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Superdestructibility: A Dual to Laver's Indestructibility

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Abstract. After small forcing, any $<\kappa$-closed forcing will destroy the supercompactness and even the strong compactness of κ.

In a delightful argument, Laver [L78] proved that any supercompact cardinal κ can be made indestructible by $<\kappa$-directed closed forcing. This indestructibility, however, is evidently not itself indestructible, for it is always ruined by small forcing: in [H96] the first author recently proved that small forcing makes any cardinal superdestructible; that is, any further $<\kappa$-closed forcing which adds a subset to κ will destroy the measurability, even the weak compactness, of κ. What is more, this property holds higher up: after small forcing, any further $<\kappa$-closed forcing which adds a subset to λ will destroy the λ-supercompactness of κ, provided λ is not too large (his proof needed that $\lambda < \aleph_{\kappa+\delta}$, where the small forcing is $<\delta$-distributive). In this paper, we happily remove this limitation on λ, and show that after small forcing, the supercompactness of κ is destroyed by any $<\kappa$-closed forcing. Indeed, we will show that even the strong compactness of κ is destroyed. By doing so we answer the questions asked at the conclusion of [H96], and obtain the following attractive complement to Laver indestructibility:

Main Theorem. *After small forcing, any $<\kappa$-closed forcing will destroy the supercompactness and even the strong compactness of κ.*

We will provide two arguments. The first, similar to but generalizing the Superdestruction Theorem of [H96], will show that supercompactness is destroyed; the second, by a different technique, will show fully that strong compactness is destroyed. Both arguments will rely fundamentally on the Key Lemma, below, which was proved in [H96]. Define that a set or sequence is *fresh* over V when it is not in V but every initial segment of it is in V.

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Key Lemma. *Assume that $|\mathbb{P}| = \beta$, that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leq\beta$-closed, and that $\mathrm{cof}(\lambda) > \beta$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ adds no fresh subsets of λ, and no fresh λ-sequences.*

While in [H96] it is proved only that no fresh sets are added, the following simple argument shows that no fresh sequences can be added: given a sequence in δ^λ, code it in the natural way with a binary sequence of length $\delta\lambda$, by using λ many blocks of length δ, each with one 1. The binary sequence corresponds to a subset of the ordinal $\delta\lambda$, which, since $\mathrm{cof}(\delta\lambda) = \mathrm{cof}(\lambda)$, cannot be fresh. Thus, the original λ-sequence cannot be fresh.

Let us give now the first argument. We will use the notion of a θ-club to extend the inductive proof of the Superdestruction Theorem [H96] to all values of λ.

Theorem. *After small forcing, any $<\kappa$-closed forcing which adds a subset to λ will destroy the λ-supercompactness of κ.*

Proof: Suppose that $|\mathbb{P}| < \kappa$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $<\kappa$-closed. Suppose that $g * G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is V-generic, and that $\mathbb{Q} = \dot{\mathbb{Q}}_g$ adds a new subset $A \subseteq \lambda$, with λ minimal, so that $A \in V[g][G]$ but $A \notin V[g]$. By the closure of $\mathbb{Q}$, we know that $\mathrm{cof}(\lambda) \geq \kappa$. Suppose, towards a contradiction, that κ is λ-supercompact in $V[g][G]$. Let $P_\kappa\lambda$ denote $(P_\kappa\lambda)^{V[g][G]}$, which is also $(P_\kappa\lambda)^{V[g]}$.

Lemma. *Every normal fine measure on $P_\kappa\lambda$ in $V[g][G]$ concentrates on $(P_\kappa\lambda)^V$.*

Proof: Let us begin with some definitions. Fix a regular cardinal θ such that $|\mathbb{P}| < \theta < \kappa$. A set $C \subseteq P_\kappa\lambda$ is *unbounded* iff for every $\sigma \in P_\kappa\lambda$ there is $\tau \in C$ such that $\sigma \subseteq \tau$. A set $D \subseteq P_\kappa\lambda$ is *θ-directed* iff whenever $B \subseteq D$ and $|B| < \theta$ then there is some $\tau \in D$ such that $\sigma \subseteq \tau$ for every $\sigma \in B$. The set C is *θ-closed* iff every θ-directed $D \subseteq C$ with $|D| < \kappa$ has $\cup D \in C$. Finally, C is a *θ-club* iff C is both θ-closed and unbounded.

