

Toward categoricity for classes with no maximal models

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printed: September 15, 2020

Abstract

We provide here the first steps toward a Classification Theory of Abstract Elementary Classes with no maximal models, plus some mild set theoretical assumptions, when the class is categorical in some λ greater than its Löwenheim-Skolem number. We study the degree to which amalgamation may be recovered, the behaviour of non μ -splitting types. Most importantly, the existence of saturated models in a strong enough sense is proved, as a first step toward a complete solution to the Loś Conjecture for these classes. Further results are in preparation.

* The first author's research was partially supported by 'BSF' (USA-Israel); Publication no 635 in the first author's publication list. Done 12/96-1/97/4-97

Annotated Content

§0 Introduction.

[We link the present work to previous articles in the same field, and provide a large-scale picture in which the results of this paper fit.]

§1 How much amalgamation is left?

[Following [Sh 88], we prove that although amalgamation is not assumed, amalgamation bases are ‘dense’ for our purposes. We also prove the existence of Universal Extensions over amalgamation bases.]

§2 Types and Splitting

[We provide here the right notions of type for our context, and study the behaviour of non μ -splitting.]

§3 Building the right kind of limits

[We define the classes ${}^+\mathfrak{K}_{\mu,\alpha}$ and variants, in order to study in depth different concepts of limit models, and of saturated models. We prove the existence of a good notion of saturated models.]

0 Introduction

We study the categoricity spectrum of abstract elementary classes, when amalgamation is not assumed a priori, and the only strong model theoretical assumption is the non existence of maximal models. This looks to us like quite a natural assumption, and many classes of models that appear usually in mathematics satisfy it— while they are not first order, and thus need the expansion of Classification Theory, to which this work contributes.

Previous work with similar motivation appeared in the papers [Sh 87a], [Sh 87b], [Sh 88], [MaSh 285], [KlSh 362], [Sh 394], [Sh 472], [Sh 576], where the endeavour of extending Classification Theory to more general classes of

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models was started. Of course, some additional assumption had to be used in each one of those directions. There were **set theoretical** as well as **model theoretical** assumptions.

Among those set theoretical, the main lines were opened by Makkai and Shelah in [MaSh 285], where the existence of compact cardinals was used, and the Categoricity Spectrum for the corresponding classes was studied. Then followed the work of Kolman and Shelah [KlSh 362], and Shelah [Sh 472], where the hypothesis was reduced to that of the existence of measurable cardinals. Along those lines, the Loś Conjecture is not yet fully solved.

Many of the central difficulties in those papers had to do with pinning down the right kinds of types (when there is no compactness, the formula-based definition of types is no longer a good one), and with proving that the amalgamation property for the class still holds. In [MaSh 285] compactness was still the central tool, and the definition of types did not present a problem. The compactness also eased out in a crucial way the proof of amalgamation as well as the study of the categoricity spectrum. Of course, the price for the relative smoothness was high; thence the natural motivation of looking for results with more modest assumptions: reducing the large cardinal assumption to the existence of a measurable cardinal. This was worked out in [KlSh 362] and [Sh 472]. A considerable amount of work was then needed to pin down a notion of ‘good’ extensions. The lack of compactness was partially supplied for by the use of Generalised Ultrapowers (of structures). Their existence uses in a crucial way the measurability, and was central to the proof of the Categoricity Theorem there.

Among the model theoretical assumptions, the main references are at this point [Sh 394], where the amalgamation property is **the main** assumption. In this context, an extensive use of various kinds of Ehrenfeucht-Mostowski models is the central tool for constructing models in the proofs.

This paper could be thought of as ‘branching off from [Sh 394]’ (here, the amalgamation property is replaced by the weaker model theoretical assumption of the non-existence of maximal models). But this is not a completely accurate description of where this paper fits in the large picture: our set theoretical assumptions are definitely stronger than those of [Sh 394]: we use GCH in large chunks of Card , as well as diamonds and weak diamonds. Nevertheless, we do not use large cardinals, and in this relative sense, this paper ‘improves’ [KlSh 362] and [Sh 472].

We plan to continue along this line of research. The forthcoming paper

[ShVi 648] is the next stage.

We shall make free use of EM-models for abstract elementary classes, throughout the paper.

We wish to thank Mirna Džamonja for her useful comments on some aspects of this paper.

1 How much amalgamation is left?

This first section provides the basic framework for the work —we study the extent to which amalgamation may be recovered under our assumptions, as well as the existence of Universal Extensions. We also provide the main basic definitions.

1.1 A word about the hypotheses to be used — Abstract Elementary Classes.

The main **model theoretical** hypothesis at work here is, as indicated in the title, the non-existence of maximal models in the class. The main **set theoretical** assumption here is the GCH, or at least the existence of weak diamonds over the relevant cardinals.

Additionally, we will assume in many parts of this work that the classes

(1) have a Löwenheim-Skolem number $LS(\mathfrak{K})$, and

(2) that they are **categorical** for some λ ,

with λ high enough compared with $LS(\mathfrak{K})$, or at least that the number of models of cardinality λ in \mathfrak{K} is $< \mu_{\mathfrak{K}}(\lambda)$, modulo isomorphism. $\mu_{\mathfrak{K}}(\lambda)$ is often equal to 2^λ (in this case, the assumption is just that the class does not have the maximum possible number of models in λ), but in other cases may be ‘a bit less’ than 2^λ . For more details on the relationship between $\mu_{\mathfrak{K}}(\lambda)$ and 2^λ , the reader is referred to [Sh 576, §1]. There, our $\mu_{\mathfrak{K}}(\lambda)$ is called $\mu_{\text{wd}}(\lambda)$; the definition provided there is much more general than what we need here; we roughly describe $\mu_{\mathfrak{K}}(\lambda)$ as ‘the covering number for the weak diamond ideal on λ ’.

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In some portions of the work, certain versions of $\diamond_{S_{\text{cf}(\mu)}^{\mu+}}$ for $\mu \in [LS(\mathfrak{K}), \lambda)$ are used. The full power of GCH is not really needed throughout the paper; still it is essential for the proof of the local character of non μ -splitting of types, a central notion in this work. Up to some point, the set theoretical assumption GCH ‘provides’ here what otherwise is missing as model theoretical assumptions, when we compare our hypotheses to those of [Sh 394] (specifically, the assumption there that all models in \mathfrak{K} are amalgamation bases).

Definition 1.1.1 (Abstract Elementary Classes)

(1) $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an **abstract elementary class** iff \mathfrak{K} is a class of models of some fixed vocabulary $\tau = \tau_{\mathfrak{K}}$ and $\leq_{\mathfrak{K}}$ is a two place relation on K , satisfying the following axioms

Ax 0: If $M \in \mathfrak{K}$, then all τ -models isomorphic to M are also in K .
The relation $\leq_{\mathfrak{K}}$ is preserved under isomorphisms,

Ax I: If $M \leq_{\mathfrak{K}} N$, then M is a submodel of N ,

Ax II: $\leq_{\mathfrak{K}}$ is an order on K ,

Ax III: The union of a $\leq_{\mathfrak{K}}$ -increasing continuous chain \vec{M} of elements of \mathfrak{K} is an element of \mathfrak{K} ,

Ax IV: The union of a $\leq_{\mathfrak{K}}$ -increasing continuous chain \vec{M} of elements of \mathfrak{K} is the lub of \vec{M} under $\leq_{\mathfrak{K}}$,

Ax V: If $M_{\ell} \leq_{\mathfrak{K}} N$ for $\ell \in \{0, 1\}$ and M_0 is a submodel of M_1 , then $M_0 \leq_{\mathfrak{K}} M_1$,

Ax VI: There is a cardinal κ such that for every $M \in \mathfrak{K}$ and $A \subset |M|$, there is $N \leq_{\mathfrak{K}} M$ such that $A \subset |N|$ and $\|N\| \leq \kappa \cdot |A|$. The least such κ is denoted by $LS(\mathfrak{K})$ and called the **Löwenheim-Skolem number of \mathfrak{K}** .

(2) If λ is a cardinal and \mathfrak{K} an abstract elementary class, we denote by \mathfrak{K}_{λ} the family of all elements of \mathfrak{K} whose cardinality is λ . We similarly define $\mathfrak{K}_{<\lambda}$

(3) Suppose that \mathfrak{K} is an abstract elementary class.

- (a) \mathfrak{K} is said to have the **joint embedding property** (*'JEP'*) iff for any $M_1, M_2 \in \mathfrak{K}$, there is $N \in \mathfrak{K}$ such that M_1, M_2 are $\leq_{\mathfrak{K}}$ -embeddable into N .
- (b) \mathfrak{K} is said to have **amalgamation** iff for all $M_0, M_1, M_2 \in \mathfrak{K}$ and $\leq_{\mathfrak{K}}$ -embeddings $g_l : M_0 \rightarrow M_l$ for $l \in \{1, 2\}$, there is $N \in \mathfrak{K}$ and $\leq_{\mathfrak{K}}$ -embeddings $f_l : M_l \rightarrow N$ such that $f_1 \circ g_1 = f_2 \circ g_2$.

(4) For $\mathfrak{K}^1 \subset \mathfrak{K}$, let

$$(\mathfrak{K}^1)^{am} = \left\{ M_0 \in \mathfrak{K}^1 \left| \begin{array}{l} \text{if } M_1, M_2 \in \mathfrak{K}^1, g_1, g_2 \text{ are as in (3)(b),} \\ \text{then there are } N \in \mathfrak{K}^1, \text{ and } f_1, f_2 \\ \text{such that } f_1 \circ g_1 = f_2 \circ g_2 \end{array} \right. \right\}.$$

The main point here is to get the amalgamation **inside** the class.

1.2 Density of Amalgamation Bases

To ease the reading of this paper, we shall (sometimes redundantly) endeavour to spell out the hypotheses used, at the beginning of each section.

Hypothesis 1.2.1 \mathfrak{K} is an abstract elementary class with no maximal model in $\mathfrak{K}_{<\lambda}$, categorical in λ .

The content and the proof of Theorem 1.2.4 below are basic in subsequent work. They are akin to those in [Sh 87a, Theorem 1.3] and [Sh 88]. Still, for the sake of completeness, we provide the argument. See more in [Sh 576], for a study of Weak Diamond principles and their relation to model theoretical properties of Abstract Elementary Classes.

Before looking at the next central questions in this context, namely the density of amalgamation bases and the existence of universal extensions over every model in \mathfrak{K}_{κ} , we need some additional results.

Fact 1.2.2 Suppose that there are no maximal models in $\mathfrak{K}_{\leq\lambda}$. Fix cardinals χ, μ such that $LS(\mathfrak{K}) \leq \mu \leq \chi$. Then

- 1) if $M \in \mathfrak{K}_{<\chi}$ and $\|M\| + LS(\mathfrak{K}) \leq \mu \leq \chi$, then there is N such that $M \leq_{\mathfrak{K}} N \in \mathfrak{K}_{\mu}$.

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2) If \mathfrak{K} is categorical in χ (in particular, if $\chi = \lambda$), then $\mathfrak{K}_{\leq \chi}$ has the JEP (joint embedding property).

PROOF 1) is easy, from a repeated use of the Löwenheim-Skolem theorem (in this context), and the non existence of maximal models: axiom III guarantees that unions of $\leq_{\mathfrak{K}}$ -increasing continuous chains of elements of \mathfrak{K} are in \mathfrak{K} .

2) is also easy to see, by embedding the models into extensions of size χ . $\square_{1.2.2}$

The main tool to construct models which have useful homogeneity properties is in this context the use of generalised Ehrenfeucht-Mostowski models. These were developed for the context of abstract elementary classes by Saharon Shelah in [Sh 88]. The following fact asserts that they exist in this context.

