrum as a useful dividing line in model theory.

Further positive consistency results appear e.g in S. Shelah's [Sh 100] where the consistent existence of a universal linear order at $\aleph_1$ with the negation of CH is shown, and in S. Shelah's [Sh 175], [Sh 175a] where the consistency of the existence of a universal graph at λ for which there is κ satisfying $\kappa = \kappa^{<\kappa} < \lambda < 2^\kappa = \mathrm{cf}(2^\kappa)$, is proved. The latter result was continued by Alan Mekler in [Me], where [Sh 175] was extended to a larger class of models.

Relating to our negative consistency result, the problem of universality has been extensively studied in functional analysis, most often for classes of Banach spaces. Probably the earliest result here is one of Stefan Banach himself in [Ba] in which he showed that $C[0,1]$ is isometrically universal for separable Banach spaces. Another well known result is that of Wiesław Szlenk, showing that there is no universal separable reflexive Banach space, [Sz]. Jean Bourgain expanded on these ideas to build a body of work. The combinatorial approach to the problem of universality in spaces coming from functional analysis is used in Stevo Todorčević's [To].

Model theory as an approach to study of Banach spaces has been extensively used, for example by Jean-Louis Krivine in [Kr] and C. Ward Henson in [He]. See Jacques Stern's [St] for an account on the early history of this interaction and [Io1] for a more recent history. Of the work of this area which is being currently carried on, we mention a systematic attempt to a classification theory for Banach spaces by José Iovino, see e.g [Io2], [Io3], which also give historical remarks.

Convention 0.1. (1) We make the standard assumption that the family of forcing names that we use is full, i.e. if $p \Vdash$ "$(\exists x)[\varphi(x)]$", then there is a name $\underset{\sim}{\tau}$ such that $p \Vdash$ "$[\varphi(\underset{\sim}{\tau})]$".
(2) If $\kappa = \mathrm{cf}(\kappa) < \alpha$, we let

$$S^\alpha_\kappa \stackrel{\text{def}}{=} \{\beta < \alpha : \mathrm{cf}(\beta) = \kappa\}.$$

(3) χ is throughout assumed to be a large enough regular cardinal. $<^*_\chi$ stands for a fixed well ordering of the set of all sets hereditarily of size $< \chi$, namely $\mathcal{H}(\chi)$.
(4) lub stands for the "least upper bound", i.e. M is the lub of a set $\mathcal{M}$ in the order $\leq$ iff it is its unique least upper bound, which means that M is an upper bound of $\mathcal{M}$ and for every M^* such that $(\forall N \in \mathcal{M})\,[N \leq M^*]$, we have $M \leq M^*$.

(5) For a model M, we use $|M|$ to denote the underlying set of M, and hence $||M||$ to denote the cardinality of $|M|$.

1 Approximation families.

Definition 1.1. [Sh 457] Given λ an infinite cardinal, and $u_1, u_2 \subseteq \lambda^+$.

A function $h : u_1 \to u_2$ is said to be *lawful* iff it is 1-1 and for all $\alpha \in u_1$ we have $h(\alpha) + \lambda = \alpha + \lambda$.

Notation 1.2. (1) For $A \subseteq \lambda^+$, let

$$\iota(A) \stackrel{\text{def}}{=} \min\{\delta : A \subseteq \delta \;\&\; \lambda|\delta\}.$$

If M is a model, we let $\iota(M) \stackrel{\text{def}}{=} \iota(|M|)$.
(2) In the following, we shall use the notation $M \restriction \delta$ for $M \restriction \tau_{\iota(M\cap\delta)} \restriction \delta$ (the meaning of τ and M will be described in the following definition).
(3) $\mathrm{Ev} \stackrel{\text{def}}{=} \{2\beta : \beta < \lambda^+\}$.

Remark 1.3. The notion of divisibility of ordinals used here is that $\lambda|\delta$ means that $\delta = \lambda \cdot \xi$ [not $\delta = \xi \cdot \lambda$] for some ξ. The intuition behind the definition of a lawful function is that one regards λ^+ as partitioned into blocks of length λ, and then a function is lawful iff it acts by permuting within each block. Then the function $\iota(A)$ simply measures how far the blocks go that meet A.

Definition 1.4. [Sh 457] Let λ be an infinite cardinal.

(1) Pair $K_{\text{ap}} = (K_{\text{ap}}, \leq_{K_{\text{ap}}})$ is *a weak λ-approximation family* iff for some (not necessarily strictly) increasing sequence[2]

$$\bar\tau = \langle \tau_i : i < \lambda^+ \;\&\; \lambda|i\rangle$$

of finitary vocabularies, each of size $\leq \lambda$ we have

[2]For the applications mentioned in this paper, in the following definitions readers can restrict their attention to the situation of $\tau_i = \tau_0$ for all i.

(a) $K_{\rm ap}$ is a set partially ordered by $\leq_{K_{\rm ap}}$, and such that

$$M \in K_{\rm ap} \Longrightarrow M \text{ is a } \tau_{\iota(M)}\text{-model.}$$

(b) If $M \in K_{\rm ap}$, then $|M| \in [\lambda^+]^{<\lambda}$ and $M \leq_{K_{\rm ap}} N \Longrightarrow M \subseteq N$.

(c) If $M \in K_{\rm ap}$ and $\lambda|\delta$, then $M \restriction \delta \in K_{\rm ap}$ and $M \restriction \delta \leq_{K_{\rm ap}} M$. Also [3], $\emptyset = M \restriction 0 \in K_{\rm ap}$. If $M, N \in K_{\rm ap}$ and $\lambda|\delta$, while $M \leq_{K_{\rm ap}} N$, then $M \restriction \delta \leq_{K_{\rm ap}} N \restriction \delta$.

(2) With $K_{\rm ap}$ as in (1), a function h is said to be a $K_{\rm ap}$-*isomorphism* from M to N iff $\mathrm{Dom}(h) = M, \mathrm{Rang}(h) = N$ are both in $K_{\rm ap}$, and h is a $\tau_{\iota(M)}$-isomorphism.

(3) A weak λ-approximation family $(K_{\rm ap}, \leq_{K_{\rm ap}})$ is said to be a *strong* λ-*approximation family* iff in addition to (a)–(c) above, it satisfies:

(d) [Union] Suppose that $i^* < \lambda$.

If $\bar{M} = \langle M_i : i < i^* \rangle$ is a $\leq_{K_{\rm ap}}$-increasing sequence in $K_{\rm ap}$, then we have that $\bigcup_{i<i^*} M_i$ is an element of $K_{\rm ap}$, and it is the $\leq_{K_{\rm ap}}$-lub of $\bar{M}$.

(e) [End extension/Amalgamation] If $0 < \delta < \lambda^+$ is divisible by λ, and $M_0, M_1, M_2 \in K_{\rm ap}$ are such that $M_2 \restriction \delta = M_0 \leq_{K_{\rm ap}} M_1$ and $|M_1| \subseteq \delta$, then M_1 and M_2 have a $\leq_{K_{\rm ap}}$-upper bound M_3 such that $M_3 \restriction \delta = M_1$.

If M_0, M_1, M_2, δ are as above and $M_1, M_2 \leq M$, then there is $M_3 \leq M$ such that $M_3 \geq M_1, M_2$ and $M_3 \restriction \delta = M_1$.

(f) [Local Cardinality] For $\alpha < \lambda^+$, the set $\{M \in K_{\rm ap} : |M| \subseteq \alpha\}$ has cardinality $\leq \lambda$.

(g) [Uniformity] For $M_1, M_2 \in K_{\rm ap}$, we call $h : M_1 \to M_2$ a *lawful isomorphism* iff h is a lawful function and a $K_{\rm ap}$-isomorphism. We demand

[3]The following contradicts the usual notation of model theory of forbidding empty models, as in such a situation we cannot interpret individual constants. However, the meaning of $\emptyset$ we use is clear.

(α) if $M \in K_{\rm ap}$ and h is a lawful mapping from $|M|$ onto some $u \subseteq \lambda^+$, then for some $M' \in K_{\rm ap}$ we have that $|M'| = u$ and h is a lawful $\tau_{\iota|M|}$-isomorphism from M onto M'.

(β) lawful $K_{\rm ap}$-isomorphisms preserve $\leq_{K_{\rm ap}}$.

(h) [Density] For every β in λ^+, and $M \in K_{\rm ap}$, there is $M' \in K_{\rm ap}$ such that $M \leq_{K_{\rm ap}} M'$ and $\beta \in |M'|$.

(i) [Amalgamation] Assume $M_l \in K_{\rm ap}$ for $l < 3$ and $M_0 \leq_{K_{\rm ap}} M_l$ for $l = 1, 2$. Then for some lawful function f and $M \in K_{\rm ap}$, we have $M_1 \leq_{K_{\rm ap}} M$, the domain of f is M_2, the restriction $f \restriction |M_0|$ is the identity, and f is a $\leq_{K_{\rm ap}}$-embedding of M_2 into M, i.e. $f(M_2) \leq_{K_{\rm ap}} M$. If $M_1 \cap M_2 = M_0$, we can assume that $f = \mathrm{id}$.

Remark 1.5. (1) There is no contradiction concerning vocabularies in (g)(α) of Definition 1.4(3): if $K_{\rm ap}$ is a weak λ-approximation family, while $M \in K_{\rm ap}$ and h is a lawful mapping from $|M|$ onto some u, then $\iota(u) = \iota(|M|)$ (so saying that h gives rise to a $K_{\rm ap}$-isomorphism makes sense).

[Why? Letting $\delta \stackrel{\rm def}{=} \sup(u)$, if $\gamma < \delta$, we can find $\alpha \in |M|$ such that $h(\alpha) \in (\gamma, \delta)$. Hence

$$\gamma < \gamma + \lambda \leq h(\alpha) + \lambda = \alpha + \lambda < \sup(|M|).$$

So, $\delta \leq \sup(|M|)$, and the other side of the inequality is shown similarly.]

(2) If $\bar{M} = \langle M_i : i < i^* \rangle$ is a $\leq_{K_{\rm ap}}$-increasing sequence, and $\lambda | \delta$, then $\langle M_i \restriction \delta : i < i^* \rangle$ is $\leq_{K_{\rm ap}}$-increasing, by Definition 1.4(1)(c), and if $i^* < \lambda$,

$$\bigcup_{i<i^*} (M_i \restriction \delta) = (\bigcup_{i<i^*} M_i) \restriction \delta$$

is the $\leq_{K_{\rm ap}}$-lub of $\langle M_i \restriction \delta : i < i^* \rangle$, by (3)(d) in Definition 1.4.

(3) Suppose M_l for $l < 3$ are as in Definition 1.4(3)(i) (amalgamation). Then we can without loss of generality assume that $M \restriction \mathrm{Ev} = M_1 \restriction \mathrm{Ev}$, as clearly there is a lawful mapping $g : M \to M^*$ extending id_{M_1} for some M^* with $M^* \restriction \mathrm{Ev} = M_1 \restriction \mathrm{Ev}$.

(4) Suppose that M_0, M_1 and M_2 are as in Definition 1.4(3)(e) (end extension/amalgamation). Then we can assume $M_3 \subseteq \iota(M_2)$, as by Definition 1.4(1)(c) we can replace M_3 by $M_3 \restriction \iota(M_2)$.

Notation 1.6. Suppose that $K_{\rm ap}$ is a weak λ-approximation family and $\bar\tau$ is a sequence of vocabularies as in Definition 1.4(1)(a). We say that $K_{\rm ap}$ *is written in* $\bar\tau$.

Definition 1.7. [Sh 457]

(1) Let $(K_{\rm ap}, \le_{K_{\rm ap}})$ be a weak λ-approximation family and $\Gamma \subseteq K_{\rm ap}$. We say that Γ is $(<\lambda)$-*closed* iff for every $\le_{K_{\rm ap}}$-increasing chain of size $<\lambda$ of elements of Γ, the lub of the chain is in Γ.

(2) Suppose that $(K_{\rm ap}, \le_{K_{\rm ap}})$ is a weak λ-approximation family. We let

$$K^-_{\rm md} = K^-_{\rm md}[K_{\rm ap}] \stackrel{\rm def}{=} \left\{ \Gamma : \begin{array}{l} \text{(i) } \Gamma \text{ is a } (<\lambda)\text{-closed subset of } K_{\rm ap}, \\ \text{(ii) } \Gamma \text{ is } \le_{K_{\rm ap}} \text{-directed}, \\ \text{(iii) for cofinally many } \beta<\lambda^+ \text{ we have} \\ (\exists M \in \Gamma)(\exists \gamma \in |M|)\iota(\gamma) = \iota(\beta) \\ \quad \text{(e.g.} \gamma = \beta) \end{array} \right\}.$$

We let $K_{\rm md} = K_{\rm md}[K_{\rm ap}] \stackrel{\rm def}{=}$

$$\left\{ \Gamma \in K^-_{\rm md} : \begin{array}{c} \text{(iv) } (M \in \Gamma \;\&\; M \le_{K_{\rm ap}} M_1) \Longrightarrow \\ (\exists M_2 \in \Gamma)(\exists h \text{ lawful})[h : M_1 \to M_2 \\ \text{embedding over } M] \\ \text{(v) } M \in \Gamma \;\&\; N \le M \Longrightarrow N \in \Gamma \end{array} \right\}.$$

(3) If $K_{\rm ap}$ is as above and $\alpha < \lambda^+$, we define $K^-_{\rm md}[K^\alpha_{\rm ap}]$ as the set of $\Gamma \subseteq K_{\rm ap}$ such that

(a) $M \in \Gamma \Longrightarrow |M| \subseteq \alpha$,

(b) Γ satisfies (i) –(ii) from (2) above.

Similarly for $K_{\rm md}[K^\alpha_{\rm ap}]$.

Claim 1.8. Suppose that $\Gamma \in K_{\rm md}[K_{\rm ap}]$, while $N \in \Gamma$ and $h : N \to M$ is a lawful embedding. Then there is $N' \in \Gamma$ and a lawful embedding $g : M \to N'$ such that for $x \in N$ we have $g(h(x)) = x$.

Proof of the Claim. There is a lawful isomorphism $f : M \to M'$ for some $M' \geq N$ such that $f(h(x)) = x$ for all $x \in N$. Then by (iv) in the definition of K_{md}, there is $N' \in \Gamma$ and a lawful embedding $g' : M' \to N'$ such that $g' \restriction N = \mathrm{id}_N$.

Let $g : M \to N'$ be given by letting $g(x) = g'(f(x))$, so g is a lawful embedding and for $x \in N$ we have $g(h(x)) = g'(f(h(x)) = g'(x) = x$. $\bigstar_{1.8}$

Definition 1.9. (1) $\mathcal{K} = (K, \leq_{\mathcal{K}})$ is an *abstract elementary class* iff K is a class of models of some fixed vocabulary $\tau = \tau_{\mathcal{K}}$ and $\leq_{\mathcal{K}} = \leq_K$ is a two place relation on K, satisfying the following axioms:

- Ax 0: If $M \in K$, then all τ-models isomorphic to M are also in K. The relation $\leq_K$ is preserved under isomorphisms,
- Ax I: If $M \leq_K N$, then M is a submodel of N,
- Ax II: $\leq_K$ is a partial order on K,
- Ax III, IV: The union of a $\leq_K$-increasing continuous chain $\bar{M}$ of elements of K is an element of K, and the lub of $\bar{M}$ under $\leq_K$,
- Ax V: If $M_l \leq_K N$ for $l \in \{0, 1\}$ and M_0 is a submodel of M_1, then $M_0 \leq_K M_1$,
- Ax VI: There is a cardinal κ such that for every $M \in \mathcal{K}$ and $A \subseteq |M|$, there is $N \leq_K M$ such that $A \subseteq |N|$ and $||N|| \leq \kappa \cdot (|A| + 1)$. The least such κ is denoted by $\mathrm{LS}(\mathcal{K})$ and called the Löwenheim-Skolem number of $\mathcal{K}$.

(2) If λ is a cardinal and $\mathcal{K}$ an abstract elementary class, we denote by $\mathcal{K}_\lambda$ the family of all elements of $\mathcal{K}$ whose cardinality is λ.
(3) For $\mathcal{K}$ an abstract elementary class, and λ a cardinal, we say that $\mathcal{K}_\lambda$ *has a universal* iff there is $M^* \in \mathcal{K}_\lambda$ such that for all $M \in \mathcal{K}_\lambda$ we have that some M' which is isomorphic to M satisfies $M' \leq_{\mathcal{K}} M^*$. Such M^* is called *universal for* $\mathcal{K}_\lambda$.
(4) Suppose that $\mathcal{K}$ is an abstract elementary class. We shall say that a member M of $\mathcal{K}$ is $\leq_{\mathcal{K}}$-*embeddable* in a member N of $\mathcal{K}$ iff there is an isomorphism between M and some $M' \in \mathcal{K}$ satisfying $M' \leq_{\mathcal{K}} N$.

