Possible Size of an Ultrapower of ω [1]

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Abstract

Let ω be the first infinite ordinal (or the set of all natural numbers) with the usual order $<$. In §1 we show that, assuming the consistency of a supercompact cardinal, there may exist an ultrapower of ω, whose cardinality is (1) a singular strong limit cardinal, (2) a strongly inaccessible cardinal. This answers two questions in [1], modulo the assumption of supercompactness. In §2 we construct several λ-Archimedean ultrapowers of ω under some large cardinal assumptions. For example, we show that, assuming the consistency of a measurable cardinal, there may exist a λ-Archimedean ultrapower of ω for some uncountable cardinal λ. This answers a question in [8], modulo the assumption of measurability.

0. On Notation and Boolean Algebras

An important way of constructing a desired ultrafilter on κ is to use the construction of an ultrafilter $\mathcal{E}$ on the Boolean algebra $\mathbb{B} = \mathcal{P}(\kappa)/\mathcal{D}$ for some filter $\mathcal{D}$ on κ as an intermediate step. The construction of $\mathcal{E}$ has a great deal of flexibility when $\mathbb{B}$ contains a large free (or κ-free) subalgebra. In this paper we always construct an ultrafilter $\mathcal{E}$ on $\mathbb{B}$ first such that $\omega^{\mathbb{B}}/\mathcal{E}$, the Boolean ultrapower of ω modulo $\mathcal{E}$, has some desired properties, and then use $\mathcal{E}$ to define an ultrafilter $\mathcal{F}$ on κ so that $\omega^{\kappa}/\mathcal{F}$, the ultrapower of ω modulo $\mathcal{F}$, is isomorphic to $\omega^{\mathbb{B}}/\mathcal{E}$. In each case a large cardinal is used to construct $\mathcal{D}$ so that $\mathbb{B} = \mathcal{P}(\kappa)/\mathcal{D}$ always contains a large free (or κ-free) subalgebra.

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[15] is recommanded for the general theory of the Boolean ultrapower of arbitrary models. Here we give the definitions and facts needed in this paper to keep it self-contained.

Throughout this paper we use κ, λ, η, etc. for infinite cardinals, α, β, γ, etc. for ordinals, and k, l, m, n, etc. for natural numbers. Let A and B be two sets. We denote by A^B for the set of all functions from B to A (except in the case when B is a Boolean algebra). We also write κ^λ for exponents in cardinal arithmetic, and this should be clear from the context. Let $\mathcal{P}_\kappa(\lambda)$ be the set of all subsets of λ of size $< \kappa$. Let $\mathcal{D}, \mathcal{E}, \mathcal{F}$, etc. denote filters or ultrafilters, and let $\mathbb{B}, \mathbb{C}$, etc. denote Ba or cBa, *i.e.* Boolean algebras or complete Boolean algebras.

We shall not distinguish a Ba $(\mathbb{B}; \vee, \wedge, -, 0, 1)$ from its base set $\mathbb{B}$. For any $S \subseteq \mathbb{B}$ let $\bigvee S$ ($\bigwedge S$) be the least upper bound (greatest lower bound) of S in $\mathbb{B}$, provided it exists. By an anti-chain in $\mathbb{B}$ we mean a subset $A \subseteq \mathbb{B}$ such that for any $a, b \in A$, $a \neq b$ implies $a \wedge b = 0$. A Ba $\mathbb{B}$ has κ-c.c. iff every anti-chain A in $\mathbb{B}$ has size $< \kappa$. ω_1-c.c. is called also c.c.c.

We write $\mathbb{C} \subseteq \mathbb{B}$ to denote that $\mathbb{C}$ is a subalgebra of $\mathbb{B}$ and for any $S \subseteq \mathbb{C}$, if $\bigvee S = c$ in $\mathbb{C}$ and $\bigvee S = b$ in $\mathbb{B}$, then $b = c$. Hence we shall not distinguish $\bigvee$ and $\bigwedge$ in $\mathbb{C}$ or in $\mathbb{B}$.

A $\mathbb{B}$ is called a κ-complete Ba or a κ-cBa iff for any $S \subseteq \mathbb{B}$, $|S| < \kappa$ implies $\bigvee S \in \mathbb{B}$. $\mathbb{B}$ is complete iff it is κ-complete for every κ. ω_1-complete is also called countably complete. Given $a \in \mathbb{B}$, let a^0 denote a and a^1 denote $-a$. Let $\mathbb{C} \subseteq \mathbb{B}$. Then a sequence $\{a_\alpha : \alpha < \lambda\}$ in $\mathbb{B}$ is called κ-independent over $\mathbb{C}$ iff for any $\sigma \in \mathcal{P}_\kappa(\lambda)$, for any $h \in 2^\sigma$ and for any $c \in \mathbb{C} \smallsetminus \{0\}$ one has

$$c \wedge (\bigwedge \{a_\alpha^{h(\alpha)} : \alpha \in \sigma\}) \neq 0.$$

A sequence in $\mathbb{B}$ is κ-independent iff it is κ-independent over $\{0,1\}$. A sequence $\{\mathbb{C}_\alpha : \alpha < \lambda\}$ of subalgebras of $\mathbb{B}$ is called κ-independent iff for any $\sigma \in \mathcal{P}_\kappa(\lambda)$ and for any $a_\alpha \in \mathbb{C}_\alpha \setminus \{0\}$

$$\bigwedge_{\alpha\in\sigma} a_\alpha \neq 0.$$

We shall omit κ when $\kappa = \omega$. Let $\mathbb{B}$ be κ-complete. Then for any $S \subseteq \mathbb{B}$ we denote by $\langle S\rangle_\kappa \subseteq \mathbb{B}$ the κ-complete subalgebra of $\mathbb{B}$ generated by S. For any $\mathbb{B}$ let $\bar{\mathbb{B}}$ denote the completion of $\mathbb{B}$. A Ba $\mathbb{B}$ is called κ-free iff there exists a κ-independent sequence $\{a_\alpha : \alpha < \lambda\}$ in $\mathbb{B}$ such that

$$\mathbb{B} = \langle\{a_\alpha : \alpha < \lambda\}\rangle_\kappa.$$

The cardinal λ above is called the dimension of $\mathbb{B}$. Note that if $\kappa^{<\kappa} = \kappa$, then a κ-free Ba has κ^+-c.c.. A Ba is free if it is ω-free. Given two Ba's $\mathbb{B} \subseteq \mathbb{C}$, a homomorphism[4] $r : \mathbb{C} \mapsto \mathbb{B}$ is called a retraction iff $r\restriction\mathbb{B}$ is an identity map.

Let $\mathbb{B}$ be ω_1-complete. Let

$$\omega^{\mathbb{B}} = \{t : t \text{ is a function from } \omega \text{ to } \mathbb{B} \text{ such that}$$

$$(1)\ \forall n \neq m\ (t(n) \wedge t(m) = 0) \text{ and } (2)\ \bigvee\{t(n) : n \in \omega\} = 1.\}.$$

For any ultrafilter $\mathcal{E}$ on $\mathbb{B}$ let $\sim_{\mathcal{E}}$ denote the equivalence relation on $\omega^{\mathbb{B}}$ such that $s \sim_{\mathcal{E}} t$ iff $\bigvee_{n\in\omega}(s(n) \wedge t(n)) \in \mathcal{E}$ for any $s, t \in \omega^{\mathbb{B}}$. For any $t \in \omega^{\mathbb{B}}$ let $t_{\mathcal{E}}$ denote the equivalence class containing t. Then $\omega^{\mathbb{B}}/\mathcal{E}$ is the set $\{t_{\mathcal{E}} : t \in \omega^{\mathbb{B}}\}$ together with the total order $<_{\mathcal{E}}$, which is defined by letting $s_{\mathcal{E}} <_{\mathcal{E}} t_{\mathcal{E}}$ iff $\bigvee_{m<n}(s(m) \wedge t(n)) \in \mathcal{E}$. We shall write $<$ instead of $<_{\mathcal{E}}$ when the meaning is clear. Note that if one identifies each $n \in \omega$ with $(t_n)_{\mathcal{E}}$, where $t_n(n) = 1$ and $t_n(m) = 0$ for any $m \neq n$, then $\omega^{\mathbb{B}}/\mathcal{E}$ is just an end-extension of ω. To make it more intuitive we often write $[\![s < t]\!]$ and $[\![s = t]\!]$ for the terms $\bigvee_{m<n}(s(m) \wedge t(n))$ and $\bigvee_{n\in\omega}(s(n) \wedge t(n))$, respectively.