Fact 1.2.3 *For every linear order I , there is Φ such that $EM(I, \Phi)$ is an EM model (so, for instance, if $EM(I, \Phi) \in \mathfrak{K}$ and $J \subset I$, then $EM(J, \Phi) \leq_{\mathfrak{K}} EM(I, \Phi)$).*

PROOF Since there are no maximal models in \mathfrak{K} , there are models in \mathfrak{K}_{μ} , where $\mu = |EM(I, \Phi)| = |I| + |\tau| + \beth_{(2^{LS(\mathfrak{K})})^+}$, by [Sh 88, 1.7] (where τ is the size of the vocabulary). Now the construction of the EM models can be carried in a way similar to how it was done in [Sh:c, VII,§5].

$\square_{1.2.3}$

Theorem 1.2.4 (Density of Amalgamation Bases) *If $LS(\mathfrak{K}) < \kappa \leq \lambda$ (remember: λ is the categoricity cardinal of the hypotheses), $\exists \theta (2^\theta = 2^{< \kappa} < 2^\kappa)$, then for every $M \in \mathfrak{K}_{< \kappa}$, there is N with $M \leq_{\mathfrak{K}} N \in \mathfrak{K}_{< \kappa}^{am}$.*

PROOF Suppose M is a counterexample to this. The idea is to build a binary tree of models on top of M , in such a way that the two immediate successors of every node act as counterexamples to amalgamation over M , and then use the weak diamond at κ (whose existence is guaranteed by $2^\theta < 2^\kappa$! —for more on generalised weak diamonds, see the Appendix to the forthcoming ‘Proper Forcing’ book by the first author [Sh:f]) to get a contradiction. So, we choose by induction on $\alpha < \kappa$ models M_η , for $\eta \in {}^\alpha 2$, such that

- (a) $M_{<>} = M$
- (b) $M_\eta \in \mathfrak{K}_{<\kappa}$
- (c) $\alpha \text{ limit } \wedge \eta \in {}^\alpha 2 \Rightarrow M_\eta = \bigcup_{\beta < \alpha} M_{\eta \upharpoonright \beta}$
- (d) $\beta < \text{lg}(\eta) \Rightarrow M_{\eta \upharpoonright \beta} \leq_{\mathfrak{K}} M_\eta$
- (e) $M_{\eta \frown \langle 0 \rangle}, M_{\eta \frown \langle 1 \rangle}$ cannot be amalgamated over M_η ; i.e. there is no $N \in \mathfrak{K}_{<\kappa}$ and $\leq_{\mathfrak{K}}$ -embedding $f_\ell : M_{\eta \frown \langle \ell \rangle} \rightarrow N$ such that $f_0 \upharpoonright M_\eta = f_1 \upharpoonright M_\eta$ (so $M_\eta \neq M_{\eta \frown \langle \ell \rangle}$).

For each $\eta \in {}^\kappa 2$, $M_\eta = \bigcup_{\alpha < \kappa} M_{\eta \upharpoonright \alpha} \in \mathfrak{K}_\kappa$ hence by Fact 1.2.2(1) there is $N_\eta \in \mathfrak{K}_\lambda$ with $M_\eta \leq_{\mathfrak{K}} N_\eta$. By the categoricity in λ , there exists an isomorphism $h_\eta : N_\eta \xrightarrow{\text{onto}} N^* := EM(\lambda, \Phi)$. But then $h_\eta(M_\eta) \leq_{\mathfrak{K}} N^*$, hence M_η is $\leq_{\mathfrak{K}}$ -embedded into $EM(\alpha_\eta, \Phi)$ for some $\alpha_\eta < \kappa^+$.

Let $<^*$ be a linear order on κ isomorphic to $({}^{>\omega}\kappa, <_{lex})$, so that each $\alpha < \kappa^+$ can be embedded into it. Then $EM(\alpha_\eta, \Phi)$ is $\leq_{\mathfrak{K}}$ -embeddable into $N^* = EM((\kappa, <^*), \Phi)$. So, there is a $\leq_{\mathfrak{K}}$ -embedding $h_\eta^* : M_\eta \rightarrow N^*$.

Now use the weak diamond: since there exists θ such that $2^\theta = 2^{<\kappa} < 2^\kappa$, the weak diamond for κ holds, and thus there are distinct $\eta_1, \eta_2 \in {}^\kappa 2$ and there is $\alpha < \kappa$ such that $h_{\eta_1}^* \upharpoonright M_{\eta_1 \upharpoonright \alpha} = h_{\eta_2}^* \upharpoonright M_{\eta_2 \upharpoonright \alpha}$, and $\eta_1(\alpha) \neq \eta_2(\alpha)$. But both $M_{\eta_1 \upharpoonright \alpha+1}$ and $M_{\eta_2 \upharpoonright \alpha+1}$ embed into $EM(\kappa^+, \Phi)$. This contradicts that $M_{\eta_1 \upharpoonright \alpha}$ is not an amalgamation base!

□_{1.2.4}

So, we have density of amalgamation bases in the case mentioned above (there exists θ such that $2^\theta = 2^{<\kappa} < 2^\kappa$), but it should be made clear that the use of the weak diamond (or, a fortiori, of GCH in $[LS(\mathfrak{K}), \lambda)$), was crucial here.

1.3 Universal Extensions

At this point, we begin to include the following assumption:

GCH Hypothesis: $2^\mu = \mu^+$, for all $\mu \in [LS(\mathfrak{K}), \lambda)$. Although we stated at the outset this assumption, we repeat it now. Up to now, the weak diamond

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was enough. Nevertheless, it is worth stressing that our aim is to obtain as much stability as possible for our new contexts, and at the same time trying to use as little as possible set-theoretical assumptions. GCH does not seem too unreasonable from this point of view.

The following theorem is crucial in the study of the right kind of types in our context, and is a natural step in allowing us to build models with enough saturation. So far, we have not defined the types, and thus we concentrate on universality. It is worth noting that the existence of universal extensions here is obtained **for amalgamation bases**.

Theorem 1.3.1 (Existence of Universal Extensions) *Suppose that $\mu \in [LS(\mathfrak{K}), \lambda)$ and $M_0 \in \mathfrak{K}_\mu^{am}$. Then there is M_1 such that $M_0 \leq_{\mathfrak{K}} M_1 \in \mathfrak{K}_\mu^{am}$, M_1 is universal over M_0 (i.e. $M_0 \leq_{\mathfrak{K}} M_2 \in \mathfrak{K}_\mu \Rightarrow M_2$ is $\leq_{\mathfrak{K}}$ -embeddable into M_1 over M_0).*

PROOF Let I be a linear order of cardinality μ^+ such that $I \times (\alpha^+ + 1) \approx I$, for every $\alpha < \mu^+$, and pick $M_0 \in \mathfrak{K}_\mu^{am}$. We first move to the case of EM models, and prove the following fact.

Claim 1.3.2 *There is a $\leq_{\mathfrak{K}}$ -embedding $f : M_0 \rightarrow EM(I, \Phi)$ such that for every M_1 with $M_0 \leq_{\mathfrak{K}} M_1 \in \mathfrak{K}_\mu$ there is a $\leq_{\mathfrak{K}}$ -embedding $g : M_1 \rightarrow EM(I, \Phi)$ extending f .*

PROOF We begin by listing (note the strong use of $2^\mu = \mu^+$ here!) all the possible embeddings from M_0 into $EM(I, \Phi)$ as $\langle f_i \mid i < \mu^+ \rangle$. For every f_i let now $M_{1,i} \in \mathfrak{K}_\mu$ be a counterexample to the property we are looking for; namely, $M_0 \leq_{\mathfrak{K}} M_{1,i}$ and f_i does not ‘lift’ to an embedding from $M_{1,i}$ to $EM(I, \Phi)$. Since $M_0 \in \mathfrak{K}_\mu^{am}$, we can find $\langle M_{2,i} \mid i \leq \mu^+ \rangle$, $\leq_{\mathfrak{K}}$ -increasing, continuous, such that $M_{2,0} = M_0$, and $i < \mu^+ \Rightarrow M_{1,i}$ is embeddable into $M_{2,i+1}$ over M_0 . Now, by categoricity in λ , we know that the limit M_{2,μ^+} is embeddable into $EM(\lambda, \Phi)$, and thus into some $EM(\alpha^*, \Phi)$, $\alpha^* < \mu^{++}$, and hence into $EM(I, \Phi)$, say by g . But then $g \upharpoonright M_0$ must be $f_{i(*)}$, for some $i(*) < \mu^+$. Contradiction. $\square_{1.3.2}$

Fix now some f as in the claim, and let $I_0 \subset I$, $|I_0| = \mu$ be such that $\text{Rang}(f) \subset EM(I_0, \Phi)$. We could have chosen I from the beginning as being decomposable as

$$I = \bigcup_{\zeta < \mu^+} I_\zeta,$$

Figure 1: Lifting f , in Claim 1.3.2

for $\langle I_\zeta \rangle_{\zeta < \mu^+}$ increasing, $|I_\zeta| = \mu$, and $I_{\zeta+1}$ universal (inside I) over I_ζ (a similar construction is also used in [Sh 220], and may be obtained by taking e.g.

$$I = \{\eta \in {}^\omega(\mu^+) \mid \eta \text{ eventually } 0 \text{ but not constantly } 0\},$$

ordered lexicographically, and $I_\zeta = \{\eta \in I \mid \text{Rang}(\eta) \subset \mu \times (1 + \zeta)\}$).

Let now M_1^* correspond via an isomorphism $f^+ \supset f$,

$$f^+ : M_1^* \xrightarrow{\text{ontg}} EM(I_1, \Phi),$$

to $EM(I_1, \Phi)$. We claim that M_1^* is universal over M_0 : Let $M_0 \leq_{\mathfrak{R}} M_2 \in \mathfrak{R}_\mu$. Pick the corresponding embedding $f_2 \supset f$, $f_2 : M_2 \rightarrow EM(I, \Phi)$. As before, let $I' \subset I$, $|I'| \leq \mu$ be such that $\text{Rang}(f_2) \subset EM(I', \Phi)$. Thus, for some automorphism h of I' , such that $h|_{I_0} = \text{identity}$, $h''(I') \subset I_1$. Then h induces an automorphism \hat{h} of $EM(I, \Phi)$, and we have

$$M_2 \xrightarrow{f_2} EM(I', \Phi) \xrightarrow{\hat{h}} EM(I_1, \Phi) \xrightarrow{(f^+)^{-1}} M_1^*,$$

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and $\hat{h} \upharpoonright \text{Rang}(f) = \text{id}$, $f_2 \supset f$. So, $(f^+)^{-1} \circ \hat{h} \circ f_2$ is an isomorphism from M_2 into M_1^* , and its restriction to M_0 is the identity. $\square_{1.3.1}$

Thus, we have universal models in the right cardinals over amalgamation bases. The following definition should be regarded as a first step toward the (μ, ν) -limits and our version of saturation.

Definition 1.3.3 Let $M_0 <_{\mu}^2 M_1$ mean (for $\mu \in [LS(\mathfrak{K}), \lambda)$) that $M_0, M_1 \in \mathfrak{K}_{\mu}$, $M_0 \leq_{\mathfrak{K}} M_1$ and M_1 is universal over M_0 .

Definition 1.3.4 Let δ be a limit ordinal, $\delta \leq \mu^+$, $\mu \in [LS(\mathfrak{K}), \lambda)$. Then,

$$M <_{\mu, \delta}^3 N$$

iff there is a $\leq_{\mathfrak{K}}$ -increasing continuous sequence $\vec{M} = \langle M_i \mid i \leq \delta \rangle$ such that $M_0 = M$ and $i < \delta \Rightarrow M_i \in \mathfrak{K}_{\mu}^{am}$ and $M_i <_{\mu}^2 M_{i+1}$, and $N = \bigcup_{i < \delta} M_i$ (so $\|N\| = \mu + |\delta|$).