(a) $\mathcal{K}$ is said to have the *joint embedding property* iff for any $M_1, M_2 \in \mathcal{K}$, there is $N \in \mathcal{K}$ such that M_1, M_2 are $\leq_{\mathcal{K}}$-embeddable into N.

(b) $\mathcal{K}$ is said to have *amalgamation* iff for all $M_0, M_1, M_2 \in \mathcal{K}$ and $\leq_{\mathcal{K}}$-embeddings $g_l : M_0 \to M_l$ for $l \in \{1, 2\}$, there is $N \in \mathcal{K}$ and $\leq_{\mathcal{K}}$-embeddings $f_l : M_l \to N$ such that $f_1 \circ g_1 = f_2 \circ g_2$.

Similar definitions are made to describe when $\mathcal{K}_\lambda$ has the joint embedding property or amalgamation.

Convention 1.10. We shall only work with abstract elementary classes which have the joint embedding property and amalgamation.

Note 1.11. The following notes are not hard and the proofs are to be found in [Sh 88]. We include them here for the reader's convenience.
(1) Suppose that $\mathcal{K}$ is an abstract elementary class. If $\bar{M} = \langle M_i : i < \delta \rangle$ is a $\leq_{\mathcal{K}}$-increasing chain (not necessarily continuous), then $\bigcup_{i<\delta} M_i$ is the $\leq_{\mathcal{K}}$-lub of $\bar{M}$.

[Why? Prove this by induction on δ. The nontrivial case is when δ is a limit. Define for $i < \delta$ a model N_i to be M_i if i is non-limit, and $\bigcup_{j<i} M_j$ otherwise. Now $\bar{N} = \langle N_i : i < \delta \rangle$ is increasing continuous and $\bigcup_{i<\delta} N_i = \bigcup_{i<\delta} M_i$ is the lub of $\bar{N}$, hence of $\bar{M}$.]
(2) If $\mathcal{K}$ is an abstract elementary class, $\mathcal{K}$ is closed under unions of $\leq_{\mathcal{K}}$-directed subsets, and the union of a $\leq_{\mathcal{K}}$-directed subset of $\mathcal{K}$ is the $\leq_{\mathcal{K}}$-lub of it.

[Why? By induction on κ, we prove that for any $\mathcal{D} \subseteq \mathcal{K}$ which is $\leq_{\mathcal{K}}$-directed and has size κ, the $\leq_{\mathcal{K}}$-lub of $\mathcal{D}$ is $\bigcup \mathcal{D}$. For $\kappa \leq \aleph_0$, this is clear. If κ is a limit $> \aleph_0$, let $\langle \kappa_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be cofinal increasing to κ, each κ_α regular, and $\mathcal{D} = \bigcup_{\alpha<\mathrm{cf}(\kappa)} \mathcal{D}_\alpha$, where each $\mathcal{D}_\alpha$ is $\leq_{\mathcal{K}}$-directed and has size κ_α, and $\mathcal{D}_\alpha$'s are $\subseteq$-increasing. Now apply the induction hypothesis and (1). If $\kappa = \lambda^+$, then we can find $\langle \mathcal{D}_\alpha : \alpha < \lambda^+ \rangle$ increasing to $\mathcal{D}$, each $\leq_{\mathcal{K}}$-directed and of size $\leq \lambda$.]

Definition 1.12. Suppose that $\mathcal{K}$ is an abstract elementary class with $\tau_{\mathcal{K}} = \tau$, and K_{ap} is a weak [strong] λ-approximation family written in

$$\langle \tau_i : i < \lambda^+ \ \& \ \lambda | i \rangle,$$

such that

(1) For all i, we have $\tau \subseteq \tau_i$,

(2) $M \in K_{\mathrm{ap}} \Longrightarrow M \restriction \tau \in \mathcal{K}$,

(3) $M \leq_{K_{\mathrm{ap}}} N \Longrightarrow M \restriction \tau \leq_{\mathcal{K}} N \restriction \tau$ and

(4) For every $M \in \mathcal{K}$ with $||M|| < \lambda$ there is $N \in K_{\mathrm{ap}}$ such that

$$M \text{ is } \leq_{\mathcal{K}} \text{-embeddable into } N \restriction \tau.$$

We say that K_{ap} *tends to [strongly] λ-approximate* $\mathcal{K}$.

We may just say "K_{ap} tends to approximate $\mathcal{K}$" if the rest is clear from the context.

Observation 1.13. Suppose that K_{ap} is a strong λ-approximation family which tends to approximate $\mathcal{K}$ and $\Gamma \in K^-_{\mathrm{md}}$. Then

(1) M_Γ defined by letting
$$M_\Gamma \stackrel{\mathrm{def}}{=} \bigcup_{M \in \Gamma} M \restriction \tau$$
is an element of $\mathcal{K}$ and for every $M \in \Gamma$ we have $M \restriction \tau \leq_{\mathcal{K}} M_\Gamma$, and in fact M_Γ is the $\leq_{\mathcal{K}}$-lub of $\{M \restriction \tau : M \in \Gamma\}$.

(2) For every $\Gamma, \Gamma^* \in K^-_{\mathrm{md}}[K_{\mathrm{ap}}]$ such that $\Gamma \subseteq \Gamma^*$, we have $M_\Gamma \leq_{\mathcal{K}} M_{\Gamma^*}$.

[Why? (1) As $\{M \restriction \tau : M \in \Gamma\}$ is $\leq_{\mathcal{K}}$-directed.
(2) By (1) and Note 1.11(2).]

Notation 1.14. Suppose that an approximation family K_{ap} tends to approximate $\mathcal{K}$, while $\Gamma \in K^-_{\mathrm{md}}$. If we write M_Γ, we always mean the model obtained from Γ as in Observation 1.13.

Definition 1.15. Let K_{ap} be a strong λ-approximation family which tends to λ-approximate $\mathcal{K}$ and let $\mathcal{K}^+$ be a subclass of $\mathcal{K}_{\lambda^+}$. Assume

(∗) For every $M^* \in \mathcal{K}^+$, there is $\Gamma \in K^-_{\rm md}[K_{\rm ap}]$ with $\{|M| : M \in \Gamma\}$ a club of $[{\rm Ev}]^{<\lambda}$ such that for some M' isomorphic to M^*, we have $M' \leq_{\mathcal{K}} M_\Gamma$.

Then we say that $K_{\rm ap}$ *approximates* $\mathcal{K}^+$.

Claim 1.16. Suppose that

(1) $\lambda \leq \kappa$,

(2) $\mathcal{K}$ is an abstract elementary class,

(3) ${\rm LS}(\mathcal{K}) \leq \kappa$ and $\mathcal{K}_\kappa$ has amalgamation,

(4) $\mathcal{T} \subseteq {}^{<\lambda^+}(\lambda^+)$ ordered by $\trianglelefteq$ (i.e. being an initial segment) is a tree with each level of size $\leq \lambda^+$,

(5) For $\eta \in \mathcal{T}$ we have $M_\eta \in \mathcal{K}$, so that

$$\eta \trianglelefteq \nu \Longrightarrow M_\eta \leq_{\mathcal{K}} M_\nu,$$

(6) $\eta \in \mathcal{T} \Longrightarrow ||M_\eta|| = \kappa$.

Then there are $M^* = M^*[\mathcal{T}] \in \mathcal{K}$ and $\langle g_\eta : \eta \in \mathcal{T}\rangle$ such that

(A) For all $\eta \in \mathcal{T}$ we have that g_η is a $\mathcal{K}$-embedding from M_η into M^*,

(B) $\eta \leq \nu \Longrightarrow g_\eta \subseteq g_\nu$,

(C) $||M^*|| \leq \kappa \cdot \lambda^+$.

(The intended use of this claim is when $\kappa = \lambda$.)

Proof of the Claim. For $i^* \leq \lambda^+$, let $\mathcal{T} \restriction i^* \stackrel{\rm def}{=} \mathcal{T} \cap {}^{<i^*}\lambda^+$.

By induction on $i^* \leq$ the height of $\mathcal{T}$, we prove that $M^*[\mathcal{T} \restriction i^*]$ and $\langle g^{i^*}_\eta : \eta \in \mathcal{T} \restriction i^*\rangle$ can be defined to satisfy (A)-(C) with $\mathcal{T} \restriction i^*$ in place of $\mathcal{T}$ and $M^*[\mathcal{T} \restriction i]$ in place of M^*, and so that

$$i \leq i^* \Longrightarrow M^*[\mathcal{T} \restriction i] \leq_{\mathcal{K}} M^*[\mathcal{T} \restriction i^*],$$

$$[\eta \in \mathcal{T} \restriction i \ \& \ i \leq i^*] \Longrightarrow g^i_\eta = g^{i^*}_\eta.$$

$\underline{i^* = 0}$. Trivial.

$\underline{i^* = i + 1}$. Let $\mathrm{lev}_i(\mathcal{T}) = \{\eta_j : j < j^* \leq \lambda^+\}$. For simplicity in notation we assume that j^* is a limit ¿ 0, the other cases are similar. By induction on j we build $\langle M^*_j : j < j^* \rangle$ so that $M^*_0 = M^*[\mathcal{T} \restriction i]$ and $M_{\eta_j}, M^*_j \leq_{\mathcal{K}} M^*_{j+1}$, while $||M^*_j|| = \kappa$, and $M^*_\delta = \bigcup_{j<\delta} M^*_j$ for δ a limit. We use amalgamation and the induction hypothesis to obtain (B). Namely, to define M^*_{j+1}, let first $M'_j \overset{\mathrm{def}}{=} \bigcup\{M_\nu : \nu \lhd \eta_j\}$ and $g_j \overset{\mathrm{def}}{=} \cup\{g^i_\nu : \nu \lhd \eta_j\}$, which is well defined by the induction hypothesis. Hence $g_j : M'_j \to M^*_0 \leq_{\mathcal{K}} M^*_j$ is a $\leq_{\mathcal{K}}$-embedding, as is $\mathrm{id} : M'_j \to M_{\eta_j}$. Using amalgamation, we can find $M^*_{j+1} \in \mathcal{K}$ and $\leq_{\mathcal{K}}$-embeddings $f : M^*_j \to M^*_{j+1}$ and $g_{\eta_j} : M_{\eta_j} \to M^*_{j+1}$ such that $f \circ g_j = g_{\eta_j} \restriction M'_j$. By Ax 0 of Definition 1.9, without loss of generality we have $f = \mathrm{id}$. By Ax VI of the same Definition, we can also assume that $||M^*_{j+1}|| \leq \kappa$. Now let $M^*[\mathcal{T} \restriction i^*] \overset{\mathrm{def}}{=} \bigcup_{j<j^*} M^*_j$.

$\underline{i^* \text{ a limit}}$. $M^*[\mathcal{T}] = \bigcup_{i<i^*} M^*[\mathcal{T} \restriction i]$.

$\bigstar_{1.16}$

Observation 1.17. With the notation of Claim 1.16, if ρ is a branch of $\mathcal{T}$, then $\bar{M} = \langle M_\eta : \eta \in \mathcal{T} \ \& \ \eta \unlhd \rho \rangle$ is a $\leq_{\mathcal{K}}$-increasing chain of $\mathcal{K}$. Hence $\bigcup \bar{M}$ is the $\leq_{\mathcal{K}}$-lub of $\bar{M}$, and so $\bigcup \bar{M}$ is $\leq_{\mathcal{K}}$-embeddable into $M^*[\mathcal{T}]$.

Definition 1.18. For a strong λ-approximation family K_{ap} we say that it is *workable* iff for every $\Gamma \in K^-_{\mathrm{md}}[K_{\mathrm{ap}}]$ such that $M \in \Gamma \Longrightarrow |M| \subseteq \mathrm{Ev}$, for all $\delta_1 < \delta_2 \in S^{\lambda^+}_\lambda$ the following holds:

Suppose that for $l \in \{1, 2\}$ we are given (M_l, N_l) such that

(i) $M_l \in \Gamma$,

(ii) $M_l \leq_{K_{\mathrm{ap}}} N_l \in K_{\mathrm{ap}}$,

(iii) $|N_l| \cap \{2\beta : \beta < \lambda^+\} = |M_l|$,

(iv) $|N_1| \subseteq \delta_2$,

(v) $N_1 \restriction \delta_1 = N_2 \restriction \delta_2$,

(vi) Some h is a lawful $K_{\rm ap}$-isomorphism from $N_1 \restriction \tau(\delta_1)$ onto $N_2 \restriction \tau(\delta_1)$ mapping M_1 onto M_2,

(vii) $h \restriction (N_1 \restriction \delta_1)$ is the identity.

Then there are $M \in \Gamma$ and $N \in K_{\rm ap}$ with $M \le N$, and g_l for $l \in \{1, 2\}$ such that $M_l \le M \le N$ and g_l is a $\le_{K_{\rm ap}}$-embedding of N_l into N, with $g_l \restriction M_l = {\rm id}_{M_l}$. In addition, $|N| \cap \{2\beta : \beta < \lambda^+\} = |M|$ and $g_l \restriction (N_l \restriction \delta_l)$ is fixed.

Note 1.19. For those familiar with definitions in [Sh 457], we emphasize that smoothness was assumed throughout. That is, our definition of $K_{\rm ap}$ is less general than the one in [Sh 457], and any strong λ-approximation family in the sense of our Definition 1.7 automatically satisfies the condition which in [Sh 457] was called smoothness.

2 Universals in λ^+.

Definition 2.1. [Sh 546] Suppose that $\lambda > \aleph_0$ is a cardinal and $\varepsilon < \lambda$ a limit ordinal. A forcing notion Q satisfies $*^{\varepsilon}_{\lambda}$ iff player I has a winning strategy in the following game $*^{\varepsilon}_{\lambda}[Q]$:

Moves: The play lasts ε moves. For $\zeta < \varepsilon$, the ζ-th move is described by:

Player I: If $\zeta \ne 0$, I chooses $\langle q^{\zeta}_i : i < \lambda^+\rangle$ such that $q^{\zeta}_i \in Q$ and $q^{\zeta}_i \ge p^{\xi}_i$ for all $\xi < \zeta$, as well as a function $f_\zeta : \lambda^+ \to \lambda^+$ which is regressive on $C_\zeta \cap S^{\lambda^+}_{\lambda}$ for some club C_ζ of λ^+. If $\zeta = 0$, we let $q^{\zeta}_i \stackrel{\rm def}{=} \emptyset_Q$ and f_ζ be identically 0.

Player II: Chooses $\langle p^{\zeta}_i : i < \lambda^+\rangle$ such that $q^{\zeta}_i \le p^{\zeta}_i \in Q$ for all $i < \lambda^+$.

The Outcome: Player I wins iff:
For some club E of λ^+, for any $i < j \in E \cap S^{\lambda^+}_{\lambda}$,

$$\bigwedge_{\zeta<\varepsilon} f_\zeta(i) = f_\zeta(j) \Longrightarrow [\{p^{\zeta}_i : \zeta < \varepsilon\} \cup \{p^{\zeta}_j : \zeta < \varepsilon\} \text{ has an upper bound in } Q].$$

We say that $E \subseteq \bigcap_{\zeta<\varepsilon} C_\zeta$ is a witness that I won.

(2) A *winning strategy for I in* $*^\varepsilon_\lambda[Q]$ is a function St$=$ (St$_*$, St*) such that in any play

$$\Big\langle \langle q^\zeta_i : i < \lambda^+\rangle, f_\zeta, \langle p^\zeta_i : i < \lambda^+\rangle : \zeta < \varepsilon \Big\rangle$$

in which we have for all ζ, i

$$q^\zeta_i = \mathrm{St}_*(i, \Big\langle \langle p^\xi_j : j < \lambda^+\rangle : \xi < \zeta \Big\rangle), f_\zeta = \mathrm{St}^*(\Big\langle \langle p^\xi_j : j < \lambda^+\rangle : \xi < \zeta \Big\rangle),$$

I wins.

i.e. a winning strategy for I depends only on the moves of II and f_ζ and C_ζ can be defined from $\langle \langle p^\xi_j : j < \lambda^+\rangle : \xi < \zeta\rangle$.

Fact 2.2. [Sh 546] Suppose that $\lambda > \aleph_0$ is a cardinal satisfying $\lambda^{<\lambda} = \lambda$, and $\varepsilon < \lambda$ a limit ordinal.
(1) If P is a forcing notion satisfying $*^\varepsilon_\lambda$, then P satisfies λ^+-cc.
(2) Suppose that P is the result of an iteration of $(< \lambda)$-complete forcing satisfying $*^\varepsilon_\lambda$. Then P is $(< \lambda)$-complete and satisfies $*^\varepsilon_\lambda$.

