TOPOLOGICAL DENSITY OF CCC BOOLEAN ALGEBRAS - EVERY CARDINALITY OCCURS

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ABSTRACT. For every uncountable cardinal μ there is a ccc Boolean algebra whose topological density is μ.

1. Introduction

For a Boolean algebra B, the topological density $d(B)$ is the minimal cardinal μ such that there is a family $\{D_\xi : \xi \in \mu\}$ of ultrafilters of B with the union $B \setminus \{0\}$. Note that if $St(B)$ is the Stone space of B, then $d(B)$ is the density of $St(B)$ (as a topological space). A Boolean algebra B has the countable chain condition (ccc) if there is no uncountable collection of pairwise disjoint elements of $B \setminus \{0\}$.

The question we consider in this paper is: what cardinals are topological densities of ccc Boolean algebras? Hajnal, Juhász, and Szentmiklóssy [HJS] prove that under some mild set-theoretic assumptions every uncountable cardinal is the topological density of some ccc Boolean algebra. We prove here that the above statement is a theorem of ZFC.

Theorem 1. *For every uncountable cardinal μ there is a ccc Boolean algebra $\mathcal{B}$, such that $d(\mathcal{B}) = \mu$.*

The rest of the paper is devoted to the proof of the theorem. Let μ be an uncountable cardinal. The idea of the proof is to define $\mathcal{B}$ as a quotient of a free Boolean algebra generated by $\{x_\nu : \nu \in T\}$, where T is a set of cardinality 2^μ. The reason we index the generators by a set T, rather than 2^μ, is that an additional structure on T is helpful in defining the quotient. In particular, the quotient is defined by imposing a set of restrictions of the form $x_{\nu_0} \cap x_{\nu_1} \cap (-(x_{\eta_0} \triangle x_{\eta_1})) = 0$, for some $\nu_0, \nu_1, \eta_0, \eta_1$ in T. The definition of T is quite technical but it is the key element of the proof that the topological density of our algebra is $\geq \mu$, Lemma 4.1. The construction of T is done in section 2. In section 3 we give the definition of the algebra $\mathcal{B}$ and prove that it has the ccc. In the last section we prove that the topological density of $\mathcal{B}$ is exactly μ.

2. Preliminaries

In this section we define the set T and the set of quadruples used in the definition of $\mathcal{B}$. For a cardinal $\sigma < \mu$ let $h_\sigma : [\sigma^+]^2 \to \sigma^+$ be such that

(a) it is one-to-one,

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(b) for every $X \in [\sigma^+]^{\sigma^+}$ and $j < \sigma$ there is $j_1 \in (j, \sigma)$ and $i_0, i_1 \in X$ such that $h_\sigma(i_0, i_1) = j_1 \bmod \sigma$.

To prove that such a function exists it is enough to show that there is a function $h'_\sigma : [\sigma^+]^2 \to \sigma$ such that (b) holds. Indeed, then we can define h_σ to be any $1-1$ function from σ^+ to σ^+ such that $h_\sigma(\alpha, \beta) = h'_\sigma(\alpha, \beta) \bmod \sigma$.

In order to define h'_σ, first fix $1-1$ functions $q_\alpha : \alpha \to \sigma$ for every $\alpha < \sigma^+$. Now for $\alpha < \beta$ define $h'_\sigma(\alpha, \beta) = q_\beta(\alpha)$. To prove (b) let $X \in [\sigma^+]^{\sigma^+}$. Let $\beta \in X$ be such that the set $X_\beta = \{\alpha \in X : \alpha < \beta\}$ has cardinality σ. Since q_β is $1-1$ it follows that the image of $[X_\beta]^2$ under h'_σ is cofinal in σ, hence (b) holds.

Definition 2.1. *We define, by induction on $\alpha \leq \mu^+$, $\mathcal{T}_\alpha = (\mathcal{T}_\alpha, \sigma_\alpha, \mathcal{P}^{\updownarrow}_\alpha, \mathcal{F}^{\Updownarrow}_\alpha)$, $(l < 8, m < 2)$ such that:*

(1) *T_α is a set of finite sequences,*
(2) *$\langle P^l_\alpha : l < 8\rangle$ is a partition of T_α,*
(3) *σ_α is a function from T_α to $\{\sigma : \aleph_0 \leq \sigma \leq \mu\}$,*
(4) *F^m_α is a partial, two-place, symmetric function from P^7_α to T_α,*
(5) *$\mathcal{T}_\alpha$ is increasing, continuous in α, i.e., if $\beta < \alpha$, then*
  (a) *$T_\beta \subseteq T_\alpha$,*
  (b) *$P^l_\beta = P^l_\alpha \cap T_\beta$ for $l < 8$,*
  (c) *$\sigma_\beta = \sigma_\alpha \restriction T_\beta$,*
  (d) *$F^m_\beta = F^m_\alpha \restriction [T_\beta]^2$ for $m < 2$,*
  (e) *if α is a limit, then $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$.*

