ON THE NUMBER OF ELEMENTARY SUBMODELS OF AN UNSUPERSTABLE HOMOGENEOUS STRUCTURE

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Abstract

We show that if $\mathbf{M}$ is a stable unsuperstable homogeneous structure, then for most $\kappa < |\mathbf{M}|$, the number of elementary submodels of $\mathbf{M}$ of power κ is 2^{κ}.

Through out this paper we assume that $\mathbf{M}$ is a stable unsuperstable homogeneous model such that $|\mathbf{M}|$ is strongly inaccessible (= regular and strong limit). We can drop this last assumption if instead of all elementary submodels of $\mathbf{M}$ we study only suitably small ones. Notice also that we do not assume that $Th(\mathbf{M})$ is stable. We assume that the reader is familiar with [HS] and use all the notions and results of it freely. In [Hy1] a strong nonstructure theorem was proved for the elementary submodels of $\mathbf{M}$ assuming the existence of Skolem-functions. In this paper we drop the assumption on the Skolem-functions and prove the following nonstructure theorem.

1 Theorem. *Let λ be the least regular cardinal $\geq \lambda(\mathbf{M})$. Assume κ is an uncountable regular cardinal ($< |\mathbf{M}|$) such that $\kappa > \lambda$ and $\kappa^{\omega} = \kappa$. Then there are models (=elementary submodels of $\mathbf{M}$) $\mathcal{A}_i$, $i < 2^{\kappa}$, such that for all $i < 2^{\kappa}$, $|\mathcal{A}_i| = \kappa$ and for all $i < j < 2^{\kappa}$, $\mathcal{A}_i \not\cong \mathcal{A}_j$.*

See [Hy1] for nonstructure results in the case $\mathbf{M}$ is unstable.

We prove Theorem 1 in a serie of lemmas. Let λ and κ be as in Theorem 1. By λ-saturated, λ-primary etc., we mean $F^{\mathbf{M}}_{\lambda}$-saturated, $F^{\mathbf{M}}_{\lambda}$-primary etc. Notice that $\mathbf{M}$ is λ-stable.

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The notion λ-construction ($=F^{\mathbf{M}}_\lambda$-construction) is defined as general F-construction is defined in [Sh].

2 Lemma. *Assume $(C,\{a_i|\ i<\alpha\},\{A_i|\ i<\alpha\})$ is a λ-construction and σ is a permutation of α. Let $b_i=a_{\sigma(i)}$ and $B_i=B_{\sigma(i)}$. If for all $i<\alpha$, $B_i\subseteq C\cup\{b_j|\ j<i\}$, then $(C,\{b_i|\ i<\alpha\},\{B_i|\ i<\alpha\})$ is a λ-construction.*

Proof. Exactly as [Sh] IV Theorem 3.3. ▫

We write $\kappa^{\leq\omega}$ for $\{\eta:\alpha\to\kappa|\ \alpha\leq\omega\}$, $\kappa^{<\omega}$ and $\kappa^\omega=\kappa^{=\omega}$ are defined similarly (of course these have also the other meaning, but it will be clear from the context, which one we mean). Let $J\subseteq\kappa^{\leq\omega}$ be such that it is closed under initial segments. If $\eta,\xi\in J$ then by $r'(\eta,\xi)$ we mean the longest element of J which is an initial segment of both η and ξ. If $u,v\in I=P_\omega(J)$ (=the set of all finite subsets of J) then by $r(u,v)$ we mean the largest set R which satisfies

(i) $R\subseteq\{r'(\eta,\xi)|\ \eta\in u,\ \xi\in v\}$

(ii) if $u,v\in R$ and u is an initial seqment of v, then $u=v$.

We order I by $u\leq v$ if for every $\eta\in u$ there is $\xi\in v$ such that η is an initial seqment of ξ i.e. $r(u,v)=r(u,u)$ $(=\{\eta\in u|\ \neg\exists\xi\in u(\eta$ is a proper initial segment of $\xi)\})$.

