NON-EXISTENCE OF UNIVERSAL MEMBERS IN CLASSES OF ABELIAN GROUPS

Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Mathematics Department
New Brunswick, NJ USA

The Fields Institute
Toronto, Ontario
Canada

Abstract. We prove that if $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$, then there is no universal reduced torsion free abelian group of cardinality λ. Similarly if $\aleph_0 < \lambda < 2^{\aleph_0}$. We also prove that if $\beth_\omega < \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$, then there is no universal reduced separable abelian p-group in λ. We also deal with the class of $\aleph_1$-free abelian group. (Note: both results fail if (a) $\lambda = \lambda^{\aleph_0}$ or if (b) λ is strong limit, $\mathrm{cf}(\mu) = \aleph_0 < \mu$).

2000 Mathematics Subject Classification. 20K20, 03E04, 20K10
Key words and phrases: Abelian groups, Universal, pcf, $\aleph_1$-free

This research was partially supported by the German-Israel Foundation for Science
I would like to thank Alice Leonhardt for the beautiful typing.
Latest Revision - 00/Aug/21
Publ. No. 622

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§0 Introduction

We deal with the problem of the existence of a universal member in $\mathfrak{K}_\lambda$ for $\mathfrak{K}$ a class of abelian groups, where $\mathfrak{K}_\lambda$ is the class of $G \in \mathfrak{K}$ of cardinality λ; universal means that every other member can be embedded into it. We are concerned mainly with the class of reduced torsion free groups. Generally, on the history of the existence of universal members see Kojman-Shelah [KjSh 409]. From previous works, a natural division of the possible cardinals for such problems is:

Case 0: $\lambda = \aleph_0$.

Case 1: $\lambda = \lambda^{\aleph_0}$.

Case 2: $\aleph_0 < \lambda < 2^{\aleph_0}$

Case 3: $2^{\aleph_0} + \mu^+ < \lambda = \text{ cf}(\lambda) < \mu^{\aleph_0}$.

Case 4: $2^{\aleph_0} + \mu^+ + \text{ cf}(\lambda) < \lambda < \mu^{\aleph_0}$.

Case 5: $\lambda = \mu^+$, $\text{cf}(\mu) = \aleph_0, (\forall \chi < \mu)(\chi^{\aleph_0} < \mu)$.

Case 6: $\text{cf}(\lambda) = \aleph_0, (\forall \chi < \lambda)(\chi^{\aleph_0} < \lambda)$.

Subcase 6a: λ is strong limit.

Subcase 6b: Case 6 but not 6a.

Our main interest was in Case 3, originally for $\mathfrak{K} = \mathfrak{K}^{\text{rtf}}$, the class of torsion free reduced abelian groups. Note that if we omit the "reduced" then divisible torsion free abelian groups of cardinality λ are universal. A second class is $\mathfrak{K}^{\text{rs(p)}}$, the class of reduced separable p-groups (see Definition 2.3(4), more in Fuchs [Fu]) but we are interested in having methods and in the class of $\aleph_1$-free abelian groups. Kojman-Shelah [KjSh 455] show that for $\mathfrak{K} = \mathfrak{K}^{\text{rtf}}, \mathfrak{K}^{\text{rs(p)}}$ in Case 3 there is no universal member if we restrict the possible embeddings to pure embeddings. This stresses that universality depends not only on the class of structures but also on the kind of embeddings. In [Sh 456] we allow any embeddings, but restrict the class of abelian groups to $(< \lambda)$-stable ones. In [Sh 552, §1,§5] we allow any embedding and all $G \in \mathfrak{K}_\lambda$ but there is a further restriction on λ related to the pcf theory (see [Sh:g]). This restriction is weak in the following sense: it is not clear if there is any cardinal (in any possible universe of set theory) not satisfying it. We here prove the full theorem for $\lambda > \beth_\omega$ with no further restrictions:

(∗) for $\lambda > \beth_\omega$ in Case 3, $\mathfrak{K} = \mathfrak{K}^{\text{rtf}}, \mathfrak{K}^{\text{rs(p)}}$ there is no universal member in $\mathfrak{K}_\lambda$

(where we define inductively $\beth_0 = \aleph_0, \beth_{n+1} = 2^{\beth_n}, \beth_\omega = \sum_{n<\omega} 2^{\beth_n}$ and generally $\beth_\alpha = \aleph_0 + \sum_{\beta<\alpha} 2^{\beth_\beta}$).

§1 deals with $\mathfrak{K}^{\rm rtf}$ using mainly type theory. In §2, we apply combinatorial ideals whose definition has some built-in algebra and purely combinatorial ones to get results on $\mathfrak{K}^{\rm rs(p)}$; there is more interaction between algebra and combinatorics than in [Sh 552]. Similarly in §3 we work on the class of $\aleph_1$-free abelian groups.

What about the other cases? Case 4 (which is like case 3 but λ singular) for $\mathfrak{K}^{\rm rtf}_\lambda$ and pure embedding, was solved showing non-existence of universals in [KjSh 455] provided that some weak pcf assumption holds and in [Sh 552] this was done for embeddings under slightly stronger pcf assumptions. For both assumptions, it is not clear if they may fail. Note that the results on consistency of existence of universals in this case cannot be attacked as long as more basic pcf problems remain open.

Concerning Case 5 - if we try to prove the consistency of the existence of universals, it is natural first to prove the existence of the relevant club guessing; here we expect consistency results. (Of course, consistently there is club guessing (by $\bar C = \langle C_\delta : \delta \in S\rangle, S \subseteq \lambda$, $\mathrm{otp}(C_\delta) = \mu$) and then there is no universal.) Also we were first of all interested in the existence of universal reduced torsion free groups under embeddings, but later we also looked into some of the other cases here. See more in [Sh:F319].

Case 1 ($\lambda = \lambda^{\aleph_0}$). By subsequent work there is a universal member of $\mathfrak{K}^{\rm rtf}_\lambda$, and (see Fuchs [Fu]) in $\mathfrak{K}^{\rm rs(p)}_\lambda$ there is a universal member, but in $\mathfrak{K}^{\aleph_1\text{-free}}_\lambda$ there is no universal member (see forthcoming work).

Case 0 ($\lambda = \aleph_0$). In $\mathfrak{K}^{\rm rtf}_\lambda$ there is no universal member (see above or 3.17) and in $\mathfrak{K}^{\rm rs(p)}_\lambda$ there is a universal member (see Fuchs [Fu]).

Case 2 ($\aleph_0 < \lambda < 2^{\aleph_0}$). For $\mathfrak{K}^{\rm rtf}_\lambda$ we prove here that there is no universal member (by 1.2), whereas for $\mathfrak{K}^{\rm rs(p)}_\lambda$ this is consistent with and independent of ZFC (see [Sh 550, §4]).
We also deal with Case 6 ($(\forall \chi < \lambda)\chi^{\aleph_0} < \lambda, \lambda >\ \mathrm{cf}(\lambda) = \aleph_0$). There is a universal member for $\mathfrak{K}^{\rm trf}_\lambda$ and also for $\mathfrak{K}^{\rm rs(p)}_\lambda$. See [Sh:F319].

We thank two referees and Mirna Dzamonja and Noam Greenberg for many corrections.

Notation: The cardinality of a set A is $|A|$, the cardinality of a structure G is $\|G\|$. $\mathscr{H}(\lambda^+)$ is the set of sets whose transitive closure has cardinality $\leq \lambda$ and $<^*_{\lambda^+}$ denotes a fixed well order of $\mathscr{H}(\lambda^+)$.

For an ideal I, we use I^+ to denote the family of subsets of $\mathrm{Dom}(I)$ which are not in I.

§1 Non-Existence of Universals Among Reduced Torsion Free Abelian Groups

The first result (1.2) deals with λ satisfying $\aleph_0 < \lambda < 2^{\aleph_0}$ and show the non-existence of universal members in $\mathfrak{K}^{\rm trf}_\lambda$ which improves [Sh 552]. The proof is straightforward by analyzing subgroups and comparing Bauer's types.

Then we deal with $2^{\aleph_0} + \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$. We add witnesses to bar the way against "undesirable" extensions (see [Sh:F319] on classes of modules) which is a critical new point compared to [Sh 552].

1.1 Definition. Let $\mathfrak{K}^{\rm rtf}$ denote the class of torsion free reduced abelian groups G where torsion free means that $nx = 0, n \in \mathbb{Z}, x \in G \Rightarrow n = 0 \vee x = 0$ and reduced means that $(\mathbb{Q},+)$ cannot be embedded into G. The subclass of $G \in \mathfrak{K}^{\rm rtf}$ of cardinality λ is denoted by $\mathfrak{K}^{\rm rtf}_\lambda$. Moreover, $\mathfrak{K}^{\rm tf}$ is the class of torsion free abelian groups.

1.2 Claim. *1) If $\aleph_0 < \lambda < 2^{\aleph_0}$ <u>then</u> $\mathfrak{K}^{\rm rtf}_\lambda$ has no universal member.*
2) Moreover, there is no member of $\mathfrak{K}^{\rm rtf}_\lambda$ universal for $\mathfrak{K}^{\rm rtf}_{\aleph_1}$.

Proof. Let $\mathbf{P}^*$ be the set of all primes and let $\{\mathbf{Q}_i : i < 2^{\aleph_0}\}$ be a family of infinite subsets of $\mathbf{P}^*$, pairwise with finite intersection. Let $\rho_\alpha \in {}^\omega 2$ for $\alpha < \omega_1$ be pairwise distinct. Let H^* be the divisible torsion free abelian group with $\{x_\alpha : \alpha < \omega_1\}$ a maximal independent subset. For $i < 2^{\aleph_0}$ let H^*_i be the subgroup of H^* generated by

$$\begin{aligned}\{x_\alpha : \alpha < \omega_1\} \cup \{p^{-n}x_\alpha : p \in \mathbf{P}^* \backslash \mathbf{Q}_i, \alpha < \omega_1 \text{ and } n < \omega\} \\ \cup \{p^{-n}(x_\alpha - x_\beta) : \alpha, \beta < \omega_1 \text{ and } p \in \mathbf{P}^* \text{ and} \\ \rho_\alpha \restriction p = \rho_\beta \restriction p \text{ and } n < \omega\}.\end{aligned}$$

Clearly $H^*_i \in \mathfrak{K}^{\rm rtf}$ and $\|H^*_i\| = \aleph_1 \leq \lambda$. Let $G \in \mathfrak{K}^{\rm rtf}_\lambda$, and we shall prove that at most λ of the groups H^*_i are embeddable into G.

So assume $Y \subseteq 2^{\aleph_0}, |Y| > \lambda$ and for $i \in Y$ we have h_i, an embedding of H^*_i into G and we shall derive that G is not reduced; a contradiction. We choose by induction on n a set $\Gamma_n \subseteq {}^n\lambda$ and pure abelian subgroups G_η of G for $\eta \in \Gamma_n$, as follows. For $n = 0$ we let $\Gamma_0 = \{<>\}$ and let $G_{<>} = G$. For $n+1$, for $\eta \in \Gamma_n$ such that $\|G_\eta\| > \aleph_0$ we let $\Gamma_{n,\eta} = \{\eta\hat{}\langle\zeta\rangle : \zeta < \|G_\eta\|\}$, and let $\bar{G}_\eta = \langle G_{\eta\hat{}\langle\zeta\rangle} : \zeta < \|G_\eta\|\rangle$ be an increasing continuous sequence of subgroups of G_η of cardinality $< \|G_\eta\|$ with union G_η such that:

$(*)$ for $\zeta < \|G_\eta\|$ we have
$G_{\eta\hat{}\langle\zeta\rangle} = G_\eta \cap (\text{the Skolem Hull of } G_{\eta\hat{}\langle\zeta\rangle} \text{ in } (\mathscr{H}(\lambda^+), \in, <^*_{\lambda^+}, G_\eta))$.

