AFTER ALL, THERE ARE SOME INEQUALITIES WHICH ARE PROVABLE IN ZFC

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Abstract. We address ZFC inequalities between some cardinal invariants of the continuum, which turned to be true in spite of strong expectations given by [10].

1. Introduction

The present paper consists two independent sections which have two things in common: both resulted in a failure to fulfill old promises to build a specific forcing notions and in both an important role is played by cardinal invariant κ^*.

The first promise was stated in [4] and was related to cardinal invariant $\mathsf{cov}^\star(\mathcal{N})$. Let $\mathbf{B}$ denote the measure algebra adding one random real.

Definition 1.1. Let $\mathcal{N}_2$ be the ideal of measure zero subsets of $\mathbb{R}\times\mathbb{R}$ and let $\mathsf{Borel}(\mathbb{R})$ be the collection of all Borel mappings from $\mathbb{R}$ into $\mathbb{R}$. Define

$$\mathsf{cov}^\star(\mathcal{N}) = \min\Big\{|\mathcal{A}| : \mathcal{A}\subseteq\mathcal{N}_2 \ \&\ (\forall f\in\mathsf{Borel}(\mathbb{R}))(\forall B\in\mathsf{Borel}\setminus\mathcal{N})(\exists H\in\mathcal{A}) \\ (\big\{x\in B : \langle x, f(x)\rangle\in H\big\}\notin\mathcal{N})\Big\}$$

and

$$\mathsf{non}^\star(\mathcal{N}) = \min\Big\{|X| : X\subseteq\mathsf{Borel}(\mathbb{R}) \ \&\ (\forall H\in\mathcal{N}_2)(\forall B\in\mathsf{Borel}\setminus\mathcal{N})(\exists f\in X) \\ (\big\{x\in B : \langle x, f(x)\rangle\notin H\big\}\notin\mathcal{N})\Big\}.$$

Proposition 1.2. $\mathsf{cov}^\star(\mathcal{N}) = \mathsf{cov}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}}$ *and* $\mathsf{non}^\star(\mathcal{N}) = \mathsf{non}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}}$. $\square$

It has been known that (see [4], [7], [9] for more details):

1. $\max\{\mathsf{cov}(\mathcal{N})^{\mathbf{V}}, \mathfrak{b}^{\mathbf{V}}\} \leq \mathsf{cov}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}} \leq \mathsf{non}(\mathcal{M})$;
2. it is consistent that $\mathsf{cov}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}} > \max\{\mathsf{cov}(\mathcal{N})^{\mathbf{V}}, \mathfrak{b}^{\mathbf{V}}\}$;
3. it is consistent that $\mathsf{cov}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}} > \mathfrak{d}$.

And in [4, 3.11] we promised that in [10] it would be proved that

- it is consistent that $\mathsf{cov}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}} < \mathsf{non}(\mathcal{M})$,

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1991 *Mathematics Subject Classification.* 03E35.

First author partially supported by NSF grant DMS 95-05375 and Alexander von Humbold Foundation.

Second author partially supported by KBN grant 2 PO3 A 01109.

Third author partially supported by Basic Research Fundation administered by Israel Academy of Sciences. Publication 616.

being sure that using the method of norms on possibilities we could construct a forcing notion which:

a: is proper ω^ω–bounding,
b: makes ground reals meager and
c: does not add a $\mathbf{B}$–name for a random real over $\mathbf{V}^{\mathbf{B}}$.

However, when trying to fill up the details of the construction, we have discovered that *there is no such forcing notion* and found new inequalities provable in ZFC.

The second section deals with an inequality related to localizations of subsets of ω by partitions of ω. Several notions of localization and related cardinal invariants were introduced in [11]. The one we will refer to is the $R_0^\exists$–localization property.

Definition 1.3. Let $\mathbf{V} \subseteq \mathbf{V}^*$ be universes of Set Theory and let $k \in \omega$.

(1) We say that the extension $\mathbf{V} \subseteq \mathbf{V}^*$ has *the $R_k^\exists$–localization property* if in $\mathbf{V}^*$:

for every infinite co-infinite set $X \subseteq \omega$ there is a partition $\langle K_n : n \in \omega\rangle \in V$ of ω such that $|K_n| > k+1$ and

$$(\exists^\infty n \in \omega)(|X \cap K_n| \leq k).$$

(2) An infinite co-infinite set $X \subseteq \omega$, $X \in \mathbf{V}^*$ is said to be $(k,0)$*–large over* $\mathbf{V}$ if

for every sequence $\langle K_n : n \in \omega\rangle \in \mathbf{V}$ of disjoint k–element subsets of ω we have

$$(\forall^\infty n \in \omega)(K_n \cap X \neq \emptyset).$$

The following result has been shown in [11, 1.8].

Proposition 1.4. *Let $\mathbf{V} \subseteq \mathbf{V}^*$ be models of ZFC, $m \geq 2$, $k \in \omega$. Then the following conditions are equivalent:*

(1) *there is no $(m,0)$-large set in $\mathbf{V}^*$ over $\mathbf{V}$,*
(2) *there is no $(2,0)$-large set in $\mathbf{V}^*$ over $\mathbf{V}$,*
(3) *$\mathbf{V} \subseteq \mathbf{V}^*$ has the $R_0^\exists$–localization property,*
(4) *$\mathbf{V} \subseteq \mathbf{V}^*$ has the $R_k^\exists$–localization property.* $\square$

After noting that if $\mathbf{V} \cap 2^\omega$ is not meager in $\mathbf{V}^*$, $\mathbf{V} \subseteq \mathbf{V}^*$ then the extension $\mathbf{V} \subseteq \mathbf{V}^*$ has the $R_0^\exists$–localization property we promised to give in [10] an example of a forcing notion showing that the converse implication does not hold. In fact we wanted to construct a forcing notion which:

a: is proper ω^ω–bounding,
b: makes ground reals meager and
c: has the $R_0^\exists$–localization property.

Once again, we have discovered that *there is no such forcing notion* and we have established some new inequalities between relevant cardinal invariants.

Notation 1.5. We try to keep our notation standard and compatible with that of classical textbooks on Set Theory (like Jech [6] or Bartoszyński Judah [3]))

(1) Let $i, j < \omega$. The set of all integers m satisfying $i \leq m < j$ is denoted by $[i, j)$, etc.

(2) For integers $k_i, \ldots, k_j$ $(i \le j < \omega)$, $\prod\limits_{\ell=i}^{j} k_\ell$ is their Cartesian product interpreted as the collection of all finite functions τ such that $\mathsf{dom}(\tau) = [i,j]$ and $(\forall \ell \in \mathsf{dom}(\tau))(\tau(\ell) \in k_i)$.

However, we will use the same notation for the cardinality of this set, hoping that it will not cause too much confusion.

(3) For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\mathsf{lh}(\eta)$.

(4) The quantifiers $(\forall^\infty n)$ and $(\exists^\infty n)$ are abbreviations for

$$(\exists m \in \omega)(\forall n > m) \quad \text{and} \quad (\forall m \in \omega)(\exists n > m),$$

respectively.

(5) For ω–sequences η, ρ we write $\eta =^* \rho$ whenever

$$(\forall^\infty n \in \omega)(\eta(n) = \rho(n)).$$

(6) The Cantor space 2^ω and the Baire space ω^ω are the spaces of all functions from ω to 2, ω, respectively, equipped with natural (product) topology.