Proof of the Fact. (1) Suppose that $\bar p = \langle p_i : i < \lambda^+\rangle$ is a sequence of elements of P, and consider a game of $*^\varepsilon_\lambda[P]$ in which II plays $\bar p$ as the first move, and I plays according to a winning strategy. At the end of the game, let E be a club of λ^+ witnessing that I won, and let $i < j$ be in $E \cap S^{\lambda^+}_\lambda$ such that for all $\zeta < \varepsilon$ we have that $f_\zeta(i) = f_\zeta(j)$, which exists as these functions are regressive. We in particular obtain that p_i and p_j are compatible in P.
(2) We refer the reader to [Sh 546].

Theorem 2.3. Suppose that the following are satisfied in a universe V_0 of set theory:

(A) $\aleph_0 \le \lambda = \lambda^{<\lambda} < \lambda^+ = 2^\lambda < 2^{\lambda^+} \le \kappa < \mu = \mu^\kappa$,

(B) R^* is the forcing notion which adds μ many Cohen subsets $\langle \rho^*_\alpha : \alpha < \mu\rangle$ to λ^+ by conditions of size $\le \lambda$.

(C) $\mathcal{T} = {}^{<\lambda^+}(\lambda^+)$ of V_0, ordered by "being an initial segment",

(D) If $\lambda > \aleph_0$, we are given a limit ordinal $\varepsilon < \lambda$.

Then in $V \stackrel{\text{def}}{=} V_0^{R^*}$ for some P we have

(a) P is a forcing notion of cardinality μ,

(b) P is $(< \lambda)$-complete and λ^+-cc (and if $\lambda > \aleph_0$, P satisfies $*^{\varepsilon}_{\lambda}$),

(c) In V^P we have $\lambda^{<\lambda} = \lambda$ and $2^\lambda = 2^{\lambda^+} = \mu$,

(d$_0$) If $\lambda = \aleph_0$, then $MA(\aleph_1)$ holds in V^P,

(d$_1$) If $\lambda > \aleph_0$, then the following holds in V^P: if Q is a $(< \lambda)$-complete forcing notion of cardinality $< \kappa$ and satisfies $*^{\varepsilon}_{\lambda}$, and if we are given a family $\{\mathcal{I}_j : j < \lambda^+\}$ of dense subsets of Q, then for some directed $G \subseteq Q$ we have that $G \cap \mathcal{I}_j \neq \emptyset$ for all $j < \lambda^+$,

(e) In V^P, if $K = K_{\rm ap}$ is a workable strong λ-approximation family, then we can find

$$\langle \bar{\Delta}_\beta = \langle \Delta^\beta_\eta : \eta \in \mathcal{T} \rangle : \beta < \lambda^{++} \rangle$$

such that

(i) For every $\beta < \lambda^{++}$ and $\eta \in \mathcal{T}$ we have $\Delta^\beta_\eta \subseteq K^{\lambda \cdot \lg(\eta)}_{\rm ap}$ is $\leq_{\mathcal{K}_{\rm ap}}$-directed, and also for $\eta \trianglelefteq \nu \in \mathcal{T}$, we have that

$$\Delta^\beta_\eta = \{M \restriction (\lambda \cdot \lg(\eta)) : M \in \Delta^\beta_\nu\},$$

(ii) For any λ^+-branch ρ of $\mathcal{T}$ and $\beta < \lambda^{++}$, we have

$$\bigcup \{\Delta^\beta_\eta : \eta \trianglelefteq \rho\} \in K^-_{\rm md}[K_{\rm ap}],$$

(iii) For any $\Gamma \in K_{\rm md}[K_{\rm ap}]$, for some $\beta < \lambda^{++}$ we have that M_Γ is isomorphically embeddable into $M_{\bigcup_{i<\lambda^+} \Delta^\beta_{\rho \restriction i}}$ for some λ^+-branch ρ of $\mathcal{T}$ with $\rho \in V$ (for the notation see 1.14),

(f) In V^P, if $K_{\rm ap}$ is a workable strong λ-approximation family and Γ^- is an element of $K^-_{\rm md}[K_{\rm ap}]$ such that $M \in \Gamma^- \implies |M| \subseteq {\rm Ev}$, then there is $\Gamma \in K_{\rm md}[K_{\rm ap}]$ such that $\Gamma^- \subseteq \Gamma$.

Once we prove the theorem, we shall be able to draw the following

Conclusion 2.4. Suppose that V satisfies

$$\aleph_0 \leq \lambda = \lambda^{<\lambda} < \lambda^+ = 2^\lambda < 2^{\lambda^+} \leq \kappa < \mu = \mu^\kappa,$$

and if $\lambda > \aleph_0$, we are given a limit ordinal $\varepsilon < \lambda$. Then there is a cofinality and cardinality preserving forcing extension V^* of V which satisfies

(1) For every abstract elementary class $\mathcal{K}$ for which there is a workable λ-approximation family K_{ap} which approximates $\mathcal{K}$, and such that $\text{LS}(\mathcal{K}) \leq \lambda$, there are λ^{++} elements $\{M_\alpha : \alpha < \lambda^{++}\}$ of $\mathcal{K}_{\lambda^+}$ which are jointly universal for $\mathcal{K}_{\lambda^+}$,

(2) $\aleph_0 \leq \lambda^{<\lambda} = \lambda < 2^\lambda = 2^{\lambda^+} = \mu = \mu^\kappa$,

(3)(a) In the case $\lambda = \aleph_0$: $MA(\aleph_1)$ holds,

(3)(b) In the case $\lambda > \aleph_0$: if Q is a $(< \lambda)$-complete forcing notion of cardinality $< \kappa$, satisfying $*^\varepsilon_\lambda$, and we are given a family $\{\mathcal{I}_j : j < \lambda^+\}$ of dense subsets of Q, then for some directed $G \subseteq Q$ we have that $G \cap \mathcal{I}_j \neq \emptyset$ for all $j < \lambda^+$,

(4) If $\mathcal{K}$ is an abstract elementary class with $\text{LS}(\mathcal{K}) \leq \lambda$ and $\mathcal{K}^+$ is a subclass of $\mathcal{K}_{\lambda^+}$ for which there is a workable strong λ-approximation family K_{ap} which approximates $\mathcal{K}^+$, and such that for every tree $\mathcal{T}$ of the form from Claim 1.16 in which every M_η is the union of $\leq \lambda$ elements of K_{ap} we have that $M^*[\mathcal{T}] \in \mathcal{K}^+$, then there are λ^{++} elements $\{M_\alpha : \alpha < \lambda^{++}\}$ of $\mathcal{K}^+$ which are jointly universal for $\mathcal{K}^+$.

Remark 2.5. The informal plan of the proof of the theorem and the conclusion is as follows. The purpose of forcing with R^* is to make $2^{\lambda^+} = \mu$ and add μ branches through $\mathcal{T}$. Then P will be an iteration of λ^{++} blocks of μ steps each. Hence

$$P = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \lambda^{++}, \beta < \lambda^{++} \rangle$$

and for each β we have $\underset{\sim}{Q}_\beta = \langle Q^\beta_i, \underset{\sim}{R}^\beta_j : i \leq \mu, j < \mu\rangle$. Each $\underset{\sim}{R}^\beta_j$ will be one of four possible kinds (three in case $\lambda = \aleph_0$). Let us first describe the situation when $\lambda > \aleph_0$.

At kind 1 coordinates we shall be taking care of the form of Martin's Axiom given in (d) of the Theorem. Each kind 2 coordinate $\underset{\sim}{R}^{\beta+1}_j$ takes a workable strong λ-approximation family $K_{\rm ap}$ from V^{P_β} and a family of $\leq \mu$ elements of $K_{\rm ap}[K_{\rm md}]$ and introduces a tree of elements of $K_{\rm ap}$ indexed by $\mathcal{T}$, which gives $\bar{\Delta}_\beta$ as in (e)(i)-(ii) of the Theorem. This tree will also have the property that for every $\Gamma \in K_{\rm ap}[K_{\rm md}] \cap V^{P_\beta}$, there is a branch ρ of $\mathcal{T}$ with $\rho \in V$ and a tree T whose elements are pairs (N, h) with $N \in \Gamma$ and h an embedding from N into $M_{\bigcup_{i<\lambda^+} \Delta^\beta_{\rho\restriction i}}$, ordered by extension. Then for some $\beta' \in (\beta, \lambda^{++})$, there will be a forcing of the third kind that will introduce a branch through T and so have (e)(iii) of the Theorem.

At the remaining coordinates, for a workable strong λ-approximation family K introduced at some earlier stage, we embed M_{Γ^-} for some $\Gamma^- \in K^-_{\rm md}$ into M_Γ for some $\Gamma \in K_{\rm md}$.

If $\lambda = \aleph_0$, the forcing is easier because we do not need a strong chain condition in order to be able to iterate. So the kind two coordinates, which satisfy ccc but not the stronger analogue of it needed if $\lambda > \aleph_0$, are simplified and guarantee (e)(i)-(iii) immediately. This eliminates the need for kind three coordinates.

To get the conclusion for a given $\mathcal{K}$ as in (1), recall from §1 that for every $\beta < \lambda^{++}$ there is a model M^*_β in $\mathcal{K}_{\lambda^+}$ such that for every branch ρ of $\mathcal{T}$, the model read along the branch in the tree indexed by $\bar{\Delta}_\beta$, embeds into M^*_β. As $K_{\rm ap}$ approximates $\mathcal{K}$ (see Definition 1.15), for every $M \in \mathcal{K}$, there is $\Gamma^- \in K^-_{\rm md}$ (and of the kind required by the Theorem) such that M embeds into M_{Γ^-}. From the Theorem, there is $\Gamma \in K_{\rm md}$ such that M_{Γ^-} embeds into M_Γ. This Γ is in V^{P_β} for some $\beta < \lambda^{++}$, and hence some $\underset{\sim}{R}^{\beta+1}_j$ will guarantee that M_Γ embeds into M^*_β.

The *proof of the Conclusion* is given after the proof of the Theorem, close to the end of the section.

Proof of the Theorem. Let R^* be as in the statement of the Theorem, and let $V = V^{R^*}$. Then in V we clearly have $\aleph_0 \leq \lambda = \lambda^{<\lambda}$, while $2^\lambda = \lambda^+$,

$2^{\lambda^+} = \mu$ and the cardinalities and cofinalities of V_0 are preserved.

Let $\langle f^*_\alpha : \alpha < \mu\rangle$ list the λ^+-branches of $\mathcal{T}$ in V.

We make some easy observations:

Note 2.6. (1) It suffices to prove the conclusion weakened by requiring each Q being considered in $(\mathrm{d})_1$, to have the set of elements some ordinal $< \kappa$.
(2) By renaming, each K_{ap} considered in the theorem can be assumed to have its vocabulary included in $\mathcal{H}(\lambda^+)$.

Definition 2.7. We define P as $P_{\lambda^{++}}$ in the iteration

$$\bar{P} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \lambda^{++}, \beta < \lambda^{++}\rangle,$$

where

(α) $\bar{P}$ is a $(<\lambda)$-support iteration.

(β) For each $\beta < \lambda^{++}$, in V^{P_β} we have that Q_β is Q^β_μ in the iteration

$$\bar{Q}^\beta = \langle Q^\beta_i, \underset{\sim}{R}^\beta_j : i \leq \mu, j < \mu\rangle,$$

where:

(i) the iteration in $\bar{Q}^\beta$ is made with $(<\lambda)$-supports,

(ii) for each $j < \mu$ one of the following occurs:

Case 1. $\underset{\sim}{R}^\beta_j$ is a Q^β_j-name of a $(<\lambda)$-complete forcing notion which satisfies $*^\varepsilon_\lambda$ if $\lambda > \aleph_0$, and is ccc if $\lambda = \aleph_0$; and whose set of elements is some ordinal $< \kappa$.

Case 2. For some P_β-name of a workable λ-approximation family $\underset{\sim}{K}^\beta_{\mathrm{ap},j}$, abbreviated as $\underset{\sim}{K}$, and elements $\{\Gamma_\alpha = \Gamma^{\beta,j}_\alpha : \alpha < \mu\}$ of $K^{V^{P_\beta}}_{\mathrm{md}}$, we have that $\underset{\sim}{R} = \underset{\sim}{R}^\beta_j$ is defined as follows. We work in $V^{P_\beta * \underset{\sim}{Q}^\beta_j}$. For $M \in K$ we let

$$w[M] \overset{\mathrm{def}}{=} \{\gamma, \gamma+1 : M \cap [\lambda\gamma, \lambda(\gamma+1)) \neq \emptyset\}.$$

Subcase 2A. $\lambda = \aleph_0$. The elements of R are conditions of the form

$$p = \langle u^p, \langle M^p_\eta : \eta \in u^p\rangle, b^p, \langle c^p_\alpha : \alpha \in b^p\rangle, \langle (N^p_{\alpha,\iota}, h^p_{\alpha,\iota}) : \alpha \in b^p, \iota \in c^p_\alpha\rangle\rangle,$$

where

(a)[*closure under intersections*] $u = u^p \in [\mathcal{T}]^{<\lambda}$ is closed under intersections

(b) $\eta \in u \Longrightarrow M^p_\eta \in K \ \& \ |M^p_\eta| \subseteq \lambda \cdot \lg(\eta)$,

(c) If $\eta \trianglelefteq \nu$ are both in u, then $M^p_\eta = M^p_\nu \restriction \lambda \cdot \lg(\nu)$,

(d) [w-*closure*] $\eta \in u \ \& \ \beta \in w[M^p_\eta] \Longrightarrow \eta \restriction \beta \in u$,

(e) $b^p \in [\mu]^{<\lambda}$, $c^p_\alpha \in [\lambda^+]^{<\lambda}$ for $\alpha \in b^p$,

(f)$_{\mathrm{A}}$ For $\alpha \in b^p, \iota \in c^p_\alpha$ we have $f^*_\alpha \restriction \iota \in u, N^p_{\alpha,\iota} \in \Gamma_\alpha$ and $h^p_{\alpha,\iota}$ is a lawful embedding from $N^p_{\alpha,\iota}$ into $M^p_{f^*_\alpha \restriction \iota}$ (and hence $|N^p_{\alpha,\iota}| \subseteq \lambda \cdot \iota$ and $h(N^p_{\alpha,\iota}) \leq_K M^p_{f^*_\alpha \restriction \iota}$),

(g)$_{\mathrm{A}}$ If $\alpha \in b^p$ and $\iota_1 < \iota_2 \in c^p_\alpha$, then $N^p_{\alpha,\iota_1} = N^p_{\alpha,\iota_2} \restriction \lambda \cdot \iota_1$ and $h^p_{\alpha,\iota_1} = h^p_{\alpha,\iota_2} \restriction N^p_{\alpha,\iota_1}$.

The order in R is given by letting $p \leq q$ iff

(i) $u^p \subseteq u^q$,

(ii) for $\eta \in u^p$ we have $M^p_\eta \leq_K M^q_\eta$,

(iii) $b^p \subseteq b^q$,

(iv) for $\alpha \in b^p$, we have $c^p_\alpha \subseteq c^q_\alpha$,

(v)$_{\mathrm{A}}$ for $\alpha \in b^p, \iota \in c^p_\alpha$ we have $N^p_{\alpha,\iota} \leq N^q_{\alpha,\iota}$ and $h^q_{\alpha,\iota} \restriction N^p_{\alpha,\iota} = h^p_{\alpha,\iota}$.

<u>Subcase 2B</u>. $\lambda > \aleph_0$. The elements of R are conditions of the form

$$p = \langle u^p, \bar{M}^p, b^p, \bar{c}^p_\alpha, \bar{d}^p_{\alpha,\iota}, (\bar{N,h})^p_{\alpha,\iota,\Upsilon} \rangle$$

where

- $\bar{M}^p = \langle M^p_\eta : \eta \in u^p \rangle$,
- $\bar{c}^p_\alpha = \langle c^p_\alpha : \alpha \in b^p \rangle$
- $\bar{d}^p_{\alpha,\iota} = \langle d^p_{\alpha,\iota} : \alpha \in b^p, \iota \in c^p_\alpha \rangle$,
- $(\bar{N,h})^p_{\alpha,\iota,\Upsilon} = \langle (N^p_{\alpha,\iota,\Upsilon}, h^p_{\alpha,\iota,\Upsilon}) : \Upsilon \in d^p_{\alpha,\iota}, \alpha \in b^p, \iota \in c^p_\alpha \rangle$

and

(a)-(e) from Subcase 2A hold,

(f)$_{\rm B}$ for $\alpha \in b^p, \iota \in c^p_\alpha$ we have $d^p_{\alpha,\iota} \in [\lambda]^{<\lambda}$,

(h) for $\alpha \in b^p, \iota \in c^p_\alpha$ we have $f^*_\alpha \restriction \iota \in u$ and for each $\Upsilon \in d^p_{\alpha,\iota}$ we have $N^p_{\alpha,\iota,\Upsilon} \in \Gamma_\alpha$ and $h^p_{\alpha,\iota,\Upsilon} : N^p_{\alpha,\iota,\Upsilon} \to M^p_{f^*_\alpha \restriction \iota}$ is a lawful embedding (and hence $|N^p_{\alpha,\iota,\Upsilon}| \subseteq \lambda \cdot i$ and $h^p_{\alpha,\iota,\Upsilon}(N^p_{\alpha,\iota,\Upsilon}) \le M^p_{f^*_\alpha \restriction i}$),

(j) if $\alpha \in b^p$ and $\iota_1 < \iota_2 \in c^p_\alpha$ while $\Upsilon \in d^p_{\alpha,\iota_2}$, then

$$(N^p_{\alpha,\iota_2,\Upsilon} \restriction \lambda \cdot \iota_1, h^p_{\alpha,\iota_2,\Upsilon} \restriction \lambda \cdot \iota_1) = (N^p_{\alpha,\iota_1,\Upsilon'}, h^p_{\alpha,\iota_1,\Upsilon'})$$

for some $\Upsilon' \in d^p_{\alpha,\iota_1}$.