Let $\Gamma_{n+1} = \{\eta\hat{}\langle\zeta\rangle : \eta \in \Gamma_n, \|G_\eta\| > \aleph_0 \text{ with } \zeta < \|G_\eta\|\}$ and $\Gamma = \bigcup\limits_{n<\omega} \Gamma_n$.

For each $i \in Y$, let $\eta = \eta_i \in \Gamma$ be such that:

(a) $\{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i}\}$ is uncountable
(b) under (a), the cardinality of G_{η_i} is minimal.

Clearly η_i is well defined as (a) holds for $\eta = \langle\rangle$ and clearly G_{η_i} is uncountable. It is also clear that the cardinality $\|G_{\eta_i}\|$ has cofinality $\aleph_1$. Let $X_i = \{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i}\}$, and let $\beta_i < \omega_1$ be minimal such that
$\{\rho_\alpha : \alpha \in \beta_i \cap X_i\}$ is a dense subset of $\{\rho_\alpha : \alpha \in X_i\}$. Let $\zeta_i < \|G_{\eta_i}\|$ be the minimal ζ such that $\{h_i(x_\alpha) : \alpha \in \beta_i \cap X_i\} \subseteq G_{\eta\hat{}\langle\zeta\rangle}$ (exists as $\mathrm{cf}(\|G_{\eta_i}\|) = \aleph_1$). Now by clause (b) the set $X'_i = \{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i\hat{}\langle\zeta_i\rangle}\}$ is countable, and hence we can find $\alpha_i \in X_i \backslash X'_i$.

Now the number of possible sequences $\langle \eta_i, \beta_i, \zeta_i, \alpha_i, h_i(x_{\alpha_i})\rangle$ is at most $|{}^{\omega>}\lambda| \times \aleph_1 \times \lambda \times \aleph_1 \times \lambda$ (as $\Gamma \subseteq {}^{\omega>}\lambda$). So for some $\langle \eta, \beta, \zeta, \alpha, y\rangle$ and $i_0 < i_1$ from Y we have (for $\ell = 0, 1$)

$$\eta_{i_\ell} = \eta, \beta_{i_\ell} = \beta, \zeta_{i_\ell} = \zeta, \alpha_{i_\ell} = \alpha, h_{i_\ell}(x_{\alpha_\ell}) = y.$$

Now as h_{i_ℓ} embeds $H^*_{i\ell}$ into G and $h_{i_\ell}(x_\alpha) = y$, necessarily

$(**)$ if $p \in \mathbf{P}^* \backslash \mathbf{Q}_{i_\ell}$ and $n < \omega$ then in G, p^{-n} divides y.

So this holds for every $p \in (\mathbf{P}^* \backslash \mathbf{Q}_{i_0}) \cup (\mathbf{P}^* \backslash \mathbf{Q}_{i_1}) = \mathbf{P}^* \backslash (\mathbf{Q}_{i_0} \cap \mathbf{Q}_{i_1})$.

Now $\mathbf{Q}_{i_0} \cap \mathbf{Q}_{i_1}$ is finite so let $p^* \in \mathbf{P}^*$ be above its supremum. As $\{\rho_\gamma : \gamma \in X'_{i_0}\}$ is a dense subset of $\{\rho_\alpha : \alpha \in X_{i_0}\}$, there is $\gamma \in X'_{i_0}$ such that $\rho_\gamma \restriction p^* = \rho_\alpha \restriction p^* (= \rho_{\alpha_{i_0}} \restriction p^*)$. Let $h_{i_0}(x_\gamma) = y^*$, it is in $G_{\eta\hat{}\langle\zeta\rangle}$.

So in $(\mathscr{H}(\lambda^+), \in, <^*_{\lambda^+}, G_\eta)$, the following formula is satisfied (recall that G_η is a pure subgroup of G)

$$\begin{aligned}\varphi(y, y^*) = \text{“in } G_\eta, y &\text{ is divisible by } p^n \text{ when } p \in \mathbf{P}^* \;\;\&\;\; p \geq p^* \;\;\&\;\; n < \omega \\ &\text{and } y - y^* \text{ is divisible by } p^n \text{ when} \\ &p \in \mathbf{P}^* \;\;\&\;\; p < p^* \;\;\&\;\; n < \omega\text{”}.\end{aligned}$$

Hence by $(*)$, i.e. by the choice of $\langle G_{\eta\hat{}\langle\xi\rangle} : \xi < \|G_\eta\|\rangle$, necessarily for some $y' \in G_{\eta\hat{}\langle\zeta\rangle}$ we have $\varphi(y', y^*)$. Now $y \neq y'$ as $y' \in G_{\eta\hat{}\langle\zeta\rangle}, y \notin G_{\eta\hat{}\langle\zeta\rangle}$. Also $y - y'$ is divisible by p^n for $p \in \mathbf{P}^*, n < \omega$.
[Why? If $p \geq p^*$ because both y and y' are divisible by p^n and if $p < p^*$ because $y - y' = (y - y^*) - (y' - y^*)$ and both $y - y^*$ and $y' - y^*$ are divisible by p^n.]
As G is torsion free, the pure closure in G of $\langle\{y - y'\}\rangle_G$ is isomorphic to $\mathbb{Q}$, a contradiction to "G is reduced". $\square_{1.2}$

1.3 Definition. 1) Let $\mathbf{P}^*$ be the set of primes.
2) For $G \in \mathfrak{K}^{\rm rtf}$ and $x \in G \backslash \{0\}$ let

(a) $\mathbf{P}(x, G) = \{p \in \mathbf{P}^* : x \in \bigcap\limits_{n<\omega} p^n G,$ equivalently x is divisible by p^n in G for every $n < \omega\}$

(b) $\mathbf{P}^-(x, G) = \{p : p \in \mathbf{P}^*$, but $p \notin \mathbf{P}(x, G)$ and there is $y \in G \backslash \{0\}$ such that $\mathbf{P}^* \backslash \{p\} \subseteq \mathbf{P}(y, G)$ and $p \in \mathbf{P}(x - y, G)\}$.

3) $G \in \mathfrak{K}^{\rm rtf}$ is called full if: for every $x \in G \backslash \{0\}$ we have $\mathbf{P}^* = \mathbf{P}(x, G) \cup \mathbf{P}^-(x, G)$.
4) The class of full $G \in \mathfrak{K}^{\rm rtf}$ is called $\mathfrak{K}^{\rm stf}$ and $\mathfrak{K}^{\rm stf}_\lambda = \mathfrak{K}^{\rm stf} \cap \mathfrak{K}^{\rm rtf}_\lambda$, (why s? as the successor of r in the alphabet).

1.4 Fact. 1) If $G \in \mathfrak{K}^{\rm rtf}$, then for any $x \in G$ the sets $\mathbf{P}(x, G)$ and $\mathbf{P}^-(x, G)$ are disjoint subsets of $\mathbf{P}^*$.
2) If G_2 is an extension of G_1, both in $\mathfrak{K}^{\rm rtf}$ and $x \in G_1 \backslash \{0\}$ then

(a) $\mathbf{P}(x, G_1) \subseteq \mathbf{P}(x, G_2)$, with equality if G_1 is a pure subgroup of G_2
(b) $\mathbf{P}^-(x, G_1) \subseteq \mathbf{P}^-(x, G_2)$.

3) For every $G \in \mathfrak{K}^{\rm rtf}$ there is a G' such that

(a) G' is full, $G' \in \mathfrak{K}^{\rm rtf}$
(b) G is a pure subgroup of G' and $\|G'\| = \|G\|$.

Proof. 1),2) Trivial.
3) It suffices to show

$(*)$ if $G \in \mathfrak{K}^{\rm rtf}$ and $x \in G \backslash \{0\}$, and $p \in \mathbf{P}^* \backslash \mathbf{P}(x, G)$ then for some pure extension G' of G with $\mathrm{rk}(G/G') = 1$ we have: $p \in \mathbf{P}^-(x, G')$.

For proving $(*)$ for a given G, x let $\hat{G}$ be the divisible hull of G and let
$G_0 = \{y \in \hat{G} :$ for some $n > 0, p^n y \in G\}$,
$G_1 = \{y \in \hat{G} :$ for some $b \in \mathbb{Z}, b > 0$ not divisible by p we have $by \in G\}$. Clearly $G = G_0 \cap G_1$. We define the following subsets of $\hat{G} \times \mathbb{Q}$:

$$H_0 = \{(y, 0) : y \in G\} \text{ (so } G \text{ is isomorphic to } H_0)$$

$$H_1 = \{(p^n bx, p^n b) : b, n \in \mathbb{Z}\}$$

$$H_2 = \{(0, c_1/c_2) : c_1, c_2 \in \mathbb{Z} \text{ and } c_2 \text{ not divisible by } p\}.$$

Easily all three are additive subgroups of $\hat{G} \times \mathbb{Q}$ and $H_2 \cong \mathbb{Z}_{(p)}$. Let $G' = H_0 + H_1 + H_2$, a subgroup of $\hat{G} \times \mathbb{Q}$.
We claim that $G' \cap (\hat{G} \times \{0\}) = H_0$. The inclusion $\supseteq$ should be clear. For the other direction let $z \in G' \cap (\hat{G} \times \{0\})$; as $z \in G'$ there are $(y, 0) \in H_0$, (so $y \in G$), $(p^n bx, p^n b) \in H_1$ (so $b \in \mathbb{Z}, n \in \mathbb{Z}$ and $x \in G$ is the constant from $(*)$) and $(0, c_1/c_2) \in H_2$ (so $c_1, c_2 \in \mathbb{Z}$ and p does not divide c_2) and integers a_0, a_1, a_2 such that $z = a_0(y, 0) + a_1(p^n bx, p^n b) + a_2(0, c_1/c_2)$ which means $z = (a_0 y + a_1 p^n bx, a_1 p^n b + a_2 c_1/c_2)$.

As $z \in \hat{G} \times \{0\}$ clearly $a_1 p^n b + a_2 c_1/c_2 = 0$, so as p does not divide c_2, necessarily $a_1 p^n b$ is an integer, hence $a_1 p^n bx \in G$, hence as $y \in G$ clearly $a_0 y + a_1 p^n bx \in G$ and hence $z \in G \times \{0\} = H_0$ as required.
It is easy to check now that H_0 is a pure subgroup of G'.
Also letting $y^* = (0, -1)$ clearly $(x, 0) - y^*$ is divisible by p^k for every $k < \omega$ (as $(p^k x, p^k) \in H_1 \subseteq G'$ for every $k \in \mathbb{Z}$) and y^* is divisible by any integer b when b is not divisible by p (as $\frac{1}{b}\, y^* = (0, -1/b) \in H_2 \subseteq G'$).