(In this case, we use M_{δ} for $\bigcup_{i < \delta} M_i$ and call $\langle M_i : i \leq \delta \rangle$ a **witness** for $M <_{\mu, \delta}^3 N$.)

Remark 1.3.5

In previous uses of these extensions, the amalgamation property was assumed to hold in the class — here we must stress the fact that **by decree** all the levels up from M to N are amalgamation bases.

Among the basic properties of $<_{\mu, \delta}^3$, we have that

Fact 1.3.6 ($<_{\mu, \delta}^3$ and limits)

- 1) If $M_{\ell} <_{\mu, \delta}^3 N_{\ell}$ for $\ell = 1, 2$ and h is a $\leq_{\mathfrak{K}}$ -embedding of M_1 into M_2 then we can extend h to an isomorphism h^* from N_1 onto N_2 .
- 2) Moreover, if $\langle M_{\ell, i} : i \leq \delta \rangle$ witnesses $M_{\ell} <_{\mu, \delta}^3 N_{\ell}$ we can demand that h^* map $M_{1, 2i}$ into $M_{2, 2i}$ and $(h^*)^{-1}$ map $M_{2, 2i+1}$ into $M_{1, 2i+1}$.

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PROOF By induction on δ . □_{1.3.6}

Another easy fact about $\langle^3_{\mu, \delta}$ is:

Fact 1.3.7 1) $M \langle^3_{\mu, \delta} N$ iff $M \langle^3_{\mu, \text{cf}(\delta)} N$.

2) If $M \langle^3_{\mu, \delta} N$ and $u \subseteq \delta = \text{sup}(u)$, $\delta_1 = \text{otp}(u)$, then $M \langle^3_{\mu, \delta_1} N$.

Hypothesis 1.3.8 $\diamond_{S_{\text{cf}(\mu)}^{\mu^+}}$ for $\mu \in [LS(\mathfrak{K}), \lambda)$.

The use of amalgamation wherever possible, together with the existence of universal models over amalgamation bases, are the two basic tools of construction of saturated enough models. The following fact is important from that point of view.

Fact 1.3.9 If $M \langle^3_{\mu, \delta} N$ and $\delta < \mu^+$, then $N \in \mathfrak{K}_\mu^{\text{am}}$.

PROOF Easy by $\diamond_{S_{\text{cf}(\mu)}^{\mu^+}}$ (proof similar to that of 1.2.4). □_{1.3.9}

Fact 1.3.10 For every limit $\delta < \mu^+$ we have

1) If $M \in \mathfrak{K}_\mu^{\text{am}}$, then for some N we have $M \langle^3_{\mu, \delta} N$,

2) If $M \langle^3_{\mu, \delta} N$, then $N \in \mathfrak{K}_\mu^{\text{am}}$.

PROOF By induction on δ . Suppose that this is true for all limit ordinals $< \delta$. If δ is not a limit of limits, let δ_0 be the highest limit below δ . We have by induction hypothesis M_{δ_0} with $M \langle^3_{\mu, \delta_0} M_{\delta_0}$, as witnessed by some sequence $\langle M_i | i \leq \delta_0 \rangle$. Just taking a universal extension M_δ of M_{δ_0} over M does the trick: by Fact 1.3.9, M_{δ_0} is itself an amalgamation base, and thus the sequence $\langle M_i | i \leq \delta_0 \rangle \frown \langle M_\delta \rangle$ witnesses that $M \langle^3_{\mu, \delta} M_\delta$.

Now, if δ is a limit of limits, we construct a $\leq_{\mathfrak{K}}$ -increasing continuous sequence $\langle M_i | i \leq \delta \rangle$ of models **from** \mathfrak{K}_μ such that $M_{i+1} \in \mathfrak{K}_\mu^{\text{am}}$, M_{i+2} universal over M_{i+1} over $M = M_0$. We just take unions at limits, and universal extensions which are amalgamation bases at successors. This is close to what

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we need, but of course we still need to get that the M_i 's are amalgamation bases **all the way through** δ . But this is taken care of by Fact 1.3.9.

□_{1.3.10}

The following definition is crucial in the study of saturatedness in this class, and will play a central role from now on.

Definition 1.3.11 *We say that N is a (σ, δ) -limit if $M <_{\sigma, \delta}^3 N$ for some M and $\sigma \leq \delta$ is regular.*

The proofs of the previous facts essentially depended on constructions by induction on δ , as well as the use of set theoretical hypotheses. These hypotheses are not too strong (from our point of view), especially when one compares them to those that were used in the past by Makkai and Shelah in [MaSh 285] (compact cardinals) and by Kolman and Shelah in [KlSh 362] and Shelah in [Sh 472] (measurable cardinals). Of course, the ‘price to pay’ is that many notions ‘natural’ in those contexts (such as definitions of types as sets of formulas in the presence of a strongly compact cardinal in [MaSh 285] or the existence of many ultrapower operations in [KlSh 362] and [Sh 472]) are no longer ‘natural’ here, and require new ways of dealing with the categoricity problem.

2 Types and Splitting

2.1 What kind of types are good here?

We start by giving a definition of types for this context. It must be stressed that here, types are only defined over **models which are amalgamation bases**, so as to avoid confusion later. The definition of types here is essentially the same from [Sh 300, Ch.II] and [Sh 576, §0] and [Sh 394]. There is, though, a difference: in the presence of monster models (like in [Sh 394]), it is natural to construe all the automorphisms relevant to the definition of types as automorphisms of the monster. Here, in its absence, we must do with embeddings into an amalgam. Our hypotheses about amalgamation clear away the problem here. Still, the diagram chasing involved might be slightly more entangled than within monster models.

Definition 2.1.1

1) We define the **type**

$$\text{tp}(\bar{a}, M, N)$$

(when $M \leq_{\mathfrak{K}} N$, $\bar{a} \subset N$, $M, N \in \mathfrak{K}_{\mu}^{am}$) as $(\bar{a}, M, N)/E$ where E is the following equivalence relation: $(\bar{a}^1, M^1, N^1)E(\bar{a}^2, M^2, N^2)$ iff $M^\ell \leq_{\mathfrak{K}} N^\ell$, $\bar{a}^\ell \in {}^\alpha(N^\ell)$ (for some α), $M^1 = M^2$ and there is $N \in \mathfrak{K}$ satisfying $M^1 = M^2 \leq_{\mathfrak{K}} N$ and $f^\ell : N^\ell \xrightarrow{\leq_{\mathfrak{K}}} N$ over M^ℓ (i.e. $f \upharpoonright M^\ell$ is the identity) and $f^1(\bar{a}^1) = f^2(\bar{a}^2)$.

More generally, for $N \in \mathfrak{K}_{\mu}$ (not necessarily an amalgamation base) and $\bar{a} \subset N$, we define $\text{tp}(\bar{a}, M, N)$ as $\text{tp}(\bar{a}, M, N')$, with $M \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} N$, $\bar{a} \subset N'$, and N' is an amalgamation base.

- 2) We say that N is κ -**saturated** (when $\kappa > LS(\mathfrak{K})$) if $M \leq_{\mathfrak{K}} N$, $|M| < \kappa$ and $p \in \mathcal{S}^{<\omega}(M)$ (see below) imply that p is realized in M , i.e. for some $\bar{b} \subset N$, $p = \text{tp}(\bar{b}, M, N)$.
- 3) $\mathcal{S}^\alpha(M) := \{\text{tp}(\bar{a}, M, N) \mid \bar{a} \in {}^\alpha N, M \leq_{\mathfrak{K}} N\}$.
- 4) When $M \leq_{\mathfrak{K}} N$ and $p \in \mathcal{S}^\alpha(N)$, we denote by $p \upharpoonright M$ the **restriction** to M of p given by $\text{tp}(\bar{a}, M, N_1)$, where $N \leq_{\mathfrak{K}} N_1$, $p = \text{tp}(\bar{a}, N, N_1)$ and $p \leq q$.
- 5) $\mathcal{S}(M) = \mathcal{S}^1(M)$ (we could just as well use $\mathcal{S}^{<\omega}(M)$).

Remark 2.1.2

We define types on M in N under the condition that M be an amalgamation base and there be some amalgamation base $N' \supset \bar{a}$ in between M and N . Under these conditions, we may prove that E is an equivalence relation. Otherwise, the diagram chasing for the transitivity of E , which we leave to the reader, would not go through.

The following fact is basic, and is used throughout the paper.

Fact 2.1.3 (Stability below λ) *Let $\mu < \lambda$. Since \mathfrak{K} is categorical in λ , for every $N \in \mathfrak{K}_{\mu}$, $|\mathcal{S}(N)| \leq \mu$.*

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PROOF Use $EM(\lambda, \Phi)$, just as in [KlSh 362, Theorem 3.9], where Kolman and Shelah prove the existence of weakly universal models over any $N \in \mathfrak{K}_\mu$.
 $\square_{2.1.3}$

Definition 2.1.4 (1) (μ -splitting) $p \in \mathcal{S}(M)$ μ -splits over $N \leq_{\mathfrak{K}} M$ iff $|N| \leq \mu$, and there are N_1, N_2, h such that: $N_1, N_2 \in \mathfrak{K}_\mu$, h an elementary mapping from N_1 onto N_2 over N such that the types $p \upharpoonright N_1$ and $h(p \upharpoonright N_1)$ are contradictory and $N \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} M$.

(2) We say that the type $q \in \mathcal{S}(N)$ is a **stationarisation** of $p \in \mathcal{S}(M)$, $\|M\| = \mu$, $M \subset N$, iff for some $M^- \leq_{\mathfrak{K}} M$, q does not μ -split over M^- .

The next theorem marks the real beginning of the new ideas in this paper. It uses GCH in a rather strong way, and sheds light on the local character of non- μ -splitting.

2.2 The Splitting

Theorem 2.2.1 *Assume that*

- (a) $\langle M_i \mid i \leq \sigma \rangle$ is $\leq_{\mathfrak{K}}$ -increasing and continuous,
- (b) for all $i \leq \sigma$, $M_i \in \mathfrak{K}_\mu$ and M_i is an amalgamation base in \mathfrak{K}_μ ,
- (c) each M_{i+1} is universal over M_i ,
- (d) $\text{cf}(\sigma) = \sigma \leq \mu^+ \leq \lambda$, and $p \in \mathcal{S}(M_\sigma)$. (Since M_σ is an amalgamation base, ‘types’ are well-defined in this context.)

Then, for some $i < \sigma$, p does not μ -split over M_i .

Remark 2.2.2

We do not just have

$$\bigwedge_{j \in (i, \sigma)} [p \upharpoonright M_j \text{ does not split over } M_i].$$

PROOF Assume that the conclusion fails. We shall choose $\langle M_i \mid i \leq \sigma \rangle$ and p contradicting the statement, fitting into one of the following possibilities.

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- (a) $j < \sigma \Rightarrow p \upharpoonright M_j$ does not μ -split over M_0 .
- (b) else (a) is impossible, and $p \upharpoonright M_{2i+1}$ μ -splits over M_{2i} and $p \upharpoonright M_{2i+2}$ does not μ -split over M_{2i+1} .
- (c) else (a) and (b) are both impossible, and $\sigma = \mu$ (so μ is regular), and $i < \sigma \Rightarrow p \upharpoonright M_{i+1}$ μ -splits over M_i .

Without loss of generality, M_{i+1} is (μ, ω) -limit over M_i , as there is such an M'_{i+1} , $M_i \leq_{\mathfrak{K}} M'_{i+1} \leq_{\mathfrak{K}} M_{i+1}$.