Claim. *A normal fine measure on $P_\kappa\lambda$ contains every θ-club.*

Proof: Work in any model $\bar{V}$. Suppose that C is a θ-club in $P_\kappa\lambda$ and that μ is a normal fine measure on $P_\kappa\lambda$. Let $j : \bar{V} \to M$ be the ultrapower by μ. It is well known that $j\,"\,\lambda$ is a seed for μ in the sense that $X \in \mu \leftrightarrow j\,"\,\lambda \in j(X)$ for $X \subseteq P_\kappa\lambda$. By elementarity $j(C)$ is a θ-club in M and $j\,"\,C \subseteq j(C)$. (We know $j\,"\,C \in M$ because M is closed under $\lambda^{<\kappa}$ sequences in $\bar{V}$.) Also, it is easy to check that $j\,"\,C$ is θ-directed. Thus, by the definition of θ-club, we know $\cup(j\,"\,C) \in j(C)$. But

$$\cup(j\,"\,C) = \bigcup_{\sigma\in C} j(\sigma) = \bigcup_{\sigma\in C} (j\,"\,\sigma) = j\,"\,\lambda.$$

Thus, $j\,"\,\lambda \in j(C)$ and so $C \in \mu$. $\square_{\text{Claim}}$

Now let $C = (P_\kappa\lambda)^V$. We will show that C is a θ-club in $V[g][G]$. First, let us show that C is unbounded. If $\sigma \in P_\kappa\lambda$ in $V[g][G]$, then actually $\sigma \in V[g]$, and so $\sigma = \dot\sigma_g$ for some $\mathbb{P}$-name $\dot\sigma \in V$. We may assume that $[\![\, |\dot\sigma| < \check\kappa \,]\!] = 1$ and consequently $\sigma \subseteq \{\, \alpha \mid [\![\, \alpha \in \dot\sigma \,]\!] \neq 0 \,\} \in C$; so σ is covered as desired. To show that C is θ-closed, suppose in $V[g][G]$ that $D \subseteq C$ has size less than κ and is θ-directed. We have to show that $\cup D \in C$. It suffices to show that $\cup D \in V$ since $C = P_\kappa\lambda \cap V$. Since $\mathbb{Q}$ is $<\kappa$-closed, we know that $D \in V[g]$, and thus $D = \dot D_g$ for some name $\dot D \in V$. In V let $D_p = \{\, \sigma \in C \mid p \Vdash \check\sigma \in \dot D \,\}$. It follows that $D = \cup_{p \in g} D_p$. There must be some $p \in g$ such that D_p is $\subseteq$-cofinal in D; for if not, then for each $p \in g$ we may choose $\sigma_p \in D$ such that D_p contains no supersets of σ_p. Since D is θ-directed and $|g| < \theta$ there is some $\sigma \in D$ such that $\sigma_p \subseteq \sigma$ for all $p \in g$. But σ must be forced into D by some condition $p \in g$, so $\sigma \in D_p$ for some $p \in g$, contradicting the choice of σ_p. So we may fix some $p \in g$ such that D_p is $\subseteq$-cofinal in D. But in this case $\cup D_p = \cup D$ and since $D_p \in V$ we conclude $\cup D \in V$. Thus C is a θ-club in $V[g][G]$, and the lemma is proved. $\square_{\text{Lemma}}$

Let us now continue with the theorem. Since κ is λ-supercompact in $V[g][G]$ there must be an embedding $j : V[g][G] \to M[g][j(G)]$ which is the ultrapower by a normal fine measure μ on $P_\kappa\lambda$.

Lemma. $P(\lambda)^M = P(\lambda)^V$.

Proof: ($\supseteq$). By the previous lemma we know that $(P_\kappa\lambda)^V \in \mu$ and so $j\,"\,\lambda \in j((P_\kappa\lambda)^V) = (P_\kappa\lambda)^M$. Since M is transitive, it follows that $j\,"\,\lambda \in M$. And obtaining this fact was the only reason for proving the previous lemma. Now if $B \subseteq \lambda$ and $B \in V$ then $j(B) \in M$, and since B is constructible from $j(B)$ and $j\,"\,\lambda$ it follows that $B \in M$ as well.