Identifying $y \in G$ with $(y, 0) \in G$ we are done: G' is as required in $(*)$, with y^* witnessing "$p \in \mathbf{P}^-(x, G')$". $\square_{1.4}$

1.5 Claim. *If $G_1 \in \mathfrak{K}^{\rm rtf}$ is full and $G_2 \in \mathfrak{K}^{\rm rtf}$ and h is an embedding of G_1 into G_2 then:*

$$\textit{for } x \in G_1 \backslash \{0\}, \mathbf{P}(x, G_1) = \mathbf{P}(h(x), G_2).$$

Proof. Without loss of generality h is the identity, now reflect using 1.4(1), 1.4(2) and the definition of full. $\square_{1.5}$

1.6 Conclusion. Assume

(*) $2^{\aleph_0} < \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$.

Then there is no universal member in $\mathfrak{K}^{\text{rtf}}_\lambda$.

Proof. Let $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \aleph_0$ and ω^2 divides $\delta\}$ be stationary and $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$ where each η_δ is an increasing ω-sequence of ordinals $< \delta$ with limit δ such that $\eta_\delta(n) - n$ is well defined and divisible by ω; so $\delta_1 \neq \delta_2 \Rightarrow \text{Rang}(\eta_{\delta_1}) \cap \text{Rang}(\eta_{\delta_2})$ is finite. Let $\{p^*_n : n < \omega\}$ list the primes in the increasing order. Let $G^0_{\bar\eta}$ be the abelian group generated by $\{x_\alpha : \alpha < \lambda\} \cup \{y_\delta : \delta \in S\} \cup \{z_{\delta,n,\ell} : n, \ell < \omega\} \cup \{x_{\alpha,m,\ell} : \alpha < \lambda, m < \omega, \alpha \neq m \text{ mod } \omega\}$ freely except for the equations

$$p^*_n z_{\delta,n,\ell+1} = z_{\delta,n,\ell} \qquad y_\delta - x_{\eta_\delta(n)} = z_{\delta,n,0}.$$

$$p^*_m x_{\alpha,m,\ell+1} = x_{\alpha,m,\ell},\ x_{\alpha,m,0} = x_\alpha,\ \text{if } \alpha \neq m \text{ mod } \omega$$

We can check that $G^0_{\bar\eta} \in \mathfrak{K}^{\text{rtf}}_\lambda$ and $\mathbf{P}^-(y_\delta, G^0_{\bar\eta})$ is the set of all primes and $\mathbf{P}(x_\alpha, G^0_{\bar\eta}) =$ the set of primes $\neq p^*_n$ if $\alpha = n$ mod ω.

Let $G_{\bar\eta} \in \mathfrak{K}^{\text{rtf}}_\lambda$ be a pure extension of $G^0_{\bar\eta}$ which is full (one exists by 1.4(3)). So

(*) if h embeds $G_{\bar\eta}$ into $G \in \mathfrak{K}^{\text{rtf}}_\lambda$ then
$x \in G_{\bar\eta} \backslash \{0\} \Rightarrow \mathbf{P}(x, G_{\bar\eta}) = \mathbf{P}(h(x), G)$.

Hence the proof in [KjSh 455] works. $\square_{1.6}$

1.7 Remark. 1) Similarly the results on λ singular (i.e. Case 4) in [KjSh 455], hold for embedding (rather than pure embedding).
2) What about Case 5? If there is a family $\mathscr{P} \subseteq \{C \subseteq \mu^+ : \text{otp}(C) = \mu\}$ which guesses clubs (i.e. every club E of μ^+ contains one of them), the result holds.
3) On $\aleph_0 \leq \lambda < 2^{\aleph_0}$ see also in 3.17.

§2 The existence of universals for separable reduced abelian p-groups

We here eliminate the very weak *pcf* assumption from the theorem of "no universal in $\mathfrak{K}^{\mathrm{rs(p)}}_\lambda$" when $\lambda > \beth_\omega$. Note that $\mathfrak{K}^{\mathrm{rs(p)}}$ is defined in 2.3(4).

In the first section we have eliminated the very weak *pcf* assumptions for the theorem concerning $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ (though the $\lambda = \mathrm{cf}(\lambda) > \mu^+$ remains, i.e. we assume we are in Case 3). This was done using the "infinitely many primes", so in the language of e.g. [KjSh 455] the invariant refers to one element x. This cannot be generalized to $\mathfrak{K}^{\mathrm{rs(p)}}_\lambda$. However, in [Sh 552, §5] we use an invariant on e.g. suitable groups and related stronger "combinatorial" ideals. We continue this, using combinatorial ideals closer to the algebraic ones to show that the algebraic is non-trivial.

We rely on the "GCH right version" provable from ZFC, see [Sh 460] hence the condition "$\lambda > \beth_\omega$" is used.

2.1 Definition. 1) For $\bar\lambda = \langle \lambda_\ell : \ell < \omega\rangle$ and $\bar t = \langle t_\ell : \ell < \omega\rangle$ (with $1 < t_\ell < \omega$) we define $J^4_{\bar t,\bar\lambda}$.

It is the family of subsets A of $\prod\limits_{\ell<\omega} [\lambda_\ell]^{t_\ell}$ such that:

$(*)_A$ for every large enough $\ell < \omega$, for every $B \in [\lambda_\ell]^{\aleph_0}$ for some $k \in (\ell,\omega)$ we <u>cannot find</u>

$$\langle \nu_\eta : \eta \in \prod_{i\in[\ell,k)} [\omega]^{t_i}\rangle$$

such that

(a) $\nu_\eta \in A$

(b) if $\eta_1,\eta_2 \in \prod\limits_{i\in[\ell,k)} [\omega]^{t_i}, \ell \le m \le k$ and $\eta_1 \restriction [\ell,m) = \eta_2 \restriction [\ell,m)$ <u>then</u> $\nu_{\eta_1} \restriction m = \nu_{\eta_2} \restriction m$; hence $\nu_{\eta_1} \restriction \ell = \nu_{\eta_2} \restriction \ell$ for $\eta_1,\eta_2 \in \prod\limits_{i\in[\ell,k)} [\omega]^{t_i}$

(c) if $\eta_0 \in \prod\limits_{i\in[\ell,k)} [\omega]^{t_i}$ and $\ell \le m < k$ <u>then</u> for some $E \in [\lambda_m]^{\aleph_0}$ we have

$$[E]^{t_m} = \{\nu_\eta(m) : \eta \in \prod_{i\in[\ell,k)} [\omega]^{t_i} \text{ and } \eta \restriction m = \eta_0 \restriction m\}$$

and $m = \ell \Rightarrow E = B$.

2) Let

$$J^4_{\bar{t},\bar{\lambda},<\theta} = \Big\{A \subseteq \prod_{\ell<\omega}[\lambda_\ell]^{t_\ell} : \text{for some } \alpha < \theta \text{ and } A_\beta \in J^4_{\bar{t},\bar{\lambda}} \text{ for } \beta < \alpha$$

$$\text{we have } A \subseteq \bigcup_{\beta<\alpha} A_\beta\Big\}.$$

When $\theta = \kappa^+$, we may write κ instead of $< \theta$.

2.2 Fact. 1) $J^4_{\bar{t},\bar{\lambda},\theta}$ is a θ^+-complete ideal.
2) If $\ell < \omega \Rightarrow \lambda_\ell > \beth_{t_\ell-1}(\theta)$ then the ideal $J^4_{\bar{t},\bar{\lambda},\theta}$ is proper (where $\beth_0(\theta) = \theta, \beth_{n+1}(\theta) = 2^{\beth_n(\theta)}$, and for general α we have $\beth_\alpha(\theta) = \theta + \sum\limits_{\beta<\alpha} 2^{\beth_\beta(\theta)}$).

Proof. 1) Trivial.
2) Let for $\ell < \omega$

$$ERI^{t_\ell}_{\lambda_\ell} = \Big\{A \subseteq [\lambda_\ell]^{t_\ell} : \text{for some } F : [\lambda_\ell]^{t_\ell} \to \theta \text{ there is no } B \in [\lambda_\ell]^{\aleph_0}$$

$$\text{such that } F \restriction [B]^{t_\ell} \text{ is constant and } [B]^{t_\ell} \subseteq A\Big\}.$$

So this is a θ^+-complete ideal. It is non-trivial by Erdös-Rado theorem (we use it similarly in [Sh 620, §1]). Now we shall prove that the ideal $J^4_{\bar{t},\bar{\lambda},\theta}$ is proper. So assume $\prod\limits_{\ell<\omega}[\lambda_\ell]^{t_\ell} = \bigcup\limits_{i<\theta} X_i$ and $X_i \in J^4_{\bar{t},\bar{\lambda}}$ for each $i < \theta$ and we shall get a contradiction. Let

$$X^+_i = \Big\{\eta \in \prod_{\ell<\omega}[\lambda_\ell]^{t_\ell} : \text{ for every } \ell < \omega \text{ for some } \eta' \in X_i \text{ we have } \eta \restriction \ell = \eta' \restriction \ell\Big\}.$$

(i.e. the closure of X_i). So $X^+_i \subseteq \prod\limits_{\ell<\omega}[\lambda_\ell]^{t_\ell} = \prod\limits_{\ell<\omega} \mathrm{Dom}(ERI^{t_\ell}_{\lambda_\ell})$ is closed, and those ideals are θ^+-complete and $\prod\limits_{\ell<\omega} \mathrm{Dom}(ERI^{t_\ell}_{\lambda_i}) = \bigcup\limits_{i<\theta} X^+_i$. Hence (see Rubin-Shelah [RuSh 117], [Sh:f, Ch.XI,3.5(2)] with $H_\alpha = X^+_i$) we can find T such that:

(a) $T \subseteq \bigcup\limits_{m<\omega} \prod\limits_{\ell<m} [\lambda_\ell]^{t_\ell}$

(b) T is closed under initial segments

(c) $<>\in T$

(d) if $\nu \in T$ and $\ell g(\nu) = \ell$ then
$\{u \in [\lambda_\ell]^{t_\ell} : \nu\hat{\ }\langle u\rangle \in T\} \in (ERI^{t_\ell}_{\lambda_\ell})^+$

(e) for some $i < \theta$, $\lim(T) \subseteq X_i^+$.

(Here, $\lim(T) = \{\nu \in \prod\limits_{\ell<\omega} [\lambda_\ell]^{t_\ell} : (\forall m < \omega)\nu \restriction m \in T\}$).

Fix i from clause (e). We would like to prove $\neg(*)_{X_i^+}$. By the definition of the ideal $\mathrm{ERI}^{t_\ell}_{\lambda_\ell}$ we get more than required (for every k in place of "some k" in $(*)$ of Definition 2.1). $\square_{2.2}$

Remark. So we could have used the stronger ideal defined implicitly in 2.2, i.e. $J^5_{\bar t,\bar\lambda,\theta} = \{X \subseteq \prod\limits_{\ell<\omega} \lambda_\ell :$ we can find $\alpha < \theta$ and $X_i \subseteq X$ for $i < \alpha$ such that $X = \bigcup\limits_{i<\alpha} X_i$ and for each i and T satisfying clauses $(a)-(d)$ from the proof of 2.2 there is $T' \subseteq T$ satisfying clauses $(a)-(d)$ such that $\lim(T)$ is disjoint to the closure of $X_i\}$. Of course, we can also replace $\mathrm{ERI}^{t_\ell}_{\lambda_\ell}$ by various variants.