Claim 2.2.3 *One of (a), (b) and (c) is always possible.*

PROOF Assume that both (a) and (b) are impossible. Given $\vec{M} = \langle M_i \mid i \leq \sigma \rangle$ and $p \in \mathcal{S}(M_\sigma)$, we will use the fact that both (a) and (b) are impossible (for any \vec{M}) in order to produce some \vec{M}' satisfying (c). As for any $j < \sigma$, possibility (a) fails for $\langle M_{j+i} \mid i \leq \sigma \rangle$, we have that necessarily

- (*) for every $j < \sigma$, there is some $\zeta_j \in (j, \sigma)$ such that $p \upharpoonright M_{\zeta_j}$ μ -splits over M_j .

Even more so, by renaming, we can require

- (*)' $p \upharpoonright M_{i+1}$ μ -splits over M_i . [We are close here to a (c)-style sequence. What is still missing is the appropriate length.]

We can find $\langle M_{i,j} \mid j \leq \mu \rangle \leq_{\mathfrak{K}}$ -increasing continuous, $M_{i,j+1}$ (μ, ω) -limit over $M_{i,j}$, $M_{i,j}$ an amalgamation base [we freely use 1.3.10], for each $j \leq \mu$, with $M_{i,0} = M_i$, $M_{i,\mu} \leq_{\mathfrak{K}} M_{i+1}$. Now we ask, for each i ,

- \otimes_i Does $p \upharpoonright M_{i,\mu}$ μ -split over $M_{i,j}$ for every $j \leq \mu$?

If for some i , the answer is ‘yes’, then we can repeat the procedure above (applied now to $\langle M_{i,j} \mid j \leq \mu \rangle$ and $p \upharpoonright M_{i,\mu}$). So we get that (*)' holds, i.e. possibility (c) holds for $\langle M_{i,j} \mid j \leq \mu \rangle$. If, on the other hand, for every $i \leq \sigma$, the answer to \otimes_i is no, then for some $j_i < \mu$, $p \upharpoonright M_{i,\mu}$ does not μ -split over M_{i,j_i} . Consider the sequence

$$\langle M_{0,j_0}, M_{0,\mu}, M_{1,j_1}, M_{1,\mu}, \dots, M_{\sigma,j_\sigma}, M_{\sigma,\mu} \rangle.$$

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This sequence and p clearly witness case (b). □_{2.2.3}

We now come back to the proof of Theorem 2.2.1, and look at the three possible cases from the last claim.

Proof in Case (c): Under our hypotheses, we have that $2^{<\mu} = \mu$. Let $p, \langle M_i | i \leq \mu \rangle$ be as in case (c). Choose by induction on $i \leq \mu$ models N_i and sequences $\langle g_\eta^i | \eta \in {}^i 2 \rangle$ such that

- (α) $N_i \in \mathfrak{K}_\mu$, $\langle N_i | i \leq \mu \rangle$ is $\leq_{\mathfrak{K}}$ -increasing continuous,
- (β) N_{i+1} is a (μ, ω) -limit over N_i ,
- (γ) $N_0 = M_0$,
- (δ) g_η^i is an isomorphism from M_{ω^i} onto N_i ,
- (ε) $g_{\eta \upharpoonright j}^i \subset g_\eta^i$, for $j < i$,
- (ζ) $g_{\eta \frown \langle 0 \rangle}^{i+1}(p \upharpoonright M_{\omega^{i+1}}) \neq g_{\eta \frown \langle 1 \rangle}^{i+1}(p \upharpoonright M_{\omega^{i+1}})$.

The clause (ζ) is possible because \vec{M} , p witness the case (c). Having obtained these sequences and isomorphisms, we have that N_μ is a (μ, μ) -limit. For $\eta \in {}^\mu 2$, $g_\eta : M_\mu \xrightarrow{\cong} N_\mu$, and the $g_\eta(p)$, for $\eta \in {}^\mu 2$ (in $\mathcal{S}(N_\mu)$) are pairwise distinct.

So, $N_\mu \in \mathfrak{K}_\mu$ is an amalgamation base, and $|\mathcal{S}(N_\mu)| > \mu$. This contradicts the basic fact 2.1.3, and ends the proof when dealing with possibility (c).

Proof in Case (a) or (b): Choose $\vec{C} = \vec{C}^\sigma = \langle C_\alpha^\sigma | \alpha \in S_\sigma^{\mu^+} \rangle$, where

- (a) $S_\sigma^{\mu^+}$ denotes the set of ordinals $< \mu^+$ of cofinality σ , and
- (b) for every α , $C_\alpha^\sigma \subset \alpha$ is a club, $\text{otp}(C_\alpha^\sigma) = \sigma$, and for every club C of μ^+ , the set

$$\{ \delta \in S_\sigma^{\mu^+} \mid \delta = \sup[C \cap \text{nacc}(C_\delta)] \}$$

is stationary, where $\text{nacc}(X)$ is the set of nonaccumulation points of X . This is possible by [Sh:g, III].

We start with p and $\langle M_i | i \leq \sigma \rangle$ as there and choose (by induction on $\alpha < \mu^+$) $N_\alpha \in \mathfrak{K}_\mu^{\text{am}}$ such that

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- (i) $\langle N_\alpha \mid \alpha < \mu^+ \rangle$ is $\leq_{\mathfrak{K}}$ -increasing continuous,
- (ii) $N_{\alpha+1}$ is (μ, ω) -limit over N_α ,
- (iii) when $\text{cf}(\alpha) = \sigma$, then we list C_α^σ (our originally chosen club in α of order type σ) increasingly as

$$C_\alpha^\sigma = \{\beta_{\sigma, \alpha, \zeta} \mid \zeta < \sigma\}.$$

Additionally, we let $\beta_{\sigma, \alpha, \sigma} = \alpha$ and also let $\langle M_i \mid i \leq \sigma \rangle$ and $\langle N_{\beta_{\sigma, \alpha, \zeta}} \mid \zeta \leq \sigma \rangle$ be isomorphic via $g_\alpha : M_\sigma \xrightarrow{\cong} N_\alpha$ (so that $g_\alpha(M_\zeta) = N_{\beta_{\sigma, \alpha, \zeta}}$). Let $a_\alpha \in N_{\alpha+1}$ realise $g_\alpha(p)$.

So, we have $\langle N_\alpha \mid \alpha < \mu^+ \rangle$. Let $N = \bigcup_{\alpha < \mu^+} N_\alpha \in \mathfrak{K}_{\mu^+}$.

Clearly, $N \leq_{\mathfrak{K}}$ -embeds into $EM(\lambda, \Phi)$. Even more, we can use $\Phi' = \Psi \circ \Phi$ such that $EM(\mu^+, \Phi')$ is universal in \mathfrak{K}_{μ^+} , and has as many automorphisms as we will need. For more details on the theory of EM models for abstract elementary classes, see [Sh 394, I, §4].

So we have a $\leq_{\mathfrak{K}}$ -embedding $h : N \rightarrow EM(\mu^+, \Phi')$. For $\alpha \in S$ ($S := S_\sigma^{\mu^+}$; $S = \text{Dom}(\vec{C}^\sigma)$), let

$$h(a_\alpha) = \tau_\alpha(\xi_1^\alpha, \dots, \xi_{n(\alpha)}^\alpha),$$

with $\xi_1^\alpha < \dots < \xi_{m(\alpha)}^\alpha < \alpha \leq \xi_{m(\alpha)+1}^\alpha, \dots, \xi_{n(\alpha)}^\alpha < \mu^+$, and let

$$E = \left\{ \gamma < \mu^+ \mid \forall c \in N [c \in N_\gamma \leftrightarrow h(c) \in EM(\gamma, \Phi')] \text{ and } \gamma \text{ a limit ordinal} \right\}.$$

Clearly, E is a club.

We now focus on case (a): for some stationary $S^* \subset S$, $\alpha \in S^* \Rightarrow C_\alpha^\sigma \subset E$, $\tau_\alpha = \tau^*$, $n(\alpha) = n^*$, $m(\alpha) = m^*$, $\xi_1^\alpha = \xi_1^*, \dots, \xi_{m(\alpha)}^\alpha = \xi_{m^*}^*$, $\beta_{\alpha, \sigma, 0} = \beta_{*, 0}$. Let $\alpha' < \alpha''$ be in S^* . We then have that

- (i) $\text{tp}(a_{\alpha''}, N_{\alpha'}, N_{\alpha''+1})$ does not μ -split over $N_{\beta_*, 0}$
- (ii) $\text{tp}(a_{\alpha'}, N_{\alpha'}, N_{\alpha''+1})$ μ -splits over $N_{\beta_*, 0}$.

For (i), we use the choice of p and $\langle M_i \mid i \leq \sigma \rangle$ as in case (a) and choose $j < \sigma$ such that $g_{\alpha''}(M_j) \supset N_{\alpha'}$; since $a_{\alpha''}$ realises $g_{\alpha''}(p)$, we get that $\text{tp}(a_{\alpha''}, N_{\alpha'}, N_{\alpha''+1})$ does not μ -split over $N_{\beta_*, 0}$. To see (ii), we just use our original assumption about the splitting of p , and ‘translate’ it via $g_{\alpha'}$.

So, the two types must be different, and thence

$$\text{tp}(h(a_{\alpha''}), EM(\alpha', \Phi'), M_{\alpha''+1}) \neq \text{tp}(h(a_{\alpha'}), EM(\alpha', \Phi'), M_{\alpha''+1}),$$

but on the other hand, it is easily seen that $h(a_{\alpha''})$ and $h(a_{\alpha'})$ realise the same type — Φ' could have been chosen at the outset so that there is an automorphism k of $EM(\mu^+, \Phi')$ with $k \upharpoonright EM(\alpha', \Phi') = \text{identity}$ and $k(h(a_{\alpha''})) = h(a_{\alpha'})$.

We now switch to case (b): Let χ be large enough, and let $\langle \mathfrak{B}_\alpha \mid \alpha < \mu^+ \rangle$ be a $\leq_{\mathfrak{K}}$ -increasing continuous sequence of elementary submodels of $\langle \mathcal{H}(\chi), \in, <_\chi^* \rangle$, each \mathfrak{B}_α of size μ , such that Φ , $EM(\lambda^+, \Phi)$, h , $\langle M_\alpha \mid \alpha < \mu^+ \rangle$ and $\langle a_\alpha \mid \alpha \in S \rangle$ all belong to \mathfrak{B}_0 , $\langle \mathfrak{B}_\alpha \mid \alpha \leq \gamma \rangle \in \mathfrak{B}_{\gamma+1}$, and $\mathfrak{B}_\gamma \cap \mu^+$ is an ordinal. Let

$$E^* = \{ \gamma \mid \mathfrak{B}_\gamma \cap \mu^+ = \gamma \}.$$

E^* is a club of μ^+ . Also, by the choice of \vec{C} , there is α such that $C_\alpha^\sigma \subset E^*$. Now find $\zeta < \sigma$ such that $\xi_1^\alpha, \dots, \xi_{m(\alpha)}^\alpha < \beta_{\sigma, \alpha, \zeta} < \alpha$, and $p \upharpoonright M_{\zeta+1}$ does not μ -split over M_ζ .

Let now φ be a formula in the language of set theory, with parametres in $\mathfrak{B}_{\beta_{\sigma, \alpha, \zeta+1}}$, satisfied by α , and saying all the properties of α we have used so far in this proof. We can then find $\alpha' \in (\beta_{\sigma, \alpha, \zeta}, \beta_{\sigma, \alpha, \zeta+1})$ such that the terms τ_α and $\tau_{\alpha'}$ coincide, and $m(\alpha) = m(\alpha')$, $n(\alpha) = n(\alpha')$, $\langle \xi_1^{\alpha'}, \dots, \xi_{m(\alpha')}^{\alpha'} \rangle = \langle \xi_1^\alpha, \dots, \xi_{m(\alpha)}^\alpha \rangle$. For every $\xi \leq \sigma$, h maps $M_{\beta_{\sigma, \alpha, \xi}}$ into $EM(\beta_{\sigma, \alpha, \xi}, \Phi')$, because $\beta_{\sigma, \alpha, \xi} \in E^*$.