The order in R is given by letting $p \le q$ iff (i)-(iv) from Subcase 2A hold and

(v)$_{\rm B}$ for $\alpha \in b^p, \iota \in c^p_\alpha, \Upsilon \in d^p_{\alpha,\iota}$ we have $N^p_{\alpha,\iota,\Upsilon} \le N^q_{\alpha,\iota,\Upsilon'}$ and $h^p_{\alpha,\iota,\Upsilon} \subseteq h^q_{\alpha,\iota,\Upsilon'}$ for some $\Upsilon' \in d^q_{\alpha,\iota}$.

If G is R-generic, then we let for $\eta \in \mathcal{T}$

$$\Delta^{\beta,j}_\eta = \{M^p_\eta : p \in G \;\&\; \eta \in u^p\}.$$

<u>Case 3</u>. If $\lambda = \aleph_0$, this case does not occur. If $\lambda > \aleph_0$, then we are given $\alpha < \mu$ and a P_β-name $\underset{\sim}{K} = \underset{\sim}{K}_{{\rm ap},j}$ of a workable λ-approximation family such that for some $j' < j$ we have had $\underset{\sim}{K}_{{\rm ap},j'} = \underset{\sim}{K}_{{\rm ap},j}$ and the forcing $\underset{\sim}{R}^\beta_{j'}$ was defined by Case 2. In $V^{P_\beta * \underset{\sim}{Q}^\beta_{j'} * \underset{\sim}{R}^\beta_{j'}}$, let G be the generic of $R^\beta_{j'}$ over $V^{P_\beta * \underset{\sim}{Q}^\beta_{j'}}$ and let $R^\beta_j \overset{\rm def}{=}$

$$\{(N,h) : (\exists p \in G)(\exists \iota \in c^p_\alpha)(\exists \Upsilon \in d^p_{\alpha,\iota})[(N,h) = (N^p_{\alpha,\iota,\Upsilon}, h^p_{\alpha,\iota,\Upsilon})]\}$$

ordered by $(N_1,h_1) \le (N_2,h_2)$ iff $N_1 \le N_2$ and $h_1 = h_2 \restriction N_1$.

<u>Case 4</u>. For some $P_\beta * \underset{\sim}{Q}^\beta_j$-names of a workable λ-approximation family $\underset{\sim}{K} = \underset{\sim}{K}^\beta_{{\rm ap},j}$ and a member $\underset{\sim}{\Gamma}^- = \underset{\sim}{\Gamma}^-_{\beta,j}$ of $K^-_{\rm md}[\underset{\sim}{K}_{\rm ap}]$ such that

$$\Vdash_{P_\beta * \underset{\sim}{Q}^\beta_j} \text{“}\{|M| : M \in \underset{\sim}{\Gamma}^-\} \subseteq [{\rm Ev}]^{<\lambda}\text{”}$$

we have (working in $V^{P_\beta} * \underset{\sim}{Q}^\beta_j$),

$$R = \Big\{\langle M, N\rangle : M, N \in K \ \&\ M = N \restriction \mathrm{Ev} \ \&\ M \in \Gamma^-\Big\}$$

ordered by

$$\langle M_1, N_1\rangle \le \langle M_2, N_2\rangle \text{ iff } [M_1 \le M_2 \text{ and } N_1 \le N_2].$$

Discussion 2.8. We now prove a series of Claims which taken together imply the Theorem. These Claims are formulated for $\beta < \lambda^{++}, j < \mu$ and are proved by induction on β and j. Let us fix $\beta < \lambda^{++}$ and $j < \mu$ and assume that we have arrived at the induction step for (β, j). We work in $V^{P_\beta * \underset{\sim}{Q}^\beta_j}$ and let $R = R^\beta_j$.

Claim 2.9. Suppose R is defined by Case 2 of Definition 2.7.
(1) If $p \in R$, then for any $\eta \in u^p$ and $\mathcal{C} \subseteq u^p$ a chain with $\bigcup \mathcal{C} = \eta$, we have $M^p_\eta = \bigcup\{M^p_\nu : \nu \in \mathcal{C}\}$.
(2) For every $\eta \in \mathcal{T}$ and $v \in [\lambda^+]^{<\lambda}$ such that $v \subseteq \lambda \cdot \lg(\eta)$, the set

$$\mathcal{J}_{\eta,v} \stackrel{\text{def}}{=} \{p \in R : \eta \in u^p \ \&\ v \subseteq |M^p_\eta|\}$$

is a dense open subset of R.
(3) Suppose that $p \in R$ is given and that for some $\eta \in u^p$ we are given $M \ge M^p_\eta$ with $|M| \subseteq \lambda \cdot \lg(\eta)$. Then there is $q \ge p$ with $M^q_\eta = M$.
(4) Suppose that $\lambda > \aleph_0$ and that $p \in R$, $\alpha < \mu$ and $(N_1, h_1), (N_2, h_2)$ are such that for some $\iota_1, \iota_2 \in c^p_\alpha$ and $\Upsilon_l \in d^p_{\alpha,\iota_l}$ we have

$$(N_l, h_l) = (N^p_{\alpha,\iota_l,\Upsilon_l}, h^p_{\alpha,\iota_l,\Upsilon_l}) \text{ for } l \in \{1, 2\},$$

while

$$h_1 \restriction (N_1 \cap N_2) = h_2 \restriction (N_1 \cap N_2).$$

Then

$$\mathcal{D} = \{q \ge p : (\exists \iota \in c^q_\alpha)(\exists \Upsilon \in d^q_{\alpha,\iota}) \bigwedge_{l \in \{1,2\}} N^q_{\alpha,\iota,\Upsilon} \ge N^p_{\alpha,\iota_l,\Upsilon_l}, h^q_{\alpha,\iota,\Upsilon} \supseteq h^p_{\alpha,\iota_l,\Upsilon_l}\}$$

is dense above p.
(5) Suppose that $\lambda > \aleph_0$ and $\alpha < \mu$. Then for every $N \in \Gamma_\alpha$, the set of all $p \in R$ such that for some ι, Υ we have $N^p_{\alpha,\iota,\Upsilon} \ge N$ is dense.

Proof of the Claim. (1) Obvious.
(2) Clearly $\mathcal{J}_{\eta,v}$ is open, we shall show that it is dense. Given $p \in R$, we shall define $q \in \mathcal{J}_{\eta,v}$ with $q \geq p$. We do the definition in several steps.

Step I. Let $u \stackrel{\text{def}}{=} u^p \cup \{\eta\} \cup \{\eta \cap \nu : \nu \in u^p\}$.

For $\sigma \in u$ we define M_σ as follows. If for some $\nu \in u^p$ we have $\sigma \trianglelefteq \nu$, then let $M_\sigma \stackrel{\text{def}}{=} M_\nu \restriction (\lambda \cdot \lg(\sigma))$. Once this has been done, we have $\sigma = \eta \notin u^p$ and we let

$$M_\eta \stackrel{\text{def}}{=} \bigcup \{M_\tau : \tau \in u \ \&\ \tau \triangleleft \eta\}.$$

It has to be checked that this definition is valid, in particular that M_σ is well defined for σ for which there are $\nu_1 \neq \nu_2$ both in u^p with $\sigma \trianglelefteq \nu_1 \cap \nu_2$. As u^p is closed under intersections, we have in this case that $\nu \stackrel{\text{def}}{=} \nu_1 \cap \nu_2$ is in u^p and

$$M^p_{\nu_1} \restriction (\lambda \cdot \lg(\nu)) = M^p_\nu = M^p_{\nu_2} \restriction (\lambda \cdot \lg(\nu)),$$

and hence M_σ is well defined. Observe also that u is closed under intersections and that

$$\sigma \trianglelefteq \tau \in u \Longrightarrow M_\sigma = M_\tau \restriction (\lambda \cdot \lg(\sigma)),$$

while clearly $\eta \in u$. Also note that if $\eta \in u^p$ we have $u = u^p$ and $M_\sigma = M^p_\sigma$ for $\sigma \in u$.

Step II. For $\sigma \in u$, we find M'_σ with $M'_\sigma \geq_{K_{\text{ap},\alpha}} M_\sigma$ and $v \cap \lambda \cdot \lg(\sigma) \subseteq |M'_\sigma|$, while $|M'_\sigma| \subseteq \lambda \cdot \lg(\sigma)$ and such that for $\sigma \triangleleft \tau \in u$ we have $M'_\sigma = M'_\tau \restriction \lambda \cdot \lg(\sigma)$. This is done by induction on $\lg(\sigma)$. Coming to σ, if $\sigma = \bigcup \{\tau \in u : \tau \triangleleft \sigma\}$, let $M'_\sigma \stackrel{\text{def}}{=} \bigcup \{M'_\tau : \tau \in u \ \&\ \tau \triangleleft \sigma\}$. Suppose otherwise and let

$$\delta \stackrel{\text{def}}{=} \bigcup \{\lambda \cdot \lg(\tau) : \tau \in u \ \&\ \tau \triangleleft \sigma\} \text{ and } M' \stackrel{\text{def}}{=} \bigcup \{M'_\tau : \tau \in u \ \&\ \tau \triangleleft \sigma\}.$$

Since $M_\sigma \restriction \delta \leq M'$ by the inductive assumptions, by the axiom of end amalgamation in K_{ap} we have that M' and M_σ are compatible and have a common upper bound M''_σ with the property $\lambda \cdot \lg(\sigma) \supseteq |M''_\sigma|$ and such that for $\tau \in u$ with $\tau \triangleleft \sigma$, we have that $M''_\sigma \restriction \lambda \cdot \lg(\sigma) = M'_\tau$. If $v \cap \lambda \cdot \lg(\sigma) \subseteq |M''_\sigma|$ let $M'_\sigma = M''_\sigma$. Otherwise, let f_σ be a lawful embedding of M''_σ into M'''_σ with $f_\sigma \restriction \delta = \text{id}$ and $(M'''_\sigma \setminus \delta) \cap (M''_\sigma \setminus \delta) = \emptyset$, while $|M'''_\sigma| \supseteq (v \cap \lambda \cdot \lg(\sigma)) \setminus M''_\sigma$. Applying amalgamation to M''_σ, M'''_σ we can find M'_σ as required.

Step III. Now for $\sigma \in u$ let $M^q_\sigma \stackrel{\rm def}{=} M'_\sigma$. Let

$$u^q \stackrel{\rm def}{=} u \cup \{\sigma \restriction \beta : \sigma \in u \ \& \ \beta \in w[M^q_\sigma]\}.$$

For $\sigma \in u^q$, let $M^q_\sigma \stackrel{\rm def}{=} M'_\sigma$ if $\sigma \in u$, and otherwise let $M^q_\sigma = M^q_\tau \restriction (\lambda \cdot \lg(\sigma))$ for any $\tau \in u$ with $\sigma \trianglelefteq \tau$.

Subcase A. $\lambda = \aleph_0$. Let

$$q \stackrel{\rm def}{=} \langle u^q, \langle M^q_\sigma : \sigma \in u^q\rangle, b^p, \langle c^p_\alpha : \alpha \in b^p\rangle, \langle (N^p_{\alpha,\iota}, h^p_{\alpha,\iota}) : \alpha \in b^p, \iota \in c^p_\alpha\rangle\rangle.$$

Subcase B. $\lambda > \aleph_0$ Let

$$q \stackrel{\rm def}{=} \langle u^q, \langle M^q_\sigma : \sigma \in u^q\rangle, b^p, \langle c^p_\alpha : \alpha \in b^p\rangle, \langle d^p_{\alpha,\iota} : \alpha \in b^p, \iota \in c^p_\alpha\rangle, (\bar{N, h})^p\rangle.$$

It is easily seen that q is as required.
(3) All coordinates of q will be the same as the corresponding coordinates of p, with the possible exception of u^p and $\langle M^p_\sigma : \sigma \in u^p\rangle$. Let $u = u^p$. For $\sigma \in u$ we define $M^q_\sigma \geq M^p_\sigma$ by induction on $\lg(\sigma)$. The inductive hypothesis is that if $\sigma \trianglelefteq \eta$, then $M^q_\sigma = M \restriction \lambda \cdot \lg(\sigma)$. The proof is similar to that of of Step II of (2). Coming to σ, let δ and M' be defined as there. If $\sigma \trianglelefteq \eta$, then we let $M^q_\sigma = M \restriction \lambda \cdot \lg(\sigma)$, and we have $M^q_\sigma \geq M^p_\sigma$ by the choice of M.

Otherwise, let $\nu = \sigma \cap \eta$, and hence $\nu \in u$ and $\lg(\nu) < \lg(\sigma)$. We have that

$$M^q_\nu = M \restriction \lambda \cdot \lg(\eta) \geq M^p_\nu = M^p_\sigma \restriction \lambda \cdot \lg(\sigma).$$

Hence we can find $M^q_\sigma \geq M^p_\sigma$ with $M^q_\sigma \restriction \lambda \cdot \lg(\nu) = M^q_\nu$ by end amalgamation.

Once the induction is done, we define u^q exactly as in the Step III of (2), and define M^q_σ for $\sigma \in u^q$ accordingly.
(4) Let $r \geq p$, we shall find $q \in \mathcal{D}$ with $q \geq r$. For some $\iota'_1, \iota'_2, \Upsilon'_1, \Upsilon'_2$ we have $N_1 \leq N^r_{\alpha,\iota'_1,\Upsilon'_1}$ and $N_2 \leq N^r_{\alpha,\iota'_2,\Upsilon'_2}$, and similarly for h_1, h_2. For simplicity of notation, we assume $\iota'_l = \iota_l$ and $\Upsilon'_l = \Upsilon_l$ for $l \in \{1,2\}$. Let $\iota = \max\{\iota_1, \iota_2\}$ and let $\eta = f^*_\alpha \restriction \iota$. Let $M' \stackrel{\rm def}{=} M^r_\eta$, which is well defined. Hence h_1 and h_2 are lawful embeddings of N_1 and N_2 into M' respectively.

First define a lawful isomorphism g_0 from M' onto some M_1 such that for $x \in N_l$ we have $g(h_l(x)) = x$ for $l \in \{1,2\}$. This is possible because h_1 and h_2 agree on $N_1 \cap N_2$. Hence we have that $N_1 \in \Gamma$ and $N_1 \leq M_1$. As

$\Gamma \in K_{\rm md}$, there is a lawful embedding $g_1 : M_1 \to M_2$ for some $M_2 \in \Gamma$ such that g_1 is the identity on N_1. Without loss of generality, again as $\Gamma \in K_{\rm md}$, we can assume that $|M_2| \subseteq \lambda\iota$.

Now let g_2 be a lawful isomorphism between M_2 and M_3 such that $g_2 \restriction N_1 = \mathrm{id}$ and $g_2(g_1(x)) = x$ for $x \in N_2$. Then $N_2 \leq M_3$, so we can find $M_4 \in \Gamma$ and a lawful embedding $g_3 : M_3 \to M_4$ over M_3. Then $N_1, N_2 \leq M_4$. Without loss of generality we have $|M_4| \subseteq \lambda \cdot \iota$. Finally, there is a lawful isomorphism g between M_4 and some M such that $g \restriction N_l = h_l$ for $l \in \{1,2\}$ and $M' \leq M$. By (3) we can find $q' \geq r$ such that $M^{q'}_\eta = M$. We shall define $q \geq q'$ so that all the coordinates of q are the same as the corresponding coordinates of q', except that we in addition choose some

$$\Upsilon \in \lambda \setminus (d^{q'}_{\alpha,\iota} \cup \bigcup\{d^q_{\alpha,j} : j \in w[M_4]\}$$

and let $N^q_{\alpha,\iota,\Upsilon} = M_4$, while $h^q_{\alpha,\iota,\Upsilon} = g$. We let $N^q_{\alpha,j,\Upsilon} = M_4 \restriction \lambda \cdot j$ for $j \in w[M_4]$, and similarly for $h^q_{\alpha,j,\Upsilon}$. Then q is as required.
(5) Similar to (4), using (3) and (4). $\bigstar_{2.9}$

Claim 2.10. (1) If $\lambda > \aleph_0$, then R is a $< \lambda$-complete forcing.
(2) If R was defined by one of the Cases 3-4 or by Case 2 and $\lambda = \aleph_0$, then every increasing sequence in R of length $< \lambda$ has a least upper bound.
(3) Suppose R was defined by Case 2. Then, for every $\alpha < \mu$

$$\bigcup_{\gamma<\lambda^+} \Delta^{\beta,j}_{f^*_\alpha\restriction\gamma} \in K^-_{\rm md}[K_{{\rm ap},\alpha}]$$

holds in $V^{P_\beta * \underset{\sim}{Q}^\beta_j * \underset{\sim}{R}^\beta_j}$. In addition, $\bigcup\{|M| : M \in \bigcup_{\gamma<\lambda^+} \Delta^{\beta,j}_{f^*_\alpha\restriction\gamma}\} = \lambda^+$.

