THE LIFTING PROBLEM WITH THE FULL IDEAL

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Abstract. We show that there are a cardinal μ, a σ-ideal $I \subseteq \mathcal{P}(\mu)$ and a σ-subalgebra $\mathcal{B}$ of subsets of μ extending I such that $\mathcal{B}/I$ satisfies the c.c.c. but the quotient algebra $\mathcal{B}/I$ has no lifting.

0. Introduction. In the present paper we prove the following theorem.

Theorem 0.1. *For some μ (in fact, $\mu = (2^{\aleph_0})^{++}$ suffices) there is a σ-ideal I on $\mathcal{P}(\mu)$ and a σ-subalgebra $\mathfrak{B}$ of $\mathcal{P}(\mu)$ extending I such that $\mathfrak{B}/I$ satisfies the c.c.c. but $\mathfrak{B}/I$ has no lifting.*

This result answers a question of David Fremlin (see chapter on measure algebras in Fremlin [2]). Moreover, it solves the problem of topologizing a Category Base (see Detlefsen Szymański [3], Morgan [6], Shilling [11] and Szymański [12]).

Note that it is well known (Mokobodzki's theorem; see Fremlin [2]) that under CH, if $|\mathfrak{B}/I| \leq (2^{\aleph_0})^+$ then this is impossible; i.e. the quotient algebra $\mathfrak{B}/I$ has a lifting.

Toward the end we deal with having better μ.

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Notation: Our notation is rather standard. All cardinals are assumed to be infinite and usually they are denoted by λ, κ, μ.

In Boolean algebras we use $\cap$ (and $\bigcap$), $\cup$ (and $\bigcup$) and $-$ for the Boolean operations.

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1. The proof of Theorem 0.1.

Main Lemma 1.1. *Suppose that*

(a) μ, λ *are cardinals satisfying* $\mu = \mu^{\aleph_0}, \lambda \leq 2^\mu$,
(b) $\mathfrak{B}$ *is a complete c.c.c. Boolean algebra,*
(c) $x_i \in \mathfrak{B} \setminus \{0\}$ *for* $i < \lambda$,
(d) *for each sequence* $\langle (u_i, f_i) : i < \lambda \rangle$ *such that* $u_i \in [\lambda]^{\leq \aleph_0}$, $f_i \in {}^{u_i}2$ *there are* $n < \omega$ *(but* $n > 0$*) and* $i_0 < i_1 \ldots < i_{n-1}$ *in* λ *such that:*
  (α) *the functions* $f_{i_0}, \ldots, f_{i_{n-1}}$ *are compatible,*
  (β) $\mathfrak{B} \models \bigcap\limits_{\ell<n} x_{i_\ell} = 0$.

Then

($\bigoplus$) *there are a σ-ideal I on $\mathcal{P}(\mu)$ and a σ-algebra $\mathfrak{A}$ of subsets of μ extending I such that $\mathfrak{A}/I$ satisfies the c.c.c. and the natural homomorphism $\mathfrak{A} \longrightarrow \mathfrak{A}/I$ cannot be lifted.*

Proof Without loss of generality the algebra $\mathfrak{B}$ has cardinality $\lambda^{\aleph_0}$ ($\leq 2^\mu$). Let $\langle Y_b : b \in \mathfrak{B}\rangle$ be a sequence of subsets of μ such that any non-trivial countable Boolean combination of the Y_b's is non-empty (possible by [1] as $\mu = \mu^{\aleph_0}$ and the algebra $\mathfrak{B}$ has cardinality $\leq 2^\mu$; see background in [4]). Let $\mathfrak{A}_0$ be the Boolean subalgebra of $\mathcal{P}(\mu)$ generated by $\{Y_b : b \in \mathfrak{B}\}$. So $\{Y_b : b \in \mathfrak{B}\}$ freely generates $\mathfrak{A}_0$ and hence there is a unique homomorphism h_0 from $\mathfrak{A}_0$ into $\mathfrak{B}$ satisfying $h_0(Y_b) = b$.

A Boolean term σ is hereditarily countable if σ belongs to the closure Σ of the set of terms $\bigcap\limits_{i<i^*} y_i$ for $i^* < \omega_1$ under composition and under $-y$.

Let $\mathcal{E}$ be the set of all equations $\mathbf{e}$ of the form $0 = \sigma(b_0, b_1, \ldots, b_n, \ldots)_{n<\omega}$ which hold in $\mathfrak{B}$, where σ is hereditarily countable. For $\mathbf{e} \in \mathcal{E}$ let $\mathrm{cont}(\mathbf{e})$ be the set of $b \in \mathfrak{B}$ mentioned in it (i.e. $\{b_n : n < \omega\}$) and let $Z_{\mathbf{e}} \subseteq \mu$ be the set $\sigma(Y_{b_0}, Y_{b_1}, \ldots, Y_{b_n}, \ldots)_{n<\omega}$.

Let I be the σ-ideal of $\mathcal{P}(\mu)$ generated by the family $\{Z_{\mathbf{e}} : \mathbf{e} \in \mathcal{E}\}$ and let $\mathfrak{A}_1$ be the Boolean Algebra of subsets of $\mathcal{P}(\mu)$ generated by $I \cup \{Y_b : b \in \mathfrak{B}\}$.

Claim 1.1.1. $I \cap \mathfrak{A}_0 = \mathrm{Ker}(h_0)$.

Proof of the claim: Plainly $\mathrm{Ker}(h_0) \subseteq I \cap \mathfrak{A}_0$. For the converse inclusion it is enough to consider elements of $\mathfrak{A}_0$ of the form

$$Y = \bigcap_{\ell=1}^{n} Y_{b_\ell} - \bigcup_{\ell=n+1}^{2n} Y_{b_\ell}.$$

If $\mathfrak{B} \models$“$\bigcap\limits_{\ell=1}^{n} b_\ell - \bigcup\limits_{\ell=n+1}^{2n} b_\ell = 0$” then easily $h_0(Y) = 0$. So assume that

$$\mathfrak{B} \models \text{ “ } c = \bigcap_{\ell=1}^{n} b_\ell - \bigcup_{\ell=n+1}^{2n} b_\ell \neq 0 \text{ ”},$$

and we shall prove $Y \notin I$. Let $Z \in I$, so for some $\mathbf{e}_m \in \mathcal{E}$ for $m < \omega$ we have $Z \subseteq \bigcup_{m<\omega} Z_{\mathbf{e}_m}$. Let g be a homomorphism from $\mathfrak{B}$ into the 2–element Boolean Algebra $\mathfrak{B}_0 = \{0, 1\}$ such that $g(c) = 1$, and g respects all the equations $\mathbf{e}_m$ (including those of the form $b = \bigcup\limits_{k<\omega} b_k$; possible by the Sikorski theorem).

By the choice of the Y_b's, there is $\alpha < \mu$ such that:

if $b \in \{b_\ell : \ell = 1, \dots, 2n\} \cup \bigcup\limits_{m<\omega} \text{cont}(\mathbf{e}_m)$ then

$$g(b) = 1 \Leftrightarrow \alpha \in Y_b.$$

So easily $\alpha \notin Z_{\mathbf{e}_m}$ for $m < \omega$, and $\alpha \in \bigcap\limits_{\ell=1}^{n} Y_{b_\ell} \setminus \bigcup\limits_{\ell=n+1}^{2n} Y_{b_\ell}$, so Y is not a subset of Z. As Z was an arbitrary element of I we get $Y \notin I$, so we have finished proving 1.1.1.