(7) In forcing arguments, *a stronger condition is the larger one.*

2. Adding a random name for a random real

As the failure in building the forcing notion we had in mind for [4, 3.11] directly results in some properties of extensions of universes of ZFC, we will formulate the main result of the present section in this language. Further we will draw several conclusions for cardinal invariants.

The result presented in 2.3 below is of some interest *per se* if you have in mind the following theorem (see [9, 3.1]).

Theorem 2.1. *1. (Krawczyk; [7]) Suppose that $\mathbf{V} \subseteq \mathbf{V}^*$ are universes of Set Theory such that $\mathbf{V} \cap \omega^\omega$ is bounded in $\mathbf{V}^* \cap \omega^\omega$. Let r be a random real over $\mathbf{V}^*$. Then there is in $\mathbf{V}^*[r]$ a random real over $\mathbf{V}[r]$.*
2. (Pawlikowski; [9, 3.2]) Suppose that c is a Cohen real over $\mathbf{V}$ and r is a random real over $\mathbf{V}[c]$. Then, in $\mathbf{V}[c][r]$ there is no random real over $\mathbf{V}[r]$.

Definition 2.2. Let $\Phi \in \omega^\omega$ be a strictly increasing function. *A Φ–constructor* is a sequence $\langle n_i, m_i, k_i : i < \omega\rangle$ of integers defined inductively by: $n_0 = 2$ and for $i \in \omega$

$$m_i = (\prod_{j<i} m_j) \cdot 2^{3(n_i+i)}, \quad k_i = (m_i \cdot \prod_{j<i} k_j) \cdot \Phi(m_i \cdot \prod_{j<i} k_j), \quad n_{i+1} = n_i(k_i+1).$$

[So $n_i < m_i < k_i < n_{i+1}$.]

Theorem 2.3. *Suppose that $\mathbf{V} \subseteq \mathbf{V}^*$ are universes of Set Theory such that*

if r is a random real over $\mathbf{V}^*$
then in $\mathbf{V}^*[r]$ there is no random real over $\mathbf{V}[r]$.

Let $\Phi \in \omega^\omega \cap \mathbf{V}$ be strictly increasing and let $\langle n_i, m_i, k_i : i < \omega\rangle$ be the Φ–constructor.

(1) *If $\mathbf{V} \cap \omega^\omega$ is dominating in $\mathbf{V}^* \cap \omega^\omega$, then, in $\mathbf{V}^*$:*
for every function $\eta \in \prod\limits_{i\in\omega} k_i$ there are sequences $\langle X_\ell : \ell < \omega\rangle \in \mathbf{V}$ and $\langle i_m : m < \omega\rangle \in \mathbf{V}$ such that

a: $(\forall \ell \in \omega)(X_\ell \subseteq k_\ell \ \& \ |X_\ell| = m_\ell \cdot \prod_{j<\ell} k_j)$,

b: $(\forall m \in \omega)(\exists \ell \in [i_m, i_{m+1}))(\eta(\ell) \in X_\ell)$.

(2) *If $\mathbf{V} \cap \omega^\omega$ is not domination in $\mathbf{V}^* \cap \omega^\omega$, then, in $\mathbf{V}^*$:*
for every function $\eta \in \prod_{i\in\omega} k_i$ there is a sequence $\langle X_\ell : \ell < \omega\rangle \in \mathbf{V}$ such that

a: $(\forall \ell \in \omega)(X_\ell \subseteq k_\ell \ \& \ |X_\ell| = m_\ell \cdot \prod_{j<\ell} k_j)$,

b: $(\exists^\infty \ell \in \omega)(\eta(\ell) \in X_\ell)$.

Theorem 2.4. *Suppose that $\mathbf{V} \subseteq \mathbf{V}^*$ are universes of Set Theory such that*
if r is a random real over $\mathbf{V}^*$
then $\mathbf{V}[r] \cap 2^\omega$ has measure zero in $\mathbf{V}^*[r]$.
Let $\Phi \in \omega^\omega \cap \mathbf{V}$ be strictly increasing and let $\langle n_i, m_i, k_i : i < \omega\rangle$ be the Φ–constructor.

(1) *If $\mathbf{V}\cap\omega^\omega$ is bounded in $\mathbf{V}^*\cap\omega^\omega$, then there are sequences $\langle X_\ell : \ell < \omega\rangle \in \mathbf{V}^*$ and $\langle i_m : m < \omega\rangle \in \mathbf{V}^*$ such that for every function $\eta \in \prod_{i\in\omega} k_i \cap \mathbf{V}$*

a: $(\forall \ell \in \omega)(X_\ell \subseteq k_\ell \ \& \ |X_\ell| = m_\ell \cdot \prod_{j<\ell} k_j)$,

b: $(\forall^\infty m \in \omega)(\exists \ell \in [i_m, i_{m+1}))(\eta(\ell) \in X_\ell)$.

(2) *If $\mathbf{V}\cap\omega^\omega$ is unbounded in $\mathbf{V}^*\cap\omega^\omega$, then, there is a sequence $\langle X_\ell : \ell < \omega\rangle \in \mathbf{V}^*$ such that*
for every function $\eta \in \prod_{i\in\omega} k_i \cap \mathbf{V}^*$

a: $(\forall \ell \in \omega)(X_\ell \subseteq k_\ell \ \& \ |X_\ell| = m_\ell \cdot \prod_{j<\ell} k_j)$,

b: $(\exists^\infty \ell \in \omega)(\eta(\ell) \in X_\ell)$.

Proof of 2.3 We will only prove 2.3, the proof of 2.4 is obtained by dualization.

The main parts of the proofs of (1) and (2) are the same, the difference comes only at the very end. So, for a while, we will not specify which part of the theorem we are proving. We will present a construction which itself is interesting, though it is very elementary.

Let $\Phi \in \omega^\omega \cap \mathbf{V}$ be increasing and let $\langle n_i, m_i, k_i : i < \omega\rangle$ be the Φ–constructor. Letting $n_{-1} = 0$, for each $i \in \omega$ choose a sequence $\langle f^i_\ell : \ell < k_i\rangle$ of functions such that

- $f^i_\ell : 2^{[n_i, n_{i+1})} \longrightarrow 2^{[n_{i-1}, n_i)}$,
- for every sequence $\langle \nu_\ell : \ell < k_i\rangle \subseteq 2^{[n_{i-1}, n_i)}$ we have

$$|\Big\{\rho \in 2^{[n_i,n_{i+1})} : (\forall \ell < k_i)(f^i_\ell(\rho) = \nu_\ell)\Big\}| = \frac{2^{n_{i+1}-n_i}}{2^{(n_i-n_{i-1})k_i}} = 2^{k_i \cdot n_{i-1}}.$$

For $i \le j < \omega$ and $\eta \in \prod_{r=i}^{j} k_r$ let

$$f^{i,j}_\eta : 2^{[n_i,n_{j+1})} \longrightarrow 2^{[n_{i-1},n_j)} : \quad \rho \mapsto \bigcup_{r=i}^{j} f^r_{\eta(r)}(\rho{\restriction}[n_r, n_{r+1})).$$

The main point of our arguments will be done by the following combinatorial observation (which should be clear if S is thought of as a tree of independent equally distributed random variables, but still it needs some calculations).