Case 1. $\alpha = 0$. Define $T_0 = \{\langle\rangle\}$, $P^0_0 = \{\langle\rangle\}$, (so $P^l_0 = \emptyset$ for $l \neq 0$). $\sigma_0(\langle\rangle) = \mu$.
Case 2. $\alpha = 1$. Define $T_1 = T_0 \cup \{\langle\sigma\rangle : \aleph_0 \leq \sigma \leq \mu\}$, $P^1_1 = \{\langle\sigma\rangle : \aleph_0 \leq \sigma \leq \mu\}$, $\sigma_1(\langle\sigma\rangle) = \sigma$.
Case 3. α is a limit. Put $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$, $P^l_\alpha = \bigcup_{\beta<\alpha} P^l_\beta$, $\sigma_\alpha = \bigcup_{\beta<\alpha} \sigma_\beta$.
Case 4. $\alpha = \beta + 1$. Define $T_\alpha = T_\beta \cup \{T_{\alpha,l} : l < 8\}$, where

$$\begin{aligned}
T_{\alpha,0} &= \emptyset,\\
T_{\alpha,1} &= \{\eta^\frown\langle\sigma\rangle : \eta \in T_\beta \setminus (P^5_\beta \cup P^7_\beta), \eta^\frown\langle\sigma\rangle \notin T_\beta, \aleph_0 \leq \sigma \leq \sigma_\beta(\eta)\},\\
T_{\alpha,2} &= \{\eta^\frown\langle 0\rangle : \eta \in T_\beta \setminus (P^5_\beta \cup P^7_\beta), \eta^\frown\langle 0\rangle \notin T_\beta\},\\
T_{\alpha,3} &= \{\eta^\frown\langle\rho^\frown\langle 0\rangle\rangle : \eta^\frown\langle\rho\rangle \in T_\beta \setminus (P^5_\beta \cup P^7_\beta), \eta^\frown\langle\rho^\frown\langle 0\rangle\rangle \notin T_\beta\},\\
T^{4,5}_\alpha &= \{\eta^\frown\langle\rho\rangle : \eta \in T_\beta,\ \rho \text{ a sequence of limit length and}\\
&\qquad (\forall\zeta < \lg(\rho))(\eta^\frown\langle\rho \restriction \zeta\rangle \in T_\beta \text{ but } \eta^\frown\langle\rho\rangle \notin T_\beta)\},\\
T_{\alpha,4} &= \{\eta^\frown\langle\rho\rangle \in T^{4,5}_\alpha : \lg(\rho) < \sigma_\beta(\eta)^+\},\\
T_{\alpha,5} &= T^{4,5}_\alpha \setminus T_{\alpha,4},
\end{aligned}$$

$$\begin{aligned}
T_{\alpha,6} = \{\eta^\frown\langle\rho^\frown\langle\nu_0,\nu_1\rangle\rangle : \eta^\frown\langle\rho\rangle \in T_\beta \setminus (P^5_\beta \cup P^7_\beta),\\
\text{and } \eta^\frown\langle\rho\rangle \lhd \nu_l, l = 0, 1, \text{ and } \nu_0, \nu_1 \in T_\beta, \nu_0 \neq \nu_1\},\\
T_{\alpha,7} = \{\eta^\frown\langle\rho\rangle^\frown\langle i\rangle : \eta^\frown\langle\rho\rangle \in P^5_\beta, \text{ and } i < \sigma_\beta(\eta)^+, \text{ and } \eta^\frown\langle\rho\rangle^\frown\langle i\rangle \notin T_\beta\}.
\end{aligned}$$

Let $P^l_\alpha = P^l_\beta \cup T_{\alpha,l}$ for $l < 8$, and define σ_α by:

$$\sigma_\alpha(\tau) = \begin{cases} \sigma_\beta(\tau) & \text{if } \tau \in T_\beta, \\ \tau(n) & \text{if } \tau \in T_{\alpha,1}, \lg(\tau) = n+1, \\ \sigma_\beta(\tau \restriction n) & \text{if } \tau \in \{T_{\alpha,l} : l \neq 1\}, \lg(\tau) = n+1. \end{cases}$$

Finally define F^m_α, $m = 0, 1$. $F^m_\alpha(\tau_1, \tau_2)$ is well defined if $F^m_\beta(\tau_1, \tau_2)$ is well defined, or for some $\eta^\frown\langle\rho\rangle \in P^5_\beta$ for $l = 1, 2$, we have $\tau_l = \eta^\frown\langle\rho\rangle^\frown\langle i_l\rangle$, $i_1 \neq i_2$, and $\rho(\zeta)$ is a pair (ν_0, ν_1), i.e., $\eta^\frown\langle\rho \restriction \zeta^\frown\langle\nu_0, \nu_1\rangle\rangle \in P^6$, where $\zeta = h_{\sigma(\eta)}(i_1, i_2)$. In the first case define $F^m_\alpha(\tau_1, \tau_2) = F^m_\beta(\tau_1, \tau_2)$. In the second case define $F^m_\alpha(\tau_1, \tau_2) = \nu_m$.

Proposition 2.2. (1) *T_α is well defined for $\alpha \leq \mu^+$.*

(2) *Each member of T_α, ($\alpha \leq \mu^+$) is a finite sequence.*
(3) *For every $\eta \in T_{\mu^+}$, the sequence $\langle \sigma(\eta \restriction k) : k \leq \lg(\eta)\rangle$ is non-increasing.*
(4) *If $\eta^\frown\langle\rho\rangle \in P^5$ and $m_1, m_2 < 2$, $\tau_1, \tau_2, \tau_3, \tau_4 \in \{\eta^\frown\langle\rho\rangle^\frown\langle i\rangle : i < \sigma(\eta)^+\}$ and $F^{m_1}_\alpha(\tau_1, \tau_2) = F^{m_2}_\alpha(\tau_3, \tau_4)$, then $m_1 = m_2$, $\{\tau_1, \tau_2\} = \{\tau_3, \tau_4\}$. Moreover, the conclusion holds if we assume that $F^{m_1}_\alpha(\tau_1, \tau_2) \restriction \lg(\eta^\frown\rho) = F^{m_2}_\alpha(\tau_3, \tau_4) \restriction \lg(\eta^\frown\rho)$.*
(5) *If $\eta \in P^7_{\mu^+}$, then η is maximal in $(T_{\mu^+}, \lhd)$.*
(6) *$|T_{\mu^+}| = 2^\mu$.*

Proof Straightforward.