Now compare the types of $h(a_\alpha)$ and $h(a_{\alpha'})$ on $h(N_{\alpha'}) \subset EM(\alpha', \Phi)$.

The first one does not μ -split by monotonicity and the choice of ζ , whereas the second one μ -splits by the construction, as p μ -splits over M_ζ . This contradicts the fact that the two types are the same by the way α' was chosen. □_{2.2.1}

3 Building the right kind of limits

We build here from the bottom up the right kind of limit, in order to approach the construction of models with strong saturation.

Hypothesis 3.0.1

(a) $LS(\mathfrak{K}) \leq \mu$,

- (b) On μ we have the consequences of §1 and §2, namely density of amalgamation (1.2.4) and non μ -splitting (2.2.1).
- (c) Categoricity in $\lambda > \mu$, $\lambda \geq \beth_{(2^{LS(\kappa)})^+}$, or at least some consequences of this.

3.1 Good extensions. Towers for Limits.

Definition 3.1.1 For $\alpha < \mu^+$, let

$$(a) \ \mathfrak{K}_{\mu,\alpha} = \left\{ (\vec{M}, \vec{a}) \left| \begin{array}{l} \vec{M} = \langle M_i \mid i < \alpha \rangle \text{ is } \leq_{\mathfrak{K}}\text{-increasing (not necessarily)} \\ \text{continuous), } \vec{a} = \langle a_i \mid i + 1 < \alpha \rangle, a_i \in M_{i+1} \setminus M_i, \\ M_i \in \mathfrak{K}_{\mu} \end{array} \right. \right\},$$

$$(b) \ \mathfrak{K}_{\mu,\alpha}^{am} = \{ (\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu,\alpha} \mid \text{each } M_i \text{ is an amalgamation base} \},$$

$$(c) \ \mathfrak{K}_{\mu,\alpha}^{\theta} = \{ (\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu,\alpha} \mid \text{each } M_i \text{ is a } (\mu, \theta)\text{-limit} \},$$

(d)

$$\mathfrak{K}_{\mu,\alpha}^* = \bigcup_{\theta \in \mu^+ \cap \text{Reg}} \mathfrak{K}_{\mu,\alpha}^{\theta},$$

where Reg denotes the class of regular cardinals,

$$(e) \ \text{}^+\mathfrak{K}_{\mu,\alpha}^{\theta} = \left\{ (\vec{M}, \vec{a}, \vec{N}) \left| \begin{array}{l} (\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu,\alpha}^{\theta}, \vec{N} = \langle N_i \mid i + 1 < \alpha \rangle, N_i \leq_{\mathfrak{K}} M_i, \\ N_i \text{ an amalgamation base in } \mathfrak{K}_{\mu}, \\ M_i \text{ universal over } N_i, \\ \text{tp}(a_i, M_i, M_{i+1}) \text{ does not } \mu\text{-split over } N_i \end{array} \right. \right\},$$

(f)

$$\text{}^+\mathfrak{K}_{\mu,\alpha}^* = \bigcup_{\theta \in \mu^+ \cap \text{Reg}} \text{}^+\mathfrak{K}_{\mu,\alpha}^{\theta}.$$

Remark 3.1.2

- (1) It is worth noting that, unlike what was done in other treatments of the subject (see, for example, [Sh 87a], [Sh 87b], [Sh 88], [MaSh 285], [KlSh 362], [Sh 394], [Sh 472] and [Sh 576]), here from now on we mainly deal with **towers** of models. Objects akin to the towers defined here were also used in [Sh 576, §8-§10] in a different context: there full

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amalgamation is obtained, but for very few cardinals (only 3 of them!)... here, we only have amalgamation for **dense** families of models, but for many more cardinals. We aim at obtaining in subsequent papers a full description of the categoricity spectrum; in that respect, amalgamation is a central feature. On the other hand, in [Sh 576, §8-§10], the construction is used in order to get the **non-forking amalgamation**, which is far down the road yet in our situation.

- (2) What is the point of the definition of ${}^+\mathfrak{K}_{\mu,\alpha}^*$? The idea is that we intend to have a parallel to ‘the stationarisation of $\text{tp}(a_i, M_i, M_{i+1}) \in \mathcal{S}(M_i)$, whenever $M_i \leq_{\mathfrak{K}} M'_i \in \mathfrak{K}_{\mu}^{am}$ ’. We now turn to defining three orders on the previously defined classes of towers of models. With these orderings we intend to capture strong enough notions of limit.
- (3) **Continuity** is not demanded in the definitions above. One of the major aims is to show that the continuous towers are dense.

Definition 3.1.3 For $\ell = 1, 2$,

- 1) for $(\vec{M}^\ell, \vec{a}^\ell) \in \mathfrak{K}_{\mu,\alpha}$, let

$$(\vec{M}^1, \vec{a}^1) \leq_{\mu,\alpha}^a (\vec{M}^2, \vec{a}^2)$$

mean $\vec{a}^1 = \vec{a}^2$, and for all $i < \alpha$, $M_i^1 \leq_{\mathfrak{K}} M_i^2$,

- 2) for $(\vec{M}^\ell, \vec{a}^\ell) \in \mathfrak{K}_{\mu,\alpha}^{am}$, let

$$(\vec{M}^1, \vec{a}^1) \leq_{\mu,\alpha}^b (\vec{M}^2, \vec{a}^2)$$

mean $\vec{a}^1 = \vec{a}^2$, and for all $i < \alpha$, $M_i^1 = M_i^2$ or $M_i^1 \leq_{\mathfrak{K}} M_i^2$, and moreover M_i^2 is universal over M_i^1 ,

- 3) for $(\vec{M}^\ell, \vec{a}^\ell, \vec{N}^\ell) \in {}^+\mathfrak{K}_{\mu,\alpha}^*$, let

$$(\vec{M}^1, \vec{a}^1, \vec{N}^1) \leq_{\mu,\alpha}^c (\vec{M}^2, \vec{a}^2, \vec{N}^2)$$

mean $\vec{a}^1 = \vec{a}^2$, $\vec{N}^1 = \vec{N}^2$ and for all $i < \alpha$, $M_i^1 = M_i^2$ or $M_i^1 \leq_{\mathfrak{K}} M_i^2$, M_i^2 is universal over M_i^1 (in \mathfrak{K}_{μ}) and $\text{tp}(a_i^1, M_i^2, M_{i+1}^2)$ does not μ -split over N_i^1 ,

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(4) In all these cases, we say ‘strictly’ and write ‘ $\leq_{\mu,\alpha}^x$, for $x = a, b$ or c ’ if $\bigwedge_i M_i^1 \neq M_i^2$.

We have the following facts.

Fact 3.1.4 $\mathfrak{K}_{\mu,\alpha} \supset \mathfrak{K}_{\mu,\alpha}^{am} \supset \mathfrak{K}_{\mu,\alpha}^*$.

PROOF The second inclusion is due to Fact 1.3.9. □_{3.1.4}

Fact 3.1.5 1) $\leq_{\mu,\alpha}^a$ is a partial order,

2) $\leq_{\mu,\alpha}^b$ is a partial order,

3) $\leq_{\mu,\alpha}^b \subset \leq_{\mu,\alpha}^a$,

4) If $\langle (\vec{M}^\zeta, \vec{a}^\zeta) \mid \zeta < \delta \rangle$ is a $\leq_{\mu,\alpha}^b$ -increasing sequence of members of $\mathfrak{K}_{\mu,\alpha}^{am}$, δ is a limit ordinal $< \mu^+$, and $(\vec{M}, \vec{a}) = (\langle \bigcup_{\zeta < \delta} M_i^\zeta \mid i < \alpha \rangle, \vec{a}^\zeta)$, then

(a) $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu,\alpha}^{am}$,

(b) (\vec{M}, \vec{a}) is the least upper bound of $\langle (\vec{M}^\zeta, \vec{a}^\zeta) \mid \zeta < \delta \rangle$ (both in $\leq_{\mu,\alpha}^a$ and $\leq_{\mu,\alpha}^b$).

□_{3.1.5}

Remark 3.1.6

Part (4) of 3.1.5 explains why we need to have $\leq_{\mu,\alpha}^b$ in addition to $\leq_{\mu,\alpha}^a$.

This lists the main basic properties of $\leq_{\mu,\alpha}^a$, $\leq_{\mu,\alpha}^b$ and $\leq_{\mu,\alpha}^c$. It is worth mentioning here that 3.1.5 (4) has several uses in what will come next.

Fact 3.1.7 $\mathfrak{K}_{\mu,\alpha}^*$, ${}^+\mathfrak{K}_{\mu,\alpha}^*$ are both non-empty.

PROOF We construct the sequences ‘from the bottom up’. Choose (by induction on $i < \alpha$) $M_i \in \mathfrak{K}_\mu$, $\leq_{\mathfrak{K}}$ -increasing continuous, such that

M_0 is (μ, ω) -limit,

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M_{i+1} is (μ, ω) -limit and universal over M_i ,

for i limit, M_i is chosen by continuity.

Choose $a_i \in M_{i+1} \setminus M_i$, and choose N_i by using 1.3.1 and 2.2.1. It is easy to see that the resulting sequence of ‘double towers’ $\langle (M_i, a_i, N_i) \mid i < \alpha \rangle$ belongs to ${}^+ \mathfrak{K}_{\mu, \alpha}^*$, and the corresponding $\langle (M_i, a_i) \mid i < \alpha \rangle$ to $\mathfrak{K}_{\mu, \alpha}^*$. $\square_{3.1.7}$

We now get a weak form of disjoint amalgamation.

Theorem 3.1.8 *If M_0 is (μ, θ) -limit, $M_0 \leq_{\mathfrak{K}} M_\ell$, $M_\ell \in \mathfrak{K}_\mu$, for $\ell = 1, 2$ and $b \in M_1$ then we can find M_3 , with $M_1 \leq_{\mathfrak{K}} M_3 \in \mathfrak{K}_\mu$ and a $\leq_{\mathfrak{K}}$ -embedding h of M_2 into M_3 such that $b \notin h(M_2)$.*

PROOF Suppose not. Then fix M_0, M_1, M_2 as in the statement, and for $i < \mu^+$, find $N_i \in \mathfrak{K}_\mu$, $\leq_{\mathfrak{K}}$ -increasing continuous, and additionally, also find N_i^0, N_i^1, N_i^2 whenever $\text{cf}(i) = \theta$, such that every N_i is an amalgamation base, N_{i+1} is universal over N_i , and

$$\text{cf}(i) = \theta \Rightarrow N_i = N_i^0 \leq_{\mathfrak{K}} N_i^\ell \leq_{\mathfrak{K}} N_{i+1}, \ell = 1, 2,$$

and $(N_i^0, N_i^1, N_i^2, b_i) \approx (M_0, M_1, M_2, b)$, for some $b_i \in N_i^1$.