It follows from 1.1.1 that we can extend h_0 (the homomorphism from $\mathfrak{A}_0$ onto $\mathfrak{B}$) to a homomorphism h_1 from $\mathfrak{A}_1$ onto $\mathfrak{B}$ with $I = \text{Ker}(h_1)$. Let $\mathfrak{A}_2$ be the σ-algebra of subsets of μ generated by $\mathfrak{A}_1$.

Claim 1.1.2. *For every $Y \in \mathfrak{A}_2$ there is $b \in \mathfrak{B}$ such that $Y \equiv Y_b \mod I$. Consequently, $\mathfrak{A}_2 = \mathfrak{A}_1$.*

Proof of the claim: Let $Y \in \mathfrak{A}_2$. Then Y is a (hereditarily countable) Boolean combination of some Y_{b_ℓ} ($\ell < \omega$) and Z_n ($n < \omega$), where $b_\ell \in \mathfrak{B}$, $Z_n \in I$. Let $Z_n \subseteq \bigcup\limits_{m<\omega} Z_{\mathbf{e}_{n,m}}$, where $\mathbf{e}_{n,m} \in \mathcal{E}$, and say

$$Y = \sigma(Y_{b_0}, Z_0, Y_{b_1}, Z_1, \dots, Y_{b_n}, Z_n, \dots)_{n<\omega}.$$

Let $\mathbf{e}_{n,m}$ be $0 = \sigma_{n,m}(b_{n,m,0}, b_{n,m,1}, \dots)$. Then clearly $\bigcup\limits_{n,m<\omega} Z_{\mathbf{e}_{n,m}} \in I$ (use the definition of I). In $\mathfrak{B}$, let $b = \sigma(b_0, 0, b_1, 0, \dots, b_n, 0, \dots)$ and let $\sigma^* = \sigma^*(b_0, b_1, \dots, b_{n,m,\ell}, \dots)_{n,m,\ell<\omega}$ be the following term

$$\bigcup_{n,m} \sigma_{n,m}(b_{n,m,0}, b_{n,m,1}, \dots) \cup (b - \sigma(b_0, 0, b_1, 0, \dots, b_m, 0, \dots)) \cup$$
$$\cup (\sigma(b_0, 0, b_1, 0, \dots, b_n, 0, \dots) - b) \cup 0.$$

Clearly $\mathfrak{B} \models$“$0 = \sigma^*$”, so the equation $\mathbf{e}$ defined as $0 = \sigma^*$ belongs to $\mathcal{E}$, and thus $Z_{\mathbf{e}}$ is well defined. It follows from the definition of σ^* that $(Y \setminus Y_b) \cup (Y_b \setminus Y) \subseteq Z_{\mathbf{e}} \in I$.

So we can sum up:

(a) I is an $\aleph_1$-complete ideal of $\mathcal{P}(\mu)$,
(b) $\mathfrak{A}_1$ is a σ-algebra of subsets of μ,
(c) $I \subseteq \mathfrak{A}_1$,
(d) h_1 is a homomorphism from $\mathfrak{A}_1$ onto $\mathfrak{B}$, with kernel I,
(e) $\mathfrak{B}$ is a complete c.c.c. Boolean algebra.

This is exactly as required, so the "only" point left is

Claim 1.1.3. *The homomorphism h_1 cannot be lifted.*

Proof of the claim: Assume that h_1 can be lifted, so there is a homomorphism $g_1 : \mathfrak{B} \longrightarrow \mathfrak{A}_1$ such that $h_1 \circ g_1 = \mathrm{id}_{\mathfrak{B}}$.

For $i < \lambda$ let $Z_i = (g_1(x_i) - Y_{x_i}) \cup (Y_{x_i} - g_1(x_i))$, so by the assumption on g_1 necessarily $Z_i \in I$. Consequently we can find $\mathbf{e}_{i,n} \in \mathcal{E}$ for $n < \omega$ such that $Z_i \subseteq \bigcup\limits_{n<\omega} Z_{\mathbf{e}_{i,n}}$. Let $W_i = \{x_i\} \cup \bigcup\limits_{n<\omega} \mathrm{cont}(\mathbf{e}_{i,n})$, so $W_i \subseteq \mathfrak{B}$ is countable. Let $\mathfrak{B}'$ be the subalgebra of $\mathfrak{B}$ generated by $\bigcup\limits_{i<\lambda} W_i$. Clearly $|\mathfrak{B}'| = \lambda$, so there is a one–to–one function t from λ onto $\mathfrak{B}'$. Put $u_i = t^{-1}(W_i) \in [\lambda]^{\leq \aleph_0}$.

For each i there is a homomorphism f_i from $\mathfrak{B}$ into the 2-element Boolean Algebra $\{0,1\}$ such that $f_i(x_i) = 1$ and f_i respects all the equations $\mathbf{e}_{i,n}$ for $n < \omega$ (as in the proof of 1.1.1). Let $f'_i : u_i \longrightarrow \{0,1\}$ be defined by $f'_i(\alpha) = f_i(t(\alpha))$. Then by clause (d) of the hypothesis there are $n < \omega$ and $i_0 < \ldots < i_{n-1} < \lambda$ such that:

(α) the functions $f'_{i_0}, \ldots, f'_{i_{n-1}}$ are compatible,
(β) $\mathfrak{B} \models$ "$\bigcap\limits_{\ell<n} x_{i_\ell} = 0$".

Hence

$(\alpha)'$ the functions $f_{i_0} \restriction W_{i_0}, \ldots, f_{i_{n-1}} \restriction W_{i_{n-1}}$ are compatible[1], call their union g.

Now let $\alpha < \mu$ be such that:

$(\otimes_1)$ $\quad \ell < n \ \&\ b \in W_{i_\ell} \quad \Rightarrow \quad [\alpha \in Y_b \Leftrightarrow g(b) = 1]$

(it exists by the choice of the Y_b's and $(\alpha)'$).

By $(\otimes_1)$ and the choice of f_{i_ℓ} we have:

$(\otimes_2)$ $\quad \alpha \in Y_{x_{i_\ell}}$

(because $f_{i_\ell}(x_{i_\ell}) = 1$) and

$(\otimes_3)$ $\quad \alpha \notin Z_{\mathbf{e}_{i_\ell,n}}$ for $n < \omega$

(because f_{i_ℓ} respects $\mathbf{e}_{i_\ell,n}$ and $\mathrm{cont}(\mathbf{e}_{i_\ell,n}) \subseteq W_{i_\ell}$) and

$(\otimes_4)$ $\quad \alpha \notin Z_{i_\ell}$

[1] as functions, not as homomorphisms

(by $(\otimes_3)$ as $Z_{i_\ell} \subseteq \bigcup_{n<\omega} Z_{\mathbf{e}_{i_\ell,n}}$).
So $\alpha \in Y_{x_{i_\ell}} \setminus Z_{i_\ell}$ and thus $\alpha \in g_1(x_{i_\ell})$. Hence $\alpha \in \bigcap_{\ell<n} g_1(x_{i_\ell})$. Since g_1 is a homomorphism we have

$$\bigcap_{\ell<n} g_1(x_{i_\ell}) = g_1(\bigcap_{\ell<n} x_{i_\ell}) = g_1(0) = \emptyset$$

(we use clause (β) above). A contradiction. $\blacksquare_{1.1}$

Remark 1.2. (1) Concerning the assumptions of 1.1, note that they seem closely related to

$(\oplus_\mu)$ there is a c.c.c. Boolean Algebra $\mathfrak{B}$ of cardinality $\leq \lambda$ which is not the union of $\leq \mu$ ultrafilters (i.e. $d(\mathfrak{B}) > \mu$).