3 Definition. *Assume $J\subseteq\kappa^{\leq\omega}$ is closed under initial segments and $I=P_\omega(J)$. Let $\Sigma=\{A_u|\ u\in I\}$ be an indexed family of subsets of $\mathbf{M}$ of power $<|\mathbf{M}|$. We say that Σ is strongly independent if*

(i) for all $u,v\in I$, $u\leq v$ implies $A_u\subseteq A_v$,

(ii) if $u,u_i\in I$, $i<n$, and $B\subseteq\cup_{i<n}A_{u_i}$ has power $<\lambda$, then there is an automorphism $f=f^{\Sigma,B}_{(u,u_0,\dots,u_{n-1})}$ of $\mathbf{M}$ such that $f\restriction(B\cap A_u)=id_{B\cap A_u}$ and $f(B\cap u_i)\subseteq A_{r(u,u_i)}$.

The model construction in Lemma 4 belowe is a generalized version of the construction used in [Sh] XII.4.

4 Lemma. *Assume that $\Sigma=\{A_u|\ u\in I\}$, $I=P_\omega(J)$, is strongly independent. Then there are sets $\mathcal{A}_u\subseteq\mathbf{M}$, $u\in I$, such that*

(i) for all $u,v\in I$, $u\leq v$ implies $\mathcal{A}_u\subseteq\mathcal{A}_v$,

(ii) for all $u\in I$, $\mathcal{A}_u$ is λ-primary over A_u, (and so by (i), $\cup_{u\in I}\mathcal{A}_u$ is a model),

(iii) if $v\leq u$, then $\mathcal{A}_u$ is λ-atomic ($=F^{\mathbf{M}}_\lambda$-atomic) over $\cup_{u\in I}A_u$ and λ-primary over $\mathcal{A}_v\cup A_u$,

(iv) if $J'\subseteq J$ is closed under initial segments and $u\in P_\omega(J')$, then $\cup_{v\in P_\omega(J')}\mathcal{A}_v$ is λ-constructible over $\mathcal{A}_u\cup\bigcup_{v\in P_\omega(J')}A_v$.

Proof. Let $\{u_i|\ i<\alpha^*\}$ be an enumeration of I such that $u\leq v$ and $v\not\leq u$ implies $i<j$. It is easy to see that we can choose α, $\gamma_i<\alpha$ for $i<\alpha^*$, a_γ and B_γ for $\gamma<\alpha$, and $s:\alpha\to I$ so that

(a) $\gamma_0=0$ and $(\gamma_i)_{i<\alpha^*}$ is increasing and continuous,

(b) if $\gamma_i\leq\gamma<\gamma_{i+1}$, then $s(\gamma)=u_i$,

(c) for all $\gamma<\alpha$, $|B_\gamma|<\lambda$ and if we write for $\gamma\leq\alpha$, $A^\gamma_u=A_u\cup\{a_\delta|\ \delta<\gamma,\ s(\delta)\leq u\}$, then $B_\gamma\subseteq A^\gamma_{s(\gamma)}$,

(d) for all $\gamma<\alpha$, if we write $A^\gamma=\cup_{u\in I}A^\gamma_u$, then $t(a_\gamma,B_\gamma)$ λ-isolates $t(a_\gamma,A^\gamma)$,

(e) for all $i < \alpha^*$, there are no a and $B \subseteq A^{\gamma_i+1}_{u_i}$ of power $< \lambda$ such that $t(a, B)$ λ-isolates $t(a, A^{\gamma_{i+1}})$,

(f) if $a_\delta \in B_\gamma$, then $B_\delta \subseteq B_\gamma$.

For all $u \in I$, we define $\mathcal{A}_u = A^\alpha_u$. We show that these are as wanted.

(i) follows immediately from the definitions and for (ii) it is enough to prove the following claim (Claim (III) implies (ii) easily).