Without loss of generality,

$$N := \bigcup_{i < \mu^+} N_i \leq_{\mathfrak{K}} EM(\mu^+, \Phi).$$

Let $E \subset \mu^+$ be a club thin enough so that, in particular,

$$\delta \in E \Rightarrow N \cap EM(\delta, \Phi) = N_\delta.$$

Let also $b_i = \tau_i(\alpha_{i,0}, \dots, \alpha_{i,n_i-1})$, with $\alpha_{i,m_i-1} < i \leq \alpha_{i,m_i}$, and

$$\bigwedge_{\ell < n_i-1} \alpha_{i,\ell} < \alpha_{i,\ell+1}.$$

Now choose $\delta_0 \in E$, with $\text{cf}(\delta_0) = \theta$, $\delta_0 < \delta_1 \in E$. Let h be the $\leq_{\mathfrak{K}}$ -mapping, with $\text{Dom } h = EM(\delta_1, \Phi)$, induced by

$$j \mapsto \begin{cases} j & \text{if } j < \delta_0 \\ \delta_1 + j & \text{if } \delta_0 \leq j < \delta_1. \end{cases}$$

On $(N_{\delta_0}^0, N_{\delta_0}^1, N_{\delta_0}^2, b_{\delta_0})$, we get precisely the required embedding, and this contradicts the assumption of its non-existence. $\square_{3.1.8}$

Fact 3.1.9 (Existence of good extensions)

- 1) If $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^*$, and $\theta \in \mu^+ \cap \text{Reg}$, then there is (\vec{M}', \vec{a}') with $(\vec{M}, \vec{a}) <_{\mu, \alpha}^b (\vec{M}', \vec{a}')$
 $(\vec{M}', \vec{a}') \in \mathfrak{K}_{\mu, \alpha}^\theta (\subset \mathfrak{K}_{\mu, \alpha}^*)$, where $(\vec{M}, \vec{a}) <_{\mu, \alpha}^b (\vec{M}', \vec{a}')$ means $(\vec{M}, \vec{a}) \leq_{\mu, \alpha}^b (\vec{M}', \vec{a}')$
 (\vec{M}', \vec{a}') and $\bigwedge_{\beta < \alpha} [M_\beta \neq M'_\beta]$.
- 2) Similarly for ${}^+ \mathfrak{K}_{\mu, \alpha}^*$, ${}^+ \mathfrak{K}_{\mu, \alpha}^\theta$, and $\leq_{\mu, \alpha}^c$.

PROOF

- 1) Start by observing that given any $M \in \mathfrak{K}_\mu^{am}$, there is $M' \in \mathfrak{K}_\mu$ universal over M which is actually a (μ, θ) -limit over M : just apply θ many times 1.3.1 (Existence of Universal Extensions). We still need to ensure that we get the ‘weak disjoint amalgamation property’, namely $a_i \notin M_i'$. Theorem 3.1.8 exactly provides this.
- 2) Like 1), together with the existence of stationarisation of types and the locality of non- μ -splitting (2.2.1).

□_{3.1.9}

We now get even more about the least upper bounds for the order $\leq_{\mu, \alpha}^c$.

Fact 3.1.10 1) $\leq_{\mu, \alpha}^c$ is a partial order,

2) If $\langle (\vec{M}^\zeta, \vec{a}^\zeta, \vec{N}^\zeta) \mid \zeta < \delta \rangle$ is a $\leq_{\mu, \alpha}^c$ -increasing sequence of members of ${}^+ \mathfrak{K}_{\mu, \alpha}^*$, δ is a limit $< \mu^+$, and (\vec{M}, \vec{a}) is as in 3.1.5 (4), then

- (a) $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^*$, $(\vec{M}, \vec{a}, \vec{N}) \in {}^+ \mathfrak{K}_{\mu, \alpha}^*$.
- (b) (\vec{M}, \vec{a}) is the l.u.b. of $\langle (\vec{M}^\zeta, \vec{a}^\zeta) \mid \zeta < \delta \rangle$ (both in $\leq_{\mu, \alpha}^a$ and $\leq_{\mu, \alpha}^b$),
and
- (c) $(\vec{M}, \vec{a}, \vec{N})$ is also a $\leq_{\mu, \alpha}^c$ -l.u.b. of $\langle (\vec{M}^\zeta, \vec{a}^\zeta, \vec{N}^\zeta) \mid \zeta < \delta \rangle$, where $\vec{N} = \vec{N}^\zeta$, for any ζ (remember they are all equal).

PROOF

- 1) Trivial,

- 2) If the conclusion were not to hold, then we would fall into ‘possibility (a)’ of the proof of 2.2.1, namely: if $\langle M_i | i \leq \sigma \rangle$ is $\leq_{\mathfrak{K}}$ -increasing and continuous, and for all $i \leq \sigma$, $M_i \in \mathfrak{K}_{\mu}^{am}$, M_{i+1} is universal over M_i , $p \in \mathcal{S}(M_{\delta})$, and $p \upharpoonright M_i$ does not μ -split over M_0 . But then, using 2.2.1, we have that p does not μ -split over M_0 .

□_{3.1.10}

Definition 3.1.11 1) $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^*$ is **reduced** if

$$(\vec{M}, \vec{a}) \leq_{\mu, \alpha}^b (\vec{M}', \vec{a}') \Rightarrow \bigwedge_{i < \alpha} \left[M'_i \cap \bigcup_{j < \alpha} M_j = M_i \right].$$

2) $(\vec{M}, \vec{a}, \vec{N}) \in {}^+\mathfrak{K}_{\mu, \alpha}^*$ is **reduced** if

$$(\vec{M}, \vec{a}, \vec{N}) \leq_{\mu, \alpha}^c (\vec{M}', \vec{a}, \vec{N}') \Rightarrow \bigwedge_{i < \alpha} \left[M'_i \cap \bigcup_{j < \alpha} M_j = M_i \right].$$

Remark 3.1.12

Equivalently, when defining (\vec{M}, \vec{a}) is reduced, we could have used $\leq_{\mu, \alpha}^a$ instead of $\leq_{\mu, \alpha}^b$: just notice that $x \leq_{\mu, \alpha}^a x' \leq_{\mu, \alpha}^b x'' \Rightarrow x \leq_{\mu, \alpha}^b x''$ and for all x in the appropriate class of towers there exists y such that $x \leq_{\mu, \alpha}^b y$.

Fact 3.1.13 (Density of Reduced Towers)

- 1) For every $\theta \in \mu^+ \cap \text{Reg}$, for every $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^{\theta}$, there is a reduced tower $(\vec{M}', \vec{a}') \in \mathfrak{K}_{\mu, \alpha}^{\theta}$ such that $(\vec{M}, \vec{a}) \leq_{\mathfrak{K}} (\vec{M}', \vec{a}')$.
- 2) Similarly for ${}^+\mathfrak{K}_{\mu, \alpha}^*$, ${}^+\mathfrak{K}_{\mu, \alpha}^{\theta}$.

PROOF

- (1) Let $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^{\theta}$. If the conclusion fails, then we can find $(\vec{M}^i, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^{\theta}$, $\leq_{\mu, \alpha}^b$ -increasing continuous for $i < \mu^+$, such that (\vec{M}^{i+1}, \vec{a}) witnesses that (\vec{M}^i, \vec{a}) is not reduced. Now the set

$$E = \left\{ \delta < \mu^+ \mid i < \alpha \Rightarrow \left(\bigcup_{\zeta < \mu^+} M_i^{\zeta} \right) \cap \left(\bigcup_{j < \alpha} M_j^{\delta} \right) = M_i^{\delta} \right\}$$

is a club of μ^+ . For $a \in \bigcup_{\substack{\zeta < \mu^+ \\ i < \alpha}} M_i^{\zeta}$, let

$$i(a) = \min\{i < \alpha \mid a \in \bigcup_{\substack{\zeta < \mu^+ \\ j < i}} M_j^\zeta\},$$

$$\zeta(a) = \min\{\zeta \mid a \in \bigcup_{\xi < \zeta} M_{i(a)}^\xi\}.$$

So, $E' = \{\delta \mid a \in M_i^\delta, i < \alpha \Rightarrow \zeta(a) < \delta\}$ is a club and $E' \subset E$. Choose $\delta^* \in E'$: this violates that the conclusion fails. [Why? $(\vec{M}^{\delta^*}, \vec{a})$ is reduced. To see this, just let $(\vec{M}^{\delta^*}, \vec{a}) \leq_{\mu, \alpha}^b (\vec{M}', \vec{a})$. We only have to check that for every $i < \alpha$, $M'_i \cap \bigcup_{j < \alpha} M_j^{\delta^*} \subset M_i^{\delta^*}$. So let $a \in M'_i \cap \bigcup_{j < \alpha} M_j^{\delta^*}$.

Since $\delta^* \in E'$, $\zeta(a) < \delta^*$, hence $a \in \bigcup_{\xi < \zeta(a)} M_{i(a)}^\xi$. But since $\delta^* \in E$, we have that $(\bigcup_{\zeta < \mu^+} M_i^\zeta) \cap (\bigcup_{j < \alpha} M_j^{\delta^*}) = M_i^{\delta^*}$. This implies that $a \in M_i^{\delta^*}$.]

(2) Clearly similar.

□_{3.1.13}

Fact 3.1.14 1) *In 3.1.5 (4), if $\delta = \sup\{\zeta < \delta \mid (\vec{M}^\zeta, \vec{a}^\zeta) \text{ is reduced}\}$, then (\vec{M}, \vec{a}) is reduced. (In fact, it is enough to have $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^{am}$, and $(\vec{M}^\zeta, \vec{a}^\zeta) \in \mathfrak{K}_{\mu, \alpha}^{am}$ is $\leq_{\mu, \alpha}^a$ -increasing.)*

2) *In 3.1.10 (2), if $\delta = \sup\{\zeta < \delta \mid (\vec{M}^\zeta, \vec{a}^\zeta, \vec{N}^\zeta) \text{ is reduced}\}$, then $(\vec{M}, \vec{a}, \vec{N}^\zeta)$ is reduced.*

PROOF Clear from the definition of ‘reduced’.

□_{3.1.14}

Theorem 3.1.15 1) *If $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^*$ is reduced, then \vec{M} is $\leq_{\mathfrak{R}}$ -increasing and continuous.*

2) *If $(\vec{M}, \vec{a}, \vec{N}) \in {}^+\mathfrak{K}_{\mu, \alpha}^*$ is reduced, then \vec{M} is $\leq_{\mathfrak{R}}$ -increasing and continuous.*

PROOF We prove by induction on $\delta < \alpha$ limit ordinal that if $(\vec{M}, \vec{a}) \in \mathfrak{K}_{\mu, \alpha}^*$ is reduced, then $M_\delta = \bigcup_{i < \delta} M_i$. Assume then failure for δ : there exists some $b \in M_\delta \setminus \bigcup_{i < \delta} M_i$. We can find (\vec{M}^ζ, \vec{a}) reduced, for $\zeta \leq \delta$, a $\leq_{\mu, \alpha}^b$ -strictly increasing continuous chain of towers, such that $(\vec{M}^0, \vec{a}) = (\vec{M}, \vec{a})$ (it

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exists by 3.1.5 (4), and because ‘reduced towers are dense and closed under limit’, 3.1.13). Now consider the diagonal sequence $\langle M_\zeta^\zeta \mid \zeta \leq \delta \rangle$. It is $\leq_{\mathfrak{K}}$ -increasing continuous, its members are in \mathfrak{K}_μ , each M_ζ^ζ is (μ, θ_ζ) -limit, for some $\theta_\zeta \in \mu^+ \cap \text{Reg}$, and $M_{\zeta+1}^{\zeta+1}$ is universal over M_ζ^ζ . Also, $M_\delta^\delta \subset M_{\delta+1}^\delta$, and $b \in M_{\delta+1}^\delta$. So, by the main result on non- μ -splitting (Theorem 2.2.1), for some $\xi < \delta$, $\text{tp}(b, \bigcup_{\xi < \delta} M_\xi^\xi, M_\delta^\delta)$ does not μ -split over M_ξ^ξ . Let now \mathfrak{B} be (μ, μ^+) -limit, $M_\delta^\delta \leq_{\mathfrak{K}} \mathfrak{B}$.