[4] $r : \mathbb{C} \mapsto \mathbb{B}$ is a homomorphism iff $r(a \vee b) = r(a) \vee r(b)$, and $r(-a) = -r(a)$. Hence one has $r(a \wedge b) = r(a) \wedge r(b)$, $r(0) = 0$ and $r(1) = 1$. Note that $r(\bigvee S)$ may not be same as $\bigvee\{r(a) : a \in S\}$.

Suppose $\mathcal{D}$ is a filter on κ and let I be the dual ideal of $\mathcal{D}$. We often write $\mathcal{P}(\kappa)/\mathcal{D}$ instead of $\mathcal{P}(\kappa)/I$, the quotient Boolean algebra of $\mathcal{P}(\kappa)$ modulo I. For any $A \subseteq \kappa$ let $[A]_\mathcal{D}$ denote the equivalence class of A in $\mathcal{P}(\kappa)/\mathcal{D}$.

Suppose $\mathcal{D}$ is a countably complete filter on κ. Then the Boolean algebra $\mathbb{B} = \mathcal{P}(\kappa)/\mathcal{D}$ is countably complete, and for any $A_n \subseteq \kappa$ one has

$$[\bigcup_{n\in\omega} A_n]_\mathcal{D} = \bigvee_{n\in\omega} [A_n]_\mathcal{D}.$$

Let $\mathcal{E}$ be an ultrafilter on $\mathbb{B}$ and let

$$\mathcal{F} = \{A \subseteq \kappa : [A]_\mathcal{D} \in \mathcal{E}\}.$$

Then it is easy to see that $\mathcal{F}$ is an ultrafilter on κ extending $\mathcal{D}$. We want to show that $\omega^\mathbb{B}/\mathcal{E}$ and $\omega^\kappa/\mathcal{F}$ are isomorphic.

For any $f \in \omega^\kappa$ let $\hat{f}$ be a function from ω to $\mathbb{B}$ such that $\hat{f}(n) = [f^{-1}(n)]_\mathcal{D}$. Then it is easy to check that $\hat{f} \in \omega^\mathbb{B}$. Let i be the map from $\omega^\kappa/\mathcal{F}$ to $\omega^\mathbb{B}/\mathcal{E}$ such that $i(f_\mathcal{F}) = \hat{f}_\mathcal{E}$.

Lemma 0.1. *(folklore)*

The map i is an isomorphism from $\omega^\kappa/\mathcal{F}$ to $\omega^\mathbb{B}/\mathcal{E}$.

Proof: We show first that i is a well-defined one-one function, which preserves the order. This is because that for any two functions $f, g \in \omega^\kappa$, one has

$$f_\mathcal{F} < g_\mathcal{F} \Leftrightarrow \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in \mathcal{F} \Leftrightarrow$$

$$\bigcup_{m<n} (f^{-1}(m) \cap g^{-1}(n)) \in \mathcal{F} \Leftrightarrow [\bigcup_{m<n} (f^{-1}(m) \cap g^{-1}(n))]_\mathcal{D} \in \mathcal{E}.$$

But one has also

$$[\bigcup_{m<n} (f^{-1}(m) \cap g^{-1}(n))]_\mathcal{D} = \bigvee_{m<n} ([f^{-1}(m)]_\mathcal{D} \wedge [g^{-1}(n)]_\mathcal{D}) = \bigvee_{m<n} (\hat{f}(m) \wedge \hat{g}(n)).$$

So it is true that

$$f_\mathcal{F} < g_\mathcal{F} \Leftrightarrow \hat{f}_\mathcal{E} < \hat{g}_\mathcal{E} \Leftrightarrow i(f_\mathcal{F}) < i(g_\mathcal{F}).$$

Then we show that i is onto. Given any $t \in \omega^{\mathbb{B}}$, by countable completeness we could choose $A_n \subseteq \kappa$ inductively such that $\{A_n : n \in \omega\}$ is a partition of κ and $[A_n]_{\mathcal{D}} = t(n)$. Define an $f \in \omega^\kappa$ such that $f(\alpha) = n$ iff $\alpha \in A_n$. Clearly, $i(f_{\mathcal{F}}) = t_{\mathcal{E}}$. $\square$

Remark 0.2. *By this lemma to find an ultrafilter $\mathcal{F}$ on κ such that $\omega^\kappa/\mathcal{F}$ has some desired properties, it suffices to find a countably complete filter $\mathcal{D}$ on κ and an ultrafilter $\mathcal{E}$ on $\mathbb{B} = \mathcal{P}(\kappa)/\mathcal{D}$ such that $\omega^{\mathbb{B}}/\mathcal{E}$ has such properties.*

1. Strong Limit Cardinal as the Size of an Ultrapower of ω

An ultrafilter $\mathcal{F}$ on κ is called uniform iff $|A| = \kappa$ for every $A \in \mathcal{F}$. An ultrafilter $\mathcal{F}$ on κ is called regular iff there exists a family $\mathcal{C} \subseteq \mathcal{F}$ such that $|\mathcal{C}| = \kappa$ and for any infinite subfamily $\mathcal{C}_0 \subseteq \mathcal{C}$, $\bigcap \mathcal{C}_0 = \emptyset$.

The study of the cardinality of ultrapowers started in the 1960's. The subject is very closely related to the regularity of ultrafilters. Regular ultrafilters were introduced in [4] and [7]. It was proved there that if $\mathcal{F}$ is a regular ultrafilter on κ, then $\omega^\kappa/\mathcal{F}$ has size 2^κ. So it is natural to ask whether it is possible to have an ultrafilter $\mathcal{F}$ on κ such that the size of $\omega^\kappa/\mathcal{F}$ does not have the form 2^κ for any κ. In fact, Chang and Keisler asked this in the following forms (see page 252 of [1]).

Question 1.1. *Is it possible that the cardinality of an ultrapower of ω is a singular (strong limit) cardinal?*

Question 1.2. *Is it possible that the cardinality of an ultrapower of ω is a strongly inaccessible cardinal?*

The original form of Question 1.1 in [1] has no requirement for the cardinality to be a strong limit cardinal. Since the cardinal 2^κ could be singular, we would like to make it more specific by requiring the singular cardinality be also a strong limit.