If $\langle \eta_i : i < i^* < \lambda\rangle$ is a $\trianglelefteq$-increasing sequence of elements of $\mathcal{T}$, and $M_i \in \Delta^{\beta,j}_{\eta_i}$ for $i < i^*$, then $\bigcup_{i<i^*} M_i \in \Delta^{\beta,j}_{\bigcup_{i<i^*}\eta_i}$.

Proof of the Claim. (1) If R is defined by Case 1, this follows by the definition of that case. For Cases 3-4 the conclusion follows by (2). We give the proof for Case 2. We deal with the situation $\lambda > \aleph_0$. The other case is trivial and in that case we actually obtain the existence of lubs.

Suppose that $\bar{q} = \langle q_i : i < i^* < \lambda\rangle$ is an increasing sequence in R. Without loss of generality, i^* is a limit ordinal. Let $b \stackrel{\text{def}}{=} \bigcup_{i<i^*} b^{q_i}$ and for $\alpha \in b$ let $c_\alpha = \bigcup\{c^{q_i}_\alpha : i < i^* \ \&\ \alpha \in b^{q_i}\}$. Let $u = \bigcup_{i<i^*} u^{q_i}$ and for every $\eta \in u$ let $M_\eta = \bigcup_{i<i^*, \eta\in u^{q_i}} M^{q_i}_\eta$.

Let $\theta = |i^*|$, and so $\theta < \lambda$. For $\alpha \in b$ and $\iota \in c_\alpha$, $\Upsilon \in \bigcup_{i<i^*} d^{q_i}_{\alpha,\iota}$ we let

$$(N_{\alpha,\iota,\theta\cdot i+\Upsilon}, h_{\alpha,\iota,\theta\cdot i+\Upsilon}) = (N^{q_i}_{\alpha,\iota,\Upsilon}, h^{q_i}_{\alpha,\iota,\Upsilon})$$

if this is defined. Let

$$d_{\alpha,\iota} = \{\theta\cdot i + \Upsilon : i < i^* \text{ and } N_{\alpha,\iota,\theta\cdot i+\Upsilon} \text{ well defined}\}.$$

Let

$$q = \langle u, \langle M_\eta : \eta \in u\rangle, b, \langle c_\alpha : \alpha \in b\rangle, \langle d_{\alpha,\iota} : \alpha \in b, \iota \in c_\alpha\rangle, (\bar{N,h})^q\rangle$$

where $(\bar{N,h})^q = \langle (N_{\alpha,\iota,\Upsilon}, h_{\alpha,\iota,\Upsilon}) : \Upsilon \in d_{\alpha,\iota}, \iota \in c_\alpha, \alpha \in b\rangle$. It is easily seen that q is an upper bound of all q_i (although not a least upper bound, which may not exist).

(2) The Case 2, Subcase $\lambda = \aleph_0$ is trivial. We distinguish Cases 3 and 4, according to Definition 2.7.

Suppose that $\bar{q} = \langle q_i : i < i^* < \lambda\rangle$ is an increasing sequence in R. We shall define the lub q of $\bar{q}$.

<u>Case 3</u>. Let $j' < j$, $\alpha < \mu$ and G be as in the definition of the forcing and let $q_i = (N_i, h_i)$ and $p_i \in G$ for $i < i^*$ be such that $\alpha \in b^p$ and for some $\iota_i \in c^p_\alpha$ and $\Upsilon_i \in d^p_{\alpha,\iota}$ we have

$$(N_i, h_i) = (N^{p_i}_{\alpha,\iota_i,\Upsilon_i}, h^{p_i}_{\alpha,\iota_i,\Upsilon_i}).$$

Without loss of generality, i^* is a limit ordinal. As we know by (1) of the induction hypothesis that $R^\beta_{j'}$ is $(< \lambda)$-complete, there is $p \in G$ with $p \geq p_i$ for all $i < i^*$. In $V^{P_\beta * \underset{\sim}{Q}^\beta_{j'}}$ let $\eta = \bigcup\{\nu \in u^p : \nu \trianglelefteq f^*_\alpha\}$ and let $M = \bigcup\{M_\nu : \nu \in u^p \ \&\ \nu \trianglelefteq f^*_\alpha\}$. Letting $N = \bigcup_{i<i^*} N_i$ and $h = \bigcup_{i<i^*} h_i$ we obtain that h is a lawful embedding from N into M. As $\Gamma_\alpha \in K_{\text{md}}$, by the $(< \lambda)$-closure of Γ_α we obtain $N \in \Gamma_\alpha$. Consider the set $\mathcal{D}$ defined by

$$\{q \geq p : \eta \in u^q \ \&\ M^q_\eta \geq M \ \&\ (\exists\iota \in c^q_\alpha)(\exists\Upsilon \in d^q_{\alpha,\iota})N^q_{\alpha,\iota,\Upsilon} = N \ \&\ h^q_{\alpha,\iota} = h\}.$$

Note that $\mathcal{D} \in V^{P_\beta * \underset{\sim}{Q}^\beta_{j'}}$ as all forcings involved are $(< \lambda)$-closed by the induction hypothesis.

Subclaim 2.11. $\mathcal{D}$ is dense above p in the forcing $R^\beta_{j'}$.

Proof of the Subclaim. Let $r \geq p$ be given. For every $\nu \in u^p$ with $\nu \trianglelefteq f^*_\alpha$ we have $M^r_\nu \geq M^p_\nu$. Hence if $\eta \in u^r$ we have $M^r_\eta = \bigcup\{M^r_\nu : \nu \trianglelefteq \eta\} \geq M$. By Claim 2.9(2) we can without loss of generality assume that this is the case. Let $u^q = u$ and for $\nu \in u^q$ let $M^q_\nu = M^r_\nu$.

Let $\iota = \lg(\eta)$, and hence $|M| \subseteq \lambda \cdot \iota$. Further let $c^q_\alpha = c^r_\alpha \cup \{\iota\}$. Let Υ be such that $\Upsilon \notin \bigcup_{\iota' \leq \iota} d^r_{\alpha,\iota'}$ and for $\iota' \leq \iota$ with $\iota' \in c^q_\alpha$ let

$$(N^q_{\alpha,\iota',\Upsilon}, h^q_{\alpha,\iota',\Upsilon}) = (N \restriction \lambda \cdot \iota', h \restriction \lambda \cdot \iota').$$

We complete the definition of q in the obvious fashion. Hence $q \geq r$ and $q \in \mathcal{D}$. $\bigstar_{2.11}$

By the Subclaim it follows that there is $q \in G \cap \mathcal{D}$, and this q witnesses that $(N, h) \in R$. Obviously, (N, h) is the lub of $\langle (N_i, h_i) : i < i^* \rangle$.

Case 4. If $\langle \langle M_i, N_i \rangle : i < i^* < \lambda \rangle$ is increasing in R then clearly $\langle \bigcup_{i<i^*} M_i, \cup_{i<i^*} N_i \rangle$ is the lub.

(3) We first prove the second statement. So, let $\langle \eta_i : i < i^* \rangle$ be as in the Claim. Without loss of generality, i^* is a limit ordinal. Let $\eta \stackrel{\text{def}}{=} \bigcup_{i<i^*} \eta_i$. For $i < i^*$ let $p_i \in G = G_{R^\beta_j}$ be such that $\eta_i \in u^{p_i}$ and $M_i = M^{p_i}_{\eta_i}$. Let p be an upper bound of $\langle p_i : i < i^* \rangle$ with $p \in G$, which exists by (1). Now let $q \in G$ be such that $\eta \in u^q$ and $p \leq q$, which exists by Claim 2.9. Note that we have $M^q_\eta = \bigcup_{i<i^*} M^q_{\eta_i}$. Let now r be defined by $u^r = u^p \cup \{\eta\}$, and for $\nu \in u^p$ we have $M^r_\nu = M^p_\nu$, while $M^r_\eta = \bigcup_{i<i^*} M_i$. We also redefine $c_\alpha, d_{\alpha,\iota}$ and $(\bar{N}, h)^q$ to accommodate the fact that we have shrunk u^q, for example by using the corresponding coordinates of p. This gives us a well defined condition r. We now claim that $r \leq q$. We only need to show that $M^r_\eta \leq M^q_\eta$, which follows by Definition 1.4 (1)(c). As G is generic, and $q \in G$, we have $r \in G$.

For the first statement, suppose that $M_i \, (i < i^* < \lambda)$ are in $\bigcup_{\gamma<\lambda^+} \Delta^{\beta,j}_{f^*_\alpha \restriction \gamma}$ and let γ_i for $i < i^*$ be such that $M_i \in \Delta^{\beta,j}_{f^*_\alpha \restriction \gamma}$. Let $\eta_i \stackrel{\text{def}}{=} f^*_\alpha \restriction \gamma_i$ for $i < i^*$. Let $\eta \stackrel{\text{def}}{=} \bigcup_{i<i^*} \eta_i$. Now proceed as above. This proves that $\bigcup_{\gamma<\lambda^+} \Delta^{\beta,j}_{f^*_\alpha \restriction \gamma}$ is a

$(<\lambda)$-closed subset of K_{ap} and it is equally easy to see that it is directed. To see that $\bigcup\{|M| : M \in \bigcup_{i<\lambda^+} \Delta^{\beta,j}_{f^*_\alpha \restriction \gamma}\} = \lambda^+$, apply Claim 2.9, and this of course implies that (iii) of Definition 1.7(2) holds.

$\bigstar_{2.10}$

Notation 2.12. The upper bound q of $\bar{q}$ that is constructed as in the proof of Claim 2.10(1) will be called a canonical upper bound (cnub) of $\bar{q}$.

Note 2.13. The same proof given above shows that if $\underset{\sim}{R}^\beta_j$ is defined by Case 2 of Definition 2.7 and $\eta \in \mathcal{T}$, then in $V^{P_\beta * \underset{\sim}{Q}^\beta_j * \underset{\sim}{R}^\beta_j}$ we have that $\Delta^{\beta,j}_\eta$ is an element of $K^-_{\mathrm{md}}[K^{\lambda \cdot \lg(\eta)}_{\mathrm{ap}}]$, where $K_{\mathrm{ap}} = K^{\beta,j}_{\mathrm{ap}}$.

Claim 2.14. If $\lambda > \aleph_0$, then R satisfies $*^\varepsilon_\lambda$.

If $\lambda = \aleph_0$, then R satisfies ccc.

Proof of the Claim. We distinguish various cases of Definition 2.7.

Case 1. R is defined by Case 1 of Definition 2.7. The conclusion follows by the assumptions.

Case 2. (main case) $R = R^\beta_j$ is defined by Case 2 of Definition 2.7. As Subcase B is more difficult, we start by it.

Subcase B. $\lambda > \aleph_0$. Let $K = K^{\beta,j}_{\mathrm{ap}}$, and let us follow the rest of the notation of Definition 2.7 as well. By our assumptions we have $|\mathcal{T}| = \lambda^+$ and by Claim 2.10, the equality $(\lambda^+)^{<\lambda} = \lambda^+$ holds. Also, for every $j < \lambda^+$ we have that $K \restriction j \stackrel{\mathrm{def}}{=} \{M \in K : |M| \subseteq j\}$ has cardinality $\leq \lambda$.

We first define several auxiliary functions. Let $g^*_0 : \mathcal{T} \to \lambda^+$ be a bijection and let

$$g^*_1 : \lambda^+ \to {}^\lambda 2$$

be a 1-1 function.

Subclaim 2.15. There is a function $g^*_2 : K \to \lambda$ such that for every $N_1, N_2 \in K$ we have

$$g^*_2(N_1) = g^*_2(N_2) \;\&\; \beta \in w[N_1] \cap w[N_2] \Longrightarrow N_1 \restriction \lambda\beta = N_2 \restriction \lambda\beta.$$

Proof of the Subclaim. For $N \in K$ define

$$o[N] \stackrel{\text{def}}{=} \{\lambda\gamma : \gamma \in w[N]\} \cup |N|,$$

$$\xi[N] \stackrel{\text{def}}{=} \min\{\zeta < \lambda : \beta \neq \gamma \in o[N] \Longrightarrow g_1^*(\beta) \restriction \zeta \neq g_1^*(\gamma) \restriction \zeta\},$$

$$\Xi_N \stackrel{\text{def}}{=} \{g_1^*(\beta) \restriction \xi[N] : \beta \in o[N]\}.$$

Note that $\xi[N]$ is well defined because g_1^* is 1-1 and $|o[N]| < \lambda$.

For $\alpha < \lambda^+$, let $g_\alpha : \alpha \to \lambda$ be one to one. For $N \in K$ let $\mathfrak{A}_N$ be a model with universe included in Ξ_N such that the function

$$\beta \mapsto g_1^*(\beta) \restriction \xi[N]$$

is an isomorphism from N onto $\mathfrak{A}_N$. Let $<_N$ be a well ordering of Ξ_N such that $\beta \mapsto g_1^*(\beta) \restriction \xi[N]$ is an isomorphism from $(o[N], <)$ onto $(\Xi_N, <_N)$. Let

$$R^N \stackrel{\text{def}}{=} \{(g_1^*(\beta) \restriction \xi[N], g_1^*(\gamma) \restriction \xi[N], g_\gamma(\beta)) : \beta < \gamma \text{ both in } o[N]\}.$$

Notice that $(i, j, k_1), (i, j, k_2) \in R^N \Longrightarrow k_1 = k_2$ by the choice of $\xi[N]$. Let $\mathcal{H}^* : \mathcal{H}(\lambda) \to \lambda$ be one to one, which exists as $\lambda^{<\lambda} = \lambda$. We define

$$g_2^*(N) \stackrel{\text{def}}{=} \mathcal{H}^*(\langle \xi[N], \Xi_N, \mathfrak{A}_N, <_N, R^N \rangle).$$

Clearly g_2^* is a well defined function from K to λ. Let us show that it has the required properties.

Suppose $g_2^*(N_1) = g_2^*(N_2)$ and $\beta^* \in w[N_1] \cap w[N_2]$. Firstly, we have that $\xi[N_1] = \xi[N_2] = \xi$ and the functions

$$f_1 : \beta \mapsto g_1^*(\beta) \restriction \xi[N_1] \text{ for } \beta \in o[N_1]$$

and

$$f_2 : \beta \mapsto g_1^*(\beta) \restriction \xi[N_2] \text{ for } \beta \in o[N_2]$$

are one to one and onto the same set $\Xi_{N_1} = \Xi_{N_2} = \Xi$. Furthermore, both f_1 and f_2 are order preserving and $<_{N_1} = <_{N_2}$. Hence there is a one to one $<$-preserving function $g : o[N_1] \to o[N_2]$ given by $g(\beta) = f_2^{-1}(f_1(\beta))$.

We claim that for every $\beta \in w[N_1] \cap w[N_2]$ we have $g(\lambda\beta) = \lambda\beta$. Namely suppose not, say $g(\lambda\beta) = \gamma$ and $\lambda\beta < \gamma$. Then $f_2(g(\lambda\beta)) = f_2(\gamma) > f_2(\lambda\beta)$, and hence $f_1(\lambda\beta) > f_2(\lambda\beta)$, which means that $g_1^*(\lambda\beta) \restriction \xi >_{N_1} g_1^*(\lambda\beta) \restriction \xi$, a contradiction. A similar contradiction can be obtained by assuming that $g(\lambda\beta) < \lambda\beta$.