Claim 2.4.1. *Suppose that $0 < i \leq j < \omega$ and $\emptyset \neq S \subseteq \prod\limits_{r=i}^{j} k_r$ is such that for each $\eta \in S$ and $r \in [i,j)$:*

$$|\{\tau(r) : \tau \in S \ \&\ \tau{\restriction}r = \eta{\restriction}r\}| = m_r.$$

Then

$$\left|\left\{\rho \in 2^{[n_i,n_{j+1})} \colon (\exists \sigma \in 2^{[n_{i-1},n_j)}) \left(\frac{7}{2^{n_j - n_{i-1}}} < \frac{|\{\tau \in S \colon f^{i,j}_\tau(\rho) = \sigma\}|}{|S|}\right)\right\}\right| < \frac{1}{2} \cdot 2^{n_{j+1} - n_i}.$$

Proof of the claim: Fix $r \in [i,j]$ and $\tau^* \in \prod\limits_{\ell \in [i,r)} k_\ell$ (so if $r = i$ then $\tau^* = \langle\rangle$) such that there is $\tau \in S$ with $\tau^* \vartriangleleft \tau$. Let

$$A^r_{\tau^*} \stackrel{\text{def}}{=} \Bigg\{\rho \in 2^{[n_r,n_{r+1})} : \ \text{for some } \sigma \in 2^{[n_{r-1},n_r)}$$
$$\left|\frac{|\{\tau(r) : \tau \in S \ \&\ \tau^* \vartriangleleft \tau \ \&\ f^r_{\tau(r)}(\rho) = \sigma\}|}{m_r} - \frac{1}{2^{n_r - n_{r-1}}}\right| \geq \frac{1}{2^{n_r - n_{r-1}} \cdot 2^r}\Bigg\}.$$

By Bernoulli's law of large numbers and by the definition of the m_i's we know that

$$\frac{|A^r_{\tau^*}|}{2^{n_{r+1} - n_r}} \leq 2^{n_r - n_{r-1}} \cdot \frac{1}{4 \cdot m_r \cdot (2^{-(n_r - n_{r-1} + r)})^2} = \frac{1}{4 \cdot \prod\limits_{\ell < r} m_\ell} \cdot 2^{-(3n_{r-1} + r)}.$$

Let

$$A \stackrel{\text{def}}{=} \left\{\rho \in 2^{[n_i,n_{j+1})} \colon (\exists r \in [i,j])(\exists \tau^* \in \prod_{\ell \in [i,r)} k_\ell)\big((\exists \tau \in S)(\tau^* \vartriangleleft \tau) \ \&\ \rho{\restriction}[n_r, n_{r+1}) \in A^r_{\tau^*}\big)\right\}.$$

Note that

$$\frac{|A|}{2^{n_{j+1} - n_i}} \leq \sum_{r \in [i,j]} \sum \left\{\frac{|A^r_{\tau^*}|}{2^{n_{r+1} - n_r}} : \tau^* \in \prod_{\ell \in [i,r)} k_\ell \ \&\ (\exists \tau \in S)(\tau^* \vartriangleleft \tau)\right\} \leq$$
$$\sum_{r \in [i,j]} \left(\frac{1}{4 \cdot \prod\limits_{\ell < r} m_\ell} \cdot 2^{-(3n_{r-1} + r)} \cdot \prod_{\ell < r} m_\ell\right) \leq \frac{1}{4} \sum_{r \in [i,j]} 2^{-r} < \frac{1}{2}.$$

Suppose now that $\rho \in 2^{[n_i,n_{j+1})} \setminus A$. Let $\sigma \in 2^{[n_{i-1},n_j)}$. We know that for each $r \in [i,j]$ and $\tau^* \in \prod\limits_{\ell \in [i,r)} k_\ell$ such that $(\exists \tau \in S)(\tau^* \vartriangleleft \tau)$ we have $\rho{\restriction}[n_r, n_{r+1}) \notin A^r_{\tau^*}$ and therefore

$$\frac{|\{\tau(r) : \tau \in S \ \&\ \tau^* \vartriangleleft \tau \ \&\ f^r_{\tau(r)}(\rho{\restriction}[n_r, n_{r+1})) = \sigma{\restriction}[n_{r-1}, n_r)\}|}{m_r} < (1 + \frac{1}{2^r}) \cdot \frac{1}{2^{n_r - n_{r-1}}}$$

(just look at the definition of the set $A^r_{\tau^*}$). Hence

$$\frac{|\{\tau \in S : f^{i,j}_\tau(\rho) = \sigma\}|}{|S|} < \prod_{r=i}^{j} (1 + \frac{1}{2^r}) \frac{1}{2^{n_r - n_{r-1}}} = \frac{1}{2^{n_j - n_{i-1}}} \cdot \prod_{r=i}^{j} (1 + \frac{1}{2^r}) <$$
$$\frac{1}{2^{n_j - n_{i-1}}} \cdot e^{2^{1-i}} < \frac{7}{2^{n_j - n_{i-1}}}.$$

This finishes the proof of the claim.

Now define a function

$$F : \prod_{i\in\omega} k_i \times 2^\omega \longrightarrow 2^\omega : \ (\eta,\rho) \mapsto \bigcup_{i\in\omega} f^i_{\eta(i)}(\rho{\restriction}[n_i,n_{i+1})).$$

It should be clear that F is well defined (look at the choice of the f^i_ℓ's) and its definition (or rather its code) is in $\mathbf{V}$. The function F is continuous and we have the following claim.

Claim 2.4.2. *If $\eta_0,\eta_1 \in \prod_{i\in\omega} k_i$, $\rho_0,\rho_1 \in 2^\omega$ and $\eta_0 =^* \eta_1$, $\rho_0 =^* \rho_1$ then $F(\eta_0,\rho_0) =^* F(\eta_1,\rho_1)$.*

Proof of the claim: Should be clear.

Before we continue with the proof of the theorem let us introduce some more notation. For a tree $T \subseteq 2^{<\omega}\times 2^{<\omega}$ and integers $\ell, i<\omega$ we let $T_i \stackrel{\mathsf{def}}{=} T\cap(2^{n_{i+1}}\times 2^{n_i})$ and

$$T^{[\ell]}_i = \big\{(\nu_0,\nu_1)\in 2^{n_{i+1}}\times 2^{n_i} : \text{ if } \ell<i \text{ then there are } (\nu_0',\nu_1')\in T_i \text{ such that } \nu_0'{\restriction}[n_{\ell+1},n_{i+1}) = \nu_0{\restriction}[n_{\ell+1},n_{i+1}) \quad\text{and}\quad \nu_1'{\restriction}[n_\ell,n_i) = \nu_1{\restriction}[n_\ell,n_i)\big\}.$$

If $\ell<i<\omega$ then we may treat members of $T^{[\ell]}_i$ as elements of $2^{[n_{\ell+1},n_{i+1})}\times 2^{[n_\ell,n_i)}$ (as only this part carries any information). Thus if $\rho_0\in 2^{[n_{\ell+1},n_{i+1})}$, $\rho_1\in 2^{[n_\ell,n_i)}$ then $(\rho_0,\rho_1)\in T^{[\ell]}_i$ means that there is $(\nu_0,\nu_1)\in T^{[\ell]}_i$ such that $\nu_0{\restriction}[n_{\ell+1},n_{i+1})=\rho_0$, $\nu_1{\restriction}[n_\ell,n_i)=\rho_1$.