We choose by induction on $i \leq \delta$ models N_i and functions h_i such that

- * if $i \leq \xi + 1$, then $N_i = M_i^\delta$ and $h_i = \text{id}_{N_i}$,
- * if $i \in (\xi + 1, \delta]$, then $N_i \leq_{\mathfrak{K}} \mathfrak{B}$ is a (μ, θ_i) -limit model, and
 1. $\langle N_i \mid i \leq \delta \rangle$ is $\leq_{\mathfrak{K}}$ -increasing, continuous,
 2. N_{i+1} is universal over N_i ,
 3. $N_{\xi+2} \supset M_\delta^\delta$,
 4. h_i is a $\leq_{\mathfrak{K}}$ -embedding of M_i^δ into N_i ,
 5. h_{i+1} maps $M_{i+1}^\delta \setminus M_i^\delta$ into $N_{i+1} \setminus N_i$,
 6. $\langle h_i \mid i \leq \delta \rangle$ is increasing continuous,
 7. $\text{tp}(b, h_i(M_i^\delta), \mathfrak{B})$ does not μ -split over $N^* := M_\xi^\xi = h_i(M_\xi^\xi)$, for $i \geq \xi$.

For $i \leq \xi + 1$, this is trivial. For $i \in (\xi + 1, \delta)$ successor, by the claim 3.1.16 below. For $i \in (\xi + 1, \delta]$ limit, use 2.2.1 for the last clause (remember, by the induction hypothesis, $\langle M_i^\delta \mid i < \delta \rangle$ is continuous, and by definition, M_i^δ is (μ, θ) -limit, hence an amalgamation base). We also have that $\text{tp}(b, \bigcup_{i < \delta} h_i(M_i^\delta), \mathfrak{B})$ does not μ -split over N^* . So,

$$h^* = \bigcup_{i < \delta} h_i \cup \{(b, b)\}$$

is a ‘legal’ map. For some $N_{\delta+1} \leq_{\mathfrak{K}} \mathfrak{B}$, (μ, ω) -limit over N_δ , we can extend h^* to $h^+ \in \text{AUT}(N_{\delta+1})$. Let for $i \leq \delta$, $M_i^\otimes = (h^+)^{-1}(N_i)$. We then have

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$$\otimes_b \langle M_i^\delta | i \leq \delta \rangle \leq_{\mu, \delta}^b \langle M_i^\otimes | i \leq \delta \rangle$$

[Why? On the one hand, $i \leq \xi \Rightarrow h^+ \supset h_i = id_{M_i^\delta} = id_{N_i}$, and thus $M_i^\otimes = N_i = M_i^\delta$. On the other hand, if $i \in (\xi, \delta)$, then $h^+ \supset h^* \supset h_i$ and $h_i \leq_{\mathfrak{K}} M_i^\delta$ into N_i . We thus have that $M_i^\delta \leq_{\mathfrak{K}} M_i^\otimes$. If $i = \delta$, then clearly $M_i^\delta \leq_{\mathfrak{K}} M_i^\otimes$.

In the $\mathfrak{K}_{\mu, \alpha}^*$ case, we still need to show why $i < \delta \Rightarrow a_i \notin M_i^\otimes$: if $i \leq \xi$, this is trivial; if $i > \xi$, as h_{i+1} maps $M_{i+1}^\delta \setminus M_i^\delta$ into $N_{i+1} \setminus N_i$, and thus $h_{i+1}(a_i) \notin N_i$, hence $a_i \notin (h^+)^{-1}(N_i) = M_i^\otimes$.]

Figure 2: The diagonal and h .

So, \otimes_b holds, and as earlier we can define M_i^\otimes , for $i \in (\delta, \alpha)$ such that $\langle M_i^\delta | i < \alpha \rangle \leq_{\mu, \delta}^b \langle M_i^\otimes | i < \alpha \rangle$. But then the place of b drops: now, $b \in M_\delta \leq_{\mathfrak{K}} N_{\xi+1}$, and $h^+(b) = b$, so $b \in M_{\xi+1}^\otimes$. Contradiction. This finishes the proof of (1).

The proof of (2) is similar: We are now in the ${}^+\mathfrak{K}_{\mu, \alpha}^*$ case, and we need (in addition to what has already proved) to prove that the non- μ -splitting holds. This is, $\text{tp}(a_i, M_i^\otimes, M_{i+1}^\otimes)$ does not μ -split over N_i^δ . By definition of ${}^+\mathfrak{K}_{\alpha, \mu}$, we already have that $\text{tp}(a_i, M_i^\delta, M_{i+1}^\delta)$ does not μ -split over N_i^δ . But $h^+ \in \text{AUT}(N_{\delta+1})$; hence, $\text{tp}(h^+(a_i), h^+(M_i^\delta), h^+(M_{i+1}^\delta))$ does not μ -split over $h^+(N_i^\delta)$. But then, pulling back again this type, we have that $\text{tp}(a_i, M_i^\otimes, M_{i+1}^\otimes)$ does not μ -split over N_i^δ .

The proof is finished just like in case (1); we use the fact

$$\otimes_c \langle M_i^\delta, N_i^\delta | i \leq \delta \rangle \leq_{\mu, \delta}^c \langle M_i^\otimes, N_i^\delta | i \leq \delta \rangle$$

the same way as there. □_{3.1.15}

Claim 3.1.16 *Assume that M_0 is (μ, θ_0) -limit over N_0^* , $N_0^* \in \mathfrak{K}_\mu^{am}$, M_ℓ is (μ, θ_1) -limit and universal over M_0 , for $\ell = 1, 2$, $b \in M_1$, $\text{tp}(b, M_0, M_1)$ does not μ -split over N_0^* . Then, we can find $M_3 \in \mathfrak{K}_\mu^{am}$ and h such that $M_1 \leq_{\mathfrak{K}} M_3$, h is a $\leq_{\mathfrak{K}}$ -embedding of M_2 into M_3 over M_0 , $\text{tp}(b, h(M_2), M_3)$ does not μ -split over N_0^* .*

PROOF We can first find M'_3, h' such that $M_2 \leq_{\mathfrak{K}} M'_3 \in \mathfrak{K}_\mu^{am}$, h' is a $\leq_{\mathfrak{K}}$ -embedding of M_1 into M'_3 over M_0 , and $p' = \text{tp}(h'(b), M_2, M'_3)$ does not μ -split over N_0^* . This follows by the definition of type and the existence of $p' \in \mathcal{S}(M_2)$, $p' \geq \text{tp}(b, M_0, M_1)$, not μ -splitting over N_0^* : hence, for some M''_3 , $M_2 \leq_{\mathfrak{K}} M''_3 \in \mathfrak{K}_\mu^{am}$, and $b' \in M''_3$ realises p . So there are $M'_3, h', M''_3 \supset M''_3$ as required: send b to b' via h' , and extend the identity on M_0 . □_{3.1.16}

3.2 Toward the uniqueness of limits

We need a refined concept of type in order to obtain the right kind of towers later ('full' towers). The following definition specifies this refinement: in addition to just 'describing elements,' like we do when defining types of various sorts, we look both at the 'elements' themselves and at witnesses of the specific 'way they do not μ -split'. In principle, this provides a tighter description of the element, since it provides along with it the specific submodel over which the type does not μ -split.

Definition 3.2.1 *For M a (μ, θ) -limit model, let*

- 1) $\mathfrak{St}(M) = \{(p, N) | N \leq_{\mathfrak{K}} M \text{ is } (\mu, \theta)\text{-limit, } M \text{ universal over } N, p \in \mathcal{S}(M) \text{ does not } \mu\text{-split over } N\}$,
- 2) for $(p_\ell, N_\ell) \in \mathfrak{St}(M)$, for $\ell = 1, 2$, let $(p_1, N_1) \approx (p_2, N_2)$ iff for every $M', M \leq_{\mathfrak{K}} M' \in \mathfrak{K}_\mu^{am}$, there is $q \in \mathcal{S}(M')$ extending p_1 and p_2 , not μ -splitting over N_1 or over N_2 .

Fact 3.2.2 1) \approx is an equivalence relation on $\mathfrak{St}(M)$,

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- 2) If $M' \in \mathfrak{K}_\mu^{am}$ is universal over M (in 3.2.1 (2)), the existence of q for this M' suffices,
- 3) $|\mathfrak{St}(M)/\approx| \leq \mu$.

PROOF

- 1) A diagram chase which we leave to the reader,
- 2) By Universality + Preservation by $\leq_{\mathfrak{K}}$ -embeddings,
- 3) Since by 3.2.2 (2), there is $M' \in \mathfrak{K}_\mu^{am}$ universal over M in which we may check all the instances of \approx -equivalence, we have $|\mathfrak{St}(M)/\approx| \leq |\mathcal{S}(M')| \leq \mu$.

□_{3.2.2}

Remarks:

- 1) It is worth noting here that perhaps \approx is the equality. We do not know yet; but for our purposes, it is OK to use \approx .
- 2) In the definition of $(\vec{M}, \vec{a}, \vec{N}) \in {}^+\mathfrak{K}_{\mu,\alpha}^*$, it is just the $\mathfrak{St}(M)$ -equivalence class $(\text{tp}(b_i, M_i, M_{i+1}), N_i)/\approx$ that matters, and not N_i itself.

And now, we can provide a crucial notion for towers (see also [Sh 576]).

Definition 3.2.3 We say that $(\vec{M}, \vec{a}, \vec{N}) \in {}^+\mathfrak{K}_{\mu,\alpha}^*$ is **full** iff

- (a) μ divides α (if μ is regular, if it is singular, μ^ω divides α).
- (b) if $\beta < \alpha$ and $(p, N^*) \in \mathfrak{St}(M_\beta)$, then for some $i < \beta + \mu$, we have that $(\text{tp}(b_{\beta+i}, M_{\beta+i}, M_{\beta+i+1}), N_{\beta+i}) \approx (p, N^*)$. (Formally, it is equivalent to the stationarisation of (p, N^*) .)

We are approaching one of our main goals (‘uniqueness of limits’) with the following theorems.

Theorem 3.2.4 If $(\vec{M}, \vec{a}, \vec{N}) \in {}^+\mathfrak{K}_{\mu,\alpha}^*$ is full, and \vec{M} is continuous, then $\bigcup_{i < \alpha} M_i$ is $(\mu, \text{cf } \alpha)$ -limit over M_0 .

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PROOF Let $\langle M'_i \mid i \leq \alpha \rangle$ be \leq_{\aleph} -increasing continuous, with each model in the tower in \aleph_{μ}^{am} , and such that each M'_{i+1} is universal over M'_i , $M'_0 \stackrel{h_0}{\approx} M_0$. List $|M_i|$ as $\{a_{i,\zeta} \mid \zeta < \mu\}$ and $|M'_i|$ as $\{a'_{i,\zeta} \mid \zeta < \mu\}$. Let $g : \alpha \rightarrow \bigcup_{i \leq \alpha} M'_i = M'_\alpha$ satisfy $g(i) \in M'_i$ and

$$\otimes \left[\begin{array}{l} \text{if } \beta < \alpha, b \in M'_\beta, (p, N) \in \mathfrak{St}(M_\beta), \text{ then} \\ \mu = \text{otp} \left\{ \gamma \mid \begin{array}{l} \beta < \gamma < \beta + \mu \\ (p, N) \approx (\text{tp}(a_\gamma, M_\gamma, M_{\gamma+1}), N_\gamma) \upharpoonright M_\beta \\ g(\gamma) = b \end{array} \right\} \end{array} \right].$$

This makes sense: $\|M'_\beta\| \leq \mu$ and by 3.2.2, $|\mathfrak{St}(M)/\approx| \leq \mu$.

Figure 3: The definition of g .