Obviously, a positive answer to either Question 1.1 or Question 1.2 would imply the existence of a uniform non-regular ultrafilter. Since the existence of a uniform non-regular ultrafilter was unclear in the early 1970's, when the first edition of [1] was published, people were more interested in a general question of Chang and Keisler concerning the existence of uniform non-regular ultrafilters. A lot of work has been done since then for solving the general question. On one hand, Prikry [17], Ketonen [9], Donder [2], etc. showed that one may need to assume the consistency of some large cardinals to construct a uniform non-regular ultrafilter. For example, Donder proved that if there is no inner model containing a measurable cardinal, then (1) every uniform ultrafilter on a singular cardinal is regular, (2) every uniform ultrafilter on a regular cardinal κ with $(\kappa^+)^K = \kappa^+$ is regular, where K is the Dodd-Jensen core model. On the other hand, Magidor [14], Laver [13], Foreman, Magidor and Shelah [3], etc. showed that it is possible to construct a uniform non-regular ultrafilter with the help of some large cardinal axioms. For example, in [3] it is proved that, assuming κ is a huge cardinal and $\mu < \kappa$ is a regular cardinal, there is a forcing extension preserving every cardinal $\leqslant \mu$, in which there exists a uniform (fully) non-regular ultrafilter on μ^+. However, as far as we know, neither Question 1.1 nor Question 1.2 has been answered yet. In this section we are going to give positive answers to both questions by assuming the consistency of a supercompact cardinal. First, we need to introduce the Laver-indestructibility of a supercompact cardinal. See [5] or [6] for the basic facts of supercompact cardinals

A supercompact cardinal κ is called Laver-indestructible iff κ remains supercompact in any κ-directed closed forcing extension. The reader should consult [12] to see how one makes a supercompact cardinal Laver-indestructible by a κ-c.c. forcing of size κ.

Theorem 1.3. *Suppose κ is a Laver-indestructible supercompact cardinal. Let $\eta \geqslant \kappa$ be such that $\eta^{<\kappa} = \eta$ and let $\lambda \leqslant 2^\eta$ be such that $\lambda^\kappa = \lambda$. Then there exists an ultrafilter $\mathcal{F}$ on η such that $|\omega^\eta/\mathcal{F}| = \lambda$.*

Remark 1.4. *Note that λ could be a singular strong limit cardinal, say, $\lambda = \eta = \beth_{\kappa^+}(\kappa)$. So Theorem 1.3 answers Question 1.1. If one assumes that there is a strongly inaccessible cardinal λ above κ, then Question 1.2 could also be answered. But the next theorem shows that this extra assumption is not necessary.*

Theorem 1.5. *Suppose κ is a Laver-indestructible supercompact cardinal. Then there exists an ultrafilter $\mathcal{F}$ on κ such that $|\omega^\kappa/\mathcal{F}| = \kappa$.*

Let κ be a strongly compact cardinal and $\lambda \geqslant \kappa$. Then it is well-known (Lemma 33.1 of [5]) that any κ-complete filter on λ could be extended to a κ-complete ultrafilter on λ. Note that a supercompact cardinal is strongly compact.

Lemma 1.6. *Suppose κ is a Laver-indestructible supercompact cardinal. Let $\eta \geqslant \kappa$ be such that $\eta^{<\kappa} = \eta$ and let $\kappa \leqslant \lambda \leqslant 2^\eta$. Then there exists a κ-complete filter $\mathcal{D}$ on η such that $\mathbb{B} = \mathcal{P}(\eta)/\mathcal{D}$ has a κ-free dense subalgebra $\mathbb{C} \subseteq \mathbb{B}$ of dimension λ.*

Proof: Let $\mathbb{P} = Fn(\lambda, 2, \kappa)$ (see page 211 of [11] for the definition) be the forcing notion for adding λ Cohen subsets of κ. Note that $\mathbb{P}$ is κ-directed closed. By Laver-indestructibility κ is still supercompact in $V^{\mathbb{P}}$. Suppose $G \subseteq \mathbb{P}$ is a V-generic filter and $g = \bigcup G$. Then g is a function from λ to 2 in $V[G]$. In V one can choose a sequence $\{A_\alpha : \alpha < \lambda\}$ of subsets of η such that for any $\sigma \in \mathcal{P}_\kappa(\lambda)$ and for any $h \in 2^\sigma$ one has

$$|\bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)}| = \eta,$$

where we denote $A^0 = A$ and $A^1 = \eta \setminus A$ for $A \subseteq \eta$. The existence of such a sequence is guaranteed by $\eta^{<\kappa} = \eta$ (see page 288 of [11]). In $V^{\mathbb{P}}$ the set $\{A_\alpha^{g(\alpha)} : \alpha < \lambda\}$ forms a κ-complete filter base. Hence there exists an κ-complete ultrafilter $\mathcal{H}_g$ on η extending $\{A_\alpha^{g(\alpha)} : \alpha < \lambda\}$. Now back in V we define

$$\mathcal{D} = \{A \subseteq \eta : \Vdash A \in \mathcal{H}_{\dot{g}}\}.$$

It is clear that $\mathcal{D}$ is a κ-complete filter in V. Let I be the dual ideal of $\mathcal{D}$.

Claim 1.6.1. For any $\sigma \in \mathcal{P}_\kappa(\lambda)$ and for any $h \in 2^\sigma$ one has $\bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)} \notin I$.

Proof of Claim 1.6.1: Since for any $\alpha \in \sigma$

$$h \Vdash A_\alpha^{h(\alpha)} \in \mathcal{H}_{\dot{g}},$$

Then one has

$$h \Vdash \bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)} \in \mathcal{H}_{\dot{g}}.$$

Hence one has

$$\nVdash \eta \setminus \bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)} \in \mathcal{H}_{\dot{g}}. \quad \square\text{(Claim 1.6.1)}$$

Claim 1.6.2. If $A \notin I$, then there exists a $\sigma \in \mathcal{P}_\kappa(\lambda)$ and an $h \in 2^\sigma$ such that $\bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)} \setminus A \in I$.

Proof of Claim 1.6.2: Suppose $A \notin I$. Then

$$\nVdash \eta \setminus A \in \mathcal{H}_{\dot{g}}.$$

So there exists an $h \in \mathbb{P}$ such that

$$h \Vdash \eta \setminus A \notin \mathcal{H}_{\dot{g}}.$$

This means that

$$h \Vdash A \in \mathcal{H}_{\dot{g}}.$$

We now want to show that $\bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)} \setminus A \in I$, where $\sigma = dom(h)$. Suppose not. Then

$$\not\Vdash \eta \setminus (\bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)} \setminus A) \in \mathcal{H}_{\dot g}.$$

Hence there exists $h' \in \mathbb{P}$ such that

$$h' \Vdash \bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)} \setminus A \in \mathcal{H}_{\dot g}.$$

This implies

$$h' \Vdash \bigcap_{\alpha\in\sigma} A_\alpha^{h(\alpha)} \in \mathcal{H}_{\dot g} \text{ and } h' \Vdash A \notin \mathcal{H}_{\dot g}.$$

Let $\sigma' = dom(h')$. For any $\alpha \in \sigma' \cap \sigma$ one has

$$h' \Vdash A_\alpha^{h(\alpha)} \in \mathcal{H}_{\dot g} \text{ and } h' \Vdash A_\alpha^{h'(\alpha)} \in \mathcal{H}_{\dot g}.$$

Hence $h(\alpha) = h'(\alpha)$. So $h \cup h' \in \mathbb{P}$. But this contradicts

$$h \Vdash A \in \mathcal{H}_{\dot g}. \quad \square\text{(Claim 1.6.2)}$$

Let $a_\alpha = [A_\alpha]_{\mathcal{D}}$. By Claim 1.6.1 the sequence $\{a_\alpha : \alpha < \lambda\}$ is κ-independent. Let $\mathbb{C} = \langle\{a_\alpha : \alpha < \lambda\}\rangle_\kappa$. Then by Claim 1.6.2 $\mathbb{C}$ is dense in $\mathbb{B} = \mathcal{P}(\eta)/\mathcal{D}$. $\square$

Now we are ready to prove the theorems.