Claim 2.4.3. *Suppose that $\eta\in\prod\limits_{i\in\omega} k_i\cap\mathbf{V}^*$. Then there is a tree $T\subseteq 2^{<\omega}\times 2^{<\omega}$, $T\in\mathbf{V}$ such that*

(i) *$\mu^2([T])>0$*
(where $[T]$ is the set of all infinite branches through T,

$$[T] = \{(\rho,\sigma)\in 2^\omega\times 2^\omega : (\forall n\in\omega)((\rho{\restriction}n,\sigma{\restriction}n)\in T)\},$$

and μ^2 stands for the Lebesgue measure on the plane $2^\omega\times 2^\omega$),

(ii) *for each $\ell<\omega$*

$$\mu\big(\{\rho\in 2^\omega : (\forall i\in\omega)((\rho{\restriction}n_{i+1}, F(\eta,\rho){\restriction}n_i)\in T^{[\ell]}_i)\}\big)=0.$$

Proof of the claim: Let r be a random real over $\mathbf{V}^*$. By the assumptions of the theorem we know that $F(\eta,r)$ is not a random real over $\mathbf{V}[r]$. Every Borel null subset of 2^ω from $\mathbf{V}[r]$ is the section at r of a Borel null subset of $2^\omega\times 2^\omega$ from $\mathbf{V}$. Consequently we find a Borel null set $B\subseteq 2^\omega\times 2^\omega$ coded in $\mathbf{V}$ and such that $(r,F(\eta,r))\in B$. We may additionally require that B is invariant under rational translations, i.e. that

$$(\rho_0,\rho_1)\in B \ \&\ \rho_0 =^* \rho_0' \ \&\ \rho_1 =^* \rho_1' \quad\Rightarrow\quad (\rho_0',\rho_1')\in B.$$

In $\mathbf{V}$ take a closed subset of $2^\omega\times 2^\omega$ of positive measure disjoint from B. This gives a tree $T\in\mathbf{V}$, $T\subseteq 2^{<\omega}\times 2^{<\omega}$ such that $\mu^2([T])>0$ and

$$(\oplus) \qquad (\forall \ell\in\omega)(\exists i\in\omega)((r{\restriction}n_{i+1}, F(\eta,r){\restriction}n_i)\notin T^{[\ell]}_i).$$

We want to argue that this T is as required and for this we need to check the demand **(ii)**. Let $\ell < \omega$. Look at the set

$$Y \stackrel{\text{def}}{=} \{\rho \in 2^\omega : (\forall i \in \omega)((\rho{\restriction}n_{i+1}, F(\eta,\rho){\restriction}n_i) \in T_i^{[\ell]})\}.$$

It is a closed subset of 2^ω coded in $\mathbf{V}^*$. Assume that $\mu(Y) > 0$. Then some finite modification r^* of the random real r is in Y. By 2.4.2 we know that $F(\eta, r) =^* F(\eta, r^*)$. Take $\ell_0 > \ell$ so large that

$$F(\eta,r){\restriction}[n_{\ell_0},\omega) = F(\eta,r^*){\restriction}[n_{\ell_0},\omega) \quad \text{and} \quad r{\restriction}[n_{\ell_0},\omega) = r^*{\restriction}[n_{\ell_0},\omega).$$

Now note that

$$r^* \in Y \quad\Rightarrow\quad (\forall i \in \omega)((r^*{\restriction}n_{i+1}, F(\eta,r^*){\restriction}n_i) \in T_i^{[\ell]} \quad\Rightarrow$$
$$\Rightarrow\quad (\forall i \in \omega)((r{\restriction}n_{i+1}, F(\eta,r){\restriction}n_i) \in T_i^{[\ell_0]}$$

and the last contradicts $(\oplus)$ above, finishing the claim.

Claim 2.4.4. *Suppose that $T \subseteq 2^{<\omega} \times 2^{<\omega}$ is a tree, $1 < i < j < \omega$ and*

$(\otimes^i_j)$ $\qquad \frac{63}{64} \le \frac{|T_j^{[i-1]}|}{2^{n_{j+1}+n_j}}$.

Let

$$W = \left\{ \tau \in \prod_{\ell=i}^{j} k_\ell : \frac{|\{\rho \in 2^{[n_i,n_{j+1})} : (\rho, f^{i,j}_\tau(\rho)) \in T_j^{[i-1]}\}|}{2^{n_{j+1}-n_i}} < \frac{1}{64} \right\}.$$

Then there are sets $X_i \subseteq k_i,\ X_{i+1} \subseteq k_{i+1}, \dots, X_j \subseteq k_j$ such that

(α) $\quad |X_\ell| \le m_\ell \cdot \prod\limits_{r<\ell} k_r$ *for each* $\ell = i, \dots, j$ *and*

(β) $\quad (\forall \tau \in W)(\exists \ell \in [i,j])(\tau(\ell) \in X_\ell)$.

Proof of the claim: Assume not. Then we may find a set $S \subseteq \prod\limits_{\ell=i}^{j} k_\ell$ such that $S \subseteq W$ and for every $\tau_0 \in S$ and every $\ell \in [i,j]$

$$|\{\tau(\ell) : \tau \in S \ \&\ \tau{\restriction}\ell = \tau_0{\restriction}\ell\}| = m_\ell.$$

How? For $\ell \in [i,j]$ let $W^\ell = \{\tau{\restriction}\ell : \tau \in W\}$ (so $W^i = \{\langle\rangle\}$). Now we choose inductively sets $X_\ell \subseteq k_\ell$ and $Y_\ell \subseteq W^\ell$ for $\ell = j, \dots, i$. First we let

$$Y_j = \{\nu \in W^j : |\{\tau(j) : \nu \lhd \tau \in W\}| < m_j\}, \quad X_j = \bigcup_{\nu \in Y_j} \{\tau(j) : \nu \lhd \tau \in W\}.$$

By its definition we have $|X_j| \le m_j \cdot |Y_j| \le m_j \cdot \prod\limits_{r<j} k_r$. Suppose that $i \le \ell < j$ and we have defined $Y_{\ell+1} \subseteq W^{\ell+1}$ already. Let

$$Y_\ell = \{\nu \in W^\ell : |\{\tau(\ell) : \nu \lhd \tau \in W^{\ell+1} \setminus Y_{\ell+1}\}| < m_\ell\}, \qquad \text{and}$$

$$X_\ell = \bigcup_{\nu \in Y_\ell} \{\tau(\ell) : \nu \lhd \tau \in W^{\ell+1} \setminus Y_{\ell+1}\}.$$

Note that $|X_\ell| \le m_\ell \cdot |Y_\ell| \le m_\ell \cdot \prod\limits_{r<\ell} k_r$.