(See the proof of 1.7 below).

(2) Concerning $(\oplus_\mu)$, by [8], if $\lambda = \mu^+$, $\mu = \mu^{\aleph_0}$ then there is no such Boolean algebra. By [9], it is consistent then $\lambda = \mu^{++} \leq 2^\mu$, $\aleph_0 < \mu = \mu^{<\mu}$ and $(\oplus_\mu)$ above holds using (see below) a Boolean algebra of the form $BA(W)$, $W \subseteq [\lambda]^3$, $(\forall u_1 \neq u_2 \in W)(|u_1 \cap u_2| \leq 1)$. Hajnal, Juhasz and Szentmiklossy [5] prove the existence of a c.c.c. Boolean algebra $\mathfrak{B}$ with $d(\mathfrak{B}) = \mu$ of cardinality 2^μ when there is a Jonsson algebra on μ (or μ is a limit cardinal) using $BA(W)$, $W \subseteq [\lambda]^{<\aleph_0}$, $u \neq v \in W \Rightarrow |u \cap v| < |u|/2$. The claim we need is close to this. On the existence of Jonson cardinals (and its history) see [10]. Of course, also in 1.7 if μ is not strong limit, instead "M is a Jonsson algebra on μ" it suffices that "M is not the union of $< \mu$ subalgebras". Rabus Shelah [7] prove the existence of a c.c.c. Boolean Algebra $\mathfrak{B}$ with $d(\mathfrak{B}) = \mu$ for every μ.

Definition 1.3. (1) For a set u let

$$\mathrm{pfil}(u) \stackrel{\mathrm{def}}{=} \{w : \ w \subseteq \mathcal{P}(u),\ u \in w,\ w \text{ is upward closed and if } (u_1, u_2) \text{ is a partition of } u \text{ then } u_1 \in w \text{ or } u_2 \in w\}$$

[pfil stands for "pseudo-filter"].

(2) The canonical (pfil) w of u for a finite set u is

$$\mathrm{half}(u) = \{v \subseteq u : |v| \geq |u|/2\}.$$

(3) We say that $(W, \mathbf{w})$ is a λ-candidate if:
   (a) $W \subseteq [\lambda]^{<\aleph_0}$,
   (b) $\mathbf{w}$ is a function with domain W,
   (c) $\mathbf{w}(u) \in \mathrm{pfil}(u)$ for $u \in W$
   (d) if $v \in [\lambda]^{<\aleph_0}$ then $\mathrm{cl}_{(W,\mathbf{w})}(v) \stackrel{\mathrm{def}}{=} \{u \in W : u \cap v \in \mathbf{w}(u)\}$ is finite.

(4) We say W is a λ-candidate if $(W, \mathrm{half}\restriction W)$ is a λ-candidate.
(5) Instead of λ we can use any ordinal (or even set).
(6) We say that $\mathcal{U} \subseteq \lambda$ is $(W, \mathbf{w})$–closed if for each $u \in W$

$$u \cap \mathcal{U} \in \mathbf{w}(u) \quad \Rightarrow \quad u \subseteq \mathcal{U}.$$

Definition 1.4. (1) For a λ–candidate $(W, \mathbf{w})$ let $BA(W, \mathbf{w})$ be the Boolean algebra generated by $\{x_i : i < \lambda\}$ freely except

$$\bigcap_{i \in u} x_i = 0 \qquad \text{for} \qquad u \in W.$$

(2) For a λ-candidate W, let

$$BA(W) = BA(W, \mathrm{half}\restriction W).$$

(3) For a λ-candidate $(W, \mathbf{w})$ let $BA^c(W, \mathbf{w})$ be the completion of $BA(W, \mathbf{w})$; similarly $BA^c(W)$.

Proposition 1.5. *Let $(W, \mathbf{w})$ be a λ-candidate. Then the Boolean algebra $BA(W, \mathbf{w})$ satisfies the c.c.c. and has cardinality λ, so $BA^c(W, \mathbf{w})$ satisfies the c.c.c. and has cardinality $\leq \lambda^{\aleph_0}$.*

Proof Let $b_\alpha = \sigma_\alpha(x_{i_{\alpha,0}}, \ldots, x_{i_{\alpha,n_\alpha-1}})$ be nonzero members of $BA(W, \mathbf{w})$ (for $\alpha < \omega_1$ and σ_α a Boolean term). Without loss of generality $\sigma_\alpha = \sigma$, $n_\alpha = n(*)$ and $i_{\alpha,0} < i_{\alpha,1} < \ldots < i_{\alpha,n_\alpha-1}$, and $\langle\langle i_{\alpha,\ell} : \ell < n(*)\rangle : \alpha < \omega_1\rangle$ forms a Δ-system, so

$$i_{\alpha_1,\ell_1} = i_{\alpha_2,\ell_2} \ \&\ \alpha_1 \neq \alpha_2 \quad \Rightarrow \quad \ell_1 = \ell_2 \ \&\ (\forall \alpha < \omega_1)(i_{\alpha,\ell_1} = i_{\alpha_1,\ell_1}).$$

Also we can replace b_α by any nonzero $b'_\alpha \leq b_\alpha$, so without loss of generality for some $s_\alpha \subseteq n(*)$ $(= \{0, \ldots, n(*) - 1\})$ we have

$$b_\alpha = \bigcap_{\ell \in s_\alpha} x_{i_{\alpha,\ell}} \cap \bigcap_{\ell \in n(*) \setminus s_\alpha} (-x_{i_{\alpha,\ell}}) > 0$$

and without loss of generality $s_\alpha = s$. Put (for $\alpha < \omega_1$)

$$\mathbf{u}_\alpha \stackrel{\mathrm{def}}{=} \{u \in W : u \cap \{i_{\alpha,\ell} : \ell \in s\} \in \mathbf{w}(u)\}$$

and note that these sets are finite (remember 1.3(3d)). Hence the sets

$$u_\alpha = \bigcup \{u : u \in \mathbf{u}_\alpha\}$$

are finite. Without loss of generality $\langle \{i_{\alpha,\ell} : \ell < n(*)\} \cup u_\alpha : \alpha < \omega_1\rangle$ is a Δ-system. Now let $\alpha \neq \beta$ and assume $b_\alpha \cap b_\beta = 0$. Clearly we have

$$b_\alpha \cap b_\beta = \bigcap_{\ell \in s} (x_{i_{\alpha,\ell}} \cap x_{i_{\beta,\ell}}) \cap \bigcap_{\ell \in n(*) \setminus s} (-x_{i_{\alpha,\ell}} \cap -x_{i_{\beta,\ell}}).$$

Note that, by the Δ-system assumption, the sets $\{i_{\alpha,\ell}, i_{\beta,\ell} : \ell \in s\}$, $\{i_{\alpha,\ell}, i_{\beta,\ell} : \ell \in n(*) \setminus s\}$ are disjoint. So why is $b_\alpha \cap b_\beta$ zero? The only possible reason is that for some $u \in W$ we have $u \subseteq \{i_{\alpha,\ell}, i_{\beta,\ell} : \ell \in s\}$. Thus

$$u = (u \cap \{i_{\alpha,\ell} : \ell \in s\}) \cup \{u \cap \{i_{\beta,\ell} : \ell \in s\})$$

and without loss of generality $u \cap \{i_{\alpha,\ell} : \ell \in s\} \in \mathbf{w}(u)$. Hence $u \in \mathbf{u}_\alpha$ and therefore $u \subseteq u_\alpha$. Now we may easily finish the proof. $\blacksquare_{1.5}$