We recall from [Sh 552, 5.1]

2.3 Definition. ([Sh 552, 5.1]) 1) For $\bar\mu = \langle \mu_n : n < \omega\rangle$ let $B_{\bar\mu}$ be the following direct sum of cyclic p-groups. Let K^n_α be a cyclic group of order p^{n+1} generated by x^n_α and let $B^n_{\mu_n} = \oplus_{\alpha<\mu_n} K^n_\alpha$ and $B_{\bar\mu} = \oplus_{n<\omega} B^n_{\mu_n}$, i.e. $B_{\bar\mu}$ is the abelian group generated by $\{x^n_\alpha : n < \omega, \alpha < \mu_n\}$ freely except that $p^{n+1}\, x^n_\alpha = 0$.
Moreover, let $B_{\bar\mu\restriction n} = \oplus\{K^m_\alpha : \alpha < \mu_m, m < n\} \subseteq B_{\bar\mu}$
(these groups are in $\mathfrak{K}^{\mathrm{rs(p)}}_{\le\sum\limits_n \mu_n}$).

Let $\hat B_{\bar\mu}$ be the p-torsion completion of $B_{\bar\mu}$ (i.e. completion under the norm $\|x\| =\ \min\{2^{-n} : p^n$ divides $x\}$ but putting only the torsion elements, see Fuchs [Fu]. Note that $\hat B_{\bar\mu}$ is the torsion part of the p-adic completion of $B_{\bar\mu}$).
2) Let $I^1_{\bar\mu,<\theta} = I^1_{\bar\mu,<\theta}[p]$ be the ideal on $\hat B_{\bar\mu}$ (depending on the choice of $\langle K^n_\alpha : \alpha < \mu_n, n < \omega\rangle$ or actually $\langle B_{\bar\mu\restriction n} : n < \omega\rangle$) consisting of unions of $< \theta$ members of $I^0_{\bar\mu}$, where

$$I^0_{\bar\mu} = I^0_{\bar\mu}[p] = \big\{A \subseteq \hat B_{\bar\mu} : \text{for every large enough } n, \text{ we have } c\ell_{\hat B_{\bar\mu}}(\langle A\rangle_{\hat B_{\bar\mu}}) \cap B_{\bar\mu} \subseteq B_{\bar\mu\restriction n}\big\}$$

($c\ell_{\hat{B}_{\bar\mu}}$ is defined in part 3) below).
When $\theta = \kappa^+$ instead of $< \theta$ we may write κ. If $\mu_n = \mu$, we may write μ instead of $\bar\mu$.
3) For $X \subseteq \hat{B}_{\bar\mu}$, recall $\langle X \rangle_{\hat{B}_{\bar\mu}}$ is the subgroup of $\hat{B}_{\bar\mu}$ which X generates and

$$c\ell_{\hat{B}_{\bar\mu}}(X) = \big\{x : (\forall n)(\exists y \in X)(x - y \in p^n \hat{B}_{\bar\mu})\big\}.$$

4) Let $\mathfrak{K}^{\mathrm{rs(p)}}$ be the family of pure subgroups of some $\hat{B}_{\bar\mu}$.
5) If p is not clear from the context, we may write $B_{\bar\mu}[p], \hat{B}_{\bar\mu}[p]$, etc.

2.4 Claim. *Assume $\bar\mu = \langle \mu_n : n < \omega \rangle, \bar t = \langle t_\ell : \ell < \omega \rangle, t_\ell = p$ and the ideal $J^4_{\bar t,\bar\mu,\theta}$ is proper (so $\mu_n \ge \beth_{p-1}(\theta)^+$ is enough by 2.2(2)). Then the ideal $I^1_{\bar\mu,\theta}$ is proper (and $I^1_{\bar\mu,\theta}$ is a θ^+-complete ideal).*

Proof. We define a function h from $\prod\limits_{\ell<\omega} [\lambda_\ell]^{t_\ell}$ into $\hat{B}_{\bar\mu}$. We let

$$h(\eta) = \Sigma\{p^n\, x^n_\beta : \beta \in \eta(n) \text{ and } n < \omega\} \in \hat{B}_{\bar\mu}[p].$$

Clearly h is one to one and it suffices to prove

(∗) if $X \in (J^4_{\bar t,\bar\mu,\theta})^+$ then $h''(X)$ belongs to $(I^1_{\bar\mu,\theta})^+$.

So assume $X \in (J^4_{\bar t,\bar\lambda,\theta})^+$ is given and suppose toward contradiction that $h''(X) \in I^1_{\bar\mu,\theta}$. So we can find $\langle Y_i : i < \theta \rangle$ such that for such $i < \theta$ we have $Y_i \in I^0_{\bar\mu}$ and $h(X) \subseteq \bigcup\limits_{i<\theta} Y_i$. Let $X_i = h^{-1}(Y_i)$. So $h(X_i) \subseteq Y_i \in I^0_{\bar\mu}$ and hence $h(X_i) \in I^0_{\bar\mu}$, but as $J^4_{\bar t,\bar\lambda,\theta}$ is θ^+-complete and $X \in (J^4_{\bar t,\bar\lambda,\theta})^+$ necessarily for some $i < \theta, X_i \in (J^4_{\bar t,\bar\lambda,\theta})^+$, so without loss of generality $h''(X) \in I^0_{\bar\mu}$. By the definition of $I^0_{\bar\mu}$, for some $n(*) < \omega$ we have

(∗) $B_{\bar\mu} \cap c\ell_{\hat{B}_{\bar\mu}}(\langle h''(X) \rangle_{\hat{B}_{\bar\mu}}) \subseteq B_{\bar\mu \restriction n(*)}$.

On the other hand, as $X \in (J^4_{\bar t,\bar\mu,\theta})^+$, it is $\notin J^4_{\bar t,\bar\mu}$ so from definition 2.1(1) of $J^4_{\bar t,\bar\mu}$ we can find $\langle B_n : n \in \Gamma \rangle$ such that:

(a) $\Gamma \in [\omega]^{\aleph_0}$ and $B_n \in [\lambda_n]^{\aleph_0}$
(b) for $n \in \Gamma$, for every $k \in (n,\omega)$ we can find $\langle \nu^{n,k}_\eta : \eta \in \prod\limits_{\ell \in [n,k)} [\omega]^{t_\ell} \rangle$ as in (a)-(c) of Definition 2.1(1) with n, B_n, k here standing for ℓ, B, k there.

For $m \in (n,k]$ and $\eta \in \prod\limits_{\ell \in [n,m)} [\omega]^{t_\ell}$ we let $\nu^{n,k}_\eta$ be $\nu^{n,k}_{\eta_1} \restriction m$ whenever $\eta \triangleleft \eta_1 \in \prod\limits_{\ell \in [n,k)} [\omega]^{t_\ell}$ (by clause (b) in $(*)$ of 2.1 it is well defined). Fix $n \in \Gamma$ and $k \in [n,\omega)$ for awhile. Let $A_\eta = A^{n,k}_\eta \in [\lambda_m]^{\aleph_0}$ be such that $\{\nu^{n,k}_{\eta\hat{\ }\langle u\rangle}(m) : u \in [\omega]^{t_m}\} = [A_\eta]^{t_m}$ and without loss of generality (otp stands for "the order type")

$(*)$ $\mathrm{otp}(A_\eta) = \omega$ and $\nu^{n,k}_{\eta\hat{\ }\langle u\rangle}(m) = \mathrm{OP}_{A_\eta,\omega}(u)$

(where $OP_{A_\eta,\omega}(i) = \alpha$ iff $i = \mathrm{otp}(A_\eta \cap \alpha)$).
Now for $m \in (n,k]$ and $\eta \in \prod\limits_{\ell \in [n,m)} [\omega]^{t_\ell}$ we define

$$y_\eta = y^{n,k}_\eta = \sum \{h(\nu^{n,k}_\rho) : \eta \trianglelefteq \rho \in \prod_{\ell \in [n,k)} [\omega]^{t_\ell} \text{ and } (\forall \ell)[\ell g(\eta) \le \ell < k \to \rho(\ell) \subseteq [0,t_\ell]\}$$

where $\trianglelefteq$ denotes being an initial segment. So $y_\eta \in \hat{B}_{\bar{\mu}}$ and we shall prove by downward induction on $m \in (n,k]$ that for every $\eta \in \prod\limits_{\ell \in [n,m)} [\omega]^{t_\ell}$ we have ($\sum\limits_{\ell<m}$ means $\sum\limits_{\ell \in [n,m)}$)

$$\boxtimes_m \ \ y_\eta = \big(\prod_{\ell=m}^{k-1} (t_\ell + 1)\big) \times \big(\sum_{\ell<m} \sum_{\alpha \in \nu^{n,k}_\eta(\ell)} p^\ell x^\ell_\alpha\big) \bmod p^k \hat{B}_{\bar{\mu}}.$$

<u>Case 1</u>: $m = k$.

In this case the product $\prod\limits_{\ell=m}^{k-1}(t_\ell+1)$ is just 1, so the equation says $y_\eta = \sum\limits_{\ell<m} \sum\limits_{\alpha \in \nu^{n,k}_\eta(\ell)} p^\ell x^\ell_\alpha \bmod p^k \hat{B}_{\bar{\mu}}$.
Now the expression for y_η is

$$\begin{aligned} &\sum \{h(\nu^{n,k}_\rho) : \eta \trianglelefteq \rho \in \prod_{\ell \in [n,k)} [\omega]^{t_\ell} \text{ and } (\forall \ell)[m \le \ell < k \Rightarrow \rho(\ell) \subseteq [0,t_\ell]]\} \\ &\qquad = h(\nu^{n,k}_\eta) = \sum_{\ell<\omega} \sum_{\alpha \in \nu^{n,k}_\eta(\ell)} p^\ell x^\ell_\alpha \\ &\qquad = \sum_{\ell<m} \sum_{\alpha \in \nu^{n,k}_\eta(\ell)} p^\ell x^\ell_\alpha + p^k \big(\sum_{\ell \in [k,\omega)} \sum_{\alpha \in \nu^{n,k}_\eta(\ell)} p^{\ell-k} x^\ell_\alpha\big) \end{aligned}$$

so the equality is trivial.

Case 2: $n < m < k$.