Let $T = T_{\mu^+}$, $P^l = P^l_{\mu^+}$ for $l < 8$, $\sigma = \sigma_{\mu^+}$ and $F_m = F^m_{\mu^+}$, $m = 0, 1$.

Definition 2.3. (1) *We say that $X \subseteq T$ is 1-closed if:*

(a) *$\langle\rangle \in X$,*
(b) *if $\eta \lhd \eta_1$, $\eta_1 \in X$, then $\eta \in X$,*
(c) *if $\eta^\frown\langle\rho\rangle \in P^5$ and for $k = 1, 2$, $\tau_k = \eta^\frown\langle\rho\rangle^\frown\langle i_k\rangle \in X$, $i_1 \neq i_2$ and $m < 2$, then $F_m(\tau_1, \tau_2) \in X$ if it is well defined.*

(2) *We say that $X \subseteq T$ is 2-closed if it is 1-closed and:*

(d) *if $\eta^\frown\langle\rho^\frown\langle\nu_0, \nu_1\rangle\rangle \in P^6 \cap X$, then $\nu_0, \nu_1 \in X$,*
(e) *if $\eta^\frown\langle\rho_1\rangle$, $\eta^\frown\langle\rho_2\rangle \in X$, $\zeta = \sup\{\xi : \rho_1 \restriction \xi = \rho_2 \restriction \xi\}$, then $\eta^\frown\langle\rho_1 \restriction (\zeta+1)\rangle \in X$ if $\zeta < \lg(\rho_1)$, and $\eta^\frown\langle\rho_2 \restriction (\zeta+1)\rangle \in X$ if $\zeta < \lg(\rho_2)$.*

Proposition 2.4. (1) *T_α is closed for $\alpha \leq \mu^+$.*

(2) *The family of k-closed sets is closed under intersections, $k = 1, 2$.*
(3) *If $X \subseteq T$ is finite, then $\mathrm{cl}_k(X)$ is finite, $k = 1, 2$.*

Proof (1), (2) are straightforward. To prove (3), prove by induction on α, that if $X \subseteq T_\alpha$ is finite, then $\mathrm{cl}_k(X)$ is finite.

3. Definition of the algebra, and the ccc

In this section we, first, define the algebra, and second, prove that it has the ccc. The proof is preceded by two propositions, which give a sufficient condition for an element of the algebra to be non-zero.

Definition 3.1. (1) *$\mathcal{B}_T$ is the Boolean algebra generated by $\{x_\eta : \eta \in T\}$ freely, except the equations in the following set:*

$$\Gamma = \{\mathbf{e}_{\tau_1,\tau_2} = [\mathbf{x}_{\tau_1} \cap \mathbf{x}_{\tau_2} \cap (-(\mathbf{x}_{\mathbf{F_0}(\tau_1,\tau_2)} \bigtriangleup \mathbf{x}_{\mathbf{F_1}(\tau_1,\tau_2)})) = \mathbf{0}] : \tau_1, \tau_2 \in T, \text{ and } F_m(\tau_1, \tau_2) \text{ is well defined, } m = 0, 1.\}$$

(2) *For $X \subseteq T$ let*

$$\Gamma_X = \{\mathbf{e}_{\tau_1,\tau_2} : \tau_1, \tau_2 \in X, \text{ and } F_l(\tau_1,\tau_2) \text{ is well defined } l = 0, 1.\}$$

(3) *For $\alpha < \mu$ define $\mathcal{B}_{\mathcal{T}_\alpha}$ to be the subalgebra of $\mathcal{B}_\mathcal{T}$ generated by $\{x_\eta : \eta \in T_\alpha\}$.*
(4) *$\mathcal{B}_\prime$ is the trivial Boolean algebra with the universe $\{0, 1\}$.*

Note: for $\eta \in T$ we consider x_η to be an element of $\mathcal{B}_T$, i.e., it is an equivalence class of the element x_η.

Proposition 3.2. *For a Boolean term $\mathbf{t} = \mathbf{t}(\mathbf{y_0}, \dots, \mathbf{y_{n-1}})$ and $\eta_0, \dots, \eta_{n-1} \in T$, $\mathcal{B}_\mathcal{T} \models \sqcup(\S_{\eta_\prime}, \dots, \S_{\eta_{\setminus -\infty}}) > \prime$ if and only if there is a function $f : T \to \{0,1\}$ such that $\mathcal{B}_\prime \models \sqcup(\{(\eta_\prime), \dots, \{(\eta_{\setminus -\infty})) = \infty$ and $(*)_{f,T}$ holds, where for $X \subseteq T$ we define:*

$(*)_{f,X}$ *If $\mathbf{e}_{\tau_1,\tau_2} \in \Gamma_X$ and $f(\tau_1) = 1 = f(\tau_2)$, then $f(F_0(\tau_1,\tau_2)) \neq f(F_1(\tau_1,\tau_2))$.*

Proof (1) Assume that $f : T \to \{0,1\}$ is such that $\mathcal{B}_\prime \models \sqcup(\{(\eta_\prime), \dots, \{(\eta_{\setminus -\infty})) = \infty$ and $(*)_{f,T}$ holds. Note that $\mathcal{B}_T \models t(x_{\eta_0}, \dots, x_{\eta_{n-1}}) > 0$ if and only if there is a homomorphism $h : \mathcal{B}_T \to \mathcal{B}_0$ such that $\mathcal{B}_0 \models h(t(x_{\eta_0}, \dots, x_{\eta_{n-1}}))= 1$. The function $f : T \to \{0,1\}$ defines a homomorphism $\bar{f}$ from a free algebra generated by $\{x_\eta : \eta \in T\}$ into $\mathcal{B}_0$. Such homomorphism induces an homomorphism of $\mathcal{B}$ into $\mathcal{B}_0$ if and only if $\bar{f}(x_{\tau_1} \cap x_{\tau_2} \cap (-(x_{F_0(\tau_1,\tau_2)} \vartriangle x_{F_1(\tau_1,\tau_2)}))) = 0$ for τ_1, τ_2 such that $F_m(\tau_1,\tau_2)$ are well-defined. Clearly this is equivalent to $(*)_{f,T}$.