If $\gamma \in N_1 \restriction \lambda\beta^*$ then $g(\gamma) < g(\lambda\beta^*) = \lambda\beta^*$. By the definition of g we have $g_1^*(\gamma) \restriction \xi = g_1^*(g(\gamma)) \restriction \xi$. Hence, $\lambda\beta^*, g(\gamma) \in o[N_2]$ and $g(\gamma) < \lambda\beta^*$. So $(g_1^*(g(\gamma)) \restriction \xi, g_1^*(\lambda\beta^*) \restriction \xi, g_{\lambda\beta^*}(g(\gamma))) \in R^{N_2} = R^{N_1}$. As also

$$(g_1^*(\gamma) \restriction \xi, g_1^*(\lambda\beta^*) \restriction \xi, g_{\lambda\beta^*}(\gamma)) \in R^{N_1},$$

we have that $g_{\lambda\beta^*}(g(\gamma)) = g_{\lambda\beta^*}(\gamma)$ and hence $g(\gamma) = \gamma$. In particular $\gamma \in o[N_2]$. As we have $\gamma \in N_1$, we have $g_1^*(\gamma) \restriction \xi \in \mathfrak{A}_{N_1}$, and hence $g_1^*(\delta) \restriction \xi = g_1^*(\gamma) \restriction \xi = g_1^*(g(\gamma)) \restriction \xi$ for some $\delta \in N_2$. As $\xi = \xi[N_2]$, we have that $\delta = g(\gamma) = \gamma$, so $\gamma \in N_2 \restriction \lambda\beta^*$.

This argument shows that $N_1 \restriction \lambda\beta^* \subseteq N_2 \restriction \lambda\beta^*$, and it can be shown similarly that $N_1 \restriction \lambda\beta^* = N_2 \restriction \lambda\beta^*$ as sets and as models. $\bigstar_{2.15}$

For $p \in R$, let $\langle(\eta(p,i) : i < i(p)\rangle$ list u^p with no repetitions, and let $\xi(p)$ be the minimal $\xi < \lambda$ such that

$$\langle g_1^*(g_0^*(\eta(p,i))) \restriction \xi : i < i(p)\rangle$$

is without repetitions (which exists as g_0^* and g_1^* are 1-1 and $\langle(\eta(p,i) : i < i(p)\rangle$ is without repetitions). Let

$$g_3^* : R \to \lambda$$

be such that for $p, q \in R$ with $g_3^*(p) = g_3^*(q)$ we have

(a) $i(p) = i(q)$,

(b) the mapping defined by sending $\eta(p,i) \mapsto \eta(q,i)$ preserves

$$\text{“}\nu \trianglelefteq \eta\text{”}, \text{“}\neg(\nu \trianglelefteq \eta)\text{”}, \text{“}\nu_1 \cap \nu_2 = \nu\text{”}, \text{“}\neg(\nu_1 \cap \nu_2 = \nu)\text{”},$$

(c) $\xi(p) = \xi(q)$,

(d) for $i < i(p)$ we have $g_1^*(g_0^*(\eta(p,i))) \restriction \xi(p) = g_1^*(g_0^*(\eta(q,i))) \restriction \xi(p)$ (recall that $|{}^{\lambda>}2| = \lambda$),

(e) for $i < i(p)$ we have $g_2^*(M^p_{\eta(p,i)}) = g_2^*(M^q_{\eta(q,i)})$.

The existence of such a function can be shown by counting.

Subclaim 2.16. If $g_3^*(p) = g_3^*(q)$, then the mapping sending $\eta(p,i)$ to $\eta(q,i)$ for $i < i(p) = i(q)$, is the identity on $u^p \cap u^q$.

Proof of the Subclaim. Suppose that $\eta \in u^p \cap u^q$. Let i be such that $\eta = \eta(p,i)$. Letting $\xi \stackrel{\text{def}}{=} \xi(p) = \xi(q)$,we have

$$g_1^*(g_0^*(\eta)) \restriction \xi = g_1^*(g_0^*(\eta(q,i))) \restriction \xi.$$

By the definition of ξ and the fact that $\eta \in u^q$, we must have $\eta(q,i) = \eta$. $\bigstar_{2.16}$

Let us also fix a bijection

$$F : \lambda \times {}^{\lambda >}([\lambda^+]^{<\lambda}) \to \lambda^+$$

and let C be a club of λ^+ such that for every $j \in S^{\lambda^+}_\lambda \cap C$ we have

$$\beta < \lambda \ \&\ u \in {}^{\lambda >}([j]^{<\lambda}) \Longrightarrow F((\beta, u)) < j.$$

We describe a winning strategy for I in $*^{\varepsilon}_{\lambda}[R]$. Given $0 < \zeta < \varepsilon$ and suppose that

$$\Big\langle (\langle q^\xi_s : s < \lambda^+ \rangle, f_\xi), \langle p^\xi_s : s < \lambda^+ \rangle : \xi < \zeta \Big\rangle$$

have been played so far and I has played according to the strategy. By Claim 2.10(1), we can let player I choose q^ζ_s as a cnub of $\langle p^\xi_s : \xi < \zeta \rangle$. Next we describe the choice of f_ζ. Let $C_\zeta \stackrel{\text{def}}{=} C$ and define g_ζ which to an ordinal $j \in S^{\lambda+}_\lambda$ assigns

$$(g_3^*(q^\zeta_j), \langle w[M^{q^\zeta_j}_{\eta(q^\zeta_j, i)}] \cap j : i < i(q^\zeta_j) \rangle).$$

Then let

$$f_\zeta \stackrel{\text{def}}{=} (F \circ g_\zeta) \restriction (C_\zeta \cap S^{\lambda^+}_\lambda) \cup 0_{\lambda^+ \setminus (C_\zeta \cap S^{\lambda^+}_\lambda)}.$$

Let us check that this definition is as required. It follows from the choice of C that each f_ζ is regressive on $C_\zeta \cap S^{\lambda^+}_\lambda$. Let $E \subseteq C$ be a club of λ^+ such that

$$[j \in E \cap S^{\lambda^+}_\lambda \ \&\ j' < j] \Longrightarrow (\forall \zeta < \varepsilon)(\forall i < i(q^\zeta_{j'}))\,[w[M^{q^\zeta_{j'}}_{\eta(q^\zeta_{j'},i)}] \subseteq j].$$

Suppose that $j' < j \in E \cap S^{\lambda^+}_\lambda$ are such that

$$\bigwedge_{\zeta<\varepsilon} f_\zeta(j') = f_\zeta(j).$$

We define an upper bound to

$$\{p^\zeta_{j'} : \zeta < \varepsilon\} \cup \{p^\zeta_j : \zeta < \varepsilon\}.$$

As we have $q^{\zeta+1}_s \geq p^\zeta_s$ for all $\zeta < \varepsilon$ and $s < \lambda^+$, and ε is a limit ordinal, it suffices to define an upper bound to

$$\{q^\zeta_{j'} : \zeta < \varepsilon\} \cup \{q^\zeta_j : \zeta < \varepsilon\}.$$

We first define q_l as a cnub of $\{q^\zeta_l : \zeta < \varepsilon\}$ for $l \in \{j', j\}$, and we shall now describe an upper bound r of $q_{j'}$ and q_j. Notice that $u^{q_l} = \bigcup_{\zeta<\varepsilon} u^{q^\zeta_l}$ for $l \in \{j', j\}$.

Let

$$u \stackrel{\text{def}}{=} u^{q_{j'}} \cup u^{q_j} \cup \{\eta \cap \nu : \eta \in u^{q_{j'}} \ \&\ \nu \in u^{q_j}\}.$$

Clearly $|u| < \lambda$ and u is closed under intersections. For $\eta \in u$, let

$$M_\eta \stackrel{\text{def}}{=} M^{q^\zeta_l}_\nu \restriction \lambda \cdot \lg(\eta)$$

for any $l \in \{j', j\}$, $\zeta < \varepsilon$ and $\nu \in u^{q^\zeta_l}$ for which $\eta \trianglelefteq \nu$.

Subclaim 2.17. For $\eta \in u$ the model M_η is well defined and $|M_\eta| \subseteq \lambda \cdot \lg(\eta)$. For every $l \in \{j, j'\}$ for which $\eta \in u^{q_l}$ we have $M^{q_l}_\eta = M_\eta$.

Proof of the Subclaim. Firstly, note that for any $\eta \in u$ we have $\eta \trianglelefteq \nu$ for some $\nu \in \bigcup_{l\in\{j',j\},\zeta<\varepsilon} u^{q^\zeta_l}$. Suppose $\eta \trianglelefteq \nu_1, \nu_2$ for some $\nu_1, \nu_2 \in \bigcup_{l\in\{j',j\},\zeta<\varepsilon} u^{q^\zeta_l}$

such that $\nu_k \in u^{q^{\zeta_k}_{l_k}}$ for $k \in \{1,2\}$, and $M^{q^{\zeta_1}_{l_1}}_{\nu_1} \restriction \lambda \cdot \lg(\eta) \neq M^{q^{\zeta_2}_{l_2}}_{\nu_2} \restriction \lambda \cdot \lg(\eta)$. By taking the larger of ζ_1, ζ_2, we may assume that $\zeta_1 = \zeta_2 = \zeta$. By the closure under intersections, we can also assume that $l_1 \neq l_2$, so without loss of generality we have $l_1 = j'$ and $l_2 = j$. Let $\beta \leq \lg(\eta)$ be minimal such that

$$M^{q^{\zeta}_{j'}}_{\nu_1} \restriction \lambda \cdot \beta \neq M^{q^{\zeta}_{j}}_{\nu_2} \restriction \lambda \cdot \beta.$$

By the minimality of β, we have $\beta = \gamma + 1$ for some $\gamma \in w[M^{q^{\zeta}_{j'}}_{\nu_1}] \cap w[M^{q^{\zeta}_{j}}_{\nu_2}]$. As $j \in E$ we have that $w[M^{q^{\zeta}_{j'}}_{\nu_1}] \subseteq j$, so $\gamma \in w[M^{q^{\zeta}_{j}}_{\nu_2}] \cap j$. As $f_\zeta(j) = f_\zeta(j')$, there is $\nu \in u^{q^{\zeta}_{j'}}$ such that for some $i < i(q^{\zeta}_{j}) = i(q^{\zeta}_{j'})$ we have $\nu_2 = \eta(q^{\zeta}_{j}, i)$, $\nu = \eta(q^{\zeta}_{j'}, i)$ and $w[M^{q^{\zeta}_{j}}_{\nu_2}] \cap j = w[M^{q^{\zeta}_{j'}}_{\nu}] \cap j'$. Hence we have $g^*_2(M^{q^{\zeta}_{j'}}_{\nu}) = g^*_2(M^{q^{\zeta}_{j}}_{\nu_2})$, and as $\gamma, \beta \in w[M^{q^{\zeta}_{j'}}_{\nu}] \cap w[M^{q^{\zeta}_{j}}_{\nu_2}]$, we have $M^{q^{\zeta}_{j'}}_{\nu} \restriction \lambda\beta = M^{q^{\zeta}_{j}}_{\nu_2} \restriction \lambda\beta$. We have not arrived at a contradiction yet, as we do not know the relationship between ν and ν_1.

As $\beta \leq \lg(\eta)$, we have $\rho \stackrel{\rm def}{=} \nu_1 \restriction \beta = \nu_2 \restriction \beta$. Since $\beta \in w[M^{q^{\zeta}_{j'}}_{\nu_1}]$, we have $\rho = \nu_1 \restriction \beta \in u^{q^{\zeta}_{j'}}$ and similarly $\rho \in u^{q^{\zeta}_{j}}$. Let o be such that $\rho = \eta(q^{\zeta}_{j'}, o)$. By Subclaim 2.15 we have $\rho = \eta(q^{\zeta}_{j}, o)$. Since we have $\rho \leq \nu_2$ by the choice of g^*_3, we have $\rho \trianglelefteq \nu$. So $\rho \trianglelefteq \nu_1 \cap \nu$, and as we have $\nu_1 \cap \nu \in u^{q^{\zeta}_{j'}}$, we obtain

$$M^{q^{\zeta}_{j'}}_{\nu_1} \restriction \lambda\beta = M^{q^{\zeta}_{j'}}_{\nu_1} \restriction \lambda \lg(\rho) = M^{q^{\zeta}_{j'}}_{\nu} \restriction \lambda \lg(\rho) = M^{q^{\zeta}_{j}}_{\nu_2} \restriction \lambda \lg(\rho) = M^{q^{\zeta}_{j}}_{\nu_2} \restriction \lambda\beta,$$

a contradiction. This proves the first part of the statement. If $\eta \in u$ and $l \in \{j, j'\}$ is such that $\eta \in u^{q_l}$, then we have $M_\eta = M^{q_l}_\eta$, as is clear from the definition. $\bigstar_{2.17}$

Now we let

$$u^r \stackrel{\rm def}{=} u \cup \{w[M_\sigma] : \sigma \in u\},$$

and define M^r_σ for $\sigma \in u^r$ accordingly, which is done as in Step III of the Proof of Claim 2.9(2).

We let $b^r = b^{q_j} \cup b^{q_{j'}}$ and for $\alpha \in b$ we let $c^r_\alpha = c^{q_j}_\alpha \cup c^{q_{j'}}_\alpha$. For $\alpha \in b, \iota \in c^r_\alpha$ we let $d^r_{\alpha,\iota} = \{2\Upsilon : \Upsilon \in d^{q_{j'}}_{\alpha,\iota}\} \cup \{2\Upsilon + 1 : \Upsilon \in d^{q_j}_{\alpha,\iota}\}$ and

$(N^r_{\alpha,\iota,2\Upsilon}, h^r_{\alpha,\iota,2\Upsilon}) = (N^{q_{j'}}_{\alpha,\iota,\Upsilon}, h^{q_{j'}}_{\alpha,\iota,\Upsilon})$ while $(N^r_{\alpha,\iota,2\Upsilon+1}, h^r_{\alpha,\iota,2\Upsilon+1}) = (N^{q_j}_{\alpha,\iota,\Upsilon}, h^{q_j}_{\alpha,\iota,\Upsilon})$.

We have now completed the proof of Subcase B of Case 2 of the Claim.
Subcase A. $\lambda = \aleph_0$. We have to prove that R satisfies ccc. Let functions g_0^*, g_1^*, g_2^* and g_3^* be as in the proof of Subcase B, and let the function F and the club C be given as in that proof.

Suppose that we are given a sequence $\langle q_s : s < \omega_1 \rangle$ of conditions in R. Let $E \subseteq C$ be a club of ω_1 such that

$$j \in S^{\omega_1}_{\aleph_0} \ \&\ j' < j \Longrightarrow (\forall i < i(q, j'))(w[M^{q_{j'}}_{\eta(q_{j'}, i)}] \subseteq j).$$

Let g be a function that to an ordinal $j \in S^{\omega_1}_{\aleph_0}$ assigns

$$(g_3^*(q_j), \langle w[M^{q_j}_{\eta(q_j, i)}] \cap j : i < i(q_j) \rangle)$$

and let

$$f = (F \circ g) \restriction (E \cap S^{\omega_1}_{\aleph_0}) \cup 0_{\omega_1 \setminus (E \cap S^{\omega_1}_{\aleph_0})}.$$

Exactly as in Subcase B, it follows that whenever

$$j' < j \in S^{\omega_1}_{\aleph_0} \cap E \ \&\ \ f(j') = f(j)$$

then letting

$$u = u^{q_{j'}} \cup u^{q_j} \cup \{\eta \cap \nu : \eta \in u^{q_j}, \nu \in u^{q_j}\}$$

and for $\eta \in u$

$$M_\eta = M^{q_l}_\eta \restriction \lambda \cdot \lg(\eta)$$

for any $l \in \{j, j'\}$ for which $\eta \in u^{q_l}$, we obtain a well defined sequence $\langle M_\eta : \eta \in u \rangle$ of elements of K with the property that for any $l \in \{j, j'\}$, $\eta \in u^{q_l}$ we have $M^{q_l}_\eta \le M_\eta$. Let $S \subseteq S^{\omega_1}_{\aleph_0} \cap E$ be stationary such that $f(j)$ is fixed for $j \in S$. We apply the Δ-system lemma to $\{b^{q_j} : j \in S\}$ and obtain $A \in [S]^{\aleph_1}$ and $b^* \in [\mu]^{<\aleph_0}$ such that for every $j \neq j' \in A$ we have $b^{q'_j} \cap b^{q_j} = b^*$. If $b^* = \emptyset$, then for every $j, j' \in A$, the condition

$$\langle u, \langle M_\eta : \eta \in u \rangle, b^{q_{j'}} \cup b^{q_j}, \langle c^{q_l}_\alpha : \alpha \in b^{q_l} \rangle, \langle (N^{q_l}_{\alpha,\iota}, h^{q_l}_{\alpha,\iota}) : \alpha \in b^{q_l}, \iota \in c^{q_l}_\alpha \rangle \rangle$$

is a common upper bound of $q_{j'}$ and q_j where $u, \langle M_\eta : \eta \in u \rangle$ are defined above.