Here (with equalities in the equation being in $\hat{B}_{\bar{\mu}}$, modulo $p^k\,\hat{B}_{\bar{\mu}}$), we have:

$$
\begin{aligned}
y_\eta = \\
&\text{[by the definition of } y_\eta, y_{\eta\char`\^\langle u\rangle}] \\
&= \sum\{y_{\eta\char`\^\langle u\rangle} : u \in [\{0,\dots,t_m\}]^{t_m}\} = \\
&\text{[by the induction hypothesis]} \\
&= \sum\{(\prod_{\ell=m+1}^{k-1}(t_\ell+1))(\sum_{\ell<m+1}\ \sum_{\alpha\in\nu^{n,k}_{\eta\char`\^\langle u\rangle}(\ell)} p^\ell\, x^\ell_\alpha) : u \in [\{0,\dots,t_m\}]^{t_m}\}
\end{aligned}
$$

[by dividing the sum $\sum_{\ell<m+1}$ into $\sum_{\ell<m}$ and $\sum_{\ell=m}$ and noting what $\nu^{n,k}_{\eta\char`\^\langle u\rangle}(m)$ is]

$$
\begin{aligned}
&= \sum\Big\{(\prod_{\ell=m+1}^{k-1}(t_\ell+1))\big(\sum_{\ell<m}\ \sum_{\alpha\in\nu^{n,k}_{\eta\char`\^\langle u\rangle}(\ell)} p^\ell\, x^\ell_\alpha\big) : u \in [\{0,\dots,t_m\}]^{t_m}\Big\} \\
&\quad + \sum\Big\{(\prod_{\ell=m+1}^{k-1}(t_\ell+1))\sum_{\alpha\in OP_{\omega,A_\eta}(u)} p^m\, x^m_\alpha : u \in [\{0,\dots,t_m\}]^{t_m}\Big\} = \\
&\qquad\text{[in the second sum, we collect together the terms with } x^m_\alpha] \\
&= \sum\Big\{(\prod_{\ell=m+1}^{k-1}(t_\ell+1))\big(\sum_{\ell<m}\ \sum_{\alpha\in\nu^{n,k}_{\eta}(\ell)} p^\ell\, x^\ell_\alpha\big) : u \in [\{0,\dots,t_m\}]^{t_m}\Big\} \\
&\quad + \sum\Big\{(\prod_{\ell=m+1}^{k-1}(t_\ell+1))(p^m\, x^m_\alpha)|\{u : u \in [\{0,\dots,t_m\}]^{t_m} \text{ and } |\alpha\cap A_\eta| \in u\}| : \\
&\qquad\qquad \alpha \text{ is a member of } A_\eta,\ \text{moreover } |\alpha\cap A_\eta| \le t_m\Big\} \\
&= (\prod_{\ell=m+1}^{k-1}(t_\ell+1))\big(\sum_{\ell<m}\ \sum_{\alpha\in\nu^{n,k}_{\eta}(\ell)} p^\ell\, x^\ell_\alpha\big) \times |\{u : u \in [\{0,\dots,t_m\}]^{t_m}\}|
\end{aligned}
$$

$$+\sum\Big\{\big(\prod_{\ell=m+1}^{k-1}(t_\ell+1)\big)(p^m\, x^m_\alpha)\cdot((t_m+1)-1) : \alpha\in A_\eta, |\alpha\cap A_\eta|\le t_m\Big\} =$$

[remember $t_m = p$ and $p^{m+1}\, x^m_\alpha = 0$]

$$= (t_m+1)\big(\prod_{\ell=m+1}^{k-1}(t_\ell+1)\big)\sum_{\ell<m}\ \sum_{\alpha\in\nu^{n,k}_\eta(\ell)} p^\ell\, x^\ell_\alpha + 0$$

$$= \big(\prod_{\ell=m}^{k}(t_\ell+1)\big)\big(\sum_{\ell<m}\ \sum_{\alpha\in\nu^{n,k}_\eta(\ell)} p^\ell\, x^\ell_\alpha\big).$$

Hence we have finished the proof of $\boxtimes_m$.
Now as $t_\ell+1 = p+1$ and $pp^\ell x^\ell_\alpha = 0$ in $\hat B_{\bar\mu}$ we get

$\boxtimes'_m$ $\quad y_\eta = \sum\limits_{\ell<m}\ \sum\limits_{\alpha\in\nu^{n,k}_\eta(\ell)} p^\ell x^\ell_\alpha \mod p^k\hat B_{\bar\mu}$.

Note that for $m = n+1$, the sum $\sum\limits_{\ell<m}$ is just $\sum\limits_{\ell=n}$. So, as for $n\in\Gamma, B_n$ serves for every $k\in(n,\omega)$, if $u_1,u_2\in[B_n]^{t_n}$ are distinct then, for $k\in(n,\omega)$ we have $y_{\langle u_1\rangle}-y_{\langle u_2\rangle} = \sum\limits_{\ell<m}\ \sum\limits_{\alpha\in\nu^{n,k}_{\langle u_1\rangle}(\ell)} p^\ell x^\ell_\alpha - \sum\limits_{\ell<m}\ \sum\limits_{\alpha\in\nu^{n,k}_{\langle u_2\rangle}(\ell)} p^\ell x^\ell_\alpha \mod p^k\hat B_{\bar\mu}$. As this holds for every $k\in(n,\omega)$ we get equality. By the demands on $\nu^{n,k}_\eta$ (see clause (b) above so Definition 2.1(1)) we have $y_{<u_1>}-y_{<u_2>}\notin B_{\bar\mu\restriction n}$ but by the last sentence $y_{<u_1>}-y_{<u_2>}\in B_{\bar\mu\restriction(n+1)}$ contradicting $(*)$. $\square_{2.4}$

Recall

2.5 Definition. 1) Let I be an ideal on κ (or just $I\subseteq\mathscr P(\kappa)$ closed downward, $I^+ = \mathscr P(\kappa)\backslash I$), then we let:

$$\mathbf U_I(\lambda) = \text{Min}\big\{|\mathscr P| : \mathscr P\subseteq[\lambda]^{\le\kappa} \text{ and for every } f\in {}^\kappa\lambda \text{ for some } a\in\mathscr P \text{ we have } \{i<\kappa : f(i)\in a\}\in I^+\big\}.$$

2) For $\sigma\le\theta\le\mu\le\lambda$ let $\text{cov}(\lambda,\mu,\theta,\sigma) =$ Min$\{\lambda+|\mathscr P| : \mathscr P$ is a family of subsets of λ each of cardinality $<\mu$ such that any $X\subseteq\lambda$ of cardinality $<\theta$ is included in the union of $<\sigma$ members of $\mathscr P\}$.

2.6 Claim. *1) For every $\lambda \geq \beth_\omega$, for some $\theta < \beth_\omega$ for every $\mu \in (\beth_{p-1}(\theta), \beth_\omega)$ we have (letting $\mu_n = \mu$) $\mathbf{U}_{I^1_{\mu,\theta}}(\lambda) = \lambda$ (hence $\mathbf{U}_{I^0_{\bar\mu}}(\lambda) = \lambda$).*
2) If cf$(\lambda) > \aleph_0$, then for some $\theta < \beth_\omega$, for every $\mu \in (\beth_{p-1}(\theta), \beth_\omega)$ and $\lambda' < \lambda$ we have $\mathbf{U}_{I^1_{\mu,\theta}}(\lambda') < \lambda$.

Proof. By 2.4, $I_{\mu,\theta}$ is a θ-complete proper ideal on a set of cardinality $\mu^{\aleph_0}$, for any μ, θ as in the assumptions. By [Sh 460] for each $\lambda' \leq \lambda$ for some $\theta = \theta[\lambda'] < \beth_\omega$ for every $\mu \in (\theta, \beth_\omega)$ we have $\text{cov}(\lambda', \mu^+, \mu^+, \theta) = \lambda'$, i.e. there is $\mathscr{P}_\mu \subseteq [\lambda']^\mu$ of cardinality $\leq \lambda'$ such that: if $Y \in [\lambda']^{\leq \mu}$ then Y is included in the union of $< \theta$ members of $\mathscr{P}_\mu$. As $I^1_{\mu,\theta}$ is a θ^+-complete ideal on a set of cardinality μ it follows that $\mathbf{U}_{I^1_{\mu,\theta}}(\lambda') \leq \lambda' \times |\mathscr{P}_\mu| = \lambda'$ (and trivially $\mathbf{U}_{I^1_{\bar\mu,\theta}}(\lambda) \geq \lambda$). This proves part (1). For part (2) we are assuming $\text{cf}(\lambda) > \aleph_0$ so for some $\theta < \beth_\omega$, for arbitrarily large $\lambda' < \lambda, \theta[\lambda'] \leq \theta$; clearly we are done. $\square_{2.6}$

2.7 Conclusion. If $\beth_\omega \leq \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$, <u>then</u> in $\mathfrak{K}^{\text{rs(p)}}_\lambda$ there is no universal member.

Proof. By 2.6 and [Sh 552, 5.9].

Moreover

2.8 Claim. *Assume*

(a) $\prod\limits_{\ell < \omega} \kappa_\ell < \mu < \lambda = \text{cf}(\lambda) \leq \lambda' < \mu^{\aleph_0}$

(b) *$\mu^+ < \lambda$ or at least for some $\mathscr{P}$ we have*

$$(*)_{\mathscr{P}} \quad |\mathscr{P}| = \lambda \ \& \ (\forall a \in \mathscr{P})(a \subseteq \lambda \ \& \ \text{otp}(a) = \mu) \\ \& \ (\forall E)(E \text{ a club of } \lambda \rightarrow (\exists a \in \mathscr{P})(a \subseteq E))$$

(c) *$\lambda' = \mathbf{U}_{I^0_{\bar\kappa}}(\lambda) < \mu^{\aleph_0}$ where $\bar\kappa = \langle \kappa_\ell : \ell < \omega \rangle$ and note that $I^0_{\bar\kappa}$ depends on the prime p.*

<u>Then</u> we can find reduced separable abelian p-groups, $G_\alpha \in \mathfrak{K}^{\text{rs(p)}}_\lambda$ for $\alpha < \mu^{\aleph_0}$ such that for every reduced separable abelian p-group G of cardinality λ' we have:

> *some G_α is not embeddable into G; also the number of ordinals $\alpha < \mu^{\aleph_0}$ such that G_α is embeddable into G is $\leq \lambda'$*

Moreover, each G_α *is slender, i.e. there is no homomorphism from* $\mathbb{Z}^\omega$ *into* G_α *with range of infinite rank.*

Proof. Same proof as that of [Sh 552, 5.9], [Sh 552, 7.5].

§3 Non-existence of universals for $\aleph_1$-free abelian groups

The first section dealt with $\mathfrak{K}^{\rm rtf}_\lambda$ improving [Sh 552]. But the groups used there are "almost divisible". So what occurs if we replace $\mathfrak{K}^{\rm rtf}$ by a variant avoiding this? We suggest to consider the $\aleph_1$-free abelian groups where type arguments like those in §1 break down. So the proof of [Sh 552] becomes relevant and it is natural to improve it as in §2 (which deals with $\mathfrak{K}^{\rm rs(p)}$), for diversity we use a stronger ideal. We have not looked at the problem for $\aleph_1$-free abelian groups of cardinality λ when $\aleph_0 < \lambda < 2^{\aleph_0}$".
So we concentrate here on torsion free (abelian) groups.

3.1 Definition. 1) Let $\bar t = \langle t_\ell : \ell < \omega\rangle, 2 \le t_\ell < \omega$. For abelian group H, the $\bar t$-valuation is

$$\|x\|_{\bar t} = \ \mathrm{Inf}\{2^{-m} : \prod_{\ell<m} t_\ell \text{ divide } x \text{ (in } G)\}.$$

This is a semi-norm. Remember $d_{\bar t}(x,y) = \|x-y\|_{\bar t}$. This semi-norm induces a topology which is called the $\bar t$-adic topology.