(2) Assume $\mathcal{B}_\mathcal{T} \models \sqcup(\S_{\eta_\prime}, \dots, \S_{\eta_{\setminus -\infty}}) > \prime$. Without loss of generality $t(x_{\eta_0}, \dots, x_{\eta_{n-1}}) = \bigcap_{l<n} x_{\eta_l}^{\epsilon(l)}$, where $\epsilon : n \to \{0,1\}$, and $x^1 = x$, and $x^0 = -x$. Moreover, we can assume that $\{\eta_0, \dots, \eta_{n_1}\}$ is 1-closed. Define $f : T \to \{0,1\}$ by: $f(\eta_l) = \epsilon(l)$ for $l < n$, and $f(\rho) = 0$ for $\rho \notin \{\eta_0, \dots, \eta_{n_1}\}$. Clearly $\mathcal{B}_0 \models t(f(\eta_0), \dots, f(\eta_{n-1})) = 1$ and $(*)_{f,T}$ holds.

Proposition 3.3. (1) *If $X \subseteq T$ is 1-closed, $f : X \to \{0,1\}$, $(*)_{f,X}$ holds, $\eta_0, \dots, \eta_{n-1} \in X$, and $t(y_0, \dots, y_{n-1})$ is a Boolean term such that $\mathcal{B}_\prime \models \sqcup(\{(\eta_\prime), \dots, \{(\eta_{\setminus -\infty})) = \infty$, then $\mathcal{B}_\mathcal{T} \models \sqcup(\S_{\eta_\prime}, \dots, \S_{\eta_{\setminus -\infty}}) > \prime$.*
(2) *For $\eta \neq \nu$ in T, $\mathcal{B}_\mathcal{T} \models \S_\eta \neq \S_\nu$, moreover $\mathcal{B}_\mathcal{T} \models (\S_\eta \setminus \S_\nu) > \prime$.*
(3) *If $X \subseteq Y \subseteq T$, $\mathrm{cl}_1(X) \subseteq Y$, $\eta_0, \dots, \eta_{n-1} \in Y$, $f : Y \to \{0,1\}$, $X \subseteq f^{-1}(\{1\})$, $t(y_0, \dots, y_{n-1})$ is a Boolean term such that $\mathcal{B}_\prime \models \sqcup(\{(\eta_\prime), \dots, \{(\eta_{\setminus -\infty})) = \infty$, and $\mathbf{e}_{\tau_1,\tau_2} \in \Gamma_X$ implies that $|\{F_0(\tau_1,\tau_2), F_1(\tau_1,\tau_2)\} \cap f^{-1}(\{1\})| = 1$, then $\mathcal{B}_\mathcal{T} \models \sqcup(\S_{\eta_\prime}, \dots, \S_{\eta_{\setminus -\infty}}) > \prime$.*

Proof (1) Use 3.2 for $f \cup 0_{T \setminus Y}$. For (2) use part (1) with $X = T$, $f(\eta) = 1$ and $f(\rho) = 0$ for $\rho \neq \eta$. (3) is similar to (1).

Lemma 3.4. *$\mathcal{B}_\mathcal{T}$ satisfies the ccc., in fact a strong version of the ccc: for every collection of $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ elements of $\mathcal{B}_\mathcal{T}$, there is a subcollection of size κ which generates a filter.*

Proof Let $\mathcal{B}_\mathcal{T} \models \dashv_\alpha > \prime$ for $\alpha < \kappa$. Let $a_\alpha = t_\alpha(x_{\eta_{\alpha,0}}, \dots, x_{\eta_{\alpha,n_\alpha - 1}})$, each t_α is a Boolean term. Without loss of generality we can assume that:

(1) $\{\eta_{\alpha,0}, \dots, \eta_{\alpha,n_\alpha - 1}\}$ is 2-closed for each α,
(2) $t_\alpha = t$, $n_\alpha = n$,
(3) $\langle \{\eta_{\alpha,k} : k < n\} : \alpha < \kappa \rangle$ is a Δ-system, i.e., for some $n(*) \leq n$ we have: if $k < n(*), \alpha < \kappa$ then $\eta_{\alpha,k} = \eta_k$, and $\langle \{\eta_{\alpha,k} : k \in [n(*), n)\} : \alpha < \kappa \rangle$ is a sequence of pairwise disjoint sets.

We can assume that $n > n(*)$, as otherwise $a_\alpha = a_0$ for every $\alpha < \kappa$, and we are done. Let $f_\alpha : T \to \{0,1\}$ be such that $(*)_{f_\alpha,T}$ holds and $\mathcal{B}_/ \vDash \sqcup(\{_\alpha(\eta_{\alpha,/}),\dots,\{_\alpha(\eta_{\alpha,\backslash-\infty})) = \infty$. Without loss of generality we can assume that:

(4) $f_\alpha(\eta_{\alpha,k}) = \mathbf{t_k}$, i.e., does not depend on α,
(5) the truth values of "$\eta_{\alpha,k_1} \lhd \eta_{\alpha,k_2}$", "$\lg(\eta_{\alpha,k}) = m$", "$\eta_{\alpha,k} \in P^l$" do not depend on α.
(6) if $\eta^\frown\langle\rho\rangle \in P^5 \cap \{\eta_k : k < n(*)\}$, $\tau_1 = \eta^\frown\langle\rho\rangle^\frown\langle i_1\rangle = \eta_{\alpha,k_1}$, $k_1 \in [n(*),n)$, and $\tau_2 = \eta^\frown\langle\rho\rangle^\frown\langle i_2\rangle = \eta_{\alpha,k_2}$, $k_2 \in [n(*),n)$, $i_1 \neq i_2$, *then* $F_0(\tau_1,\tau_2) \notin \{\eta_k : k < n(*)\}$, moreover $F_0(\tau_1,\tau_2) \restriction \lg(\eta^\frown\langle\rho\rangle) \notin \{\eta_k : k < n(*)\}$,
(7) if $\eta^\frown\langle\rho\rangle \in P^5 \cap \{\eta_k : k < n(*)\}$, $\alpha < \beta < \kappa$, $\tau_1 = \eta^\frown\langle\rho\rangle^\frown\langle i_1\rangle = \eta_{\alpha,k_1}$, $k_1 \in [n(*),n)$, $\tau_2 = \eta^\frown\langle\rho\rangle^\frown\langle i_2\rangle = \eta_{\beta,k_2}$, $k_2 \in [n(*),n)$, $i_1 \neq i_2$, *then* $F_l(\tau_1,\tau_2) \restriction \lg(\eta^\frown\langle\rho\rangle) \notin \{\eta_{\alpha,k} : k < n, \alpha < \kappa\}$ for $l = 0,1$.

It is easy to satisfy (4) – (6). To satisfy (7) note that the function $F(*,*) \restriction \lg(\eta^\frown\rho)$ is $1-1$ by 2.2(4). Therefore we can choose a required sequence by induction of length κ.

Now we will show that if $\alpha_0,\dots,\alpha_{m(*)} < \kappa$, then $\mathcal{B}_\mathcal{T} \vDash \bigcap_{\Updownarrow\leq\Updownarrow(*)} \dashv_\Updownarrow > /$. It is enough to define $f : T \to \{0,1\}$ such that $(*)_{f,T}$ holds and

$$\mathcal{B}_/ \vDash \sqcup(\{(\eta_{\alpha_\Updownarrow,/}),\dots,\{(\eta_{\alpha_\Updownarrow,\backslash-\infty})) = \infty \text{ for } \Updownarrow \leq \Updownarrow(*).$$

Define $f(\nu) = 1$ if and only if one of the following occurs:

(a) $\nu \in \{\eta_{\alpha_m,k} : k < n, m \leq m(*),\}$ and $f_{\alpha_m}(\nu) = 1$.
(b) For some $\eta^\frown\langle\rho\rangle \in P^5 \cap \{\eta_k : k < n(*)\}$ and $m_1 \neq m_2$, and $k_1,k_2 \in [n(*),n)$, we have $\eta^\frown\langle\rho\rangle^\frown\langle i_1\rangle = \eta_{\alpha_{m_1},k_1}$, $\eta^\frown\langle\rho\rangle^\frown\langle i_2\rangle = \eta_{\alpha_{m_2},k_2}$ and $\nu = F_0(\eta_{\alpha_{m_1},k_1},\eta_{\alpha_{m_2},k_2})$.

Lemma 3.5. (1) *$\mathcal{B}_/ \vDash \sqcup(\{(\eta_{\alpha_\Updownarrow,/}),\dots,\{(\eta_{\alpha_\Updownarrow,\backslash-\infty})) = \infty$ for $m \leq m(*)$.*
(2) *$(*)_{f,T}$ holds.*

Proof (1) It suffices to prove that $f \restriction \{\eta_{\alpha_m,0},\dots,\eta_{\alpha_m,n-1}\} \subseteq f_{\alpha_m}$. Assume first that $f_{\alpha_m}(\eta_{\alpha_m,k}) = 1$. By the definition of f we have $f(\eta_{\alpha_m,k}) = 1$. Now assume that $f(\eta_{\alpha_m,k}) = 1$. Hence one of the cases (a) or (b) holds. If case (a) holds we are done. So suppose that (b) holds. By (7) it follows that $\eta_{\alpha_m,k} \notin \{\eta_{\alpha,k} : \alpha < \kappa, k < n\}$, a contradiction.

(2) Assume that $(*)_{f,T}$ fails. Then there is $\mathbf{e}_{\nu_1,\nu_2} \in \Gamma_T$ such that $\nu_1 = \eta^\frown\langle\rho\rangle^\frown\langle i_1\rangle$. $\nu_2 = \eta^\frown\langle\rho\rangle^\frown\langle i_2\rangle$, $i_1 \neq i_2$ and $f(\nu_1) = 1 = f(\nu_2)$ and $f(F_0(\nu_1,\nu_2)) = f(F_1(\nu_1,\nu_2))$. Working toward a contradiction we consider three cases.

Case 1. $\nu_1,\nu_2 \in \{\eta_{\alpha_m,k} : m \leq m(*), k < n\}$. Hence there is $m_1,m_2 \leq m(*)$, end $k_1,k_2 < n$ such that $\nu_1 = \eta_{\alpha_{m_1},k_1}$, $\nu_2 = \eta_{\alpha_{m_2},k_2}$.

If $m_1 = m_2$, then as $\{\eta_{\alpha_{m_1},k} : k < n\}$ is 1-closed, we have $\nu_1,\nu_2,F_0(\nu_1,\nu_2)$, $F_1(\nu_1,\nu_2) \in \{\eta_{\alpha_{m_1},k} : k < n\}$. Since $(*)_{f_{\alpha_{m_1}},T}$ holds we get a contradiction.