($\subseteq$). Now we prove the converse. By induction we will show that $P(\delta)^M \subseteq V$ for all $\delta \leq \lambda$. Suppose that $B \subseteq \delta$ and $B \in M$ and every initial segment of B is in V. By the Key Lemma it follows that $B \in V$ unless $\mathrm{cof}(\delta) < \kappa$. So suppose $\mathrm{cof}(\delta) < \kappa$. By the closure of $\mathbb{Q}$ we know in this case that $B \in V[g]$ and so $B = \dot B_g$ for some name $\dot B \in V$. We may view $\dot B$ as a function from δ to the set of antichains of $\mathbb{P}$. Since $\dot B$ may be coded with a subset of δ, we know $\dot B \in M$ by the previous direction of this lemma. Thus, both B and $\dot B$ are in M and g is M-generic. Since $B = \dot B_g$ in $M[g]$ there is in M a condition $p \in g$ such that $p \Vdash \dot B = \check B$. That is, p decides every antichain of $\dot B$ in a way that makes it agree with B. Use p to decide $\dot B$ in V and conclude that $B \in V$. This completes the induction. $\square_{\text{Lemma}}$

Now we are nearly done. Consider again the new set $A \subseteq \lambda$ such that $A \in V[g][G]$ but $A \notin V[g]$. Since j is a λ-supercompact embedding, we know $A \in M[g][j(G)]$. Since the $j(G)$ forcing is $<j(\kappa)$-closed, we know $A \in M[g]$. Therefore $A = \dot{A}_g$ for some name $\dot{A} \in M$. Viewing $\dot{A}$ as a function from λ to the set of antichains in $\mathbb{P}$, we can code $\dot{A}$ with a subset of λ, and so by the last lemma we know $\dot{A} \in V$. Thus, $A = \dot{A}_g \in V[g]$, contradicting the choice of A. $\square_{\text{Theorem}}$

Corollary. *By first adding in the usual way a generic subset to β and then to λ, where $\operatorname{cof}(\lambda) > \beta$, one destroys all supercompact cardinals between β and λ.*

In fact, one does not even need to add them in the usual way. This is because the proof of the theorem does not really use the full $<\kappa$-closure of $\mathbb{Q}$. Rather, if $\mathbb{P}$ has size β, then we only need that $\mathbb{Q}$ is $\leq\beta$-closed and adds no new elements of $P_\kappa\lambda$. Thus, we have actually proved the following theorem.

Theorem. *After any forcing of size $\beta < \kappa$, any further $\leq\beta$-closed forcing which adds a subset to λ but no elements to $P_\kappa\lambda$ will destroy the λ-supercompactness of κ.*

This improvement is striking when β is small, having the consequence that after adding a Cohen real, any countably-closed forcing which adds a subset to some minimal λ destroys all supercompact cardinals up to λ.

Let us now give the second argument, which will improve the previous results with a different technique and establish fully that strong compactness is destroyed.

Theorem. *After small forcing, any $<\kappa$-closed forcing which adds a λ-sequence will destroy the λ-strong compactness of κ.*

Proof: Define that a cardinal κ is λ-*measurable* iff there is a κ-complete (non κ^+-complete) uniform measure on λ. Necessarily $\kappa \leq \operatorname{cof}(\lambda)$. This notion is studied in [K72].

Lemma. *Assume that $|\mathbb{P}| < \kappa \leq \lambda$, that $\dot{\mathbb{Q}}$ adds a new λ-sequence over $V^{\mathbb{P}}$, λ minimal, and that κ is λ-measurable in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ must add a fresh λ-sequence over V.*

Proof: This lemma is the heart of the proof. Assume the hypotheses of the lemma. So $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \dot{s}$ is a λ-sequence of ordinals not in $V^{\mathbb{P}}$, and $\dot{\mu}$ is a κ-complete uniform measure on λ. Without loss of generality, we may assume that $\Vdash_{\mathbb{P}}\dot{\mathbb{Q}}$ is a complete boolean algebra on an ordinal. Suppose now that $g * G$ is V-generic for $\mathbb{P} * \dot{\mathbb{Q}}$. Let $\mathbb{Q} = \dot{\mathbb{Q}}_g$, and $s = \dot{s}_{g*G}$.

In $V[g]$, let $T = \{\, u \in \text{ORD}^{<\lambda} \mid [\![\, \check{u} \subseteq \dot{s} \,]\!]^{\mathbb{Q}} \neq 0 \,\}$. Thus, under inclusion, T is a tree with λ many levels, and $\mathbb{Q}$ adds the λ-branch s. For $u \in T$, let $b_u = [\![\, u \subseteq \dot{s} \,]\!]^{\mathbb{Q}}$.