Now look at the sets $X_i, \dots, X_j$. By our assumption we know that there is $\tau_0 \in W$ such that $(\forall \ell \in [i,j])(\tau_0(\ell) \notin X_\ell)$. This implies that $\langle\rangle \notin Y_i$. [Why? If $\langle\rangle \in Y_i$ then, as $\tau_0(i) \notin X_i$, we have $\langle \tau_0(i) \rangle \in Y_{i+1}$. Suppose have already shown that $\langle \tau_0(i), \dots, \tau_0(\ell) \rangle \in Y_{\ell+1}$, $i \le \ell < j-1$. Since $\tau_0(\ell+1) \notin X_{\ell+1}$ we conclude $\langle \tau_0(i), \dots, \tau_0(\ell), \tau_0(\ell+1) \rangle \in Y_{\ell+2}$. Thus, by induction, $\langle \tau_0(i), \dots, \tau_0(j-1) \rangle \in Y_j$

and $\tau_0(j) \in X_j$, a contradiction.]
Now we define the set $S \subseteq W$. We do this choosing inductively a finite tree $S^* \subseteq \bigcup\limits_{\ell=i}^{j} \prod\limits_{r=i}^{\ell} k_r$ in which maximal nodes will be elements of W. First we declare that $\langle\rangle \in S^*$ and since $\langle\rangle \notin Y_i$ we may choose a set $S_i^{\langle\rangle} \subseteq \{\tau(i) : \tau \in W^{i+1} \setminus Y_{i+1}\}$ of size m_i. We declare that $\{\langle z\rangle : z \in S_i^{\langle\rangle}\} \subseteq S^*$. Note that $\langle z\rangle \in W^{i+1} \setminus Y_{i+1}$ for $z \in S_i^{\langle\rangle}$. Suppose that we have decided that a sequence $\nu \in \prod\limits_{r=i}^{\ell} k_r$ is in S, $i \leq \ell < j-1$ and we know that $\nu \in W^{\ell+1} \setminus Y_{\ell+1}$. By the definition of $Y_{\ell+1}$ we may choose a set $S^\nu_{\ell+1} \subseteq \{\tau(\ell+1) : \nu \lhd \tau \in W^{\ell+2} \setminus Y_{\ell+2}\}$ of size $m_{\ell+1}$. We declare that all the sequences $\nu^\frown\langle z\rangle$ for $z \in S^\nu_{\ell+1}$ are in S^*. Note that we are sure that $\nu^\frown\langle z\rangle \in W^{\ell+2} \setminus Y_{\ell+2}$ (for $z \in S^\nu_{\ell+1}$). Finally, having decided that a sequence $\nu \in W^j \setminus Y_j$ is in S^* we choose a set $S^\nu_j \subseteq \{\tau(j) : \nu \lhd \tau \in W\}$ of size m_j and we declare $\nu^\frown\langle z\rangle \in S^*$ for $z \in S^\nu_j$. Immediately by the construction of S^* we see that the set $S = S^* \cap \prod\limits_{\ell=i}^{j} k_\ell$ is as required.
Define:

$$u_0 \stackrel{\text{def}}{=} \left\{\rho \in 2^{[n_i,n_{j+1})} : \frac{|\{\tau \in S : (\rho, f^{i,j}_\tau(\rho)) \in T^{[i-1]}_j\}|}{|S|} \geq \frac{1}{8}\right\},$$

$$u_1 \stackrel{\text{def}}{=} \left\{\rho \in 2^{[n_i,n_{j+1})} : \frac{|\{\sigma \in 2^{[n_{i-1},n_j)} : (\rho, \sigma) \in T^{[i-1]}_j\}|}{2^{n_j-n_{i-1}}} \leq \frac{7}{8}\right\},$$

$$u_2 \stackrel{\text{def}}{=} \left\{\rho \in 2^{[n_i,n_{j+1})} : (\exists \sigma \in 2^{[n_{i-1},n_j)}) \left(\frac{7}{2^{n_j-n_{i-1}}} < \frac{|\{\tau \in S : f^{i,j}_\tau(\rho) = \sigma\}|}{|S|}\right)\right\}.$$

Since $S \subseteq W$, by Fubini theorem, we have that

$$\frac{|u_0|}{2^{n_{j+1}-n_i}} < \frac{1}{8}.$$

Now look at the assumption $(\otimes^i_j)$ on T: it implies that, by Fubini theorem once again,

$$\frac{|u_1|}{2^{n_{j+1}-n_i}} \leq \frac{1}{8}.$$

Finally, by 2.4.1, we know that

$$\frac{|u_2|}{2^{n_{j+1}-n_i}} \leq \frac{1}{2}.$$

Consequently we find a sequence $\rho \in 2^{[n_i,n_{j+1})} \setminus (u_0 \cup u_1 \cup u_2)$. Since $\rho \notin u_0 \cup u_1$ we know that in the sequence

$$\langle f^{i,j}_\tau(\rho) : \tau \in S\rangle$$

less than $\frac{1}{8} \cdot 2^{n_j-n_{i-1}}$ many values (from $2^{[n_{i-1},n_j)}$) appear more than $\frac{7}{8} \cdot |S|$ times. This implies that there is one value $\sigma \in 2^{[n_{i-1},n_j)}$ which appears in this sequence more than $\frac{7}{2^{n_j-n_{i-1}}} \cdot |S|$ times and therefore $\rho \in u_2$, a contradiction finishing the proof of the claim.

Now we may prove the theorem.

(1) Assume that $\mathbf{V}\cap\omega^\omega$ is dominating in $\mathbf{V}^*\cap\omega^\omega$. Let $\langle n_i,m_i,k_i : i<\omega\rangle$ be the Φ–constructor and let $F:\prod\limits_{i\in\omega} k_i\times 2^\omega\longrightarrow 2^\omega$ be as defined above. Suppose $\eta\in\prod\limits_{i\in\omega} k_i$.
By Claim 2.4.3 we find a tree $T\subseteq 2^{<\omega}\times 2^{<\omega}$ from $\mathbf{V}$ satisfying the demands (i) and (ii) of 2.4.3. Let $\varphi\in\omega^\omega\cap\mathbf{V}^*$ be such that for each $i\in\omega$

$$i<\varphi(i)\quad\text{and}\quad \frac{|\{\rho\in 2^{[n_i,n_{\varphi(i)+1})}:(\rho,f^{i,\varphi(i)}_{\eta\restriction[i,\varphi(i)]}(\rho))\in T^{[i-1]}_{\varphi(i)}\}|}{2^{n_{\varphi(i)+1}-n_i}}<\frac{1}{64}.$$

Since $\mathbf{V}\cap\omega^\omega$ is dominating in $\mathbf{V}^*\cap\omega^\omega$ we find an increasing sequence of integers $\langle i_m : m\in\omega\rangle\in\mathbf{V}$ such that

($\otimes$) $\quad \frac{63}{64}\leq\frac{|T^{[i_0-1]}_j|}{2^{n_{j+1}+n_j}}$ for each $i_0<j<\omega$,

($\otimes^+$) for each $m\in\omega$

$$\frac{|\{\rho\in 2^{[n_{i_m},n_{i_{m+1}})}:(\rho,f^{i_m,i_{m+1}-1}_{\eta\restriction[i_m,i_{m+1})}(\rho))\in T^{[i_m-1]}_{i_{m+1}-1}\}|}{2^{n_{i_{m+1}}-n_{i_m}}}<\frac{1}{64}.$$

[Note that to get ($\otimes^+$) it is enough to require $\varphi(i_m)<i_{m+1}$ for each $m\in\omega$, what is easy to get as $\mathbf{V}\cap\omega^\omega$ is dominating.]
Now we construct, in $\mathbf{V}$, a sequence $\langle X_\ell : \ell<\omega\rangle$.
Fix $m\in\omega$ for a moment. Note that ($\otimes$) implies $(\otimes^{i_m}_{i_{m+1}-1})$ of 2.4.4. Let

$$W_m=\left\{\tau\in\prod_{\ell=i_m}^{i_{m+1}-1}k_\ell:\frac{|\{\rho\in 2^{[n_{i_m},n_{i_{m+1}})}:(\rho,f^{i_m,i_{m+1}-1}_\tau(\rho))\in T^{[i_m-1]}_{i_{m+1}-1}\}|}{2^{n_{i_{m+1}}-n_{i_m}}}<\frac{1}{64}\right\}.$$

It follows from 2.4.4 that there are sets $X_{i_m}\subseteq k_{i_m},\dots,X_{i_{m+1}-1}\subseteq k_{i_{m+1}-1}$ such that

(α) $\quad |X_\ell|\leq m_\ell\cdot\prod\limits_{r<\ell}k_r$

(β) $\quad (\forall\tau\in W_m)(\exists\ell\in[i_m,i_{m+1}))(\tau(\ell)\in X_\ell)$.