Proof of Theorem 1.3: Let $\lambda \leqslant 2^\eta$ be such that $\lambda^\kappa = \lambda$. Let $\mathcal{D}$ be a κ-complete filter on η obtained in Lemma 1.6 such that the Ba $\mathbb{B} = \mathcal{P}(\eta)/\mathcal{D}$ has a κ-free dense subalgebra $\mathbb{C} \subseteq \mathbb{B}$ of dimension λ. It is clear that $|\mathbb{C}| = \lambda$ and every element a in $\mathbb{B}$ is the least upper bound of an anti-chain in $\mathbb{C}$. Since $\mathbb{C}$ has κ^+-c.c., every anti-chain in $\mathbb{C}$ has size $\leqslant \kappa$. Hence $|\mathbb{B}| \leqslant |\mathbb{C}|^\kappa = \lambda^\kappa = \lambda$. This shows that for any ultrafilter $\mathcal{E}$ on $\mathbb{B}$ one has $|\omega^{\mathbb{B}}/\mathcal{E}| \leqslant \lambda^\omega = \lambda$.

We now want to show that there exists an ultrafilter $\mathcal{E}$ on $\mathbb{B}$ such that $|\omega^{\mathbb{B}}/\mathcal{E}| \geqslant \lambda$. Let $\{a_{\alpha,n} : \alpha < \lambda,\ n \in \omega\}$ be a κ-independent sequence in $\mathbb{C}$ such that

$$\mathbb{C} = \langle\{a_{\alpha,n} : \alpha < \lambda, n \in \omega\}\rangle_\kappa.$$

For each $\alpha < \lambda$ let $t_\alpha \in \omega^{\mathbb{B}}$ be such that

$$t_\alpha(0) = -(\bigvee_{n\in\omega} a_{\alpha,n}) \text{ and } t_\alpha(n+1) = a_{\alpha,n} \wedge -(\bigvee_{m<n} a_{\alpha,m}).$$

Let $\mathcal{E}_0 = \{[\![t_\alpha < t_\beta]\!] : \alpha < \beta < \lambda\}$.

Claim 1.3.1. $\mathcal{E}_0$ has the finite intersection property.

Proof of Claim 1.3.1: Since $[\![t_\alpha < t_\beta]\!] \wedge [\![t_\beta < t_\gamma]\!] \leqslant [\![t_\alpha < t_\gamma]\!]$, then one needs only to show that for any $\alpha_0 < \alpha_1 < \ldots < \alpha_k$ in λ

$$\bigwedge_{n<k} [\![t_{\alpha_n} < t_{\alpha_{n+1}}]\!] \neq 0.$$

Choose any $m_0 < m_1 < \ldots < m_k$ in ω. Then one has

$$\bigwedge_{n\leqslant k} t_{\alpha_n}(m_n+1) = \bigwedge_{n<k} (t_{\alpha_n}(m_n+1) \wedge t_{\alpha_{n+1}}(m_{n+1}+1)) \leqslant \bigwedge_{n<k} [\![t_{\alpha_n} < t_{\alpha_{n+1}}]\!].$$

But one has also that

$$\bigwedge_{n\leqslant k} t_{\alpha_n}(m_n+1) = \bigwedge_{n\leqslant k} (a_{\alpha_n,m_n} \wedge (\bigwedge_{l<m_n} (-a_{\alpha_n,l}))) \neq 0$$

by the independence of $a_{\alpha,n}$'s. $\square$ (Claim 1.3.1)

Let $\mathcal{E}$ be an ultrafilter on $\mathbb{B}$ extending $\mathcal{E}_0$. Then there is a strictly increasing sequence $\{(t_\alpha)_{\mathcal{E}} : \alpha < \lambda\}$ in $\omega^{\mathbb{B}}/\mathcal{E}$. Hence $|\omega^{\mathbb{B}}/\mathcal{E}| \geqslant \lambda$. Now the theorem follows from Lemma 0.1. $\square$

Proof of Theorem 1.5: Again by Lemma 1.6[5] one can find a κ-complete filter $\mathcal{D}$ such that $\mathbb{B} = \mathcal{P}(\kappa)/\mathcal{D}$ has a κ-free, κ^+-c.c. dense subalgebra $\mathbb{C} \subseteq \mathbb{B}$ of dimension κ generated by a κ-independent sequence $\{a_{\alpha,n} : \alpha < \kappa, n \in \omega\}$. Let $t_\alpha \in \omega^{\mathbb{B}}$ be same as in the proof of Theorem 1.3 for every $\alpha < \kappa$. For any successor ordinal α let $\mathbb{C}_\alpha = \langle\{a_{\beta,n} : \beta < \alpha, n \in \omega\}\rangle_\kappa$ and for any limit ordinal $\alpha < \kappa$ let $\mathbb{C}_\alpha = \bigcup_{\beta<\alpha} \mathbb{C}_\beta$.

[5]The full strength of supercompactness is not necessary here; for example, 2^κ-supercompactness suffices.

Note that for any successor α $\mathbb{C}_\alpha$ is atomic, for any limit α $\mathbb{C}_\alpha$ is not κ-complete, and for any $\alpha < \kappa$ $|\mathbb{C}_\alpha| < \kappa$. It is easy to see that $\mathbb{C} = \bigcup_{\alpha<\kappa} \mathbb{C}_\alpha$. For each $\alpha < \kappa$ let

$$D_\alpha = \{\bigvee A : A \text{ is a maximal anti-chain in } \mathbb{C}_\alpha\}.$$

Claim 1.5.1. The set $(\bigcup_{\alpha<\kappa} D_\alpha) \cup \{[\![t_\alpha < t_\beta]\!] : \alpha < \beta < \kappa\}$ has the finite intersection property.

Proof of Claim 1.5.1: We show by induction on γ that the set

$$(\bigcup_{\alpha<\gamma} D_\alpha) \cup \{[\![t_\alpha < t_\beta]\!] : \alpha < \beta < \gamma\}$$

has the finite intersection property. This is trivial when $\gamma = 0$ or γ is a limit ordinal. Let's assume that $\gamma = \beta + 1$ for some $\beta < \kappa$. It suffices to show that for any $a \in \bigcup_{\alpha<\beta} \mathbb{C}_\alpha \setminus \{0\}$, for any set of maximal anti-chains $A_0, A_1, \ldots, A_k \subseteq \mathbb{C}_\beta$ and for any $\alpha < \beta$

$$a \wedge (\bigwedge_{n \leqslant k} \bigvee A_n) \wedge [\![t_\alpha < t_\beta]\!] \neq 0.$$

Since $\bigvee_{n\in\omega} t_\alpha(n) = 1$ there is an $m \in \omega$ such that $a \wedge t_\alpha(m) \neq 0$. It is easy to see that $a \wedge t_\alpha(m) \wedge t_\beta(m+1) \neq 0$ because $t_\beta(m+1)$ is independent over $\mathbb{C}_\beta$ and $a \wedge t_\alpha(m) \in \mathbb{C}_\beta$. Now using the maximality of A_n's one could find $a_n \in A_n$ for each $n \leqslant k$ inductively on n such that

$$a \wedge t_\alpha(m) \wedge t_\beta(m+1) \wedge \bigwedge_{n\leqslant k} a_n \neq 0.$$

This finishes the proof because

$$a \wedge t_\alpha(m) \wedge t_\beta(m+1) \wedge \bigwedge_{n\leqslant k} a_n \leqslant a \wedge (\bigwedge_{n\leqslant k} \bigvee A_n) \wedge [\![t_\alpha < t_\beta]\!].$$

$\Box$(Claim 1.5.1)

By Claim 1.5.1 one can find an ultrafilter $\mathcal{E}$ on $\mathbb{B}$ extending the set

$$(\bigcup_{\alpha<\kappa} D_\alpha) \cup \{[\![t_\alpha < t_\beta]\!] : \alpha < \beta < \kappa\}.$$

Clearly, $\{(t_\alpha)_\mathcal{E} : \alpha < \kappa\}$ is a strictly increasing sequence of length κ in $\omega^\mathbb{B}/\mathcal{E}$. Hence $|\omega^\mathbb{B}/\mathcal{E}| \geqslant \kappa$. We need to show that $|\omega^\mathbb{B}/\mathcal{E}| \leqslant \kappa$.