Claim. For all $i < \alpha^*$,

(I) $\Sigma_i = \{A^{\gamma_i}_u |\ u \in I\}$ is strongly independent, we write $f^{i,B}_{(u,u_0,\dots,u_{n-1})}$ instead of $f^{\Sigma_i,B}_{(u,u_0,\dots,u_{n-1})}$,

(II) the functions $f^{i,B}_{(u,u_0,\dots,u_{n-1})}$ can be chosen so that if $j < i$, $u, u_k \in I$, $k < n$, $B \subseteq \cup_{i<n} A^{\gamma_i}_{u_k}$ has power $< \lambda$ and $a_\gamma \in B$ implies $B_\gamma \subseteq B$ and $B' = B \cap A^{\gamma_j}$, then $f^{i,B}_{(u,u_0,\dots,u_{n-1})} \restriction B' = f^{j,B'}_{(u,u_0,\dots,u_{n-1})} \restriction B'$,

(III) if $j < i$, then $A^{\gamma_j+1}_{u_j}$ is λ-saturated,

Proof. Notice that if $a_\gamma \in A^\delta_u \cap A^\delta_v$, then $a_\gamma \in A^\delta_{r(u,v)}$. Similarly we see that the first half of (I) in the claim is always true (i.e. if $u \le v$ then for all $\delta < \alpha$, $A^\delta_u \subseteq A^\delta_v$.) We prove the rest by induction on $i < \alpha^*$. We notice first that it is enough to prove the existence of $f^{i,B}_{(u,u_0,\dots,u_{n-1})}$ only in the case when B satisfies

(*) if $a_\gamma \in B$, then $B_\gamma \subseteq B$.

For $i = 0$, there is nothing to prove. If i is limit, then the claim follows easily from the induction assumption (use (II) in the claim). So we assume that the claim holds for i and prove it for $i+1$. We prove first (I) and (II). For this let $u, u_k \in I$, $k < n$, and $B \subseteq \cup_{k<n} A^{\gamma_i+1}_{u_k}$ be of power $< \lambda$ such that (*) above is satisfied. If for all $k < n$, $s(\gamma_i) \not\le u_k$, then (I) and (II) in the claim follow immediately from the induction assumption. So we may assume that $s(\gamma_i) \le u_0$. Let $B' = B \cap (\cup_{k<n} A^{\gamma_i}_{u_k})$. By the induction assumption there is an automorphism $f = f^{i,B'}_{(u,u_0,\dots,u_{n-1})}$ of $\mathbf{M}$ such that $f \restriction (B' \cap A^{\gamma_i}_u) = id_{B' \cap A^{\gamma_i}_u}$ and $f(B' \cap A^{\gamma_i}_{u_k}) \subseteq A^{\gamma_i}_{r(u,u_k)}$. If $s(\gamma_i) \le u$, then, by (*) and (d) in the construction, we can find an automorphism $g = f^{i+1,B}_{(u,u_0,\dots,u_{n-1})}$ of $\mathbf{M}$ such that $g \restriction B' = f \restriction B'$ and $g \restriction (B - B') = id_{B-B'}$. Clearly this is as wanted.

So we may assume that $s(\gamma_i) \not\le u$. Since $s(\gamma_i) \le u_0$, $u_0 \not\le r(u, u_0)$. By the choise of the enumeration of I there is $j < i$ such that $u_j = r(u, u_0)$. Then by the induction assumption (part (III)), $A^{\gamma_i+1}_{u_j} = A^{\gamma_i}_{u_j} = A^{\gamma_j+1}_{u_j}$ is λ-saturated and by the choise of f, $f(B' \cap A^{\gamma_i}_{u_0}) \subseteq A^{\gamma_i}_{u_j}$. So by (d) in the construction and (*) above, there are no difficulties in finding the required automorphism $f^{i+1,B}_{(u,u_0,\dots,u_{n-1})}$.

So we need to prove (III): For this it is enough to show that $A^{\gamma_i+1}_{u_i}$ is λ-saturated. Assume not. Then there are a and B such that $B \subseteq A^{\gamma_i+1}_{u_i}$, $|B| < \lambda$ and $t(a, B)$ is not realized in $A^{\gamma_i+1}_{u_i}$. Since $\lambda \ge \lambda(\mathbf{M})$, there are b and C such that $B \subseteq C \subseteq A^{\gamma_i+1}_{u_i}$, $|C| < \lambda$, $t(b, B) = t(a, B)$ and $t(b, C)$ λ-isolates $t(b, A^{\gamma_i+1}_{u_i})$. But since (I) in the claim holds for $i+1$, $t(b, C)$ λ-isolates $t(b, A^{\gamma_{i+1}})$. This contradicts (e) in the construction. □ Claim

(iii) and (iv) follow immediately from the construction, Claim (III) and Lemma 2. □

Since $\mathbf{M}$ is unsuperstable, by [HS] Lemma 5.1, there are a and $\lambda(\mathbf{M})$-saturated models $\mathcal{A}_i$, $i<\omega$, such that

(i) if $j<i<\omega$, then $\mathcal{A}_j \subseteq \mathcal{A}_i$,

(ii) for all $i<\omega$, $a \not\downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$.