Remark 1.6. If we define a (λ,κ)–candidate weakening clause (d) to

$$(\mathrm{d})_\kappa \quad v \in [\lambda]^{<\aleph_0} \quad \Rightarrow \quad \kappa > |\{u \in W : u \cap v \in \mathbf{w}(u)\}|,$$

then the algebra $BA(W,\mathbf{w})$ satisfies the κ^+-c.c.c.
[Why? We repeat the proof of Proposition 1.5 replacing $\aleph_1$ with κ. There is a difference only when $\mathbf{u}_\alpha$ has cardinality $< \kappa$ (instead being finite) and (being the union of $< \kappa$ finite sets) also u_α has carinality $\mu_\alpha < \kappa$. Wlog $\mu_\alpha = \mu < \kappa$. Clearly the set

$$S \overset{\mathrm{def}}{=} \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \mu^+\}$$

is a stationary subset of κ^+, so for some stationary subset S^* of S and $\alpha(*) < \kappa$ we have:

$$(\forall \alpha \in S^*)\big(u_\alpha \cap \alpha \subseteq \alpha^* \quad \& \quad u_\alpha \subseteq \min(S^* \setminus (\alpha+1))\big).$$

Let us define $u^*_\alpha = u_\alpha \cup \{i_{\alpha,\ell} : \ell \in s\} \setminus \alpha(*)$. Wlog $\langle u^*_\alpha : \alpha \in S^*\rangle$ is a Δ–system. The rest should be clear.]

Theorem 1.7. *Assume that there is a Jonsson algebra on μ, $\lambda = 2^\mu$, and*

$$(\forall \alpha < \mu)(|\alpha|^{\aleph_0} < \mu = \mathrm{cf}(\mu)).$$

Then for some λ–candidate $(W,\mathbf{w})$ the Boolean algebra $BA^c(W,\mathbf{w})$ and λ satisfy the assumptions **(b)**—**(d)** *of 1.1.*

Proof Let $F : [\mu]^{<\aleph_0} \longrightarrow \mu$ be such that

$$(\forall A \in [\mu]^\mu)[F''([A]^{<\aleph_0} \setminus [A]^{<2}) = \mu]$$

(well known and easily equivalent to the existence of a Jonsson algebra).
Let $\langle \bar{A}^\alpha : \alpha < 2^\mu\rangle$ list the sequences $\bar{A} = \langle A_i : i < \mu\rangle$ such that

- $A_i \in [2^\mu]^\mu$,
- $(\forall i < \mu)(\exists \alpha)(A_i \subseteq [\mu \times \alpha, \mu \times \alpha + \mu))$, and
- $i < j < \mu \quad \Rightarrow \quad A_i \cap A_j = \emptyset$.

Without loss of generality we have $A^\alpha_i \subseteq \mu \times (1+\alpha)$ and each $\bar{A}$ is equal to $\bar{A}^\alpha$ for 2^μ ordinals α. Clearly $\mathrm{otp}(A^\alpha_i) = \mu$.

By induction on $\alpha < 2^\mu$ we choose pairs $(W_\alpha, \mathbf{w}_\alpha)$ and functions F_α such that

(α) $(W_\alpha, \mathbf{w}_\alpha)$ is a $\mu\times(1+\alpha)$–candidate,
(β) $\beta<\alpha$ implies $W_\beta = W_\alpha \cap [\mu\times(1+\beta)]^{<\aleph_0}$ and $\mathbf{w}_\beta = \mathbf{w}_\alpha \restriction W_\beta$,
(γ) F_α is a one-to-one function from the set

$$\{u : u\subseteq [\mu\times(1+\alpha), \mu\times(1+\alpha+1)) \text{ finite with } \geq 2 \text{ elements }\}$$

into $\bigcup\limits_{i<\mu} A^\alpha_i$,
(δ) $W_{\alpha+1} = W_\alpha \cup \{u\cup\{F_\alpha(u)\} : u\in W^*_\alpha\}$, where

$W^*_\alpha = \{u : u \text{ a subset of } [\mu\times(1+\alpha), \mu\times(1+\alpha+1)) \text{ such that } \aleph_0 > |u| \geq 2\}$,

(ε) for any (finite) $u\in W^*_\alpha$ we have

$\mathbf{w}_{\alpha+1}(u\cup\{F_\alpha(u)\}) = \{v\subseteq u\cup\{F_\alpha(u)\} : u\subseteq v \text{ or } F_\alpha(u)\in v \ \&\ v\cap u\neq\emptyset\}$,

(ζ) F_α is such that for any subset X of $J_\alpha = [\mu\times(1+\alpha), \mu\times(1+\alpha+1))$ of cardinality μ and $i<\mu$ and $\gamma\in A^\alpha_i$ for some finite subset u of X with ≥ 2 elements we have $F_\alpha(u)\in A^\alpha_i\setminus\gamma$.

There is no problem to carry out the definition so that clauses (β)–(ζ) are satisfied (to define functions F_α use the function F chosen at the beginning of the proof). Then $(W_\alpha, \mathbf{w}_\alpha)$ is defined for each $\alpha<2^\mu$ (at limit stages α we take $W_\alpha = \bigcup\limits_{\beta<\alpha} W_\beta$, $\mathbf{w}_\alpha = \bigcup\limits_{\beta<\alpha}\mathbf{w}_\beta$, of course).

Claim 1.7.1. *For each $\alpha<2^\mu$, $(W_\alpha,\mathbf{w}_\alpha)$ is a $\mu\times(1+\alpha)$–candidate.*

Proof of the claim: We should check the requirements of 1.3(3). Clauses (a), (b) there are trivially satisfied. For the clause (c) note that every element u of W_α is of the form $u'\cup\{F_\beta(u')\}$ for some $\beta<\alpha$ and $u'\in W^*_\beta$. Now, if $u = u_0\cup u_1$ then either one of u_0, u_1 contains u' or one of the two sets contains $F_\beta(u')$ and has non-empty intersection with u'. In both cases we are done. Regarding the demand (d) of 1.3(3), note that if

$$v\in[2^\mu]^{<\aleph_0},\quad u\in W_\alpha,\quad u = u'\cup\{F_\beta(u')\},\quad u'\in W^*_\beta,\quad \beta<\alpha$$

and $v\cap u\in\mathbf{w}_{\beta+1}(u)$ then $v\cap u'\neq\emptyset$ and either $u'\subseteq v$ or $F_\beta(u')\in u$. Hence, using the fact that the functions F_γ are one-to-one, we easily show that for every $v\in[2^\mu]^{<\aleph_0}$ the set

$$\{u\in W_\alpha : u\cap v\in\mathbf{w}_\alpha(u)\}$$

is finite (remember the definition of $\mathbf{w}_{\beta+1}$), finishing the proof of the claim.