Claim 1.5.2. For any maximal anti-chain $\{b_\alpha : \alpha < \kappa\}$ in $\mathbb{C}$ there exists a $\delta < \kappa$ such that $\bigvee\{b_\alpha : \alpha < \delta\} \in \mathcal{E}$.

Proof of Claim 1.5.2: Note that $|\mathbb{C}_\alpha| < \kappa$ for any $\alpha < \kappa$. Using the inaccessibility of κ one could show that there exists a $\delta < \kappa$ such that $\{b_\alpha : \alpha < \delta\}$ is a maximal anti-chain in $\mathbb{C}_\delta$. Hence $\bigvee\{b_\alpha : \alpha < \delta\} \in \mathcal{E}$. $\square$(Claim 1.5.2)

Claim 1.5.3. For any $t \in \omega^\mathbb{B}$ there exists an $s \in \omega^\mathbb{C}$ such that $t_\mathcal{E} = s_\mathcal{E}$.

Proof of Claim 1.5.3: Since $\mathbb{C}$ is dense in $\mathbb{B}$ and $\mathbb{C}$ has κ^+-c.c., then there exists a maximal anti-chain $\{b_\alpha : \alpha < \kappa\}$ in $\mathbb{C}$, which refines $\{t(n) : n \in \omega\}$, *i.e.* for any $\alpha < \kappa$ there exists an $n \in \omega$ such that $b_\alpha \leqslant t(n)$. By Claim 1.5.2 there is a $\delta < \kappa$ such that $\bigvee\{b_\alpha : \alpha < \delta\} \in \mathcal{E}$. Define $s \in \omega^\mathbb{C}$ such that

$$s(0) = (\bigvee\{b_\alpha : \alpha < \delta,\ b_\alpha \leqslant t(0)\}) \vee (-\bigvee\{b_\alpha : \alpha < \delta\})$$

and

$$s(n+1) = \bigvee\{b_\alpha : \alpha < \delta,\ b_\alpha \leqslant t(n+1)\}$$

for every $n \in \omega$. Here we use the fact that $\mathbb{C}$ is κ-complete and $\delta < \kappa$. It is now easy to check that

$$[\![t = s]\!] = \bigvee_{n \in \omega} (t(n) \wedge s(n)) \geqslant \bigvee\{b_\alpha : \alpha < \delta\} \in \mathcal{E}.$$

Hence $|\omega^\mathbb{B}/\mathcal{E}| \leqslant |\mathbb{C}|^\omega = \kappa$. Now the theorem follows from Lemma 0.1. $\square$

2. λ-Archimedean Ultrapowers

Let $\mathcal{L}$ be any first-order language including symbols for number theory, say, $\mathbb{N}, +, \cdot, <$ and S. An $\mathcal{L}$-structure $\mathfrak{A}$ is called a (standard or nonstandard) model of PA (PA stands for Peano Arithmetic) iff $(\mathcal{N}^\mathfrak{A}; +^\mathfrak{A}, \cdot^\mathfrak{A}, <^\mathfrak{A}, S^\mathfrak{A})$ is a (standard or nonstandard) model of PA (respectively). Given a cardinal λ, a model $\mathfrak{A}$ of PA is called λ-Archimedean iff $|\mathbb{N}^{\mathfrak{A}}| = \lambda$ and for every $n \in \mathbb{N}^{\mathfrak{A}}$, $|\{0, 1, \dots, n\}^{\mathfrak{A}}| < \lambda$.

In [8] Keisler and Schmerl asked whether one could find an ultrapower of a standard model of PA, which is λ-Archimedean for some cardinal $\lambda > \omega$. It is clear that the question remains same if one replaces the standard model of PA by $(\omega; <)$. If $\mathcal{F}$ is a regular ultrafilter on κ, then $\omega^\kappa/\mathcal{F}$ will never be λ-Archimedean for any $\lambda > \omega$ because there exists an $x \in \omega^\kappa/\mathcal{F}$ such that $|\{y \in \omega^\kappa/\mathcal{F} : y < x\}| = 2^\kappa = |\omega^\kappa/\mathcal{F}|$ (see, for example, [10]). So a positive answer to the question implies the existence of a uniform non-regular ultrafilter.

In this section we construct several λ-Archimedean ultrapowers under some large cardinal assumptions.

First, we list some lemmas needed in the proof of theorems.

Lemma 2.1. *(Sikorski)*

Let $\mathbb{B}$ be a cBa and $\mathbb{B} \subseteq \mathbb{C}$. Then there is a retraction r from $\mathbb{C}$ to $\mathbb{B}$. Furthermore, if I is an ideal of $\mathbb{C}$ and $I \cap \mathbb{B} = \{0\}$, then one could require that $r(a) = 0$ for every $a \in I$.

Proof: See page 70 of [16] for the first assertion. For the second assertion one considers the fact that $\mathbb{B}$ could be viewed as a subalgebra of $\mathbb{C}/I$. Then one uses the retraction $\bar{r}$ from $\mathbb{C}/I$ to $\mathbb{B}$ to induce a retraction r from $\mathbb{C}$ to $\mathbb{B}$. $\square$

Lemma 2.2. *(folklore)*

Let $\mathbb{B}$ and $\mathbb{C}$ be two ω_1-cBa's such that $\mathbb{B} \subseteq \mathbb{C}$. Let $\mathcal{E}$ be an ultrafilter on $\mathbb{B}$ and $\mathcal{E}'$ be an ultrafilter on $\mathbb{C}$ such that $\mathcal{E} \subseteq \mathcal{E}'$. Then the inclusion map i from $\omega^{\mathbb{B}}/\mathcal{E}$ to $\omega^{\mathbb{C}}/\mathcal{E}'$ such that $i(t_{\mathcal{E}}) = t_{\mathcal{E}'}$ is an elementary embedding.

See page 576 of [10] for a proof. From now on we shall view $\omega^{\mathbb{B}}/\mathcal{E}$ as a subset of $\omega^{\mathbb{C}}/\mathcal{E}'$ via the embedding i whenever $\mathcal{E} \subseteq \mathcal{E}'$.

Lemma 2.3. *(Exercise IV.3.35 of* [19] *or Lemma 1 of* [10]*)*

Let $\mathbb{B}$ and $\mathbb{C}$ be two ω_1-cBa's such that $\mathbb{B} \subseteq \mathbb{C}$. Suppose $\mathcal{E}$ is an ultrafilter on $\mathbb{B}$, $r : \mathbb{C} \mapsto \mathbb{B}$ is a retraction and $\mathcal{E}' = r^{-1}(\mathcal{E})$. Then $\omega^{\mathbb{B}}/\mathcal{E}$ is an initial segment of $\omega^{\mathbb{C}}/\mathcal{E}'$. Furthermore, for any $t \in \omega^{\mathbb{C}}$ $t_{\mathcal{E}'} \in \omega^{\mathbb{B}}/\mathcal{E}$ iff $\bigvee_{n\in\omega} r(t(n)) \in \mathcal{E}$.