Thus, b_u is an ordinal. Let $I = \{\langle \ell(u), b_u \rangle \mid u \in T\}$, where $\ell(u)$ denotes the length of u, and define $\langle \alpha, b_u \rangle \lhd \langle \alpha', b_{u'} \rangle$ when $\alpha' < \alpha$ and $b_u \leq_{\mathbb{Q}} b_{u'}$. Since $u \supset v \leftrightarrow \langle \ell(u), b_u \rangle \lhd \langle \ell(v), b_v \rangle$ it follows that $\langle T, \supset \rangle \cong \langle I, \lhd \rangle$, and consequently I is also a tree, under the relation $\lhd$, with λ many levels. Furthermore, the α^{th} level of I consists of pairs of the form $\langle \alpha, \beta \rangle$. For $p \in \mathbb{P}$ let us define that $a \lhd_p b$ when $p \Vdash a \lhd b$. Thus, $\lhd = \cup_{p \in g} \lhd_p$.

In $V[g][G]$ let $b_\gamma = \langle \gamma, b_{s \restriction \gamma} \rangle$. Thus, $b_\gamma \in I$, and if $\gamma < \zeta$ then $b_\zeta \lhd b_\gamma$ and so there is some $r \in g$ such that $b_\zeta \lhd_r b_\gamma$. Since there are fewer than κ many such r, for each γ there must be an r which works for μ-almost every ζ. But then again, since there are relatively few r, it must be that there is some $r^* \in g$ which has this property for μ-almost every γ. So, fix $r^* \in g$ such that for μ-almost every γ, for μ-almost every ζ, we have $b_\zeta \lhd_{r^*} b_\gamma$. Fix also a condition $\langle p_0, q_0 \rangle \in g * G$ forcing r^* to have this property. Let $t = \langle b_\gamma \mid \gamma < \lambda \ \& \text{ for } \mu\text{-a.e. } \zeta,\ b_\zeta \lhd_{r^*} b_\gamma \rangle$. Thus, t is a partial function from λ to pairs of ordinals, and $\mathrm{dom}(t) \in \mu$. In particular, $\mathrm{dom}(t)$ is unbounded in λ.

We will argue that t is fresh over V. First, notice that $t \notin V[g]$ since in $V[g]$ knowing t we could read off the branch s. Thus, $t \notin V$.

Nevertheless, we will argue that every initial segment of t is in V. Suppose $\delta < \lambda$, and let $t_\delta = t \restriction \delta$. By the minimality of λ it follows that $t_\delta \in V[g]$, and so there is a $\mathbb{P}$-name $\dot{t}_\delta$ and a condition $\langle p_1, q_1 \rangle \in g * G$, stronger than $\langle p_0, q_0 \rangle$, forcing this name to work. Assume towards a contradiction that $t_\delta \notin V$, and that this is forced by p_1. Then, for each $r \in \mathbb{P}$ below p_1 we may choose $\gamma_r < \delta$ such that r does not decide $t(\gamma_r)$ (or whether γ_r is in the domain of t). But, nevertheless, for each r either for μ-almost every ζ, $b_\zeta \lhd_{r^*} b_{\gamma_r}$ or else for μ-almost every ζ, $b_\zeta \not\lhd_{r^*} b_{\gamma_r}$ (but not both). In the first case it follows that $t(\gamma_r) = b_{\gamma_r}$, and in the second it follows that $\gamma_r \notin \mathrm{dom}(t)$. Since there are relatively few r, by intersecting these sets of ζ we can find a single ζ which acts, with respect to the γ_r, exactly the way μ-almost every ζ acts. Fix such a ζ. Thus, for each r we have either $b_\zeta \lhd_{r^*} b_{\gamma_r}$, and consequently $t(\gamma_r) = b_{\gamma_r}$, or else $\gamma_r \notin \mathrm{dom}(t)$ (but not both). Notice that ζ and b_ζ are just some particular ordinals. Fix some condition $\langle p^*, q^* \rangle$ below $\langle p_1, q_1 \rangle$ forcing ζ and b_ζ to have the property we mention in the sentence before last. Now we will argue that this is a contradiction. Let $\gamma = \gamma_{p^*}$. There are two cases. First, it might happen that $b_\zeta \lhd_{r^*} \langle \gamma, \beta \rangle$ for some ordinal β. Such a situation can be observed in V. In this case, $\langle p^*, q^* \rangle$ forces $\beta = b_{s \restriction \gamma}$ and therefore, by the assumption on ζ, it also forces $t(\gamma) = \langle \gamma, \beta \rangle$. Since $\dot{t}_\delta$ is a $\mathbb{P}$-name, it follows that $p^* \Vdash \dot{t}_\delta(\check{\gamma}) = \langle \check{\gamma}, \check{\beta} \rangle$, contrary to the choice of $\gamma = \gamma_{p^*}$. Alternatively, in the second case, it may happen that $b_\zeta \not\triangleleft_{r^*} \langle \gamma, \beta \rangle$ for every β. In this case, by the assumption on ζ, it must be that $\langle p^*, q^* \rangle$ forces that $\gamma \notin \mathrm{dom}(t)$. Again, since $\dot{t}_\delta$ is a $\mathbb{P}$-name, it follows that $p^* \Vdash \gamma \notin \mathrm{dom}(\dot{t}_\delta)$, contrary again to the choice of $\gamma = \gamma_{p^*}$. Thus, in either case we reach a contradiction, and so we have proven that $\mathbb{P} * \dot{\mathbb{Q}}$ must add a fresh λ-sequence. $\square_{\text{Lemma}}$