Let $W = \bigcup\limits_\alpha W_\alpha$, $\mathbf{w} = \bigcup\limits_\alpha \mathbf{w}_\alpha$, $\mathfrak{B} = BA^c(W,\mathbf{w})$. It follows from 1.7.1 that $(W,\mathbf{w})$ is a λ–candidate. The main point of the proof of the theorem is clause **(d)** of the assumptions of 1.1. So let $f_\alpha : u_\alpha \longrightarrow \{0,1\}$ for $\alpha<2^\mu$, $u_\alpha\in[2^\mu]^{\leq\aleph_0}$, be given. For each $\alpha<2^\mu$, by the assumption that

$(\forall \beta < \mu)[|\beta|^{\aleph_0} < \mu = \mathrm{cf}(\mu)]$ and by the Δ-lemma, we can find $X_\alpha \in [\mu]^\mu$ such that $\langle f_{\mu\times\alpha+\zeta} : \zeta \in X_\alpha\rangle$ forms a Δ-system with heart f^*_α. Let

$$G = \{g : g \text{ is a partial function from } 2^\mu \text{ to } \{0,1\} \text{ with countable domain}\}.$$

For each $g \in G$ let $\langle \gamma(g,i) : i < i(g)\rangle$ be a maximal sequence such that $g \subseteq f^*_{\gamma(g,i)}$ and

$$\mathrm{Dom}(f^*_{\gamma(g,i)}) \cap \mathrm{Dom}(f^*_{\gamma(g,j)}) = \mathrm{Dom}(g) \qquad \text{for } j < i$$

(just choose $\gamma(g,i)$ by induction on i).

By induction on $\zeta \le \omega_1$, we choose $Y_\zeta, G_\zeta, Z_\zeta$ and $U_{\zeta,g}$ such that

(a) $Y_\zeta \in [2^\mu]^{\le\mu}$ is increasing continuous in ζ,
(b) $Z_\zeta \overset{\mathrm{def}}{=} \bigcup\{\mathrm{Dom}(f_\gamma) : (\exists \alpha \in Y_\zeta)[\mu\times\alpha \le \gamma < \mu\times(\alpha+1)]\}$,
(c) $G_\zeta = \{g \in G : \mathrm{Dom}(g) \subseteq Z_\zeta\}$,
(d) for $g \in G_\zeta$ we have: $U_{\zeta,g}$ is $\{i : i < i(g)\}$ if $i(g) < \mu^+$ and otherwise it is a subset of $i(g)$ of cardinality μ such that

$$j \in U_{\zeta,g} \quad \Rightarrow \quad \mathrm{Dom}(f^*_{\gamma(g,j)}) \cap Z_\zeta = \mathrm{Dom}(g),$$

(e) $Y_{\zeta+1} = Y_\zeta \cup \{\gamma(g,j) : g \in G_\zeta \text{ and } j \in U_{\zeta,g}\}$.

Let $Y = Y_{\omega_1}$. Let $\{(g_\varepsilon, \xi_\varepsilon) : \varepsilon < \varepsilon(*)\}$, $\varepsilon(*) \le \mu$, list the set of pairs (g,ξ) such that $\xi < \omega_1$, $g \in G_\xi$ and $i(g) \ge \mu^+$. We can find $\langle \zeta_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ such that $\langle \gamma(g_\varepsilon,\zeta_\varepsilon) : \varepsilon < \varepsilon(*)\rangle$ is without repetition and $\zeta_\varepsilon \in U_{g_\varepsilon,\xi_\varepsilon}$. Then for some $\alpha < 2^\mu \setminus Y_{\omega_1}$ we have

$$(\forall \varepsilon < \varepsilon(*))(A^\alpha_\varepsilon = \{\mu\times\gamma(g_\varepsilon,\zeta_\varepsilon) + \Upsilon : \Upsilon \in X_{\gamma(g_\varepsilon,\zeta_\varepsilon)}\}).$$

Now let $g = f^*_\alpha \restriction Z_{\omega_1}$. Then for some $\zeta_0(*) < \omega_1$ we have $g \in G_{\zeta_0(*)}$ and thus $U_{g,\zeta} \subseteq i(g)$ for $\zeta \in [\zeta_0(*), \omega_1)$ and $\langle \gamma(g,i) : i < i(g)\rangle$ are well defined. Now, α exemplifies that $i(g) < \mu^+$ is impossible (see the maximality of $i(g)$, as otherwise $i < i(g) \quad \Rightarrow \quad \gamma(g,i) \in Y_{\zeta_0(*)+1} \subseteq Y_{\omega_1}$).

Next, for each $\gamma \in X_\alpha$, $\mathrm{Dom}(f_{\mu\times\alpha+\gamma})$ is countable and hence for some $\zeta_{1,\gamma}(*) < \omega_1$ we have $\mathrm{Dom}(f_{\mu\times\alpha+\gamma}) \cap Z_{\omega_1} \subseteq Z_{\zeta_{1,\gamma}(*)}$. As $\mathrm{cf}(\mu) > \aleph_1$ necessarily for some $\zeta_1(*) < \omega_1$ we have that $X'_\alpha \overset{\mathrm{def}}{=} \{\gamma \in X_\alpha : \zeta_{1,\gamma}(*) \le \zeta_1(*)\} \in [\mu]^\mu$, and without loss of generality $\zeta_1(*) \ge \zeta_0(*)$.

So for some $\varepsilon < \varepsilon(*) \le \mu$ we have $g_\varepsilon = g$ & $\xi_\varepsilon = \zeta_1(*) + 1$. Let $\Upsilon_\varepsilon = \gamma(g_\varepsilon, \zeta_\varepsilon)$. Clearly

$(*)_1$ $f^*_\alpha, f^*_{\Upsilon_\varepsilon}$ are compatible (and countable),
$(*)_2$ $\langle f_{\mu\times\alpha+\gamma} : \gamma \in X'_\alpha\rangle$ is a Δ-system with heart f^*_α.

So possibly shrinking X'_α without loss of generality

$(*)_3$ if $\gamma \in X'_\alpha$ then $f_{\mu\times\alpha+\gamma}$ and $f^*_{\Upsilon_\varepsilon}$ are compatible.

For each $\gamma \in X'_\alpha$ let

$$t_\gamma = \{\beta \in X_{\Upsilon_\varepsilon} : f_{\mu \times \Upsilon_\varepsilon + \beta} \text{ and } f_{\mu \times \alpha + \gamma} \text{ are incompatible}\}.$$

As $\langle f_{\mu \times \Upsilon_\varepsilon + \beta} : \beta \in X_{\Upsilon_\varepsilon} \rangle$ is a Δ-system with heart $f^*_{\Upsilon_\varepsilon}$ (and $(*)_3$) necessarily

$(*)_4$ $\gamma \in X'_\alpha$ implies t_γ is countable.

For $\gamma \in X'_\alpha$ let

$$s_\gamma \overset{\text{def}}{=} \bigcup \{u : \begin{array}{l} u \text{ is a finite subset of } X'_\alpha \text{ and} \\ F_\alpha(\{\mu \times \alpha + \beta : \beta \in u\}) \text{ belongs to } t_\gamma\}. \end{array}$$

As F_α is a one-to-one function clearly

$(*)_5$ s_γ is a countable set.

Hence without loss of generality (possibly shrinking X'_α), as $\mu > \aleph_1$,

$(*)_6$ if $\gamma_1 \neq \gamma_2$ are from X'_α then $\gamma_1 \notin s_{\gamma_2}$.