But now we easily finish notifying that ($\otimes^+$) implies that

$$(\forall m\in\omega)(\eta\restriction[i_m,i_{m+1})\in W_m).$$

(2) We repeat the arguments from the first case, but now we cannot require ($\otimes^+$). Still, as $\mathbf{V}\cap\omega^\omega$ is unbounded in $\mathbf{V}^*\cap\omega^\omega$ we may demand that the sequence $\langle i_m : m\in\omega\rangle\in\mathbf{V}$ satisfies ($\otimes$) and

($\otimes^-$) for infinitely many $m\in\omega$

$$\frac{|\{\rho\in 2^{[n_{i_m},n_{i_{m+1}})}:(\rho,f^{i_m,i_{m+1}-1}_{\eta\restriction[i_m,i_{m+1})}(\rho))\in T^{[i_m-1]}_{i_{m+1}-1}\}|}{2^{n_{i_{m+1}}-n_{i_m}}}<\frac{1}{64}.$$

Then, defining W_m as above, we will have

$$(\exists^\infty m\in\omega)(\eta\restriction[i_m,i_{m+1})\in W_m),$$

and this is enough to get the conclusion of (2). $\square$

Corollary 2.5. *Suppose that $\mathbf{V}\subseteq\mathbf{V}^*$ are universes of Set Theory such that*
if r is a random real over $\mathbf{V}^*$
then in $\mathbf{V}^*[r]$ there is no random real over $\mathbf{V}[r]$.

Let $H \in \omega^\omega \cap \mathbf{V}$ be an increasing function. Then:

$$\mathbf{V}^* \models (\forall f \in \prod_{\ell\in\omega} H(\ell))(\exists g \in \prod_{\ell\in\omega} H(\ell) \cap \mathbf{V})(\exists^\infty \ell \in \omega)(g(\ell) = f(\ell)).$$

Proof Define inductively a sequence $\langle n_i, m_i, x_i, y_i, k_i : i \in \omega\rangle \in \mathbf{V}$:

$$n_0 = 2, \quad m_0 = 64, \quad x_0 = 64, \quad y_0 = 64 + \prod_{\ell<64} H(\ell), \quad k_0 = 64 \cdot y_0,$$

$$n_{i+1} = n_i \cdot (k_i + 1), \quad m_{i+1} = \big(\prod_{j\le i} m_j\big) \cdot 2^{3(n_{i+1}+i+1)}, \quad x_{i+1} = x_i + m_{i+1} \cdot \prod_{j\le i} k_j,$$

$$y_{i+1} = y_i + x_{i+1} + \prod_{\ell\in[x_i,x_{i+1})} H(\ell), \quad k_{i+1} = y_{i+1} \cdot \big(m_{i+1} \cdot \prod_{j\le i} k_j\big).$$

Note that $y_{i+1} - y_i > x_{i+1} > m_{i+1} \cdot \prod\limits_{j\le i} k_j - m_i \cdot \prod\limits_{j<i} k_j$. Consequently we may choose a strictly increasing function $\Phi \in \omega^\omega \cap \mathbf{V}$ such that $(\forall i \in \omega)(\Phi(m_i \cdot \prod\limits_{j<i} k_j) = y_i)$. Now look at the definition of the sequence $\langle n_i, m_i, k_i : i \in \omega\rangle$ – clearly it is the Φ–constructor.

For $i \in \omega$ we have $\prod\limits_{\ell\in[x_{i-1},x_i)} H(\ell) \le k_i$ (we let $x_{-1} = 0$ here). So we may take a one–to–one function $\pi_i : \prod\limits_{\ell\in[x_{i-1},x_i)} H(\ell) \longrightarrow k_i$.

Now suppose $f \in \prod\limits_{\ell\in\omega} H(\ell) \cap \mathbf{V}^*$. Define $\eta \in \prod\limits_{i\in\omega} k_i \cap \mathbf{V}^*$ by

$$(\forall i \in \omega)(\eta(i) = \pi_i(f{\restriction}[x_{i-1}, x_i))).$$

By 2.3(2) we find a sequence $\langle X_\ell : \ell \in \omega\rangle \in \mathbf{V}$ satisfying 2.3(2)(a),(b) (for our η). Using the sequence $\langle X_\ell : \ell \in \omega\rangle$ (and working in $\mathbf{V}$) we define a function $g \in \prod\limits_{r\in\omega} H(r) \cap \mathbf{V}$. Fix $\ell \in \omega$ and look at the set

$$Y_\ell \stackrel{\text{def}}{=} \left\{\tau \in \prod_{r\in[x_{\ell-1},x_\ell)} H(r) : \pi_\ell(\tau) \in X_\ell\right\}.$$

Since $|Y_\ell| \le m_\ell \cdot \prod\limits_{j<\ell} k_j = x_\ell - x_{\ell-1}$, we find $\sigma_\ell \in \prod\limits_{r\in[x_{\ell-1},x_\ell)} H(r)$ such that

$$(\forall \tau \in Y_\ell)(\exists r \in [x_{\ell-1}, x_\ell))(\sigma_\ell(r) = \tau(r)).$$

Next let $g \in \prod\limits_{r\in\omega} H(r) \cap \mathbf{V}$ be such that $g{\restriction}[x_{\ell-1}, x_\ell) = \sigma_\ell$ (for $\ell \in \omega$). We finish noting that if $\eta(\ell) \in X_\ell$ then $f{\restriction}[x_{\ell-1}, x_\ell) \in Y_\ell$ and therefore for some $r \in [x_{\ell-1}, x_\ell)$ we have $g(r) = f(r)$. $\square$

3. $\mathsf{cov}^*(\mathcal{N})$ and other cardinal invariants

Results of the previous section allow us to compare $\mathsf{cov}^*(\mathcal{N})$ to other cardinal invariants.

We will need several definitions. Let $f, g \in \omega^\omega$ be two nondecreasing functions such that $0 < g(n) < f(n)$ for every n. Let $S_{f,g} = \prod_n [f(n)]^{g(n)}$ and $S^*_{f,g} = \prod_n [f(n)]^{g(n)} \times [\omega]^\omega$. Define relations $R^\forall_{f,g}, R^\exists_{f,g}$ as

$$\eta R^\exists_{f,g} S \iff \exists^\infty n\ \eta(n) \in S(n)$$

$$\eta R^{\forall}_{f,g} S \iff \forall^\infty n\ \eta(n) \in S(n)$$

for $\eta \in \prod_n f(n)$ and $S \in S_{f,g}$. In case when $g(n) = 1$ for all n we will drop subscript g and define

$$\eta_0 R^{\exists}_f \eta_1 \iff \exists^\infty n\ \eta_0(n) = \eta_1(n)$$

for $\eta_0, \eta_1 \in S_f$. The dual relation $R^{\forall}_f$ is not very interesting, so we consider the following weaker relations $R^{**}_{f,g}$ and R^{**}_f defined as

$$\eta R^{**}_{f,g}(S, K) \iff \forall^\infty n\ \exists m \in [k_n, k_{n+1})\ \eta(m) \in S(m),$$

for $\eta \in S_f$, $S \in S_{f,g}$ and $K = \{k_0 < k_1 < \dots\} \in [\omega]^\omega$. Finally define for a relation $R \subseteq A \times B$,

$$\mathfrak{b}(R) = \min\{|X| : X \subseteq A\ \&\ \forall y \in B\ \exists x \in X\ \neg x R y\}$$
$$\mathfrak{d}(R) = \min\{|Y| : Y \subseteq B\ \&\ \forall x \in A\ \exists y \in Y\ x R y\}.$$

For various independence results and techniques connected with these invariants see [10].