Hence $m_1 \neq m_2$, and $k_1,k_2 \in [n(*),n)$. By the definition of f we have $f(F_0(\nu_1,\nu_2)) = 1$, so it suffices to show that $f(F_1(\nu_1,\nu_2)) = 0$. Assume to the contrary that $f(F_1(\nu_1,\nu_2)) = 1$. Hence, as $F_1(\nu_1,\nu_2) \notin \{\eta_{\alpha,k} : \alpha < \kappa, k < n\}$ by (7), case (b) must hold. So there is $\eta^{*\frown}\langle\rho^*\rangle \in P^5 \cap \{\eta_k : k < n(*)\}$ and τ_1,τ_2 such that

(i) $\tau_1 = \eta^{*\frown}\langle\rho^*\rangle^\frown\langle i_3\rangle = \eta_{\alpha_{m_3},k_3}$,

(ii) $\tau_2 = \eta^{*\frown}\langle\rho^*\rangle^\frown\langle i_4\rangle = \eta_{\alpha_{m_4},k_4}$,
(iii) $k_3, k_4 \in [n(*), n)$,
(iv) $m_3 \neq m_4$,
(v) $F_1(\nu_1,\nu_2) = F_0(\tau_1,\tau_2)$.

Note that $\eta^{*\frown}\langle\rho^*\rangle \neq \eta^\frown\langle\rho\rangle$. On the other hand (5) implies that $\eta = \eta^*$. Let $\zeta = \sup\{\xi : \rho^* \restriction \xi = \rho \restriction \xi\}$. Since $\{\eta_k : k < n(*)\}$ is 2-closed, it follows that $\eta^\frown\langle\rho^* \restriction \zeta\rangle = \eta^\frown\langle\rho \restriction \zeta\rangle \in \{\eta_k : k < n(*)\}$. And moreover both $\eta^\frown\langle\rho^* \restriction (\zeta+1)\rangle$ and $\eta^\frown\langle\rho \restriction (\zeta+1)\rangle$ are in $\{\eta_k : k < n(*)\}$.

Note that $\lg(F_1(\tau_1,\tau_2)(\lg(\eta^\frown\langle\rho\rangle))) = \zeta$. Otherwise $\{F_0(\tau_1,\tau_2), F_1(\tau_1,\tau_2,)\} \cap \{F_0(\nu_1,\nu_2), F_1(\nu_1,\nu_2,)\} = \emptyset$. Hence $F_0(\tau_1,\tau_2) \restriction \lg(\eta^\frown\langle\rho\rangle) = \eta^\frown\langle\rho \restriction \zeta\rangle \in \{\eta_k : k < n(*)\}$, contradicting (7).

Case 2. $\{\nu_1,\nu_2\} \cap \{\eta_{\alpha_m,k} : m \leq m(*), k < n\}$ is a singleton. By symmetry assume that $\nu_1 \in \{\eta_{\alpha_m,k} : m \leq m(*), k < n\}$. Hence there are $\eta^{*\frown}\langle\rho^*\rangle \in P^5 \cap \{\eta_k : k < n(*)\}$, τ_1,τ_2 and $m \leq m(*)$, $i_3 \neq i_4$, $k, k_3, k_4 < n$ such that

(a) $\nu_1 = \eta_{\alpha_m,k}$,
(b) $\nu_2 = F_0(\tau_1,\tau_2)$,
(c) $\tau_1 = \eta^{*\frown}\langle\rho^*\rangle^\frown\langle i_3\rangle$,
(d) $\tau_2 = \eta^{*\frown}\langle\rho^*\rangle^\frown\langle i_4\rangle$.

It follows that $\eta^{*\frown}\langle\rho^* \restriction \zeta\rangle \lhd \nu_2 = \eta^\frown\langle\rho\rangle^\frown\langle i_2\rangle$, where $\zeta = h_{\sigma(\eta^*)}(\tau_1,\tau_2)$.

As the last element of the sequence $\eta^{*\frown}\langle\rho^* \restriction \zeta\rangle$ has length ζ we must have: $\eta^{*\frown}\langle\rho^* \restriction \zeta\rangle \lhd \eta$. Hence $\eta^{*\frown}\langle\rho^* \restriction \zeta\rangle \in \{\eta_{\alpha_m,k} : k < n\}$. Since $\{\eta_{\alpha_m,k} : k < n\}$ is 2-closed and it follows by 2.3(2)(e) that $\nu_2 \in \{\eta_{\alpha_m,k} : k < n\}$, contradiction.

Case 3. $\{\nu_1,\nu_2\} \cap \{\eta_{\alpha_m,k} : m < m(*), k < n\} = \emptyset$. By the definition there are τ_l, m_l, k_l i_l for $l = 1,\ldots,4$ such that $\nu_1 = F_0(\tau_1,\tau_2)$, and $\nu_2 = F_0(\tau_3,\tau_4)$, and

(a) $\tau_1 = \eta_1^\frown\langle\rho_1\rangle^\frown\langle i_1\rangle = \eta_{\alpha_{m_1},k_1}$, $\tau_2 = \eta_1^\frown\langle\rho_1\rangle^\frown\langle i_2\rangle = \eta_{\alpha_{m_2},k_2}$,
(b) $m_1 \neq m_2$, $i_1 \neq i_2$,
(c) $\tau_3 = \eta_2^\frown\langle\rho_2\rangle^\frown\langle i_3\rangle = \eta_{\alpha_{m_3},k_3}$, $\tau_4 = \eta_2^\frown\langle\rho_2\rangle^\frown\langle i_4\rangle = \eta_{\alpha_{m_4},k_4}$,
(d) $m_3 \neq m_4$, $i_3 \neq i_4$,
(e) $\eta_1^\frown\langle\rho_1\rangle, \eta_2^\frown\langle\rho_2\rangle \in P^5 \cap \{\eta_k : k \leq n(*)\}$.