By the choice of F_α for some finite subset u of X'_α with at least two elements, letting $u' \overset{\text{def}}{=} \{\mu \times \alpha + j : j \in u\}$ we have

$$\beta \overset{\text{def}}{=} F_\alpha(u') \in \{\mu \times \gamma(g_\varepsilon, \zeta_\varepsilon) + \gamma : \gamma \in X_{\gamma(g_\varepsilon, \zeta_\varepsilon)}\}$$

(remember $\Upsilon_\varepsilon = \gamma(g_\varepsilon, \zeta_\varepsilon)$), so $u' \cup \{\beta\} \in W$. Thus it is enough to show that $\{f_{\mu \times \alpha + j} : j \in u\} \cup \{f_\beta\}$ are compatible. For this it is enough to check any two. Now, $\{f_{\mu \times \alpha + j} : j \in u\}$ are compatible as $\langle f_{\mu \times \alpha + j} : j \in X_\alpha \rangle$ is a Δ-system. So let $j \in u$, why are $f_{\mu \times \alpha + j}$, f_β compatible? As otherwise $\beta - (\mu \times \Upsilon_\varepsilon) \in t_j$ and hence u is a subset of s_j. But u has at least two elements, so there is $\gamma \in u \setminus \{j\}$. Now u is a subset of X'_α and this contradicts the statement $(*)_6$ above, finishing the proof. $\blacksquare_{1.7}$

Remark 1.8. In 1.7, we can also get $d(BA(W, \mathbf{w})) = \mu$, but this is irrelevant to our aim. E.g. in this case let for $i < \mu$, h_i be a partial function from 2^μ to $\{0, 1\}$ such that $\text{Dom}(h_i) \cap [\beta, \beta + \mu)$ is finite for $\beta < 2^\mu$ and such that every finite such function is included in some h_i. Choosing the $(W_\alpha, \mathbf{w}_\alpha)$ preserve:

$$\{x_\beta : h_i(\beta) = 1\} \cup \{-x_\beta : h_i(\beta) = 0\} \text{ generates a filter of } BA(W_\alpha, \mathbf{w}_\alpha).$$

Conclusion 1.9. Theorem 0.1 holds.

PROOF By 1.1, 1.7. $\blacksquare_{2.1}$

2. Getting the example for $\mu = (\aleph_2)^{\aleph_0}$, $\lambda = 2^{\aleph_2}$. Our aim here is to show that there are I, $\mathfrak{B}$ as in 0.1 for $\mu = (\aleph_2)^{\aleph_0}$. For this we shall weaken the conditions in the Main Lemma 1.1 (see 2.1 below) and then show that we can get it in a variant of 1.7 (see 2.2 below). More fully, by 2.2 there is a $2^{\aleph_2}$–candidate $(W, \mathbf{w})$ satisfying the assumptions of 2.1 except possibly clause **(a)**, so μ is irrelevant in the clauses **(b)**–**(f)**. Let $\mu = (\aleph_2)^{\aleph_0} = \aleph_2 + 2^{\aleph_0}$ and apply 2.2. Now we get the conclusion of 1.1 as required.

Proposition 2.1. *Assume that*

(a) $\mu = \mu^{\aleph_0}$, $\lambda \leq 2^\mu$,
(b) $\mathfrak{B}$ *is a complete c.c.c. Boolean Algebra,*
(c) $x_i \in \mathfrak{B} \setminus \{0\}$ *for* $i < \lambda$, *and* $\mathcal{S} \subseteq \{u \in [\lambda]^{\leq \aleph_0} : (\forall i \in \lambda \setminus u)(x_i \notin \mathfrak{B}_u)\}$, *where* $\mathfrak{B}_u$ *is the completion of* $\langle \{x_i : i \in u\} \rangle_{\mathfrak{B}}$ *in* $\mathfrak{B}$ *(for* $u \in [\lambda]^{\leq \aleph_0}$*),*
(d)$^-$ *if* $i \in u_i \in [\lambda]^{\leq \aleph_0}$ *for* $i < \lambda$, *then we can find* $n < \omega$, $i_0 < \ldots < i_{n-1} < \lambda$ *and* $u \in \mathcal{S} (\subseteq [\lambda]^{\leq \aleph_0})$ *such that:*
  (i) $\mathfrak{B} \models$ "$\bigcap\limits_{\ell < n} x_{i_\ell} = 0$",
  (ii) $i_\ell \in u_{i_\ell} \setminus u$ *for* $\ell < n$,
  (iii) $\langle u_{i_\ell} \setminus u : \ell < n \rangle$ *are pairwise disjoint;*
(e) $u \in \mathcal{S}$ & $i \in \lambda \setminus u$ & $y \in \mathfrak{B}_u \setminus \{0, 1\} \quad \Rightarrow \quad y \cap x_i \neq 0$ & $y - x_i \neq 0$,
(f) $\mathcal{S}$ *is cofinal in* $([\mu]^{<\aleph_0}, \subseteq)$
  [actually, it follows from (d)$^-$*].*

Then there are a σ*-ideal* I *on* $\mathcal{P}(\mu)$ *and a* σ*-algebra* $\mathfrak{A}$ *of subsets of* μ *extending* I *such that* $\mathfrak{A}/I$ *satisfies the c.c.c. and the natural homomorphism* $\mathfrak{A} \longrightarrow \mathfrak{A}/I$ *cannot be lifted.*

Remark: Actually we can in clause **(e)** omit "$y - x_i \neq 0$".

PROOF Repeat the proof of 1.1 till the definition of $\mathbf{e}_{i,n}$ and W_i in the beginning of the proof of 1.1.3 (which says that h_2 cannot be lifted). Then choose $u_i \in \mathcal{S}$ such that $W_i \subseteq \mathfrak{B}_{u_i}$ (exists by clause **(f)** of our assumptions). By clause **(d)**$^-$ we can find $n < \omega$, $i_0 < \ldots < i_{n-1}$ and $u \in \mathcal{S}$ such that clauses (i),(ii),(iii) of **(d)**$^-$ hold.

Claim 2.1.1. *For* $\ell < n$, *there are homomorphisms* f_{i_ℓ} *from* $\mathfrak{B}$ *into* $\{0, 1\}$ *respecting* $\mathbf{e}_{i_\ell, m}$ *for* $m < \omega$ *and mapping* x_{i_ℓ} *to 1 such that* $\langle f_{i_\ell} \restriction (W_{i_\ell} \cap \mathfrak{B}_u) : \ell < n \rangle$ *are compatible functions.*

Proof of the claim: E.g. by absoluteness it suffices to find it in some generic extension. Let $G_u \subseteq \mathfrak{B}_u$ be a generic ultrafilter. Now $\mathfrak{B}_u \lessdot \mathfrak{B}$ and $(\forall y \in G_u)(y \cap x_{i_\ell} > 0)$ (see clause **(e)**). So there is a generic ultrafilter G_ℓ of $\mathfrak{B}$ extending G_u such that $x_{i_\ell} \in G_\ell$. Define f_{i_ℓ} by $f_{i_\ell}(y) = 1 \quad \Leftrightarrow \quad y \in G_\ell$

for $y \in u_{i_\ell}$. By Clause (iii) of $\mathbf{(d)}^-$ those functions are compatible and we finish as in 1.1.