If $t_\ell = p$ for $\ell < \omega$ we may write p instead of $\bar t$.
2) Let $c\ell_{\bar t}(A,H)$ be the closure of A in H under the $\bar t$-adic topology.
Let $PC_H(X)$ be the pure closure of X in H that is $\{x \in H :$ for some $n > 0, nx$ belongs to $\langle x\rangle_H\}$. Moreover $PC^p_H(X)$ is the p-adic closure in H of the subgroup of H which X generates.
3) Let $\mathfrak{K}^{\rm rtf}[\bar t]$ be the class of $\bar t$-reduced torsion free abelian groups, i.e. the $G \in \mathfrak{K}^{\rm rtf}$ such that $\bigcap\limits_{n<\omega} (\prod\limits_{i<n} t_i)G = \{0\}$ hence $\| - \|_{\bar t}$ induces a Hausdorff topology.
(Inversely if G is torsion free with the $\bar t$-adic topology Hausdorff then $G \in \mathfrak{K}^{\rm rtf}[\bar t]$.)
4) If the $\bar t$-adic topology is Hausdorff, then $G^{[\bar t]}$ is the completion of G by $\| - \|_{\bar t}$.
If $t_\ell = 2 + \ell$, this is the $\mathbb{Z}$-adic completion.

The following continues the analysis in [Sh 552, 1.1] (which deals with $\mathfrak{K}^{\rm rs(p)}$) and [Sh 552, 1.5] (which deals with $\mathfrak{K}^{\rm rtf}$).

3.2 Definition. We say G has $\bar t$-density μ if it has a pure subgroup of cardinality $\le \mu$ which is $\bar t$-dense, i.e. dense in the $\bar t$-adic topology, but has no such subgroup of cardinality $< \mu$.

3.3 Proposition. *Suppose that*

(α) $\mu \le \lambda \le \mu^{\aleph_0}$

(β) *G is an $\aleph_1$-free abelian group, $|G| = \lambda$*

(γ) *$\bar t$ is as in 3.1 such that $(\forall \ell)(\exists m > \ell)$ (t_ℓ divides t_m).*

Then there is an $\aleph_1$-free group H such that $G \subseteq H, |H| = \lambda$ and H has $\bar t$-density μ.

Proof. Choose $\lambda_n < \mu$ for $n < \omega$ such that $\prod\limits_{n<\omega} \lambda_n \ge \lambda, \mu \ge \sum\limits_{n<\omega} \lambda_n, 2\lambda_n < \lambda_{n+1}$ (so $\lambda_n > 0$ may be finite). Let $\{x_i : i < \lambda\}$ list the elements of G. Let $\lambda'_{n+1} = \lambda_{n+1}, \lambda'_0 = \mu$. Let $\eta_i \in \prod\limits_{n<\omega} \lambda_n$ for $i < \lambda$ be pairwise distinct such that $\eta_i(n+1) \ge \lambda_n$ and $i \ne j \Rightarrow (\exists m)(\forall n)[m \le n \Rightarrow \eta_i(n) \ne \eta_j(n)]$. Without loss of generality $\mu = \{\eta_i(n) : i < \lambda, n < \omega\}$. Let H be generated by G, x^m_i (for $i < \lambda'_m, m < \omega$), y^n_i (for $i < \lambda, n < \omega$) freely except for

(a) the equations of G

(b) $y^0_i = x_i \, (\in G)$

(c) $t_n y^{n+1}_i + x^n_{\eta_i(n)} = y^n_i$.

Fact A: H extends G and is torsion free.

Proof. H can be embedded into the divisible hull of $G \times F$, where F is the abelian group generated freely by $\{x^m_\alpha : m < \omega \text{ and } \alpha < \lambda'_m\}$.

Fact B: H is $\aleph_1$-free and moreover H/G is $\aleph_1$-free.

Proof. Let K be a countable pure subgroup of H. Now as we can increase K without loss of generality K is generated by

(i) $K_1 = \{x_i : i \in I\}$ is a pure subgroup of G, where I is some countably infinite subset of λ, and so $G \supseteq K_1$,

(ii) y^m_i, x^n_j for $i \in I, m < \omega$ and $(n, j) \in J$, where $J \subseteq \omega \times \lambda$ is countable and

$$i \in I, n < \omega \Rightarrow (n, \eta_i(n)) \in J$$

$$(n, j) \in J \Rightarrow j \in I.$$

Moreover, the equations holding among those elements are deducible from the equations of the form

$(a)^-$ equations of K_1

$(b)^-$ $y^0_i = x_i$ for $i \in I$

$(c)^-$ $t_n y^{n+1}_i + x^n_{\eta_i(n)} = y^n_i$ for $i \in I, n < \omega$.

We can find $\langle k_i : i < \omega\rangle$ such that $[i \neq j \ \ \& \ \ i \in I \ \ \& \ \ j \in I \ \ \& \ \ n \geq k_i \ \ \& \ \ n \geq k_j \ \ \& \ \ i \neq j \Rightarrow \eta_i(n) \neq \eta_j(n)]$.

Now we know that K_1 is free (being a countable subgroup of G), and it suffices to prove that K/K_1 is free. But K/K_1 is freely generated by $\{y^n_i : i \in I \text{ and } n > k_i\} \cup \{x^n_\alpha : (n,\alpha) \in J$ but for no $i \in I$ do we have $n > k_i, \eta_i(n) = \alpha\}$. So K is free.

<u>Fact C</u>: $H_0 = \langle x^n_i : n < \omega, i < \lambda'_n\rangle_H$ satisfies:

(a) $i < \lambda \Rightarrow d_{\bar t}(x_i, H_0) = \inf\{d_{\bar t}(x_i, z) : z \in H_0\} = 0$

(b) $x \in G \Rightarrow d_{\bar t}(x, H_0) = 0$

(c) $x \in H \Rightarrow d_{\bar t}(x, H_0) = 0$.

Proof. First note that

$(*)_1$ $Y = \{x \in H : d_{\bar t}(x, H_0) = 0\}$ is a subgroup of H.

Also for every $i < \lambda$ and n

$$(*)_2 \ \ y^n_i = x^n_{\eta_i(n)} + t_n y^{n+1}_i = x^n_{\eta_i(n)} + t_n x^{n+1}_{\eta_i(n+1)} + t_n t_{n+1} y^{n+2}_i$$
$$= \sum_{k=n}^{m} \Big(\prod_{\ell=n}^{k-1} t_\ell\Big) x^k_{\eta_i(k)} + \Big(\prod_{\ell=n}^{k} t_\ell\Big) y^{k+1}_i$$

(prove by induction on $m \geq n$), and note that as $(\forall \ell)(\exists m > \ell)(t_i$ divides $t_m)$ necessarily $(\forall \ell)(\exists^\infty m)(t_\ell$ divides $t_m)$ hence $(\forall k)(\exists^\infty m)(\prod_{i \leq \ell} t_\ell$ divides $\prod_{i=k}^{m} t_i)$. Now $(*)_2$ implies

$(*)_3$ $y^n_i \in Y$.

But $x_i = y_i^0$ and hence clause (a) holds, so as $\{x_i : i < \lambda\}$ is dense in G also clause (b) holds. So $G \subseteq Y$ (by clause (b)), and $x_\alpha^n \in Y$ (as $H_0 \subseteq Y$ and the choice of H_0) and $y_i^n \in Y$ (by $(*)_3$).
By $(*)_1$ clearly $Y = H$, as required in clause (c).

Fact D: $|H| = \lambda$.

Fact E: The $\bar{t}$-density of H is μ.

Proof. It is $\leq \mu$ as H_0 has cardinality μ and is $\bar{t}$-dense in H, it is $\geq \mu$, as we now show.

Define a function h with domain the generators of H listed above, into H. Let $h(x) = 0$ if $x \in G; h(x_\alpha^m) = 0$ if $m > 0 \vee \alpha < \lambda_0; h(x_\alpha^m) = x_\alpha^m$ if $m = 0 \& \lambda_0 \leq \alpha < \lambda_0'(= \mu); h(y_i^m) = 0$ if $m < \omega, i < \lambda$.
This function preserves the equations defining H and hence induces a homomorphism $\hat{h}$ from H onto $\langle \mathrm{Rang}(h) \rangle_H = \langle \{x_\alpha^0 : \alpha < \lambda_0', \alpha \geq \lambda_0\} \rangle_H$. Clearly $\hat{h}(h(x)) = \hat{h}(x)$ for the generators hence $\hat{h} \circ \hat{h} = \hat{h}$. Hence $\langle \{x_\alpha^n : \alpha < \lambda_0', \alpha \geq \lambda_0\} \rangle_H$ is a direct summand of H and hence the $d_{\bar{t}}$-density of H is at least the $d_{\bar{t}}$-density of $\langle \{x_\alpha^n : \alpha \in [\lambda_0', \lambda_0)\} \rangle_H$ which is $\lambda_0' = \mu$. $\square_{3.3}$

We define variants of Definition 2.1.

3.4 Definition. For $\bar{\lambda} = \langle \lambda_\ell : \ell < \omega \rangle, \bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$, we let

$$
\begin{aligned}
J^5_{\bar{t},\bar{\lambda}} = \Big\{ X \subseteq \prod_{\ell<\omega} [\lambda_\ell]^{t_\ell} : & \text{ we cannot find } m(*) < \omega, \bar{Y} = \langle Y_m : m < \omega \text{ and } m \geq m(*) \rangle, \\
& \bar{A}^m = \langle A_\eta : \eta \in Y_m \rangle \text{ such that:} \\
& (a) \qquad Y_m \subseteq \prod_{\ell<m} [\lambda_\ell]^{t_\ell} \\
& (b) \qquad Y_{m(*)} \subseteq \prod_{\ell<m(*)} [\lambda_\ell]^{t_\ell} \text{ is a singleton} \\
& (c) \qquad \langle A_\eta : \eta \in Y_m \rangle \text{ is a sequence of pairwise disjoint} \\
& \qquad\quad \text{subsets of } \lambda_m \text{ each of order type } \omega \\
& (d) \qquad Y_{m+1} = \{ \eta\hat{\ }\langle u \rangle : \eta \in Y_m \text{ and } u \in [A_\eta]^{t_m} \} \\
& (e) \qquad Y_m \subseteq \{ \nu \restriction m : \nu \in X \} \Big\},
\end{aligned}
$$

$J^6_{\bar t,\bar\lambda}$ is defined similarly but $m(*) = 0$,

$$J^\ell_{\bar t,\bar\lambda,<\theta} = \Big\{X : \text{ for some } \alpha < \theta \text{ and } X_\beta \in J^\ell_{\bar t,\bar\lambda} \text{ for } \beta < \alpha \text{ we have } X \subseteq \bigcup_{\beta<\alpha} X_\beta\Big\}.$$

Also let $J^\ell_{\bar t,\bar\lambda,\theta} = J^\ell_{\bar t,\bar\lambda,<\theta^+}$.

3.5 Claim. *1) $J^{i(1)}_{\bar t,\bar\lambda,<\theta_1} \subseteq J^{i(2)}_{\bar t,\bar\lambda,<\theta_2}$ when $\theta_1 \le \theta_2$ and $i(1) \le i(2)$ are among 4,5,6.*
2) $J^i_{\bar t,\bar\lambda,\theta}$ is a θ^+-complete ideal for $i = 4,5,6$.
3) If $\lambda_\ell \ge \beth_{t_\ell-1}(\theta)$ then the ideal $J^i_{\bar t,\bar\lambda,\theta}$ is proper for $i = 4,5,6$.