Lemma 2.4. *Let $\{\mathbb{B}_\alpha : \alpha < \delta\}$ be a sequence of cBa's for some limit ordinal δ and $\mathbb{B}_\delta, \mathbb{C}$ be two Ba's. Suppose $\mathbb{B}_\alpha \subseteq \mathbb{B}_\beta \subseteq \mathbb{B}_\delta \subseteq \mathbb{C}$ and $r_{\alpha,\gamma} : \mathbb{B}_\gamma \mapsto \mathbb{B}_\alpha$ are retractions such that $r_{\alpha,\beta} \circ r_{\beta,\gamma} = r_{\alpha,\gamma}$ for any $\alpha < \beta < \gamma \leqslant \delta$. Given $x \in \mathbb{C}$, let $y_\alpha \in \mathbb{B}_\alpha$ be such that*

$$\bigvee\{r_{\alpha,\delta}(a) : a \leqslant x,\ a \in \mathbb{B}_\delta\} \leqslant y_\alpha \leqslant \bigwedge\{r_{\alpha,\delta}(a) : a \geqslant x,\ a \in \mathbb{B}_\delta\}$$

and $y_\alpha = r_{\alpha,\beta}(y_\beta)$ for any $\alpha < \beta < \delta$. Then there exist retractions $p_\alpha : \langle \mathbb{B}_\delta \cup \{x\}\rangle \mapsto \mathbb{B}_\alpha$ such that $p_\alpha \restriction \mathbb{B}_\delta = r_{\alpha,\delta}$, $p_\alpha = r_{\alpha,\beta} \circ p_\beta$ and $p_\alpha(x) = y_\alpha$ for any $\alpha < \beta < \delta$.

Proof: Trivial.

Lemma 2.5. *Let $\{\mathbb{B}_\alpha : \alpha < \delta\}$, $\mathbb{B}_\delta$, $\mathbb{C}$, $r_{\alpha,\gamma}$ for any $\alpha < \gamma \leqslant \delta$ be as in Lemma 2.4. Then there exist retractions $p_\alpha : \mathbb{C} \mapsto \mathbb{B}_\alpha$ such that $p_\alpha \restriction \mathbb{B}_\delta = r_{\alpha,\delta}$ and $p_\alpha = r_{\alpha,\beta} \circ p_\beta$ for any $\alpha < \beta < \delta$.*

Proof: By Lemma 2.4 and Zorn's Lemma, it suffices to show that for any $x \in \mathbb{C}$ there exists $y_\alpha \in \mathbb{B}_\alpha$ such that

$$\bigvee\{r_{\alpha,\delta}(a) : a \leqslant x,\ a \in \mathbb{B}_\delta\} \leqslant y_\alpha \leqslant \bigwedge\{r_{\alpha,\delta}(a) : a \geqslant x,\ a \in \mathbb{B}_\delta\}$$

and $y_\alpha = r_{\alpha,\beta}(y_\beta)$ for any $\alpha < \beta < \delta$. Given any $\alpha < \delta$, let

$$u_\alpha = \bigvee\{r_{\alpha,\delta}(a) : a \leqslant x,\ a \in \mathbb{B}_\delta\}$$

and

$$v_\alpha = \bigwedge\{r_{\alpha,\delta}(a) : a \geqslant x,\ a \in \mathbb{B}_\delta\}$$

for each $\alpha < \delta$. It is easy to check that $u_\alpha \leqslant r_{\alpha,\beta}(u_\beta) \leqslant r_{\alpha,\beta}(v_\beta) \leqslant v_\alpha$ for any $\alpha < \beta < \delta$. Let $\mathcal{S}$ be the set of all sequences $\{\langle a_\alpha, b_\alpha\rangle : \alpha < \delta\}$ with $u_\alpha \leqslant a_\alpha \leqslant b_\alpha \leqslant v_\alpha$ and $a_\alpha \leqslant r_{\alpha,\beta}(a_\beta) \leqslant r_{\alpha,\beta}(b_\beta) \leqslant b_\alpha$ for any $\alpha < \beta < \delta$. Define a partial order $\leqslant_{\mathcal{S}}$ on $\mathcal{S}$ such that

$$\{\langle a_\alpha, b_\alpha\rangle : \alpha < \delta\} \leqslant_{\mathcal{S}} \{\langle a'_\alpha, b'_\alpha\rangle : \alpha < \delta\}$$

iff $a_\alpha \leqslant a'_\alpha$ and $b'_\alpha \leqslant b_\alpha$ for every $\alpha < \delta$. Suppose

$$\mathcal{T} = \{\{\langle a^i_\alpha, b^i_\alpha\rangle : \alpha < \delta\} : i \in I\}.$$

is a totally ordered subset of $\mathcal{S}$. Let $a^{\mathcal{T}}_\alpha = \bigvee\{a^i_\alpha : i \in I\}$ and $b^{\mathcal{T}}_\alpha = \bigwedge\{b^i_\alpha : i \in I\}$ for each $\alpha < \delta$. It is easy to check that the sequence $\{\langle a^{\mathcal{T}}_\alpha, b^{\mathcal{T}}_\alpha\rangle : \alpha < \delta\}$ is in $\mathcal{S}$ and is an upper bound of $\mathcal{T}$. By Zorn's Lemma there is a maximal element $\{\langle c_\alpha, d_\alpha\rangle : \alpha < \delta\}$ in $\mathcal{S}$.

Claim 2.5.1. $r_{\alpha,\beta}(c_\beta) = c_\alpha$ and $r_{\alpha,\beta}(d_\beta) = d_\alpha$ for any $\alpha < \beta < \delta$.

Proof of Claim 2.5.1: Suppose not. Without loss of generality let $\beta_0 < \delta$ be the smallest such that there exists an $\alpha_0 < \beta_0$ with $c_{\alpha_0} < r_{\alpha_0,\beta_0}(c_{\beta_0})$. Now let $\bar{c}_\alpha = c_\alpha$ when $\alpha \geqslant \beta_0$ and $\bar{c}_\alpha = r_{\alpha,\beta_0}(c_{\beta_0})$ when $\alpha < \beta_0$. Then it is easy to check that the sequence $\{\langle \bar{c}_\alpha, d_\alpha\rangle : \alpha < \delta\}$ is in $\mathcal{S}$ and is strictly greater than $\{\langle c_\alpha, d_\alpha\rangle : \alpha < \delta\}$. This contradicts the maximality of $\{\langle c_\alpha, d_\alpha\rangle : \alpha < \delta\}$. □(Claim 2.5.1)

Clearly the sequence $\{\langle c_\alpha, d_\alpha\rangle : \alpha < \delta\}$ is what we want. □

Theorem 2.6. *Assume κ is a Laver-indestructible supercompact cardinal and $\eta > \kappa$ such that $\eta^{<\kappa} = \eta$. Suppose $\lambda \leqslant 2^\eta$ such that $\theta^\kappa < \lambda$ for any $\theta < \lambda$ and $\lambda^\kappa = \lambda$. Then there exists an ultrafilter $\mathcal{F}$ on η such that $\omega^\eta/\mathcal{F}$ is λ-Archimedean.*

Proof: By Lemma 1.6 there exists a κ-complete filter $\mathcal{D}$ on η such that $\mathbb{B} = \mathcal{P}(\eta)/\mathcal{D}$ contains a κ-free dense subalgebra $\mathbb{C}$ of dimension λ. Without loss of generality let

$$\mathbb{C} = \langle \{a_{\alpha,n} : \alpha < \lambda,\ n \in \omega\} \rangle_\kappa,$$

where $\{a_{\alpha,n} : \alpha < \lambda,\ n \in \omega\}$ is a κ-independent sequence in $\mathbb{C}$. For each $\beta \leqslant \lambda$ let

$$\mathbb{C}_\beta = \langle \{a_{\alpha,n} : \alpha < \beta,\ n \in \omega\} \rangle_\kappa$$

and let $\mathbb{B}_\beta = \bar{\mathbb{C}}_\beta$. Note that $\mathbb{C}_0 = \mathbb{B}_0 = \{0,1\}$. Since $\mathbb{C}_\beta$ has κ^+-c.c., so does $\mathbb{B}_\beta$. This implies that

$$\mathbb{B} = \mathbb{B}_\lambda = \bigcup_{\alpha<\lambda} \mathbb{B}_\alpha$$

because $cf(\lambda) > \kappa$. Also for each $\beta < \lambda$ one has

$$|\mathbb{C}_\beta| \leqslant |\beta|^{<\kappa} < \lambda.$$

Hence $|\mathbb{B}_\beta| \leqslant |\mathbb{C}_\beta|^\kappa < \lambda$. Next we build an ultrafilter $\mathcal{E}_\beta$ on $\mathbb{B}_\beta$ for every $\beta \leqslant \lambda$ such that $\omega^{\mathbb{B}_\beta}/\mathcal{E}_\beta$ is a proper end-extension of $\omega^{\mathbb{B}_\alpha}/\mathcal{E}_\alpha$ when $\alpha < \beta \leqslant \lambda$, and

$$\omega^{\mathbb{B}_\lambda}/\mathcal{E}_\lambda = \bigcup_{\alpha<\lambda} \omega^{\mathbb{B}_\alpha}/\mathcal{E}_\alpha.$$