Using this terminology we can express the results of the previous section as follows.

Theorem 3.1. *There are $f, g \in \omega^\omega$ such that $\mathsf{cov}^*(\mathcal{N}) \geq \mathfrak{d}(R^{\exists}_{f,g})$. If $\mathsf{cov}^*(\mathcal{N}) \geq \mathfrak{d}$ then $\mathsf{cov}^*(\mathcal{N}) \geq \mathfrak{d}(R^{**}_{f,g})$.*

Similarly, $\mathsf{non}^*(\mathcal{N}) \leq \mathfrak{b}(R^{\exists}_{f,g})$, and if $\mathsf{non}^*(\mathcal{N}) \leq \mathfrak{b}$ then $\mathsf{non}^*(\mathcal{N}) \leq \mathfrak{b}(R^{**}_{f,g})$.

Proof This is a simple reformulation of Theorem 2.3. Fix an increasing function $\Phi \in \omega^\omega$. Let M be a model of size $\mathsf{cov}^*(\mathcal{N})$ containing a witness for $\mathsf{cov}^*(\mathcal{N})$, and containing Φ. Since $\mathsf{cov}^*(\mathcal{N}) \geq \mathfrak{b}$ we can assume that $M \cap \omega^\omega$ is an unbounded family. Let $\{n_i, m_i, k_i : i \in \omega\} \in M$ be a Φ-constructor. Define $f(n) = k_n$ and $g(n) = m_n \prod_{i<n} f(i)$. By 2.3,

$$\forall \eta \in S_f\ \exists S \in S_{f,g} \cap M\ \exists^\infty n\ \eta(n) \in S(n).$$

Thus $\mathfrak{d}(R^{\exists}_{f,g}) \leq |M| = \mathsf{cov}^*(\mathcal{N})$. Remaining parts of the theorem are proved in the same way by using 2.4. It is not very hard to see that by simple diagonalization we can show that for many triples (h, f, g) we have $\mathfrak{b}(R^{\exists}_h) \leq \mathfrak{b}(R^{\exists}_{f,g})$ and $\mathfrak{d}(R^{\exists}_h) \geq \mathfrak{d}(R^{\exists}_{f,g})$

Definition 3.2. Let

$$\kappa^* = \sup\left\{\mathfrak{d}(R^{\exists}_f) : f \in (\omega \setminus \{0\})^\omega\right\} \quad \text{and} \quad \lambda^* = \inf\left\{\mathfrak{b}(R^{\exists}_f) : f \in (\omega \setminus \{0\})^\omega\right\}.$$

Theorem 3.3. $\mathsf{cov}^*(\mathcal{N}) \geq \kappa^*$ *and* $\mathsf{non}^*(\mathcal{N}) \leq \lambda^*$.

Proof Let $f \in (\omega \setminus \{0\})^\omega$. We may assume that f is strictly increasing. Take a family $\mathcal{A} \subseteq \mathcal{N}_2$ realizing the minimal cardinality in the definition of $\mathsf{cov}^*(\mathcal{N})$ and take an unbounded family $\mathcal{F} \subseteq \omega^\omega$ of size $\mathfrak{b}$ (remember $\mathfrak{b} \leq \mathsf{cov}^*(\mathcal{N})$). Let $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be an elementary submodel of size $\mathsf{cov}^*(\mathcal{N})$ containing all members of $\mathcal{A}$ and $\mathcal{F}$ and such that $f \in N$. Now apply 2.5 to $N \subseteq \mathbf{V}$. Note that if r is a random real over $\mathbf{V}$ then in $\mathbf{V}[r]$ there is no random real over $N[r]$ (as $\mathcal{A} \subseteq N$). Moreover $N \cap \omega^\omega$ is unbounded in $\mathbf{V} \cap \omega^\omega$ (as $\mathcal{F} \subseteq N$). Consequently (in $\mathbf{V}$) we have

$$(\forall h \in \prod_{n \in \omega} f(n))(\exists g \in \prod_{n \in \omega} f(n) \cap N)(\exists^\infty n \in \omega)(g(n) = h(n)),$$

showing that $\mathfrak{d}(R^{\exists}_f) \leq |N| = \mathsf{cov}^*(\mathcal{N})$. $\square$

Definition 3.4. Suppose that $X \subseteq 2^\omega$.

(1) $X \in \mathcal{SN}$ (strong measure zero) if for every meager set $F \subseteq 2^\omega$, $X+F \neq 2^\omega$,
(2) $X \in \mathcal{SM}$ (strongly meager) if for every null set $H \subseteq 2^\omega$, $X+H \neq 2^\omega$,

Lemma 3.5. *$\lambda^* = \mathsf{non}(\mathcal{SN})$ and $\kappa^* \geq \mathsf{non}(\mathcal{SM})$.*

Proof The first equality was proved by Miller (see [8] or [3], 8.1.14).

Suppose that a family $\mathcal{F} \subseteq \prod_{n\in\omega} f(n)$ exemplifies $\mathfrak{d}(R^{\exists}_f)$. Work in the space $X = \prod_{n\in\omega} f(n)$ (for sufficiently big f) equipped with the standard product measure. Consider the set $G = \{x \in X : \exists^\infty n\ x(n) = 0\}$. It is easy to see that G is a null set and $\mathcal{F} + G = X$. Thus $\mathcal{F} \notin \mathcal{SM}$ in X (which easily translates to 2^ω). □

Corollary 3.6. $\mathsf{cov}^*(\mathcal{N}) \geq \max\{\mathfrak{b}, \mathsf{non}(\mathcal{SM})\}$ *and* $\mathsf{non}^*(\mathcal{N}) \leq \min\{\mathfrak{d}, \mathsf{non}(\mathcal{SN})\}$. □

Lemma 3.7. *If $\mathsf{cov}^*(\mathcal{N}) \geq \mathfrak{d}$ then $\mathsf{cov}^*(\mathcal{N}) = \mathsf{non}(\mathcal{M})$. If $\mathsf{non}^*(\mathcal{N}) \leq \mathfrak{b}$ then $\mathsf{non}^*(\mathcal{N}) = \mathsf{cov}(\mathcal{M})$.*

Proof We will prove only the first assertion. The other one is proved by the dual argument.