Proof. 1), 2) Easy.
3) As in 2.4. $\square_{3.5}$

3.6 Definition. Let $\bar\lambda = \langle \lambda_\ell : \ell < \omega\rangle, \bar t = \langle t_\ell : \ell < \omega\rangle$ such that $2 \le t_\ell < \omega$ and $(\forall n)(\exists m > n)(t_n | t_m)$ we define

- (A) $B^{\rm rtf}_{\bar t,\bar\lambda}$ is the free (abelian) group generated by $\{x^m_\alpha : m < \omega, \alpha < \lambda_m\}$.
- (B) Let $B^{\rm rtf}_{\bar t,\bar\lambda,n}$ be the subgroup of $B^{\rm rtf}_{\bar t,\bar\lambda}$ generated by $\{x^m_\alpha : m < n \text{ and } \alpha < \lambda_m\}$
- (C) $G^{\rm rtf}_{\bar t,\bar\lambda}$ is the pure closure in $(B^{\rm rtf}_{\bar t,\bar\lambda})^{[\bar t]}$ of the subgroup of $(B^{\rm rtf}_{\bar t,\bar\lambda})^{[\bar t]}$ generated by
$$B^{\rm rtf}_{\bar t,\bar\lambda} \cup \{\sum_{m<\omega}(\prod_{\ell<m} t_\ell)(x^m_{(\eta(\ell))(1)} - x^m_{(\eta(\ell))(0)}) : \eta \in \prod_{\ell<\omega}[\lambda_\ell]^2\}$$
(here we use the notation that if e.g. $\eta(\ell) = \{\alpha,\beta\}, \alpha < \beta$ then $(\eta(\ell))(1) = \beta, (\eta(\ell))(0) = \alpha$).
- (D) Let $\bar B^{\rm rtf}_{\bar t,\bar\lambda} = \langle B^{\rm rtf}_{\bar t,\bar\lambda,n} : n < \omega\rangle$.

To cover also the case $\neg(\forall n)(\exists m > n)(t_n | t_m)$ we can use

3.7 Definition. Let $\aleph_0 \le \lambda_\ell \le \lambda_{\ell+1}$ for $\ell < \omega$.
Let $\bar\lambda = \langle \lambda_\ell : \ell < \omega\rangle, \bar t = \langle t_\ell : \ell < \omega\rangle, 2 \le t_\ell < \omega, \aleph_0 \le \lambda_\ell \le \lambda_{\ell+1}, \neg(\forall n)(\exists m > n)(t_n | t_m)$. Let clauses (A), (B), (D) be as in Definition 3.6 but clause (C) is replaced by

$(C)'$ we choose $\bar{Y}^* = \langle Y^*_m : m < \omega \rangle$ such that $Y^*_m \subseteq \prod\limits_{\ell < m} [\lambda_\ell]^2, Y^*_0 = \{<>\}$ and for each m there is a sequence $\langle A^*_\eta : \eta \in Y^*_m \rangle$ of pairwise disjoint subsets of λ_m each of cardinality λ_m such that $Y^*_{m+1} = \cup\{[A^Y_\eta]^2 : \eta \in Y^*_m\}$. Let $Y^*_\omega = \{\eta \in \prod\limits_{\ell < \omega} [\lambda_\ell]^2$: for every $m < \omega$ we have $\eta \restriction m \in Y^*_m\}$. Let $G^{rtf}_{\bar{t},\bar{\lambda}}$ be the abelian group generated by

$$B^{rtf}_{\bar{t},\bar{\lambda}} \cup \{x_\eta, y_{\eta,\ell} : \eta \in Y^*_\omega, \ell < \omega\}$$

freely except the equations which hold in $B^{rtf}_{\bar{t},\bar{\lambda}}$ and $y_{\eta,0} = x_\eta$ and

$$t_\ell y_{\eta,\ell+1} - y_{\eta,\ell} = x^\ell_{(\eta(\ell))(1)} - x^\ell_{(\eta(\ell))(0)}.$$

3.8 Definition. Assume

$\boxtimes^{\bar{t}}_{H,\bar{H}}$ $\bar{H} = \langle H_n : n < \omega \rangle$ is an increasing sequence of abelian subgroups of H, such that $\bigcup\limits_{n < \omega} H_n$ is dense in H by the $\bar{t}$-adic topology.

Then we let

$$I^{4,\bar{t}}_{H,\bar{H}} = \Big\{X \subseteq H : \text{for some } n < \omega, \text{ the intersection of the } \bar{t}\text{-adic closure of } PC_H(X) \text{ in } H, \\ c\ell_{\bar{t}}(PC_H(X), H) \text{ with } \bigcup_{\ell < \omega} H_\ell \text{ is a subset of } H_n\Big\}$$

$$I^{4,\bar{t}}_{H,\bar{H},<\theta} = \Big\{X \subseteq H : \text{ for some } \alpha < \theta \text{ and } X_\beta \in I^{4,\bar{t}}_{H,\bar{H}} \text{ for } \beta < \alpha \text{ we have } X \subseteq \bigcup_{\beta < \alpha} X_\beta\Big\}$$

$$I^{4,\bar{t}}_{H,\bar{H},\theta} = I^{\bar{t}}_{H,\bar{H},<\theta^+}.$$

3.9 Definition. Assume $\bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$, and

$\boxtimes^{\bar{t}}_{H,\bar{H}}$ H is Hausdorff in the $\bar{t} \restriction [k,\omega)$-topology for each $k < \omega$ where $t \restriction [k,\omega) = \langle t_{k+\ell} : \ell < \omega \rangle$. Further $\bar{H} = \langle H_n : n < \omega \rangle$ is an increasing sequence of abelian groups, $\bigcup\limits_{n < \omega} H_n \subseteq H$ is dense in the $\bar{t} \restriction [k,\omega)$-adic topology for each $k < \omega$.

Then we let

1)
$$I^{5,\bar t}_{H,\bar H} = \big\{X \subseteq H : \text{for some } n(*) < \omega, \text{ for every } n \in (n(*),\omega) \text{ there is no } y \in H_{n+1} \text{ such that: } d_{\bar t \restriction [n,\omega)}(y, PC(\langle X\rangle_H)) = 0 \text{ but } d_{\bar t\restriction[n,\omega)}(y,H_n) > 0\big\}$$

$$I^{5,\bar t}_{H,\bar H,<\theta} = \big\{X : \text{ there are } \alpha < \theta \text{ and } X_\beta \in I^{5,\bar t}_{H,\bar H} \text{ for } \beta < \alpha \text{ such that } X \subseteq \bigcup_{\beta<\alpha} X_\beta\big\}.$$

Moreover $I^{5,\bar t}_{H,\bar H,\theta} = I^{5,t}_{H,\bar H,<\theta^+}$.

2) $I^{6,\bar t}_{H,\bar H}$ (and $I^{6,\bar t}_{H,\bar H,<\theta}, I^{6,\bar t}_{H,\bar H,\theta}$) are defined similarly except that we demand $n(*) = 0$.

3) $I^{i,\mathrm{rtf}}_{\bar t,\bar\lambda}$ means $I^{i,\bar t}_{G^{\mathrm{rtf}}_{\bar t,\bar\lambda},\bar B^{\mathrm{rtf}}_{\bar t,\bar\lambda}}$ where $\bar B^{\mathrm{rtf}}_{\bar t,\bar\lambda} = \langle B^{\mathrm{rtf}}_{\bar t,\bar\lambda,n} : n < \omega\rangle$.

3.10 Claim. *For $\bar\lambda, \bar t$ as in 3.4*

- (a) *we have $\boxtimes^{\bar t}_{G^{\mathrm{rtf}}_{\bar t,\bar\lambda},\bar B^{\mathrm{rtf}}_{\bar t,\bar\lambda}}$ (from 3.9)*
- (b) *$G^{\mathrm{rtf}}_{\bar t,\bar\lambda}$ is $\aleph_1$-free; moreover $G^{\mathrm{rtf}}_{\bar t,\bar\lambda}/B^{\mathrm{rtf}}_{\bar t,\bar\lambda,n}$ is $\aleph_1$-free for each $n < \omega$*
- (c) *$I^{i,\mathrm{rtf}}_{\bar t,\bar\lambda,\theta}$ are θ^+-complete ideals for $i = 4,5,6$*
- (d) *if $\boxtimes^{\bar t}_{H,\bar H}$ (from 3.9) and $i \in \{4,5,6\}$ then $I^{i,\bar t}_{H,\bar H,\theta}$ is a θ^+-complete ideal.*

Proof. Straightforward (for (6), use an argument similar to that of 3.3). $\square_{3.10}$

The following lemma connects the combinatorial ideals defined above and the more algebraic ideals defined in 3.8.

3.11 Claim. *1) Assume*

$\boxtimes_1$ $\bar t = \langle t_\ell : \ell < \omega\rangle, 2 \le t_\ell < \omega$
$\boxtimes_2$ $\bar\lambda = \langle \lambda_\ell : \ell < \omega\rangle$, *and* $\lambda_\ell > \beth_1(\theta)$ *for* $\ell < \omega$.

Then the ideal $I^{i,\mathrm{rtf}}_{\bar t,\bar\lambda,\theta}$ is proper for $i = 4,5,6$.
2) Assume $\boxtimes_1$ and

$\boxtimes'_2$ $\bar\lambda = \langle\lambda_\ell : \ell < \omega\rangle, \lambda_\ell = \aleph_0, \theta = \aleph_0$.

Then the ideal $I^{i,\mathrm{rtf}}_{\bar{t},\bar{\lambda},\theta}$ is proper.

Proof. 1) If not, we can find $X_\alpha \subseteq L =: G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ for $\alpha < \theta$ such that $G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}} = \bigcup\limits_{\alpha<\theta} X_\alpha$ and $X_\alpha \in I^{i,\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$. For $\alpha \le \omega$ and $\eta \in \prod\limits_{\ell<\alpha} [\lambda_\ell]^2$ we let

$$x_\eta = \sum_{m<\alpha} (\prod_{\ell<m} t_\ell)(x^m_{(\eta(n))(1)} - x^m_{(\eta(n))(0)}).$$

As in the proof of 2.4, we can apply a partition theorem on trees (see [Sh:f, Ch.XI,3.5]) for the ideal $J_\ell = ERI^2_\theta(\lambda_\ell)$ (this ideal is, of course, θ^+-complete and non-trivial as $\lambda_\ell > 2^\theta$).

So we can find $\langle Y_m : m < \omega\rangle, \langle A_\eta : \eta \in Y_m\rangle$ and $\alpha(*) < \theta$ such that

- (a) $Y_m \subseteq \prod\limits_{\ell<m} [\lambda_\ell]^2$
- (b) Y_0 is a singleton
- (c) $A_\eta \in (J_{\ell g(\eta)})^+$ for $\eta \in Y_m$ (so $A_\eta \subseteq [\lambda_{\ell g(\eta)}]^2$)
- (d) $Y_{m+1} = \{\eta\hat{\ }\langle u\rangle : u \in A_\eta, \eta \in Y_m\}$
- (e) if $\eta \in Y_m$ then $\eta \in \{\nu \restriction m : x_\nu \in X_{\alpha(*)}\}$.