Let $\zeta_1 = h_{\sigma(\eta_1)}(\tau_1,\tau_2)$, $\zeta_2 = h_{\sigma(\eta_2)}(\tau_3,\tau_4)$. Note that $\eta_1^\frown\langle\rho_1 \restriction \zeta_1\rangle \lhd \eta^\frown\langle\rho\rangle$ and $\eta_2^\frown\langle\rho_2 \restriction \zeta_2\rangle \lhd \eta^\frown\langle\rho\rangle$. Hence either $\eta_1^\frown\langle\rho_1 \restriction \zeta_1\rangle \lhd \eta_2^\frown\langle\rho_2 \restriction \zeta_2\rangle$ or $\eta_2^\frown\langle\rho_2 \restriction \zeta_2\rangle \lhd \eta_1^\frown\langle\rho_1 \restriction \zeta_1\rangle$. Assume the first case, the other is symmetric. If $\eta_1^\frown\langle\rho_1 \restriction \zeta_1\rangle = \eta_2^\frown\langle\rho_2 \restriction \zeta_2\rangle$, then $\zeta_1 = \zeta_2 = \sup\{\xi : \rho_1 \restriction \xi = \rho_2 \restriction \xi\}$, as $\nu_1 \neq \nu_2$. Hence both $\eta_1^\frown\langle\rho_1 \restriction \zeta_1\rangle$ and $\eta_2^\frown\langle\rho_2 \restriction \zeta_2\rangle$ are in $\{\eta_k : k < n(*)\}$, and by 2.3 $\nu_1,\nu_2 \in \{\eta_k : k < n(*)\}$, contradiction. Therefore $\eta_1^\frown\langle\rho_1 \restriction \zeta_1\rangle \lhd \eta_2$. Since $\eta_2 \in \{\eta_k : k < n(*)\}$, also $\eta_1^\frown\langle\rho_1 \restriction \zeta_1\rangle \in \{\eta_k : k < n(*)\}$. As above this implies that $\nu_1 \in \{\eta_k : k < n(*)\}$, contradiction.

4. The exact density

In this section we prove that the topological density of the algebra $\mathcal{B}$ is μ.

Lemma 4.1. $d(\mathcal{B}_{\mathcal{T}}) \geq \mu$.

Proof Assume that $d(\mathcal{B}_{\mathcal{T}}) < \mu$ Hence there is a sequence $\mathcal{D}^{\otimes} = \langle\mathcal{D}^{\otimes}_{|} : | < \lceil(\mathcal{B}_{\mathcal{T}})\rangle$ of ultrafilters of $\mathcal{B}_{\mathcal{T}}$ such that for every $a \in \mathcal{B}_{\mathcal{T}} \setminus \{\prime\}$ there is j such that

$a \in D_j^{\otimes}$. Let

$$\mathcal{F} = \{\sigma : \sigma < \mu \text{ and there are } \eta \text{ and } \bar{\mathcal{D}} \text{ such that:}$$
$$(a) \quad \eta \in T \setminus P^7, \sigma(\eta) \geq \sigma$$
$$(b) \quad \bar{\mathcal{D}} = \langle D_j : j < \sigma \rangle \text{ is a sequence of ultrafilters of } \mathcal{B}_{\mathcal{T}},$$
$$(c) \quad \text{if } \eta \lhd \nu_1, \eta \lhd \nu_2, \nu_1 \neq \nu_2, \text{ then } x_{\nu_1} \vartriangle x_{\nu_2} \in \bigcup_{j<\sigma} D_j\}$$

Note that $\mathcal{F} \neq \emptyset$. In particular $\langle\rangle$, $\mathcal{D}^{\otimes}$ witness that $d(\mathcal{B}_{\mathcal{T}}) \in \mathcal{F}$. Let $\sigma^* = \min(\mathcal{F})$ and let $\eta^*, \mathcal{D}^*$ witness that $\sigma^* \in \mathcal{F}$. Note that $\sigma^* \geq \aleph_0$. Without loss of generality $\sigma^* = \sigma(\eta^*)$, (otherwise use $\eta^{*\frown}\langle\sigma^*\rangle$ instead of σ^*.)

Now we choose by induction on $\zeta < (\sigma^*)^+$, a sequence ρ_ζ such that:

(1) $\eta^{*\frown}\langle\rho_\zeta\rangle \in T$,
(2) $\lg(\rho_\zeta) = 1 + \zeta$,
(3) $\xi < \zeta \implies \rho_\xi = \rho_\zeta \restriction (1 + \xi)$,
(4) $\eta^{*\frown}\langle\rho_{\zeta+1}\rangle \in P^6$, $\rho_{\zeta+1}(\zeta) = (\nu_{0,\zeta}, \nu_{1,\zeta})$,
(5) if $\zeta = \sigma^*\xi + j$, $j < \sigma^*$, then $(x_{\nu_{0,\zeta}} \vartriangle x_{\nu_{1,\zeta}}) \notin \bigcup_{i<j} D_i^*$

For $\zeta = 0$, let $\rho_0 = \langle\sigma^*\rangle$. For ζ limit put $\rho_\zeta = \bigcup_{\xi<\zeta} \rho_\xi$. For $\zeta = \xi + 1$, $\zeta = \sigma^*\xi + j$, if we cannot find suitable (ν_0, ν_1), then $\eta^{*\frown}\langle\rho_\xi\rangle$, $\langle D_i^* : i < j\rangle$ witness that $j \in \mathcal{F}$, contradicting the minimality of σ^*.