Thus we have finished. $\blacksquare_{2.1}$

Theorem 2.2. *In 1.7 if we let e.g. $\mu = \aleph_2$ then we can find a 2^μ-candidate $(W, \mathbf{w})$ such that $BA^c(W, \mathbf{w})$ satisfies the clauses* (b)–(f) *of 2.1.*

Proof In short, we repeat the proof of 1.7 after defining $(W, \mathbf{w})$. But now we are being given $\langle u_i : i < \lambda \rangle$, $u_i \in [2^\mu]^{\leq \aleph_0}$, $i \in u_i$. For each $\alpha < 2^\mu$ (we cannot in general find a Δ-system but) we can find u^*_α, X_α such that $X_\alpha \in [\mu]^\mu, u^*_\alpha \in \mathcal{S} \subseteq [2^\mu]^{\leq \aleph_0}$ and $\langle u_{\mu \times \alpha + i} \setminus u^*_\alpha : i \in X_\alpha \rangle$ are pairwise disjoint, and $i \in X_\alpha \quad \Rightarrow \quad \mu \times \alpha + i \in u_{\mu \times \alpha + i} \setminus u^*_\alpha$ and we continue as there (replacing the functions by the sets where instead $G_\zeta = \{g : g \in Z_\zeta, \mathrm{Dom}(g) \subseteq Z_\zeta\}$ we let h_ζ be a one-to-one function from Z_ζ onto μ and $G_\zeta = \{u \subseteq Z_\zeta : {h_\zeta}''(u) \in \mathcal{S}\}$ and instead $g = f^*_\alpha \restriction Z_{\omega_1}$ let $u^*_\alpha \cap Z_{\omega_1} \subseteq Z_{\zeta_0(*)}$, $u^*_\alpha \cap Z_{\omega_1} \subseteq v \in G_\zeta$).

Detailed Proof Let $F^* : [\mu]^{<\aleph_0} \longrightarrow \mu$ be such that

$$(\forall A \in [\mu]^\mu)[F''([A]^{<\aleph_0} \setminus [A]^{<2}) = \mu].$$

Let $\langle \bar{A}^\alpha : \alpha < 2^\mu \rangle$ list the sequences $\bar{A} = \langle A_i : i < \mu \rangle$ such that $A_i \in [2^\mu]^\mu$, $(\forall i < \mu)(\exists \alpha)(A_i \subseteq [\mu \times \alpha, \mu \times \alpha + \mu))$ and $i < j < \mu \quad \Rightarrow \quad A_i \cap A_j = \emptyset$. Without loss of generality we have $A^\alpha_i \subseteq \mu \times (1 + \alpha)$ and each $\bar{A}$ is equal to $\bar{A}^\alpha$ for 2^μ ordinals α. Clearly $\mathrm{otp}(A^\alpha_i) = \mu$.

We choose by induction on $\alpha < 2^\mu$ pairs $(W_\alpha, \mathbf{w}_\alpha)$ and functions F_α such that

- (α) $(W_\alpha, \mathbf{w}_\alpha)$ is a $\mu \times (1 + \alpha)$-candidate,
- (β) $\beta < \alpha$ implies $W_\beta = W_\alpha \cap [\mu \times (1 + \beta)]^{<\aleph_0}$, $\mathbf{w}_\beta = \mathbf{w}_\alpha \restriction W_\beta$,
- (γ) F_α is a one-to-one function from

$$\{u : u \subseteq [\mu \times (1 + \alpha), \mu \times (1 + \alpha + 1)) \text{ finite with at least two elements}\}$$

into $\bigcup\limits_{i<\mu} A^\alpha_i$,

- (δ) $W_{\alpha+1} = W_\alpha \cup \{u \cup \{F_\alpha(u)\} : u \in W^*_\alpha\}$, where $W^*_\alpha = \{u : u$ is a subset of $[\mu \times (1 + \alpha), \mu \times (1 + \alpha + 1))$ such that $\aleph_0 > |u| \geq 2\}$,
- (ε) for finite $u \in W^*_\alpha$ we have

$$\mathbf{w}(u \cup \{F_\alpha(u)\}) = \{v \subseteq u \cup \{F_\alpha(u)\} : u \subseteq v \text{ or } F_\alpha(u) \in v \ \& \ v \cap u \neq \emptyset\},$$

- (ζ) Let F_α be such that for any subset X of $J_\alpha = [\mu \times (1 + \alpha), \mu \times (1 + \alpha + 1))$ of cardinality μ and $i < \mu$ and $\gamma \in A^\alpha_i$ for some finite subset u of X we have $F_\alpha(u) \in A^\alpha_i \setminus \gamma$.

There are no difficulties in carrying out the construction and checking that it as required. Let $W = \bigcup\limits_{\alpha} W_\alpha$, $\mathbf{w} = \bigcup\limits_{\alpha} \mathbf{w}_\alpha$, $\mathfrak{B} = BA^c(W, \mathbf{w})$. Clearly $(W, \mathbf{w})$ is a λ-candidate.

Let $\mathcal{S}^* \subseteq [\mu]^{\leq \aleph_0}$ be stationary of cardinality μ. Let

$$\mathcal{S}' = \{u \in [\lambda]^{\leq \aleph_0} : \text{ if } v \in W \text{ and } v \cap u \in \mathbf{w}(v) \text{ then } v \subseteq u\}.$$

Now, clause (f) holds as $(W, \mathbf{w})$ satisfies clause (d) of Definition 1.3(3). As for clause (e) use Lemma 2.3 below.

The main point is clause **(d)**$^-$ of 2.1. So let $i \in a_i \in [\lambda^\mu]^{\leq \aleph_0}$ for $i < \lambda$ be given. For each $\alpha < \lambda$, as $\mu = \aleph_2$ we can find $X_\alpha \in [\mu]^\mu$ and $a^*_\alpha \in \mathcal{S}'$ such that $\alpha \in a^*_\alpha$ and:

$(\otimes_\alpha)$ $\zeta_1 \neq \zeta_2$ & $\zeta_1 \in X_\alpha$ & $\zeta_2 \in X_\alpha \quad \Rightarrow \quad a_{\mu\times\alpha+\zeta_1} \cap a_{\mu\times\alpha+\zeta_2} \subseteq a^*_\alpha$ and $\zeta \in X_\alpha \quad \Rightarrow \quad \mu \times \alpha + \zeta \notin a^*_\alpha$.

For each $b \in [\lambda]^{\leq \aleph_0}$ let $\langle \gamma(b,i) : i < i(g)\rangle$ be a maximal sequence such that $\gamma(b,i) < \lambda$ and $u^*_{\gamma(b,i)} \cap u^*_{\gamma(b,j)} \subseteq b$ and $\gamma(b,i) \notin b$ for $j < i$ (just choose $\gamma(b,i)$ by induction on i).

We choose by induction on $\zeta \leq \omega_1$, $Y_\zeta, h_\zeta, S_\zeta, G_\zeta, Z_\zeta$ and $U_{\zeta,g}$ such that

(a) $Y_\zeta \in [2^\mu]^{\leq \mu}$ is increasing continuous in ζ,

(b) Z_ζ is the minimal subset of λ (of cardinality $\leq \mu$) which includes

$$\bigcup \{u_\gamma : (\exists \alpha \in Y_\zeta)[\mu \times \alpha \leq \gamma < \mu \times (\alpha + 1)]\}$$

and satisfies

$$u \in W \ \& \ u \cap Z_\zeta \in \mathbf{w}(u) \quad \Rightarrow \quad u \subseteq Z_\zeta,$$

(c) h_ζ is a one-to-one function from μ onto Z_ζ, and

$$G_\zeta = \{h''_\zeta(b) : b \in \mathcal{S}\} \cup \bigcup_{\xi<\zeta} G_\xi.$$

(d) for $b \in G_\zeta$ we have $U_{\zeta,b}$ is $\{i : i < i(b)\}$ if $i(b) < \mu^+$ and otherwise is a subset of $i(b)$ of cardinality μ such that

$$j \in U_{\zeta,b} \quad \Rightarrow \quad \mathrm{Dom}(f^*_{\gamma(b,j)}) \cap Z_\zeta \subseteq b,$$

(e) $Y_{\zeta+1} = Y_\zeta \cup \{\gamma(b,j) : b \in G_\zeta \text{ and } j \in U_{\zeta,b}\}$.