We choose by induction on $i < \alpha$, h_i such that

- (a) h_i is a \leq_{\aleph} -embedding of $M_{j_i^1}$ into $M'_{j_i^2}$
- (b) $\langle h_i \rangle_{i < \alpha}$ is increasing continuous,
- (c) j_i^1, j_i^2 are increasing continuous,
- (d) $j_{i+1}^1 < j_i^1 + \mu, j_{i+1}^2 < j_i^2 + \mu$,

(e) $g(i) \in \text{Rang}(h_{i+1})$.

For $i = 0$, this is trivial. for i limit, just take unions. For $\underline{i+1}$, without loss of generality $g(i) \in M'_i \leq_{\mathfrak{K}} M'_{j_i^2}$, and we know that M_i is (μ, θ_i) -limit (say $\langle M_{i,\varepsilon} \mid \varepsilon \leq \theta_i \rangle$ witnesses this). So, for some ε_i , $\text{tp}(g(i), h_i(M_i), M'_{j_i^2})$ does not μ -split over $h_i(M_{i,\varepsilon_i})$. So there is $p_i \in \mathcal{S}(M_{j_i^1})$ such that $h_i(p_i) = \text{tp}(g(i), h_i(M_i), M'_{j_i^2})$, and thus for some $\xi \in (0, \mu)$, we have

$$(p_i, M_{i,\varepsilon_i}) \approx (\text{tp}(a_{j_i^1+\xi}, M_{j_i^1+\xi}, M_{j_i^1+\xi+1}), N_{j_i^1+\xi}).$$

Let $j_{i+1}^1 = j_i^1 + \xi + 1$, $j_{i+1}^2 = j_i^2 + 1$; there is h_{i+1} as required.

So, letting $h = \bigcup_{i < \alpha} h_i$, we have a witness for the original requirement.

For μ regular, $j_i^2 = i \leq j_i^1 \in [i, i + \mu)$, for μ singular, $j_i^2 = i$, $j_i^1 \in [i, \mu i)$. So if μ^ω divides α we catch our tail ($j_\delta^2 = j_\delta^1 = \delta$), and get the isomorphism we are looking for ($h_\delta : M_\delta \rightarrow M'_\delta$ is onto by the bookkeeping g). $\square_{3.2.4}$

3.3 Limits via sequences of different lengths

So far, ‘the limit’ has been proven to be unique, when the sequences converging to it in the various orderings defined are of same length. We are striving for more: we want to prove that even if we approach a model **via sequences of certain different lengths**, the limit may be proven to be ‘unique’ in a robust enough sense, by using a rectangle of models which will be (μ, θ_ℓ) -limits over $M_{0,0}$, for $\ell = 1, 2$, by the two sides.

Definition 3.3.1 For u an interval, and \mathfrak{U} a set of intervals, let

$${}^+\mathfrak{K}_{\mu,u}^* = \left\{ (\vec{M}, \vec{a}, \vec{N}) \left| \begin{array}{l} \vec{M} = \langle M_i \mid i \in u \rangle \text{ is } \leq_{\mathfrak{K}}\text{-increasing (not necessarily continuous), } \vec{a} = \langle a_i \mid i \in u \rangle, \\ a_i \in M_{i+1} \setminus M_i, \vec{N} = \langle N_i \mid i \in u \rangle, N_i \leq_{\mathfrak{K}} M_i, \\ N_i \text{ an amalgamation base in } \mathfrak{K}_\mu, \\ M_i \text{ universal over } N_i, \\ \text{tp}(a_i, M_i, M_{i+1}) \text{ does not } \mu\text{-split over } N_i \end{array} \right. \right\},$$

$${}^+\mathfrak{K}_{\mu,\mathfrak{U}}^* = \bigcup_{u \subset \mathfrak{U}} {}^+\mathfrak{K}_{\mu,u}^*.$$

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The right way to think about these classes is by immediate analogy to the original ${}^+\mathfrak{K}_{\mu,\alpha}^*$ classes. As there, it is natural to expect to have a $\leq^c_{\mu,\alpha}$ relation.

Definition 3.3.2 For $(\vec{M}^\ell, \vec{a}^\ell, \vec{N}^\ell) \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_\ell}^*$, $\ell = 1, 2$, let

$$(\vec{M}^1, \vec{a}^1, \vec{N}^1) \leq^c (\vec{M}^2, \vec{a}^2, \vec{N}^2)$$

mean that $(\vec{M}^1, \vec{a}^1, \vec{N}^1) \leq^c_{\mu,\alpha} (\vec{M}^2|_{\mathfrak{U}_1}, \vec{a}^2|_{\mathfrak{U}_1}, \vec{N}^2|_{\mathfrak{U}_1})$.

Now, as before for the definitions of towers, we have the following basic facts about the new ‘scattered’ towers.

Fact 3.3.3 1) Let $\mathfrak{U}_1 \subset \mathfrak{U}_2 \subset ORD$. Then,

- (a) If $(\vec{M}, \vec{a}, \vec{N}) \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_2}^*$, then $(\vec{M}, \vec{a}, \vec{N})|_{\mathfrak{U}_1} \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_1}^*$,
- (b) If $(\vec{M}, \vec{a}, \vec{N}) \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_1}^*$, then there is $(\vec{M}', \vec{a}', \vec{N}') \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_2}^*$ such that $(\vec{M}', \vec{a}', \vec{N}')|_{\mathfrak{U}_1} \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_2}^*$.

2) Let $\langle \mathfrak{U}_\varepsilon | \varepsilon < \varepsilon(*) \rangle$ be an increasing sequence of sets of ordinals such that $|\mathfrak{U}_\varepsilon| \leq \mu$ then the parallel of 3.1.10 for limits.

3) If $\mathfrak{U}_1 \subset \mathfrak{U}_2$, $(\vec{M}^\ell, \vec{a}^\ell, \vec{N}^\ell) \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_\ell}^*$, and $(\vec{M}^2, \vec{a}^2, \vec{N}^2)|_{\mathfrak{U}_1} \leq^c (\vec{M}^1, \vec{a}^1, \vec{N}^1)$, then we can find $(\vec{M}^3, \vec{a}^3, \vec{N}^3) \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_1}^*$ such that $(\vec{M}^1, \vec{a}^1, \vec{N}^1) \leq^c (\vec{M}^3, \vec{a}^3, \vec{N}^3)$ and $(\vec{M}^2, \vec{a}^2, \vec{N}^2) \leq^c (\vec{M}^3, \vec{a}^3, \vec{N}^3)$.

4) If $\mathfrak{U} \subset \mathfrak{U}_1 \cap \mathfrak{U}_2$, $(\vec{M}^\ell, \vec{a}^\ell, \vec{N}^\ell) \in {}^+\mathfrak{K}_{\mu,\mathfrak{U}_\ell}^*$, for $\ell = 1, 2$, $(\vec{M}^2, \vec{a}^2, \vec{N}^2)|_{\mathfrak{U}_1} \leq^c (\vec{M}^1, \vec{a}^1, \vec{N}^1)$, then $(\vec{M}^2, \vec{a}^2, \vec{N}^2)|_{\mathfrak{U}} \leq^c (\vec{M}^1, \vec{a}^1, \vec{N}^1)|_{\mathfrak{U}}$.

We also have that

Fact 3.3.4 In 3.3.3, if each one of the towers is reduced, then so are the limits.

Construction 3.3.5

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Fix ζ and let $\alpha < \mu^+$ be divisible by μ^ω (ordinal exponentiation). Let now $\mathfrak{u}_\varepsilon = \bigcup \{[\beta\mu\zeta, \beta\mu\zeta + \mu\varepsilon) \mid \beta < \alpha\}$, for each $\varepsilon \leq \zeta$. Now we can define the class ${}^+\mathfrak{K}_{\mu, \mathfrak{u}_\varepsilon}^*$ just like ${}^+\mathfrak{K}_{\mu, \alpha}^*$, but now using \mathfrak{u}_ε as our set of indices, instead of α . The point is that now we want to play with changing \mathfrak{u}_ε in various ways.

We choose $(\vec{M}^\varepsilon, \vec{a}^\varepsilon, \vec{N}^\varepsilon) \in {}^+\mathfrak{K}_{\mu, \mathfrak{u}_\varepsilon}^*$, $<^c$ -increasing (naturally), each one of these towers reduced. In successor stages, in the new intervals which have length μ , we put representatives to all types.

In the end, we get $\vec{M} = \langle M_i \mid i < (\beta\mu\zeta)\alpha \rangle$. So,

- \otimes_1 each M_i is $(\mu, \text{cf } \zeta)$ -limit (by $<^c$),
- \otimes_2 every $(p, N) \in \mathfrak{St}(M_i) / \approx$, up to equivalence (by 2.2.1 — we dealt with it so it appears in $j \in [i, i + \beta\mu\zeta)$),
- \otimes_3 if there exists ε such that $i \in \mathfrak{u}_\varepsilon$ and $i = \sup(\mathfrak{u}_\varepsilon \cap i)$ then $M_i = \bigcup_{j < i} M_j$.

So, reformulating ‘full \Rightarrow limit,’ we get a similar claim for \mathfrak{u} instead of μ or μ^ω .

If $\alpha = \alpha^* + 1$ is a large enough ordinal, then we can find \mathfrak{W} such that $\vec{M} \upharpoonright \mathfrak{W}$ is full, $\alpha^* = \sup \mathfrak{W} \cap \alpha^*$ and $\alpha^* \in \mathfrak{W}$. So, by \otimes_3 , we have

$$\otimes M_{\alpha^*} = \bigcup_{\beta \in \mathfrak{W}} M_\beta \text{ is } (\mu, \text{cf } \alpha^*)\text{-limit over } M_0.$$

But M_α is $(\mu, \text{cf } \zeta)$ -limit. We can arrange $\text{cf } \alpha^*$, $\text{cf } \zeta$ to be any regular $< \mu^+$.

Conclusion 3.3.6

If M_ℓ is (μ, θ_ℓ) -limit ($\ell = 1, 2$), then $M_1 \approx M_2$.

So, to speak about ‘**the μ -limit model**’ now makes sense.

Comments: A nicer construction may be obtained if we set

$$\mathfrak{u}_\varepsilon = \left\{ \mu^\omega \zeta + j \mid \begin{array}{l} j < \mu^\omega \zeta, \quad j = 0 \pmod 3 \\ \text{or } j = 1 \pmod 3 \\ \text{or } j < \mu^\omega \varepsilon \end{array} \right\}.$$

This way, the first set \mathfrak{u}_0 contains at least all the ordinals which are 0 or 1 mod 3, hence there is no problem with the limit.

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Theorem 3.3.7 *If M_i is (μ, θ_ℓ) -limit over M , for $\ell = 1, 2$, then*

$$M_1 \approx_M M_2.$$

PROOF The same proof as for Theorem 3.2.4 works, although naturally it has to be adapted to our ‘scattered tower’ situation. Without loss of generality, both 0 and $\alpha^* \in \mathfrak{u}$. We define $(\vec{M}^\zeta, \vec{a}^\zeta, \vec{N}^\zeta) \in {}^+\mathfrak{R}_{\mu, \mathfrak{u}_\zeta}^*$, $M_0^0 = M$. So, we have that $M_{\alpha^*}^\zeta$ is a $(\mu, \theta_{\text{cf}\alpha^*})$ -limit over M , and also $(\mu, \text{cf } \zeta)$ -limit over M . □_{3.3.7}

With this, we can by now conclude that **saturated models exist** in a strong enough sense. We may take as our definition of a ‘saturated model over M ’ in a cardinal μ the (by now unique because of Theorem 3.3.7) (μ, θ) -limit over M , for an arbitrary θ .

This paves the way towards a full study of the categoricity spectrum for abstract elementary classes without maximal elements. We intend to continue developing this theory in that direction, by studying the type theory for our context, non forking amalgamation, and the true role of saturation. But this will be the material of forthcoming papers.

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