Lemma. *If $\kappa \leq \mathrm{cof}(\lambda)$ and κ is λ-strongly compact, then κ is λ-measurable.*

Proof: Let $j : V \to M$ be the ultrapower map witnessing that κ is λ-strongly compact. By our assumption on $\mathrm{cof}(\lambda)$, it follows that $\sup j\,"\,\lambda < j(\lambda)$. Let $\alpha = (\sup j\,"\,\lambda) + \kappa$, and let μ be the measure germinated by the seed α. That is, $X \in \mu$ iff $\alpha \in j(X)$. Since $\alpha < j(\lambda)$ it follows that μ is a measure on λ. Since $j(\beta) < \alpha$ for all $\beta < \lambda$ it follows that μ is uniform. Since $\mathrm{cp}(j) = \kappa$ it follows that μ is κ-complete. For $\gamma < \kappa$, let $B_\gamma = \{\, \beta \mid \gamma < \mathrm{cof}(\beta) < \kappa \,\}$. Since $\mathrm{cof}(\alpha) = \kappa$ in M, it follows that $\alpha \in j(B_\gamma)$ and consequently $B_\gamma \in \mu$ for every $\gamma < \kappa$. Since $\cap_\gamma B_\gamma = \varnothing$, it follows that μ is not κ^+-complete, as desired. $\square_{\text{Lemma}}$

Remark. Ketonen [K72] has proved that if κ is λ-measurabile for every regular λ above κ, then κ is strongly compact. This cannot, however, be true level-by-level, since if $\kappa < \lambda$ are both measurable, with measures μ and ν, then $\mu \times \nu$ is a κ-complete, non-κ^+-complete, uniform measure on $\kappa \times \lambda$. Thus, in this situation, κ will be λ-measurable, even when it may not be even κ^+-strongly compact. But the previous lemma establishes that the direction we need does indeed hold level-by-level.

Let us now finish the proof of the theorem. Suppose that $V[g][G]$ is a forcing extension by $\mathbb{P} * \dot{\mathbb{Q}}$, where $|\mathbb{P}| < \kappa$ and $\mathbb{Q}$ is $<\kappa$-closed. Let λ be least such that $\mathbb{Q}$ adds a new λ-sequence not in $V[g]$. Necessarily, $\kappa \leq \lambda$ and λ is regular. By the Key Lemma $V[g][G]$ has no λ-sequences which are fresh over V. Thus, by the first lemma κ is not λ-measurable in $V[g][G]$. Therefore, by the second lemma, κ is not λ-strongly compact in $V[g][G]$. $\square_{\text{Theorem}}$

So the proof actually establishes that after small forcing of size $\beta < \kappa$, any $\leq\beta$-closed forcing which adds a new λ-sequence for some minimal λ, with $\lambda \geq \kappa$, will destroy the λ-measurability of κ. This subtlety about adding a λ-sequence as opposed to a *subset* of λ has the following intriguing consequence, which is connected with the possibilities of changing the cofinalities of very large cardinals.

Corollary. *Suppose that κ is λ-measurable. Then after forcing with $\mathbb{P}$ of size $\beta < \kappa$, any $\leq\beta$-closed $\mathbb{Q}$ which adds a λ-sequence, but no shorter sequences, must necessarily add subsets to λ.*

Proof: Such forcing will destroy the λ-measurability of κ. Hence, it must add subsets to λ. $\square_{\text{Corollary}}$

Bibliography

[H96] Joel David Hamkins, *Small Forcing Makes Any Cardinal Superdestructible*, Journal of Symbolic Logic (in press)

[K72] Jussi Ketonen, *Strong Compactness and Other Cardinal Sins*, Annals of Mathematical Logic **5** p. 47-76 (1972)

[L78] Richard Laver, *Making the Supercompactness of κ Indestructible Under κ-Directed Closed Forcing*, Isreal Journal Math **29** p. 385-388 (1978)