It is easy to see that we may choose the models $\mathcal{A}_i$ so that they are λ-saturated and of power λ. Let $\mathcal{A}_\omega$ be λ-primary over $a \cup \bigcup_{i<\omega} \mathcal{A}_i$. As in [Hy1] Chapter 1, for all $\eta \in \kappa^{\leq\omega}$, we can find $\mathcal{A}_\eta$ such that

(a) for all $\eta \in \kappa^{\leq\omega}$, there is an automorphism f_η of $\mathbf{M}$ such that $f_\eta(\mathcal{A}_{length(\eta)}) = \mathcal{A}_\eta$,

(b) if η is an initial segment of ξ, then $f_\xi \restriction \mathcal{A}_{length(\eta)} = f_\eta \restriction \mathcal{A}_{length(\eta)}$,

(c) if $\eta \in \kappa^{<\omega}$, $\alpha \in \kappa$ and X is the set of those $\xi \in \kappa^{\leq\omega}$ such that $\eta \frown (\alpha)$ is an initial segment of ξ, then

$$\cup_{\xi \in X} \mathcal{A}_\xi \downarrow_{\mathcal{A}_\eta} \cup_{\xi \in (\kappa^{\leq\omega} - X)} \mathcal{A}_\xi .$$

For all $\eta \in \kappa^\omega$, we let $a_\eta = f_\eta(a)$.

5 Lemma. *Assume $\eta \in \kappa^{<\omega}$, $\alpha \in \kappa$ and X is the set of those $\xi \in \kappa^{<\omega}$ such that $\eta \frown (\alpha)$ is an initial segment of ξ. Let $B \subseteq \cup_{\xi \in (\kappa^{\leq\omega} - X)} \mathcal{A}_\xi$ and $C \subseteq \cup_{\xi \in X} \mathcal{A}_\xi$ be of power $<\lambda$. Then there is $C' \subseteq \mathcal{A}_\eta$ such that $t(C', B) = t(C, B)$.*

Proof. By [Hy2] Lemma 8 (or [HS] Lemma 3.15 plus little work) we can find $D \subseteq \mathcal{A}_\eta$ of power $<\lambda$ such that for all $b \in B$, $t(b, \mathcal{A}_\eta \cup C)$ does not split over D. So if we choose $C' \subseteq \mathcal{A}_\eta$ so that $t(C', D) = t(C, D)$, then C' is as wanted. □

6 Lemma. *Assume $J \subseteq \kappa^{\leq\omega}$ and $I = P_\omega(J)$. For all $u \in I$, define $A_u = \cup_{\eta \in u} \mathcal{A}_\eta$. Then $\{A_u \mid u \in I\}$ is strongly independent.*

Proof. Follows immediately from Lemma 5. □

Let $S \subseteq \{\alpha < \kappa \mid cf(\alpha) = \omega\}$. By J_S we mean the set

$$\kappa^{<\omega} \cup \{\eta \in \kappa^\omega \mid \eta \text{ is strictly increasing and } \cup_{i<\omega} \eta(i) \in S\}.$$

Let $I_S = P_\omega(J_S)$ and $\mathcal{A}_S$ be the model given by Lemmas 4 and 6 for $\{A_u \mid u \in I_S\}$.