Suppose that $|b^*| = n^* > 0$. Using the Δ-system lemma n^* times if necessary, we can find $B \in [A]^{\aleph_1}$ and for $\alpha \in b^*$ a set $c^*_\alpha \in [\omega_1]^{<\aleph_0}$ such that $\alpha \in b^* \ \&\ j' < j \in B \Longrightarrow$

(i) $c_\alpha^{q_{j'}} \cap c_\alpha^{q_j} = c_\alpha^*$,

(ii) $\min(c_\alpha^{q_{j'}} \setminus c_\alpha^*) > \max\{\lambda\iota : \iota \in c_\alpha^*\}$,

(iii) $\min(c_\alpha^{q_j} \setminus c_\alpha^*) > \max\{\lambda\iota : \iota \in c_\alpha^{q_{j'}}\}$,

(iv) $\iota \in c_\alpha^* \Longrightarrow (N_{\alpha,\iota}^{q_{j'}}, h_{\alpha,\iota}^{q_{j'}}) = (N_{\alpha,\iota}^{q_j}, h_{\alpha,\iota}^{q_j})$

and for $k < n_\alpha^* \stackrel{\text{def}}{=} |c_\alpha^*|$ letting ι'_k, ι_k be the k-th element of $c_\alpha^{q_{j'}}, c_\alpha^{q_j}$ respectively, we have that $N_{\alpha,\iota'_k}^{q_{j'}}$ and $N_{\alpha,\iota_k}^{q_j}$ are isomorphic. Let $j' < j \in B$ and let $\alpha \in b^*$. Let $N'_j = \bigcup_{\iota \in c_\alpha^{q_{j'}}} N_{\alpha,\iota}^{q_{j'}}$ and $N_j = \bigcup_{\iota \in c_\alpha^{q_j}} N_{\alpha,\iota}^{q_j}$, while $h'_j = \bigcup_{\iota \in c_\alpha^{q_{j'}}} h_{\alpha,\iota}^{q_{j'}}$ and $h_j = \bigcup_{\iota \in c_\alpha^{q_j}} h_{\alpha,\iota}^{q_j}$. Then N_j and N'_j are isomorphic and h_j and h_j agree on their intersection, while there are $\delta_0 < \delta_1 < \delta_2$ divisible by λ such that

$$N_{j'} \restriction \delta_0 = N_j \restriction \delta_1 = N_j \cap N'_j$$

with $|N'_j| \subseteq \delta_1$ and $|N_j| \subseteq \delta_2$ and $\eta = f_\alpha^* \restriction \delta_2 \in u^{q_j}$. We also have that $N'_j, N_j \in \Gamma$. Then h_j is a lawful embedding of N_j into M_η and h'_j is a lawful embedding of N'_j into M_η, by the choice of S. Similarly to the proof of Claim 2.9(4), we can see that q'_j and q_j are compatible, by finding $N \in \Gamma$ with $N \geq N_1, N_2$, extending q_j, $q_{j'}$ to enlarge M_η and then taking an upper bound of the extensions.

<u>Case 3</u>. Suppose that

$$\langle (\langle q_i^\xi : i < \lambda^+ \rangle, f_\xi), \langle p_i^\xi : i < \lambda^+ \rangle : \xi < \zeta \rangle$$

have been played so far. By Claim 2.10 we can have q_i^ζ be the lub of $\langle q_i^\xi : \xi < \zeta \rangle$. Let $\langle I_\gamma : \gamma < \lambda \rangle$ list the isomorphism types of elements of K. Let F be a bijection

$$F : \lambda \times K \times \{h : h \text{ a lawful function with } \mathrm{Dom}(h) \in [\lambda^+]^{<\lambda}\} \to \lambda^+.$$

Let C be a club of λ^+ such that for every $j \in S_\lambda^{\lambda^+} \cap C$ we have

$$\gamma < \lambda, u \in [j]^{<\lambda} \ \&\ \mathrm{Dom}(h) \in [j]^{<\lambda} \Longrightarrow F((\gamma, u, h)) < j.$$

Let $C_\zeta = C$ and define g_ζ which to an ordinal $j \in S_\lambda^{\lambda^+}$ assigns

$$(\mathrm{type}(N^{q_j^\zeta}), N^{q_j^\zeta} \cap j, h^{q_j^\zeta} \restriction (N^{q_j^\zeta} \cap j)).$$

Then let $f_\zeta = (F \circ g_\zeta) \restriction (C_\zeta \cap S^{\lambda^+}_\lambda) \cup 0_{\lambda^+ \setminus (C_\zeta \cap S^{\lambda^+}_\lambda)}$. Let $E \subseteq C$ be a club of λ^+ such that

$$j \in E \cap S^{\lambda^+}_\lambda \ \&\ j' < j \Longrightarrow (\forall \zeta < j')(|N^{q^\zeta_{j'}}| \subseteq j).$$

Let (N', h') be the lub of $\{(N^{q^\zeta_{j'}}, h^{q^\zeta_{j'}}) : \zeta < \varepsilon\}$ and (N, h) the lub of $\{(N^{q^\zeta_j}, h^{q^\zeta_j}) : \zeta < \varepsilon\}$. We shall show that (N, h) and (N, h') are compatible. As $N, N' \in \Gamma_\alpha$, clearly they are compatible as elements of K. We need to show that h and h' agree on $N \cap N'$.

Suppose not and let $\zeta < \varepsilon$ be the least such that h_ζ and h'_ζ disagree on $N^{q^\zeta_{j'}} \cap N^{q^\zeta_j}$-such a ζ exists by the definition of the lub in the forcing. By the choice of E we have $|N^{q^\zeta_{j'}}| \subseteq j$ and by the choice of f_ζ we have $(N^{q^\zeta_{j'}} \cap j', h^{q^\zeta_{j'}} \restriction (N^{q^\zeta_{j'}} \cap j')) = (N^{q^\zeta_j} \cap j, h^{q^\zeta_j} \restriction (N^{q^\zeta_j} \cap j))$. Hence $N^{q^\zeta_{j'}} \cap N^{q^\zeta_j} \subseteq j'$ and $h^{q^\zeta_{j'}}$ and $h^{q^\zeta_j}$ agree on this intersection, a contradiction. We can also see that N' and N are isomorphic.

Now let G be as in the Definition 2.7 Case 3 and let $p' \in G$ witness that $(N', h') \in R$, while $p \in G$ witnesses that $(N, h) \in R$. Let $p^+ \in G$ be a common upper bound of p and p'. By Claim 2.9(4) it follows that

$$\mathcal{D} = \{p^{++} \geq p^+ : (\exists \iota \in c^{p^{++}}_\alpha)(\exists \Upsilon \in d^{p^{++}}_{\alpha,\iota})\,(N, N' \leq N^{p^{++}}_{\alpha,\iota,\Upsilon}\ \&\ h \cup h' \subseteq h^{p^{++}}_{\alpha,\iota,\Upsilon})\}$$

is dense in the forcing R' giving rise to G, which suffices.

Now suppose that we are in Case 4 and that

$$\Big\langle (\langle q^\xi_i : i < \lambda^+\rangle, f_\xi), \langle p^\xi_i : i < \lambda^+\rangle : \xi < \zeta\Big\rangle$$

have been played so far in the game $*^\varepsilon_\lambda[R]$. This is where we get to use the workability of K. As before, we let player I choose q^ζ_i as the unique least upper bound of $\langle p^\xi_i : \xi < \zeta\rangle$. Let $q^\zeta_i = \langle M^\zeta_i, N^\zeta_i\rangle$. Using $\lambda = \lambda^{<\lambda}$, we can find a regressive function f_ζ such that if $i < j$ in $S^{\lambda^+}_\lambda$ are such that $f_\zeta(i) = f_\zeta(j)$, then

(a) $N^\zeta_i \restriction i = N^\zeta_j \restriction j$,

(b) There is a $K_{\rm ap}$-isomorphism $h^\zeta_{i,j}$ from N^ζ_i onto N^ζ_j mapping M^ζ_i onto M^ζ_j, and such that $h^\zeta_{i,j} \restriction (|N^\zeta_i| \cap i)$ is the identity.

At the end, let $C \subseteq \lambda^+$ be a club such that for every $\zeta < \varepsilon$

$$i < j \ \&\ j \in C \Longrightarrow |N_i^\zeta| \subseteq j.$$

Suppose now that $i < j \in C \cap S_\lambda^{\lambda^+}$ are such that $f_\zeta(i) = f_\zeta(j)$ for all $\zeta < \varepsilon$.

For $l \in \{i, j\}$ let $M_l \stackrel{\text{def}}{=} \bigcup_{\zeta<\varepsilon} M_l^\zeta$ and $N_l \stackrel{\text{def}}{=} \bigcup_{\zeta<\varepsilon} N_l^\zeta$. Notice that $\langle M_l, N_l\rangle \in R$ and that for every $\zeta < \varepsilon$ we have $\langle M_l^\zeta, N_l^\zeta\rangle \leq \langle M_l, N_l\rangle$. Also observe that $|N_i| \subseteq j$ and that $N_i \restriction i = N_j \restriction j$. Let $h = h_{i,j} \stackrel{\text{def}}{=} \bigcup_{\zeta<\varepsilon} h_{i,j}^\zeta$. Then h is a $K_{\rm ap}$-isomorphism from N_i onto N_j mapping M_i onto M_j, and such that $h \restriction (|N_i| \cap i)$ is the identity. By the definition of workability, we can find $\langle M, N\rangle \in R$ which is stronger than both $\langle M_i, N_i\rangle$ and $\langle M_j, N_j\rangle$.

$\bigstar_{2.14}$

Claim 2.18. Suppose that R is given by Case 4 of Definition 2.7. Then, keeping the notation of Def 2.7, in $V^{P_\beta * \underset{\sim}{Q}_j^\beta * \underset{\sim}{R}_j^\beta}$ we have

$$\Gamma = \{N' : (\exists N \geq N')(\exists M)[\langle M, N\rangle \in G_R]\} \in K_{\rm md}[K_{\rm ap}]\},$$

where G_R is R-generic over $V^{P_\beta * \underset{\sim}{Q}_j^\beta}$.

Proof of the Claim. We verify that Γ satisfies the required properties (i)-(v) from Definition 1.7. As (v) is obvious, we check (i)-(iv). By Claim 2.10(2)

$$R \text{ is a } (< \lambda)\text{-complete forcing,}$$

hence Γ is $(< \lambda)$-closed, and so it satisfies (i). Property (ii) follows by genericity.

Given $\beta < \lambda^+$ such that the requirement of (iii) of Definition 1.7(2) holds for Γ^- (Γ^- comes from the definition of R by Case 4), arguing in $V^{P_\beta * \underset{\sim}{Q}_j^\beta}$ we shall show that

$$\mathcal{I} \stackrel{\text{def}}{=} \{\langle M, N\rangle \in R : (\exists \gamma \in |N|)[\beta + \lambda = \gamma + \lambda]\}$$

is dense in R. So let $\langle M, N\rangle \in R$ be given. Let $M' \in \Gamma^-$ be $\geq M$ and such that for some γ with $\gamma + \lambda = \beta + \lambda$ we have $\gamma \in |M'|$. Since $N \restriction \mathrm{Ev} = M$, we can apply amalgamation to N, M, M' to find $N' \geq_{K_{\rm ap}} N$ with $M' \leq N'$. By Remark 1.5 (3), we can assume that $N' \restriction {\rm Ev} = M'$. Hence $\langle M', N' \rangle \in R \cap \mathcal{I}$ is as required, showing (iii).

To show (iv), suppose $N \in \Gamma$ and $N \leq_{K_{\rm ap}} N'$ after we have forced by R. As the forcing with R is $(< \lambda)$-closed, we have $N' \in V^{P^\beta * \underset{\sim}{Q}^\beta_j}$ and $N \leq_{K_{\rm ap}} N'$ holds. Let M be such that $\langle M, N \rangle \in G_R$. Now observe that by amalgamation and Remark 1.5(3), the set

$$\{\langle M, N'' \rangle : (\exists \text{ lawful } h)[h : N' \to N'' \text{ embedding over } M]\}$$

is dense in R above $\langle M, N \rangle$. $\bigstar_{2.18}$

This finishes the inductive proof.

Claim 2.19. It is possible to define the iteration $\bar{P}$ so that in $V^{\bar{P}}$ we have

(1) If $\lambda > \aleph_0$ then for every $(< \lambda)$-complete forcing notion Q which satisfies $*^\varepsilon_\lambda$ and has the set of elements some ordinal $< \kappa$ and for every $\beta < \lambda^{++}$ large enough we have $\underset{\sim}{R}^\beta_j = Q$ for some $j < \mu$. If $\lambda = \aleph_0$, the analogous statement holds with ccc forcing in place of $(< \lambda)$-complete $*^\varepsilon_\lambda$ forcing,

(2) For every workable strong λ-approximation family K and a family $\bar{\Gamma} = \{\Gamma_\alpha : \alpha < \mu\}$ of elements of $K_{\rm md}$, and for every $\beta < \lambda^{++}$ large enough, there is $j < \mu$ such that R^β_j is given by Case 2 of Definition 2.7 using K, Γ^- as parameters.

(3) If $\lambda > \aleph_0$, then for every $K, \bar{\Gamma}, \beta, j$ as in (2), for every $\alpha < \mu$, there is $\beta' > \beta$ such that $R^{\beta'}_j$ is defined by Case 3 of Definition 2.7 using Γ_α and β as parameters.

(4) For every workable strong λ-approximation family K and $\Gamma^- \in K^-_{\rm md}[K]$ such that $\bigcup\{|M| : M \in \Gamma\} \subseteq {\rm Ev}$, for every j large enough there is $\beta < \lambda^{++}$ such that R^β_j is defined by Case 4 of Definition 2.7 using Γ^- as a parameter.

Proof of the Claim. We use the standard bookkeeping. As the forcing is $(<\lambda)$-closed, any workable strong λ-approximation family $K \in V^{\bar{P}}$ appears at some stage and does not gain any new members later. Also notice that being in $K^-_{\rm md}$ and $K_{\rm md}$ is absolute between $V^{\bar{P}}$ and $V^{P^\beta_j * \underset{\sim}{Q}^\beta_j}$ containing Γ for $\Gamma \subseteq K$. $\bigstar_{2.19}$

$\bigstar_{2.3}$
This finishes the proof of the Theorem.

Remark 2.20. Applying the usual proof of the consistency of $MA + \neg CH$ if we assume in Theorem 2.3 that V satisfies

$$\theta < \kappa \Longrightarrow \theta^{<\lambda} < \kappa,$$

we can drop the assumptions $|Q| < \kappa$ from (d) in the conclusion of Theorem 2.3.

Proof of the Conclusion. Let $V^* \stackrel{\rm def}{=} V^{\bar{P}}$, where $\bar{P}$ is an iteration satisfying the requirements listed in Claim 2.19.
(1) Given an abstract elementary class $\mathcal{K}$ in V^* such that there is a workable strong λ-approximation family $K_{\rm ap}$ approximating $\mathcal{K}$ and such that ${\rm LS}(\mathcal{K}) \le \lambda$ and suppose that $M \in \mathcal{K}_{\lambda^+}$. Let $\langle \bar{\Delta}_\beta : \beta < \lambda^{++}\rangle$ be as in (e) of the conclusion of Theorem 2.3, for our $K_{\rm ap}$. Let M^*_β be as in Claim 1.16, with $M_\eta \stackrel{\rm def}{=} \bigcup \Delta^\beta_\eta$ for $\eta \in \mathcal{T}$. (Note that $\eta \trianglelefteq \nu$ does imply that $M_\eta \le_{\mathcal{K}_{\rm ap}} M_\nu$). We claim that M embeds into M^*_β for some β.

By the definition of approximation, there is Γ^- which is an element of $K^-_{\rm md}[K_{\rm ap}]$, such that $M \le_{\mathcal{K}} M_{\Gamma^-}$ and $N \in \Gamma^- \Longrightarrow |N| \subseteq {\rm Ev}$. By Theorem 2.3(f), there is $\Gamma \in K_{\rm md}[K_{\rm ap}]$ such that $\Gamma^- \subseteq \Gamma$, and hence by Observation 1.13(2), we have $M_{\Gamma^-} \le_{\mathcal{K}} M_\Gamma$. Let $\beta < \lambda^{++}$ be such that $\Gamma, K_{\rm ap} \in V_\beta$, which is easily seen to exist. By (e) in the conclusion of Theorem 2.3 and its proof, there is $\alpha < \mu$ such that M_Γ is isomorphically embeddable into $M_{\bigcup_{i<\lambda^+} \Delta^\beta_{f^*_\alpha \restriction i}}$. By Observation 1.17, we have

$$M \le_{\mathcal{K}} M_{\Gamma^*} \le_{\mathcal{K}} M_\Gamma \le_{\mathcal{K}} M_{\bigcup_{i<\lambda^+} \Delta^\beta_{f^*_\alpha \restriction i}} \le_{\mathcal{K}} M^*_\beta.$$

(2) In addition to what we have already observed, we need to observe that $2^\lambda = \mu$, and this is the case because $\bar{P}$ adds a Cohen subset to λ μ many times.
(3) Follows from (1) of the Theorem.
(4) This part follows similarly to (1), using the assumptions on $\mathcal{K}^+$. $\bigstar_{2.4}$

Fact 2.21. Suppose $\lambda = \lambda^{<\lambda} \geq \aleph_0$. Each of the following classes $\mathcal{K}$ is an abstract elementary class for which there is a workable strong λ-approximation family approximating it, and the Löwenheim-Skolem number of $\mathcal{K}$ is $\leq \lambda$:

(1) The class of models of T^*_{feq}, i.e. an indexed family of independent equivalence relations, with $M \leq N$ iff M embeds into N,

(2) The class T_{trf} of triangle free graphs, with the same order as in (1),

(3) The class of models of any simple theory.