We first construct retractions $r_{\alpha,\gamma} : \mathbb{B}_\gamma \mapsto \mathbb{B}_\alpha$ such that $r_{\alpha,\gamma} = r_{\alpha,\beta} \circ r_{\beta,\gamma}$ for any $\alpha < \beta < \gamma \leqslant \lambda$. We want also $r_{\alpha,\alpha+1}(a) = 0$ for every $a \in I_\alpha$, where I_α is the ideal in $\mathbb{B}_{\alpha+1}$ generated by $\{a_{\alpha,n} : n \in \omega\} \cup \{-\bigvee_{n\in\omega} a_{\alpha,n}\}$.

Claim 2.6.1. $I_\alpha \cap \mathbb{B}_\alpha = \{0\}$.

Proof of Claim 2.6.1: Suppose $a \in I_\alpha \cap \mathbb{B}_\alpha$. Then there exists an $m \in \omega$ such that

$$a \leqslant (\bigvee_{n<m} a_{\alpha,n}) \vee (-\bigvee_{n\in\omega} a_{\alpha,n}).$$

So one has

$$a \wedge (-\bigvee_{n<m} a_{\alpha,n}) \wedge (\bigvee_{n\in\omega} a_{\alpha,n}) = 0.$$

Hence

$$a \wedge (\bigwedge_{n<m} -a_{\alpha,n}) \wedge a_{\alpha,m} = 0.$$

This implies $a = 0$ because of the independence of $\{a_{\alpha,n} : n \in \omega\}$ over $\mathbb{B}_\alpha$. $\square$(Claim 2.6.1)

We construct $r_{\alpha,\beta}$ for any $\alpha < \beta < \delta$ inductively on δ when $\delta \leqslant \lambda$. Suppose we have already $\{r_{\alpha,\beta} : \alpha < \beta < \delta\}$ and want to find $r_{\alpha,\delta}$ for every $\alpha < \delta$. It is trivial when $\delta = 0$ or 1.

Case 1: δ is a limit ordinal.

Apply Lemma 2.5 to obtain retractions $p_\alpha : \mathbb{B}_\delta \mapsto \mathbb{B}_\alpha$ such that $p_\alpha = r_{\alpha,\beta} \circ p_\beta$ for any $\alpha < \beta < \delta$. Now let $r_{\alpha,\delta} = p_\alpha$.

Case 2: $\delta = \beta + 1$.

Apply Lemma 2.1 to obtain a retraction $r : \mathbb{B}_\delta \mapsto \mathbb{B}_\beta$ such that $r(a) = 0$ for every $a \in I_\beta$. Now let $r_{\alpha,\delta} = r_{\alpha,\beta} \circ r$ for every $\alpha < \beta$ and let $r_{\beta,\delta} = r$. Clearly, the set of retractions $\{r_{\alpha,\delta} : \alpha < \delta\}$ is what we want.

Let $\mathcal{E}_0 = \{1\}$ and let $\mathcal{E}_\alpha = r^{-1}_{0,\alpha}(\mathcal{E}_\alpha)$. It is easy to see that $\mathcal{E}_\beta = r^{-1}_{\alpha,\beta}(\mathcal{E}_\beta)$ for any $\alpha < \beta \leqslant \lambda$. By Lemma 2.3 $\omega^{\mathbb{B}_\beta}/\mathcal{E}_\beta$ is an end-extension of $\omega^{\mathbb{B}_\alpha}/\mathcal{E}_\alpha$ whenever $\alpha < \beta \leqslant \lambda$. Define $f_\alpha \in \omega^{\mathbb{B}_{\alpha+1}}/\mathcal{E}_{\alpha+1}$ such that

$$f_\alpha(0) = -\bigvee_{n\in\omega} a_{\alpha,n} \text{ and } f(n+1) = a_{\alpha,n} \wedge (-\bigvee_{m<n} a_{\alpha,m}).$$

Then $\bigvee_{n\in\omega} r_{\alpha,\alpha+1}(f(n)) = 0$. Hence by Lemma 2.3

$$(f_\alpha)_{\mathcal{E}_{\alpha+1}} \in \omega^{\mathbb{B}_{\alpha+1}}/\mathcal{E}_{\alpha+1} \smallsetminus \omega^{\mathbb{B}_\alpha}/\mathcal{E}_\alpha,$$

i.e. $\omega^{\mathbb{B}_{\alpha+1}}/\mathcal{E}_{\alpha+1}$ is a proper end-extension of $\omega^{\mathbb{B}_\alpha}/\mathcal{E}_\alpha$. It is also easy to see that

$$\omega^{\mathbb{B}}/\mathcal{E} = \bigcup_{\alpha<\lambda} \omega^{\mathbb{B}_\alpha}/\mathcal{E}_\alpha,$$

where $\mathbb{B} = \mathbb{B}_\lambda$ and $\mathcal{E} = \mathcal{E}_\lambda$, because $\mathbb{B} = \bigcup_{\alpha<\lambda} \mathbb{B}_\alpha$ and $cf(\lambda) > \omega$. Hence $|\omega^{\mathbb{B}}/\mathcal{E}| \geqslant \lambda$. On the other hand, if $x \in \omega^{\mathbb{B}}/\mathcal{E}$, then there is a $\beta < \lambda$ such that $x \in \omega^{\mathbb{B}_\beta}/\mathcal{E}_\beta$. Hence

$$|\{y \in \omega^{\mathbb{B}}/\mathcal{E} : y \leqslant x\}| \leqslant |\omega^{\mathbb{B}_\beta}/\mathcal{E}_\beta| \leqslant |\mathbb{B}_\beta|^\omega < \lambda.$$

This shows that $\omega^{\mathbb{B}}/\mathcal{E}$ is λ-Archimedean. Now the theorem follows from Lemma 0.1

□

Next we show a different way to construct λ-Archimedean ultrapowers. The construction of a λ-Archimedean ultrapower in Theorem 2.8 needs only to assume a measurable cardinal.

Lemma 2.7. *Suppose $\lambda \geqslant \kappa$ and κ is a λ-supercompact cardinal in V. Let $\mathcal{U}$ be a κ-complete normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$ in V and let j be the elementary embedding from V to $M = V^{\mathcal{P}_\kappa(\lambda)}/\mathcal{U}$ induced by $\mathcal{U}$. Suppose $\mathbb{B} \subseteq V_\kappa$ is a cBa and $j(\mathbb{B}) \cong \mathbb{B} * \dot{\mathbb{C}}$,* i.e. *$\mathbb{B} \cong \{j(p) : p \in \mathbb{B}\}$ is completely embedded into $j(\mathbb{B})$. Then in $V^{\mathbb{B}}$ there exists a κ-complete filter $\mathcal{D}$ on $(\mathcal{P}_\kappa(\lambda))^V$ such that $\mathcal{P}((\mathcal{P}_\kappa(\lambda))^V)/\mathcal{D} \cong \mathbb{C}$.*

Proof: Let $G \subseteq \mathbb{B}$ be a V-generic filter and let $H \subseteq \mathbb{C}$ be an $M[G]$-generic filter. Let $\hat{j}$ be the embedding from $V[G]$ to $M[G][H]$ defined by letting $\hat{j}(x) = (j(\dot{x}))_{G*H}$. It is well-known that $\hat{j}$ is an elementary embedding. Note that $M[G] \subseteq V[G]$ but $M[G][H]$ is generally not a subclass of $V[G]$. For any $A \subseteq (\mathcal{P}_\kappa(\lambda))^V$ in $V[G]$ let

$$i(A) = [\![j''\lambda \in (j(\dot{A}))_G]\!]_{\mathbb{C}},$$

where $\dot{A}$ is a $\mathbb{B}$-name for A and $j''\lambda = \{j(\alpha) : \alpha < \lambda\} \in M^\lambda \subseteq M$.