Let $\rho^* = \rho_{(\sigma(\eta^*))^+}$. Note that for every $i_1 < i_2 < \sigma^*$, we have $x_{\eta^{*\frown}\langle\rho^*\rangle^\frown\langle i_1\rangle} \vartriangle x_{\eta^{*\frown}\langle\rho^*\rangle^\frown\langle i_2\rangle} \in \bigcup_{j<\sigma^*} D_j^*$. Hence there is $j(*) < \sigma^*$ such that $X = \{i < (\sigma^*)^+ : x_{\eta^{*\frown}\langle\rho^*\rangle^\frown\langle i\rangle} \in D_{j(*)}^*\}$ has cardinality $(\sigma^*)^+$. Now let $j_1 \in (j(*), \sigma^*)$ and $i_0, i_1 \in X$ be such that $h_{\sigma^*}(\{i_0, i_1\}) = j_1 \mod \sigma^*$, ($j_1$ exists be the definition of h_{σ^*}).

Now let $\tau_0 = \eta^{*\frown}\langle\rho^*\rangle^\frown\langle i_0\rangle$, $\tau_1 = \eta^{*\frown}\langle\rho^*\rangle^\frown\langle i_1\rangle$. Hence $F_0(\tau_0, \tau_1) = \nu_{0,\epsilon}$ and $F_1(\tau_0, \tau_1) = \nu_{1,\epsilon}$, where $\epsilon = h_{\sigma^*}(\{i_0, i_1\})$. So $\mathbf{e}_{\tau_0,\tau_1} \in \Gamma_T$. Note that $x_{\tau_0} \cap x_{\tau_1} \in D_{j(*)}^*$, and $x_{\nu_{0,\epsilon}} \vartriangle x_{\nu_{1,\epsilon}} \notin D_{j_1}^*$, as $j(*) < j_1$. Hence $-(x_{\nu_{0,\epsilon}} \vartriangle x_{\nu_{1,\epsilon}}) \in D_{j(*)}^*$. On the other hand $\mathbf{e}_{\tau_0,\tau_1} \in \Gamma_T$ implies that $x_{\tau_0} \cap x_{\tau_1} \cap -(x_{\nu_{0,\epsilon}} \vartriangle x_{\nu_{1,\epsilon}}) = 0$, contradiction.

Lemma 4.2. *$d(\mathcal{B}_{\mathcal{T}}) \leq \mu$.*

Proof The idea of the proof is to define a set $\mathcal{F} \subseteq \{\infty, -\infty\}^{\mathcal{T}}$ such that:

(1) $|\mathcal{F}| = \mu$
(2) For every $f \in \mathcal{F}$, the set $D_f = \{x_\eta^{f(\eta)} : \eta \in T\}$ generates an ultrafilter in $\mathcal{B}_{\mathcal{T}}$, where $x^1 = x$ and x^{-1} is the complement of x.
(3) For every $a \in \mathcal{B}_{\mathcal{T}}$, non-zero, there is $f \in \mathcal{F}$ such that a is an the ultrafilter generated by D_f.

First, divide T into three disjoint sets $T = T_0 \cup T_1 \cup T_2$ as follows. $T_0 = \bigcup\{\{\tau_0, \tau_1\} : \{\tau_0, \tau_1\} \in \mathrm{dom}(F^i), i = 0, 1\}$. Let T_1 be the image of T_0 under F^0 and F^1. It follows from the construction of T that T_0 is disjoint from T_1. Finally let $T_2 = T \setminus (T_0 \cup T_1)$.

Let $F_2 \subseteq \{1, -1\}^{T_2}$ be a set of cardinality μ such that

(a) for every finite $u \subseteq T_2$ and a function $h : u \to \{1, -1\}$ there is $f \in F_2$ such that $h \subseteq f$.

Similarly, let $F_1 \subseteq \{1, -1\}^{T_1}$ be a set of cardinality μ such that

(b) for every finite $u \subseteq T_1$ and a function $h : u \to \{1, -1\}$ there is $f \in F_1$ such that $h \subseteq f$.

Now, for every $f \in F_1$ we define a set $F_0^f \subseteq \{1,-1\}^{T_0}$ of cardinality μ such that

(c) for every $g \in F_0^f$, for every $\nu_0, \nu_1 \in T_1$, $\tau_0, \tau_1 \in T_0$, if $F^i(\tau_0, \tau_1) = \nu_i$, $i = 0, 1$ and $f(\nu_0) = f(\nu_1)$, then $(g(\tau_0), g(\tau_1)) \neq (1, 1)$

(d) F_0^f is dense with respect to (c), i.e., for every finite $u \subseteq T_0$ and a function $h : u \to \{1,-1\}$ such that the condition (c) is satisfied with h in place of g, then there is $g \in F_0^f$ such that $h \subseteq g$, i.e. $(*)_{f \cup g, T}$ holds.

Finally define $\mathcal{F} = \{\{_{\in} \cup \{_{\infty} \cup \{_{\prime} : \{_{\in} \in \mathcal{F}_{\in}, \{_{\infty} \in \mathcal{F}_{\infty}, \{_{\prime} \in \mathcal{F}_{\prime}^{\{_{\prime}}\}$.

We prove that $\mathcal{F}$ is as required. It is obvious that (1) holds, and (2) follows from the definition of $\mathcal{F}$, D_f and Proposition 3.2.

To prove (3) let $\mathcal{B}_{\mathcal{T}} \models \dashv > \prime$. Without loss of generality $a = \bigcup_{\eta \in u} x_\eta^{h(\eta)}$ for some finite $u \subseteq T$ and $h : u \to \{1,-1\}$. Let $u_i = u \cap T_i$ for $i \leq 2$. Let $f_2 \in F_2$ and $f_1 \in F_1$ be such that $h \restriction u_i \subseteq f_i$, $i = 1, 2$. Let $f_0 \in F_0^{f_1}$ be such that $h \restriction u_0 \subseteq f_0$. We have to show that f_0 exists. Note that $(*)_{h,T}$ holds since a is non-zero, hence f_0 exists by (d). It follows that $a \in D_f$, where $f = f_2 \cup f_1 \cup f_0$.

This finishes the proof of the theorem.

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