[Why? (1) and (2) were proved in [Sh 457], and (3) is proved in [Sh 500].]

3 Consistency of the non-existence of universal normed vector spaces.

Definition 3.1. Suppose that I is a linear order.
(1) We define a vector space B_I over $\mathbb{Q}$ by

$$B_I \stackrel{\text{def}}{=} \left\{ \sum_{i\in I} a_i x_i : a_i \in \mathbb{Q} \ \& \ \{i : a_i \neq 0\} \text{ finite} \right\},$$

where $\{x_i : i \in I\}$ is a set of variables that serve as a basis for B_I. The addition and scalar multiplication is defined in the obvious manner.
(2) For any I-increasing sequence $\bar{t} \in {}^{\omega>}I$, we define a functional $f_{\bar{t}} : B_I \to \mathbb{R}$ by letting

$$f_{\bar{t}}(\sum_{i\in I} a_i x_i) \stackrel{\text{def}}{=} \sum_{l<\lg(\bar{t})} \frac{1}{\ln(l+2)} a_{\bar{t}(l)}.$$

Let
$$F \stackrel{\text{def}}{=} \{f_{\bar{t}} : \bar{t} \in \bigcup_{n<\omega} {}^{n}I \text{ is } I\text{-increasing}\}.$$

For $x \in B_I$ we define $||x|| = ||x||_F \stackrel{\text{def}}{=} \sup\{|f(x)| : f \in F\}$.

Note 3.2. (1) Functionals $f_{\bar{t}}$ defined as above are linear.
(2) For every $x \in B_I$, there are only finitely many possible values of $f_{\bar{t}}(x)$. (Hence, $||x|| = \mathrm{Max}\{|f(x)| : f \in F\}$).

Claim 3.3. Suppose that I and B_I are as in Definition 3.1 and I is infinite. Then B_I is a normed vector space over $\mathbb{Q}$ with $|B_I| = |I|$.

Proof of the Claim. We prove that $||-||$ is a norm on B_I. The triangular inequality is easily verified. We need to check that for all $x \in B_I$ we have $0 \leq ||x|| < \infty$ and $||x|| = 0 \iff x = 0$. The second statement is obvious, by considering sequences $\bar{t}$ whose length is 1, and the first follows from Note 3.2(2). $\bigstar_{3.3}$

Theorem 3.4. Suppose that $\aleph_0 \leq \lambda = \lambda^{<\lambda} < \lambda^+ < \mu = \mathrm{cf}(\mu) = \mu^{\lambda^+}$.

Then for some $(< \lambda)$-complete and λ^+-cc forcing notion P of cardinality μ, we have that P forces

"$2^\lambda = \mu$ and for every normed vector space $\underset{\sim}{A}$ over $\mathbb{Q}$ of cardinality $|\underset{\sim}{A}| < \mu$, there is a normed vector space $\underset{\sim}{B}$ over $\mathbb{Q}$ of dimension λ^+ (so cardinality λ^+) such that there is no vector space embedding $h : \underset{\sim}{B} \to \underset{\sim}{A}$ with the property that for some $\underset{\sim}{c} \in \mathbb{R}^+$ for all $x \in \underset{\sim}{B}$

$$1/\underset{\sim}{c} < \frac{||h(x)||_{\underset{\sim}{A}}}{||x||_{\underset{\sim}{B}}} < \underset{\sim}{c}.\text{"} \qquad (*)$$

Proof. We deal with the situation $\lambda > \aleph_0$, and the proof for $\lambda = \aleph_0$ is similar but easier.

Definition 3.5. (1) We define an iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \mu, \beta < \mu \rangle$$

with $(< \lambda)$-supports such that for all $\beta < \mu$ we have that $\underset{\sim}{Q}_\beta$ is a P_β-name defined by

$$\underset{\sim}{Q}_\beta \stackrel{\text{def}}{=} \{(\underset{\sim}{w}, \underset{\sim}{\lneq}_w) : \underset{\sim}{w} \in [\lambda^+]^{<\lambda} \ \& \ \underset{\sim}{\lneq}_w \text{ is a linear order on } \underset{\sim}{w}\},$$

ordered by letting $(\underset{\sim}{w}, \underset{\sim}{\lneq}_w) \leq (\underset{\sim}{z}, \underset{\sim}{\lneq}_z)$ iff $\underset{\sim}{w} \subseteq \underset{\sim}{z}$ and $\underset{\sim}{\lneq}_w = \underset{\sim}{\lneq}_z \restriction (\underset{\sim}{w} \times \underset{\sim}{w})$.
(2) Let $P \stackrel{\text{def}}{=} P_\mu$.

Claim 3.6. (1) For every $\alpha < \mu$ we have

$$\Vdash_{P_\alpha} \text{“}\underset{\sim}{Q}_\alpha \text{ is } (< \lambda)\text{-complete and satisfies } *^\omega_\lambda\text{”}.$$

(2) P is λ^+-cc, $(< \lambda)$-complete and $\Vdash_P$ “$2^\lambda = \mu$”.
(3) For $\alpha < \mu$

$$\underset{\sim}{I}_\alpha \stackrel{\text{def}}{=} (\lambda^+, \bigcup\{\underset{\sim}{\lneq}_w : (\underset{\sim}{w}, \underset{\sim}{\lneq}_w) \in \underset{\sim}{G}_{\underset{\sim}{Q}_\alpha}\})$$

is a $P_{\alpha+1}$-name which is forced to be a linear order on λ^+.

Proof of the Claim. (1) The first statement is obvious, we shall prove the second one. The proof is by induction on α. Given $\alpha < \mu$, by the induction hypothesis we have in V^{P_α} that $\lambda^{<\lambda} = \lambda$. We work in V^{P_α}, and describe the winning strategy of player I in the game $*^\omega_\lambda[Q_\alpha]$. As $\lambda^{<\lambda} = \lambda$, we can fix a bijection F which to every triple $(w, \leq, \gamma)$, where $w \in [\lambda^+]^{<\lambda}$ and $\leq$ is a linear order on w, and $\gamma < \lambda^+$, assigns an element of λ^+. We can find a club E of λ^+ such that for every $j \in S^{\lambda^+}_\lambda \cap E$ and every relevant triple $(w, \leq, \gamma)$,

$$w \in [j]^{<\lambda} \ \& \ \gamma < j \Longrightarrow F((w, \leq, \gamma)) < j.$$

Suppose that $n < \omega$ and

$$\Big\langle \langle q^k_i : i < \lambda^+ \rangle, f_k, \langle p^k_i : i < \lambda^+ \rangle : k \leq n \Big\rangle$$

have been played so far, and we shall describe how to choose q^{n+1}_i and f_{n+1}. We let $q^{n+1}_i \stackrel{\text{def}}{=} p^n_i$, for $i < \lambda^+$. Suppose that $p^n_i = (w^n_i, \leq^n_i) \in Q_\alpha$ for $i < \lambda^+$.

For $j < \lambda^+$, let $\gamma(j,n) \stackrel{\rm def}{=} \sup(w^n_j \cap j)$. Note that for $j \in S^{\lambda^+}_\lambda$ we have $\gamma(j,n) < j$. Let $C_{n+1} \stackrel{\rm def}{=} E$. Define g_{n+1} which to an ordinal $j \in S^{\lambda^+}_\lambda$ assigns

$$(w^n_j \cap j, \leq^n_j \restriction (w^n_j \cap j), \gamma(j,n)).$$

Then let

$$f_{n+1} \stackrel{\rm def}{=} (F \circ g_{n+1}) \restriction (C_{n+1} \cap S^{\lambda^+}_\lambda) \cup 0_{\lambda^+ \setminus (C_{n+1} \cap S^{\lambda^+}_\lambda)}.$$

Hence f_{n+1} is regressive on $C_{n+1} \setminus \{0\}$.

At the end of the game, for $i < \lambda^+$ let $w^i \stackrel{\rm def}{=} \bigcup_{n<\omega} w^n_i$ and $\leq^i \stackrel{\rm def}{=} \bigcup_{n<\omega} \leq^n_i$. Let $C \subseteq E$ be a club such that $i < j \in C \implies w^i \subseteq j$. Suppose that $i < j \in C \cap S^{\lambda^+}_\lambda$ are such that for all $n < \omega$ we have $f_n(i) = f_n(j)$. We shall define a condition p such that $p = (z, \leq_z)$ and $z = w^i \cup w^j$, by amalgamating linear orders. For $x, y \in z$ we let $n = n(x,y)$ be the minimal n such that $x, y \in w^n_i \cup w^n_j$, and let $x \leq_z y$ iff

(i) $x, y \in w^l$ and $x \leq^l y$ for some $l \in \{i, j\}$, or

(ii) $x \in w^n_i \setminus w^n_j$ and $y \in w^n_j \setminus w^n_i$ and for some $z \in w^n_j \cap w^n_i$ we have $x \leq^i z$ and $z \leq^j y$,

(iii) $y \in w^n_i \setminus w^n_j$ and $x \in w^n_j \setminus w^n_i$ and (ii) does not hold.

It is easily seen that p is as required.

(2) That P is λ^+-cc follows from (1) by the fact that $*^\omega_\lambda$ is preserved under $(< \lambda)$-support iterations. See [Sh 546]. That $\Vdash_P$ "$2^\lambda = \mu$" is seen by observing that every $\underset{\sim}{Q}_\alpha$ adds a subset to λ.

(3) Obvious. $\bigstar_{3.6}$

Suppose that in V^P we have a normed vector space A over $\mathbb{Q}$ with $|A| < \mu$, with the universe of A a set of ordinals. Hence for some $\alpha < \mu$ and a P_α-name $\underset{\sim}{A}$ we have that $A = \underset{\sim}{A}_G$. Suppose that $h \in V^P$ is a vector space embedding from B_{I_α} into A, satisfying $(*)$ above, for some $c \in \mathbb{Q}$. Hence for some $p^* \in P/P_\alpha$ we have that p^* forces over V^{P_α} the following statement:

"$\underset{\sim}{h} : B_{\underset{\sim}{I}_\alpha} \to A$ is a normed vector space embedding satisfying $(*)$ for $\underset{\sim}{c}$."

Without loss of generality, p^* decides the value c of $\underset{\sim}{c}$. Let $0 < n^* < \omega$ be such that $c < n^*$. Let $\underset{\sim}{x}_i$ for $i < \lambda^+$ be the generators of $B_{\underset{\sim}{I}_\alpha}$. For $i < \lambda^+$ we find $p_i \in P/P_\alpha$ such that $p^* \leq p_i$ and

$$p_i \Vdash \text{“}\underset{\sim}{h}(\underset{\sim}{x}_i) = y_i\text{” for some } y_i.$$

Let us now work in V^{P_α}. Let $p_i(\alpha) = (w_i, <_i)$, for $i < \lambda^+$. Without loss of generality we have $i \in w_i$ for all i.

By a Δ-system argument, noting that $\lambda^{<\lambda} = \lambda$ holds in V^{P_α}, we can find $Y \in [\lambda^+]^{\lambda^+}$ such that

(a) for some w^* we have that $w_i \cap w_j = w^*$, for all $i \neq j \in Y$, and $<_i \restriction (w^* \times w^*)$ is constant,

(b) If $i < j$ are both in Y, then

$$\sup(w_i) < \min(w_j \setminus w^*), \text{ while } \sup(w^*) < \min(w_i \setminus w^*),$$

(c) $Y \cap w^* = \emptyset$,

(d) For $i < j$ both in Y, there is an isomorphism $h_{i,j}$ mapping $(w_i, <_i)$ onto $(w_j, <_j)$ such that $h_{i,j}(i) = j$. (Note that $i \in w_i \setminus w^*$ and $j \in w_j \setminus w^*$.)

Observation 3.7. The series

$$\sum_{l \geq 1} \frac{1}{(l+1)\ln(l+2)}$$

diverges, while the sum

$$\sum_{l=1}^{n} \frac{1}{(n-l+1)\ln(l+2)}$$

is uniformly and strictly bounded by 4.

Proof of the Observation. The first statement follows by comparison with $\int_1^\infty \frac{1}{(x+1)\ln(x+2)}dx$. The second statement follows from the following estimate:

$\sum_{l=1}^{n} \frac{1}{(n-l+1)\ln(l+2)} \leq \sum_{l=1}^{[n/2]} \frac{1}{(n-l+1)\ln(l+2)} + \sum_{l=[n/2]+1}^{n} \frac{1}{(n-l+1)\ln(l+2)} \leq$
$\frac{1}{[n/2]+1} \sum_{l=1}^{[n/2]} \frac{1}{\ln(l+2)} + \sum_{l=1}^{n-[n/2]} \frac{1}{l\ln(n-l+3)} \leq$
$\frac{1}{[n/2]+1} \cdot [n/2] \cdot \frac{1}{\ln 3} + \frac{1}{\ln([n/2]+2)} \sum_{l=1}^{n-[n/2]} \frac{1}{l} \leq$
$\frac{1}{\ln 3} + \frac{1}{\ln([n/2]+2)}(1 + \int_1^{n-[n/2]} \frac{1}{x}dx) \leq \frac{1}{\ln 3} + \frac{1}{\ln([n/2]+2)} \cdot (1 + \ln(n - [n/2]))$
$\leq \frac{1}{\ln 3} + \frac{1}{\ln 2} + 1 < 4.$

$\bigstar_{3.7}$

By Observation 3.7, we can choose m large enough such that

$$\sum_{l=1}^{m} \frac{1}{(l+1)\ln(l+2)} \geq 4(n^*)^2.$$

Let us choose $i_1 < \cdots < i_m \in Y$.

Claim 3.8. We can find q' and q'' in Q_α, both extending all $p_{i_l}(\alpha)$ for $1 \leq l \leq m$, and such that

$$q' \Vdash_{Q_\alpha} \text{“}\langle i_1, \ldots, i_m\rangle \text{ is increasing in } \underset{\sim}{I}_\alpha\text{”}$$

and

$$q'' \Vdash_{Q_\alpha} \text{“}\langle i_1, \ldots, i_m\rangle \text{ is decreasing in } \underset{\sim}{I}_\alpha\text{.”}$$

Proof of the Claim. Notice that for no $1 \leq l_1 < l_2 \leq m$ and $i \in Y$ do we have that p_i decides the order between i_{l_1} and i_{l_2} in I_α, by the choice of Y (this is elaborated below). The proof can proceed by induction on m. The inductive step is as in the proof of $*^\omega_\lambda$. The only constraint we could have to letting $i_{l_1} \leq i_{l_2}$ (for q') or $i_{l_2} \leq i_{l_1}$ (for q'') would be if some $z \in w^*$ would prevent this, but this does not happen. For example, if we could not let $i_{l_1} \leq i_{l_2}$ in q' then this would mean that $i_{l_1} \geq i_{l_2}$ would have to hold. By the choice of Y and since $i_{l_1} \in w_{i_{l_1}} \setminus w^*$ and similarly for i_{l_2}, this could only be the case if for some $z \in w^*$ it would hold that $i_{l_1} \geq_{w_{i_{l_1}}} z$ while $i_{l_2} \leq_{w_{i_{l_2}}} z$. However, this would contradict item (d) in the choice of Y. $\bigstar_{3.8}$

Back in V^{P_α}, let $z \overset{\text{def}}{=} \sum_{l=1}^m \frac{1}{l+1} x_{i_l}$. Let $a \overset{\text{def}}{=} \left|\left| \sum_{l=1}^m \frac{1}{l+1} y_{i_l} \right|\right|_A$. Hence

$$q' \Vdash \text{“}||\underset{\sim}{z}||_{B_{\underset{\sim}{I}_\alpha}} \geq 4(n^*)^2\text{”},$$

and so $a \geq 4(n^*)^2/n^* = 4n^*$. On the other hand,

$$q'' \Vdash \text{“}||\underset{\sim}{z}||_{B_{\underset{\sim}{I}_\alpha}} < 4\text{”},$$

and hence $a < 4n^*$, a contradiction. $\bigstar_{3.4}$

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