Claim 2.7.1. i is a Boolean homomorphism from $\mathcal{P}((\mathcal{P}_\kappa(\lambda))^V)$ to $\mathbb{C}$ in $V[G]$.

Proof of Claim 2.7.1: It suffices to show that i is well-defined function. The rest follows from the fact that j is an elementary embedding. Suppose $\dot{A}_1$ and $\dot{A}_2$ are two

$\mathbb{B}$-names for same set A in $V[G]$. Then there exists a $p \in G$ such that

$$p \Vdash_{\mathbb{B}} \dot{A}_1 = \dot{A}_2.$$

By the fact that $j(p) = p$ one has

$$p \Vdash_{\mathbb{B}*\mathbb{C}} j(\dot{A}_1) = j(\dot{A}_2)$$

or

$$p \Vdash_{\mathbb{B}} \Vdash_{\mathbb{C}} j(\dot{A}_1) = j(\dot{A}_2).$$

Since $p \in G$, then

$$V[G] \models \Vdash_{\mathbb{C}} (j(\dot{A}_1))_G = (j(\dot{A}_2))_G.$$

Hence in $V[G]$

$$[\![j''\lambda \in (j(\dot{A}_1))_G]\!]_{\mathbb{C}} = [\![j''\lambda \in (j(\dot{A}_2))_G]\!]_{\mathbb{C}}. \quad \square((\text{Claim } 2.7.1)$$

Let's define a filter $\mathcal{D}$ in $V[G]$ by letting

$$\mathcal{D} = \{A \subseteq (\mathcal{P}_\kappa(\lambda))^V : i(A) = 1_{\mathbb{C}}\}.$$

It is easy to see that $\mathcal{D} \in V[G]$ is a κ-complete filter on $(\mathcal{P}_\kappa(\lambda))^V$.

Claim 2.7.2. $\mathcal{P}((\mathcal{P}_\kappa(\lambda))^V)/\mathcal{D} \cong \mathbb{C}$ in $V[G]$.

Proof of Claim 2.7.2: It suffices to show that i is an onto map. Given any $a \in \mathbb{C}$, let a be expressed as $[F]_{\mathcal{U}}$ for some $F : \mathcal{P}_\kappa(\lambda) \mapsto \mathbb{B}$ in V such that F is not equal to any single $p \in \mathbb{B}$ modulo $\mathcal{U}$. Let $\dot{A}$ be a $\mathbb{B}$-name for a subset of $\mathcal{P}_\kappa(\lambda)$ in V such that for each $\sigma \in \mathcal{P}_\kappa(\lambda)$ one has $[\![\sigma \in \dot{A}]\!]_{\mathbb{B}} = F(\sigma)$. Since $j''\lambda = [d]_{\mathcal{U}}$, where d is the identity function from $\mathcal{P}_\kappa(\lambda)$ to $\mathcal{P}_\kappa(\lambda)$,

$$[\![j''\lambda \in (j(\dot{A}))_G]\!]_{\mathbb{C}} = [\![[d]_{\mathcal{U}} \in (j(\dot{A}))_G]\!]_{\mathbb{C}} = [\langle [\![\sigma \in \dot{A}]\!]_{\mathbb{B}} : \sigma \in \mathcal{P}_\kappa(\lambda)\rangle]_{\mathcal{U}} =$$

$$[\langle F(\sigma) : \sigma \in \mathcal{P}_\kappa(\lambda)\rangle]_{\mathcal{U}} = [F]_{\mathcal{U}} = a.$$

Hence i is an onto map from $\mathcal{P}((\mathcal{P}_\kappa(\lambda))^V)$ to $\mathbb{C}$. $\square$

Theorem 2.8. *Suppose κ is a measurable cardinal such that $\theta^\omega < 2^\kappa$ for any $\theta < 2^\kappa$. Let $\mathbb{B}_\kappa$ be the Boolean algebra for adding κ Cohen reals or adding κ random reals. Then in $V^{\mathbb{B}_\kappa}$ there exists an ultrafilter $\mathcal{F}$ on κ such that $\omega^\kappa/\mathcal{F}$ is 2^κ-Archimedean.*

Proof: Obviously, κ is κ-supercompact. Let j be the elementary embedding induced by a κ-complete normal ultrafilter $\mathcal{U}$ on κ. For any λ let $\mathbb{B}_\lambda$ denote the cBa for adding λ Cohen reals (or random reals). Then $j(\mathbb{B}_\kappa) = \mathbb{B}_{j(\kappa)} = \mathbb{B}_\kappa * \dot{\mathbb{C}}$, where $\mathbb{C}$ is isomorphic to the cBa for adding 2^κ Cohen (or random) reals in $V^{\mathbb{B}_\kappa}$ because $|j(\kappa) \smallsetminus \kappa| = 2^\kappa$. By Lemma 2.7 there exists a κ-complete filter $\mathcal{D}$ on κ such that

$$\mathcal{P}(\kappa)/\mathcal{D} \cong \mathbb{C}$$

in $V^{\mathbb{B}_\kappa}$. For any λ let $\mathbb{B}'_\lambda$ denote the cBa for adding λ Cohen reals (or random reals) in $V^{\mathbb{B}_\kappa}$. Note that $\mathbb{C} \cong \mathbb{B}'_{2^\kappa}$. Since $\mathbb{B}'_\lambda$ has c.c.c. for any λ, then $\mathbb{B}'_{2^\kappa} = \bigcup_{\alpha<2^\kappa} \mathbb{B}'_{\alpha\cdot\omega}$. By the facts that $|\mathbb{B}'_{\alpha\cdot\omega}|^\omega < 2^\kappa$ and there exists an independent sequence $\{a_{\alpha,n} : n \in \omega\}$ in $\mathbb{B}'_{\alpha\cdot\omega+\omega}$ over $\mathbb{B}'_{\alpha\cdot\omega}$ for every $\alpha < 2^\kappa$, then, by a similar argument in the proof of Theorem 2.6, one can construct an $\mathcal{F}$ such that $\omega^\kappa/\mathcal{F}$ is 2^κ-Archimedean. $\square$

Remark 2.9. *(1) Note that $\kappa = 2^\omega$ in $V^{\mathbb{B}_\kappa}$ and it is impossible to have a 2^ω-Archimedean ultrapower by ω_1-saturation. (2) We could have a more general theorem with a much more general $\mathbb{B}_\kappa$. Restricting $\mathbb{B}_\kappa$ to be a cBa for adding κ Cohen or random reals is just for simplicity to illustrate the idea.*

Theorem 2.10. *Suppose κ is a supercompact cardinal and $\mathbb{B}_\kappa$ is as in Theorem 2.8. Then in $V^{\mathbb{B}_\kappa}$ there exists an ultrafilter $\mathcal{F}$ on λ for every $\lambda > \kappa$ with $\lambda^{<\kappa} = \lambda$ and $\theta^\omega < 2^\lambda$ for every $\theta < 2^\lambda$ such that $\omega^\lambda/\mathcal{F}$ is 2^λ-Archimedean.*

Proof: Almost identical to the proof of Theorem 2.8. $\square$

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