It is well known (see [3], 2.4.7, 2.4.1) that

$$\mathsf{non}(\mathcal{M}) = \min\{|F| : F \subseteq \omega^\omega \ \&\ \forall g \in \omega^\omega\ \exists f \in F\ \exists^\infty n\ f(n) = g(n)\}$$

and

$$\mathsf{cov}(\mathcal{M}) = \min\{|F| : F \subseteq \omega^\omega \ \&\ \forall g \in \omega^\omega\ \exists f \in F\ \forall^\infty n\ f(n) \neq g(n)\}$$

Let $F \subseteq \omega^\omega$ be a dominating family of size $\mathfrak{d}$. For each $f \in F$ choose a witness $X_f \subseteq S_f$ of size $\mathfrak{d}(R^{\exists}_f)$. Let $X = \bigcup_{f\in F} X_f$. It is clear that $|X| = \max\{\mathfrak{d}, \kappa^*\} \leq \mathsf{cov}^*(\mathcal{N})$ and

$$\forall g \in \omega^\omega\ \exists f \in F\ \exists x_f \in X_f\ \exists^\infty n\ g(n) = x_f(n).$$

Thus, $\mathsf{non}(\mathcal{M}) \leq \mathsf{cov}^*(\mathcal{N})$. To see that $\mathsf{cov}^*(\mathcal{N}) \leq \mathsf{non}(\mathcal{M})$ in we need the following lemma:

Lemma 3.8. $\mathsf{cov}^\star(\mathcal{N}) \leq \mathsf{non}(\mathcal{M})$ *and* $\mathsf{non}^*(\mathcal{N}) \geq \mathsf{cov}(\mathcal{M})$.

Proof We have the following $\mathsf{cov}^\star(\mathcal{N}) = \mathsf{cov}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}} \leq \mathsf{non}(\mathcal{M})^{\mathbf{V}^{\mathbf{B}}} = \mathsf{non}(\mathcal{M})$. The first equality is by 1.2, the second is well known, and for the third one see [3] or [4]. □

Corollary 3.9. *There is no proper forcing notion $\mathcal{P}$ such that*

(1) *is proper ω^ω–bounding,*
(2) *makes ground reals meager and*
(3) *does not add a $\mathbf{B}$–name for a random real over $\mathbf{V}^{\mathbf{B}}$.*

4. Adding a $(2,0)$–large set.

Theorem 4.1. *Assume that $V \subseteq \mathbf{V}^*$ are universes of Set Theory. Let $h \in \omega^\omega \cap \mathbf{V}$ be a strictly increasing function. Suppose that*

$$\mathbf{V}^* \models (\exists \eta \in \prod_{n\in\omega} h(n))(\forall \rho \in \prod_{n\in\omega} \cap \mathbf{V})(\forall^\infty n \in \omega)(\rho(n) \neq \eta(n)).$$

Then there is a set $X \in [\omega]^\omega \cap \mathbf{V}^*$ such that

$$\mathbf{V}^* \models (\forall f \in \omega^\omega \cap \mathbf{V})\big(\ (\forall n \in \omega)(n < f(n)) \quad \Rightarrow \quad |\{m \in X : f(m) \in X\}| < \omega\big)$$

(so in particular the set $\omega \setminus X$ is $(2,0)$–large over $\mathbf{V}$*).*

PROOF Let $\langle n_i : i \in \omega\rangle$ be defined by

$$n_0 = 0, \qquad n_{i+1} = n_i + \prod_{k \le n_i} h(k).$$

Let $H : \bigcup_{i\in\omega} \prod_{k\le n_i} h(k) \xrightarrow{1-1} \omega$ be a bijection such that for each $i \in \omega$

$$H\left[\prod_{k\le n_i} h(k)\right] = [n_i, n_{i+1}).$$

For a function $f \in \omega^\omega$ define $\rho_f \in \prod_{k\in\omega} h(k)$ by

$$\rho_f(k) = \begin{cases} H^{-1}(f(k))(k) & \text{if } n_i \le k < n_{i+1} \text{ and } n_{i+1} \le f(k) \\ 0 & \text{otherwise.} \end{cases}$$

Note that the mapping $f \mapsto \rho_f$ is coded in $\mathbf{V}$.
Let $X = \{H(\eta{\restriction}n_i) : i \in \omega\}$ (so it is an infinite subset of ω from $\mathbf{V}^*$). Suppose that $f \in \omega^\omega \cap \mathbf{V}$ is such that $(\forall n \in \omega)(n < f(n))$. Look at ρ_f. We know that $\rho_f \in \prod_{k\in\omega} h(k) \cap \mathbf{V}$. So, by the assumptions on η, we find $i_0 \in \omega$ such that

$$(\forall i \ge i_0)(\eta(i) \ne \rho_f(i)).$$

Suppose now that $i \ge i_0$ and $f(H(\eta{\restriction}n_i)) \in X$. Then $f(H(\eta{\restriction}n_i)) = H(\eta{\restriction}n_j)$ for some $j > i$. But this means that

$$\rho_f(H(\eta{\restriction}n_i)) = H^{-1}(H(\eta{\restriction}n_j))(H(\eta{\restriction}n_i)) = \eta(H(\eta{\restriction}n_i)),$$

a contradiction with the choice of i_0. □

Definition 4.2. Let $\mathfrak{d}(R_0^\exists)$ be the minimal size of a family $\mathcal{K}$ of partitions $\langle K_n : n \in \omega\rangle$ of ω into sets of size ≥ 2 such that for every infinite co-infinite subset X of ω we have

$$(\exists \langle K_n : n \in \omega\rangle \in \mathcal{K})(\exists^\infty n \in \omega)(K_n \cap X = \emptyset).$$

In [11, 3.1] we remarked that $\mathfrak{b} \le \mathfrak{d}(R_0^\exists) \le \mathsf{non}(\mathcal{M})$. Now we may add:

Corollary 4.3. $\kappa^* \le \mathfrak{d}(R_0^\exists)$.

PROOF It follows from 4.1 (compare the proof of 3.6); remember 1.4. □

References

[1] Tomek Bartoszyński, *Additivity of measure implies additivity of category*, Transactions of the American Mathematical Society **281** (1984), no. 1, 209–213.
[2] ———, *Combinatorial aspects of measure and category*, Fundamenta Mathematicae **127** (1987), no. 3, 225–239.
[3] Tomek Bartoszyński and Haim Judah, *Set Theory: the structure of the real line*, A K Peters, Wellesley, Massachusetts, 1995.
[4] Tomek Bartoszyński, Andrzej Rosłanowski and Saharon Shelah, *Adding one random real*, The Journal of Symbolic Logic, **61** (1996), 80–90
[5] David H. Fremlin, *Cichoń's diagram*, 1983/1984, no. 5, p. 13, presented at the Séminaire Initiation à l'Analyse, G. Choquet, M. Rogalski, J. Saint Raymond, at the Université Pierre et Marie Curie, Paris, 23e année.
[6] Thomas Jech, *Set Theory*, Academic Press, New York, 1978.

[7] Adam Krawczyk, *Dominating reals add random reals*, 1985, unpublished notes.
[8] Arnold W. Miller, *Some properties of measure and category*, Transactions of the American Mathematical Society **266** (1981), no. 1, 93–114.
[9] Janusz Pawlikowski, *Why Solovay real produces Cohen real*, The Journal of Symbolic Logic **51** (1986), no. 4, 957–968.
[10] Andrzej Rosłanowski and Saharon Shelah, *Norms on possibilities I: forcing with trees and creatures*, Memoirs of AMS, to appear (`http://math.rutgers.edu/~shelah/`
[11] Andrzej Rosłanowski and Saharon Shelah, *Localizations of infinite subsets of* ω, Archive for Mathematical Logic, **35** (1996), 315–339.
[12] Saharon Shelah, *Proper and Improper forcing*, Perspectives in Logic, Springer Verlag, 1998

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