We now prove by induction on $k < \omega$ that

$(*)_k$ for any $m < \omega$, if $\eta \in Y_m$ and $A \subseteq A_\eta$ is from $(J_m)^+$ then for some infinite $A' \subseteq \lambda_m$ for any $\alpha < \beta$ from A' we have
$$\otimes^k_{\alpha,\beta} \quad (\prod_{\ell<m} t_\ell)(x^m_\beta - x^m_\alpha) \in c\ell_{\bar{t}}(\langle X_{\alpha(*)}\rangle, L) + (\prod_{\ell<m+k} t_\ell)L.$$

For $k = 0$ this is trivial: the element $(\prod\limits_{\ell<m} t_\ell)(x^m_\beta - x^m_\alpha)$ belongs to $(\prod\limits_{\ell<m+k} t_\ell)L$.
For $k+1$, to prove $(*)_{k+1}$ we are given $m < \omega, \eta \in Y_m$ and $A' \subseteq A_\eta, A' \in (J_\eta)^+$, and have to find $\{\alpha,\beta\} \in A'$ such that $\otimes^{k+1}_{\alpha,\beta}$ holds. For $\ell \in [m,\omega)$, as J_ℓ is an ideal we can find $A''_\nu \in (J_\ell)^+$ for $\nu \in Y_\ell$ such that $A''_\nu \subseteq A_\nu$ and the statement $\otimes^k_{\alpha,\beta}$ holds for every $\{\alpha,\beta\} \in A''_\nu$ or for no $\{\alpha,\beta\} \in A''_\nu$ and $\nu = \eta \Rightarrow A''_\nu \subseteq A'_\nu$. As we are assuming $(*)_k$ necessarily $\{\alpha,\beta\} \in A''_\nu \Rightarrow \otimes^k_{\alpha,\beta}$. By renaming without loss of generality $A''_\nu = A_\nu$. As $A_\eta \in (J_m)^+$, by the choice of J_m we can let $\gamma_0 < \gamma_1 < \gamma_2 < \dots$ be in A_η. So for each $j < \omega$, let $\eta_j \in Y_{m+k+1}$, (yes, not $\eta_j \in Y_{m+1}$!) be such that

$\eta_j \restriction m = \eta, \eta_j(m) = \{\gamma_j, \gamma_{j+1}\}$. By clause (e) above we know that there are ν_j such that $\eta_j \triangleleft \nu_j \in \prod\limits_{\ell<\omega} [\lambda_\ell]^2$ and

(i) $x_{\nu_j} \in X_{\alpha(*)}$.
Now by the definitions of x_{η_j}, x_{ν_j}

(ii) $x_{\eta_j} = x_{\nu_j} \bmod (\prod\limits_{\ell<m+k+1} t_\ell)L$

(iii) if $\ell \in [m+1, m+k+1)$ and $j < \omega$ then
$x_{\eta_j \restriction (\ell+1)} - x_{\eta_j \restriction \ell} \in c\ell_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod\limits_{i<\ell+k} t_i)L \subseteq c\ell_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod\limits_{i<m+k+1} t_i)L$

[why? the first inclusion, by the induction hypothesis as the difference is $(\prod\limits_{i<m+\ell} t_i)(x^\ell_{(\eta_j(\ell))(1)} - x^\ell_{(\eta_j(\ell))(0)})$, the second inclusion as $m+1 \le \ell$]

(iv) $x_{\eta_j} - x_{\eta_j \restriction (m+1)} \in c\ell_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod\limits_{i<m+k+1} t_i)L$
[why? use (iii) for $\ell = m+1, \dots, m+k$, noting that $\ell g(\eta_j) = m+k+1$.]

(v) $x_{\eta_j \restriction (m+1)} \in c\ell_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod\limits_{i<m+k+1} t_i)L$
[why? by (i) + (ii) + (iv)]

(vi) $\sum\{x_{\eta_j \restriction (m+1)} : j < \prod\limits_{i<m+k+1} t_i\} \in c\ell_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod\limits_{i<m+k+1} t_i)L$
[why? by (v)]

(vii) $x^m_{\gamma_{j(*)}} - x^m_{\gamma_0} \in c\ell_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod\limits_{i<m+k+1} t_i))L$ for $j(*) = \prod\limits_{i<m+k+1} t_i$
[why? by (vi) because

$$\begin{aligned}
&\sum\{x_{\eta_j \restriction (m+1)} : j < \prod_{i<m+k+1} t_i\} \\
&= \sum\{x_{\eta_j \restriction m} + (\prod_{i<m} t_i)(x^m_{\gamma_{j+1}} - x^m_{\gamma_j}) : j < \prod_{i<m+k+1} t_i\} \\
&= \sum\{x_{\eta_j \restriction m} : j < \prod_{i<m+k+1} t_i\} + (\prod_{i<m} t_i) \sum\{(x^m_{\gamma_{j+1}} - x^m_{\gamma_j}) : j < \prod_{i<m+k+1} t_i\}
\end{aligned}$$

[as $\eta_j \restriction m$ does not depend on j and obvious arithmetic]

$$= (\prod_{i<m+k+1} t_i) \cdot x_{\eta_{j(*)} \restriction m} + (\prod_{i<m} t_i)(x^m_{\gamma_{j(*)}} - x^m_{\gamma_0}) \in$$

$$(\prod_{i<m} t_i)(x^m_{\gamma_{j(*)}} - x^m_{\gamma_0}) + (\prod_{i<m+k+1} t_i)L]$$

$(viii)$ if $\rho \in Y_m$ and $\alpha < \beta$ are in A_η then
$(\prod\limits_{i<m+1} t_i)(x^m_\beta - x^m_\alpha) \in c\ell_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod\limits_{i<m+k+1} t_i)L$
[why? by (vii) and the choice of the $Y_m, A_\eta(\eta \in Y_m, m < \omega)$.]

So we have carried the induction on k.
2) Easier. $\square_{3.11}$

3.12 Claim. *Assume*

$\boxtimes_1$ $\bar t = \langle t_\ell : \ell < \omega\rangle$ *and* $2 \le t_\ell < \omega$

$\boxtimes_2$ $\lambda_\ell > \beth_1(\theta)$

$\boxtimes_3$ $cov(\lambda, (\prod\limits_{\ell<\omega} \lambda_\ell)^+, (\prod\limits_{\ell<\omega} \lambda_\ell)^+, \theta^+) \le \lambda$.

Then $\mathbf{U}_{J^6_{\bar t,\bar\lambda,\theta}}(\lambda) = \lambda$ *and* $\mathbf{U}_{I^6_{\bar t,\bar\lambda,\theta}}(\lambda) = \lambda$.

Proof. By the previous claims 3.10, 3.11 (and the relevant definitions 3.6 - 3.9.

3.13 Conclusion. For every $\lambda \ge \beth_\omega$ for some $\theta < \beth_\omega$, for every $\kappa \in (\beth_1(\theta), \beth_\omega)$ for every $\lambda_n \in [\beth_\omega(\theta), \kappa]$ we have

$$\mathbf{U}_{I^6_{\bar t,\bar\lambda,\theta}}(\lambda) = \lambda = \mathbf{U}_{J^6_{\bar t,\bar\lambda,\theta}}(\lambda).$$

Proof. By the previous claim and [Sh 460] (similar to 2.6). $\square_{3.13}$

3.14 Claim. *Assume*

(a) $\prod\limits_{\ell<\omega} \lambda_\ell < \mu < \lambda = cf(\lambda) \le \lambda' \le \lambda'' < \mu^{\aleph_0}$

(b) $\mu^+ < \lambda$ *or at least for some* $\mathscr{P}$,

$(*)_{\mathscr{P}}$ $|\mathscr{P}| = \lambda$ *and* $(\forall a \in \mathscr{P})(a \subseteq \lambda \ \& \ \mathrm{otp}(a) = \mu)$
and $(\forall E)(E$ *a club of* $\lambda \rightarrow (\exists a \in \mathscr{P})(a \subseteq E))$

(c) $\lambda'' = \mathbf{U}_{I^6_{\bar t,\bar\lambda}}(\lambda') < \mu^{\aleph_0}$ *where* $t_m = \prod_{\ell<m} \ell!$ *or at least* $\lambda'' = \mathbf{U}_{J^6_{\bar\lambda,\bar t}}(\lambda')$

(d) *cov*$(\lambda'', \lambda^+, \lambda^+, \lambda) < \mu^{\aleph_0}$ *or at least* $\mathbf{U}_{\mathrm{id}^a(\mathscr{P})}(\lambda'') < \mu^{\aleph_0}$ *where* $\mathscr{P}$ *satisfies the demand* $(*)_{\mathscr{P}}$.

__Then__ we can find $\aleph_1$*-free abelian groups* G_α *of cardinality* λ *for* $\alpha < \mu^{\aleph_0}$ *such that for every* $\aleph_1$*-free abelian group* G *of cardinality* λ *or just* $G \in \mathfrak{K}^{\mathrm{rtf}}_\lambda$ *we have:*

> *some* G_α *is not embeddable into* G*; also the number of ordinals* $\alpha < \mu^{\aleph_0}$ *for which* G_α *is embeddable into* G *is at most cov*$(\lambda'', \lambda^+, \lambda^+, \lambda)$ *(or* $\leq \mathbf{U}_{\mathrm{id}^a(\mathscr{P})}(\lambda'')$ *at least)*

Proof. Like 2.8, note that "$\aleph_1$-free" implies $\| - \|_{\bar t}$ is a norm.

3.15 Conclusion. If $\beth_\omega \leq \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ __then__ in $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ there is no member universal even just for $\mathfrak{K}^{\aleph_1\text{-free}}_\lambda$.

Proof. Straightforward.

3.16 Remark. In §2 we can use the parallel of 3.11.

3.17 Remark. If $\lambda = \aleph_0$ there is no universal member in $\mathfrak{K}^{\mathrm{rtf}}_\lambda$. In fact for any $\mathbf{Q} \subset \mathbf{P}^*$ let $G_{\mathbf{Q}}$ be the subgroup of $\mathbb{Q}x \oplus \bigoplus_p \{\mathbb{Q}x_p : p \in \mathbf{P}^* \backslash \mathbf{Q}\}$ generated by

$$\begin{aligned} &\{p^{-n}x : p \in \mathbf{Q}\} \cup \{q^{-n}x_p : p \in \mathbf{P}^*\backslash\mathbf{Q} \text{ and } n < \omega, \text{ and } q \in \mathbf{P}^*\backslash\{p\}\} \\ &\qquad \cup \{p^{-n}(x - x_p) : n < \omega \text{ and } p \in \mathbf{Q}\}. \end{aligned}$$

So $G_{\mathbf{Q}} \in \mathfrak{K}^{\mathrm{rtf}}_{\aleph_0}$, and (see Definition 1.3) $\mathbf{P}(x, G_{\mathbf{Q}}) = \mathbf{Q}$ and $\mathbf{P}^-(x, G_{\mathbf{Q}}) = \mathbf{P}^*\backslash\mathbf{Q}$ hence (see 1.4) if h embeds $G_{\mathbf{Q}}$ into $G \in K^{\mathrm{trf}}$ then $\mathbf{P}(h(x), G) = \mathbf{Q}$. As the number of possible $\mathbf{Q}$'s is $2^{\aleph_0}$ we are easily done. This proof gives an alternative proof to 1.2, but the proof there looks more promising for generalization.

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