7 Lemma.

(i) Assume $\eta \in \kappa^{<\omega}$, $u \in I_S$, $\alpha < \kappa$, $\{\eta\} \leq u$ and $\{\eta \frown (\alpha)\} \not\leq u$. Let X be the set of theose $\xi \in J_S$ such that $\eta \frown (\alpha)$ is an initial segment of ξ. Then

$$\cup_{\xi \in X} \mathcal{A}_\xi \downarrow_{\mathcal{A}_u} \cup_{\xi \in J_S - X} \mathcal{A}_\xi .$$

(ii) Assume $\alpha \in \kappa$, $u \in I_S$ and $v \in P_\omega(J_S \cap \alpha^{\leq\omega})$ is maximal such that $v \leq u$. Then

$$\mathcal{A}_u \downarrow_{\mathcal{A}_v} \cup_{w \in P_\omega(J_S \cap \alpha^{\leq\omega})} \mathcal{A}_w .$$

Proof. (i): Let $C = \cup_{\xi \in X} \mathcal{A}_\xi$. By (c) in the definition of $\mathcal{A}_\xi$, $\xi \in \kappa^{\leq\omega}$, there is C' such that $t(C', \cup_{\xi \in J_S - X} \mathcal{A}_\xi) = t(C, \cup_{\xi \in J_S - X} \mathcal{A}_\xi)$ and $C' \downarrow_{\mathcal{A}_\eta} \mathcal{A}_u \cup \bigcup_{\xi \in J_S - X} \mathcal{A}_\xi$. So the claim follows from the first half of Lemma 4 (iii).

(ii): By (i), $A_u \downarrow_{\mathcal{A}_v} \cup_{w \in P_\omega(J_S \cap \alpha^{\leq\omega})} A_w$ from which the claim follows by Lemma 4 (iii) and (iv). □

8 Lemma. *Assume* $S, R \subseteq \{\alpha < \kappa | \, cf(\alpha) = \omega\}$ *are such that* $(S-R) \cup (R-S)$ *is stationary. Then* $\mathcal{A}_S$ *is not isomorphic to* $\mathcal{A}_R$.

Proof. Assume not. Let $f : \mathcal{A}_S \to \mathcal{A}_R$ be an isomorphism. We write I^{α}_S for the set of those $u \in I_S$, which satisfy that for all $\xi \in u$, $\cup_{i<length(\xi)} \xi(i) < \alpha$. I^{α}_R is defined similarly. Then we can find α and a_i, $i < \omega$, such that $\eta = (a_i)_{i<\omega}$ is strictly increasing, for all $i < \omega$, $f(\cup_{u \in I^{\alpha_i}_S} \mathcal{A}_u) = \cup_{u \in I^{\alpha_i}_R} \mathcal{A}_u$ and $\alpha = \cup_{i<\omega} \alpha_i \in (S-R) \cup (R-S)$. Without loss of generality we may assume that $\alpha \in S - R$, and so $\eta \in J_S - J_R$. Let $\mathcal{A}^{\alpha_i}_S = \cup_{u \in I^{\alpha_i}_S} \mathcal{A}_u$ and $\mathcal{A}^{\alpha_i}_R = \cup_{u \in I^{\alpha_i}_R} \mathcal{A}_u$. Then it easy to see that for all $i < \omega$, $a_\eta \not\downarrow_{\mathcal{A}^{\alpha_i}_S} \mathcal{A}^{\alpha_{i+1}}_S$ (use [HS] Lemma 3.8 (iii)). So there is $u \in I_R$ such that for all $i < \omega$, $\mathcal{A}_u \not\downarrow_{\mathcal{A}^{\alpha_i}_R} \mathcal{A}^{\alpha_{i+1}}_R$. Since $\alpha \notin R$, this contradicts Lemma 7 (ii). $\square$

We can now prove Theorem 1: By [Sh] Appendix 1 Theorem 1.3 (2) and (3), there are stationary $S_i \subseteq \{\alpha < \kappa | \, cf(\alpha) = \omega\}$, $i < \kappa$, such that for all $i < j < \kappa$, $S_i \cap S_j = \emptyset$. For all $X \subseteq \kappa$, let $\mathcal{A}_X = \mathcal{A}_{\cup_{i \in X} S_i}$. Then by Lemma 8, if $X \neq X'$, then $\mathcal{A}_X$ is not isomorphic to $\mathcal{A}_{X'}$. Since $\kappa^\omega = \kappa$, $|\mathcal{A}_X| = \kappa$. $\square$ Theorem 1.

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