Again, there is no problem to carry out the definition (e.g. $|Z_\zeta| \leq \mu$ by clause (d) of 1.3(3)). Let $Y = Y_{\omega_1}$. Let $\{(b_\varepsilon, \xi_\varepsilon) : \varepsilon < \varepsilon(*) \leq \mu\}$ list the set of pairs (b, ξ) such that $\xi < \omega_1$, $b \in G_\xi$ and $i(b) \geq \mu^+$. We can find $\langle \zeta_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ such that $\langle \gamma(b_\varepsilon, \zeta_\varepsilon) : \varepsilon < \varepsilon(*)\rangle$ is without repetition and $\zeta_\varepsilon \in U_{b_\varepsilon, \xi_\varepsilon}$, $\varepsilon(*) \leq \mu$. So for some $\alpha < 2^\mu \setminus Y_{\omega_1}$ we have

$$(\forall \varepsilon < \varepsilon(*))(A^\alpha_\varepsilon = \{\mu \times \gamma(b_\varepsilon, \zeta_\varepsilon) + \Upsilon : \Upsilon \in X_{\gamma(b_\varepsilon, \zeta_\varepsilon)}\}.$$

Now, let $b_0 = a^*_\alpha \cap Z_{\omega_1}$, so for some $\zeta_0(*) < \omega_1$ we have $b_0 \subseteq Z_{\zeta_0(*)}$. As a^*_α is countable and $G_\zeta \subseteq [Z_\zeta]^{\leq \aleph_0}$ is stationary (and the closure property of Z_ζ) there is $b^* \in \mathcal{S}'$ such that $b \stackrel{\text{def}}{=} b^* \cap Z_{\zeta_0(*)}$ belongs to G_ζ and $a^*_\alpha \subseteq b^*$ and so $U_{b,\zeta} \subseteq i(b)$ for $\zeta \in [\zeta_0(*), \omega_1)$ and $\langle \gamma(b,i) : i < i(b) \rangle$ are well defined. Now α exemplified $i(b) < \mu^+$ is impossible (see the maximality as otherwise $i < i(b) \quad \Rightarrow \quad \gamma(b,i) \in Z_{\zeta_0(*)+1} \subseteq Z_{\omega_1}$).

As for each $\gamma \in X_\alpha$, the set $a_{\mu \times \alpha + \gamma}$ is countable, for some $\zeta_{1,\gamma}(*) < \omega_1$ we have $a_{\mu \times \alpha + \gamma} \cap Z_{\omega_1} \subseteq Z_{\zeta_{1,\gamma}(*)}$. Since $\mathrm{cf}(\mu) > \aleph_1$ necessarily for some $\zeta_1(*) < \omega_1$ we have

$$X'_\alpha \stackrel{\text{def}}{=} \{\gamma \in X_\alpha : \zeta_{1,\gamma}(*) \leq \zeta_1(*)\} \in [\mu]^\mu$$

and without loss of generality $\zeta_1(*) \geq \zeta_0(*)$. Thus for some $\varepsilon < \mu$ we have $b_\varepsilon = b$ & $\xi_\varepsilon = \zeta_1(*) + 1$. Let $\Upsilon_\varepsilon = \gamma(b_\varepsilon, \zeta_\varepsilon)$. Clearly

$(*)_1$ $a^*_\alpha, a^*_{\Upsilon_\varepsilon}$ are countable,

$(*)_2$ $\gamma \in X'_\alpha \quad \Rightarrow \quad \mu \times \alpha + \gamma \in a_\gamma$,

$(*)_3$ $\gamma_1 \neq \gamma_2$ & $\gamma_1 \in X'_\alpha$ & $\gamma_2 \in X'_\alpha \quad \Rightarrow \quad a_{\mu \times \alpha + \gamma_1} \cap a_{\mu \times \alpha + \gamma_2} \subseteq b^*$.

So possibly shrinking X'_α without loss of generality

$(*)_4$ if $\gamma \in X'_\alpha$ then $a_{(\mu \times \alpha + \gamma)} \cap a^*_{\Upsilon_\varepsilon} \subseteq b^*$.

For each $\gamma \in X'_\alpha$ let

$$t_\gamma = \{\beta \in X_{\Upsilon_\varepsilon} : a_{(\mu \times \Upsilon_\varepsilon + \beta)} \cap a_{(\mu\alpha + \gamma)} \not\subseteq b^*\}.$$

As $\langle f_{(\mu \times \Upsilon_\varepsilon + \beta)} : \beta \in X_{\Upsilon_\varepsilon} \rangle$ was chosen to satisfy $(\otimes_{\Upsilon_\varepsilon})$ (and $(*)_3$) necessarily

$(*)_5$ $\gamma \in X'_\alpha$ implies t_γ is countable.

For $\gamma \in X'_\alpha$ let

$$s_\gamma \stackrel{\text{def}}{=} \bigcup \{u : \begin{array}{l} u \text{ is a finite subset of } X'_\alpha \text{ and} \\ F_\alpha(\{\mu \times \alpha + \beta : \beta \in u\}) \text{ belongs to } t_\gamma\}. \end{array}$$

As F_α is a one-to-one function clearly

$(*)_6$ s_γ is a countable set.

So without loss of generality (possibly shrinking X'_α using $\mu > \aleph_1$)

$(*)_7$ if $\gamma_1 \neq \gamma_2$ are from X'_α then $\gamma_1 \notin s_{\gamma_2}$.

By the choice of F_α, for some finite subset u of X'_α with at least two elements, letting $u' \stackrel{\text{def}}{=} \{\mu \times \alpha + j : j \in u\}$ we have

$$\beta \stackrel{\text{def}}{=} F_\alpha(u') \in \{\mu \times \gamma(b_\varepsilon, \zeta_\varepsilon) + \gamma : \gamma \in X_{\gamma(b_\varepsilon, \zeta_\varepsilon)}\}.$$

Hence $u' \cup \{\beta\} \in W$, so it is enough to show that $\{a_{\mu \times \alpha + j} : j \in u\} \cup \{a_\beta\}$ are pairwise disjoint outside b^*. For the first it is enough to check any two. Now, $\{f_{\mu \times \alpha + j} : j \in u\}$ are O.K. by the choice of $\langle f_{\mu \times \alpha + j} : j \in X_\alpha \rangle$. So let

$j \in u$. Now, $a_{\mu\times\alpha+j}$, a_β are O.K., otherwise $\beta - (\mu \times \Upsilon_\varepsilon) \in t_j$ and hence u is a subset of s_j but u has at least two elements and is a subset of X'_α and this contradicts the statement $(*)_6$ above and so we are done. $\blacksquare_{2.2}$

Lemma 2.3. *Let* $(W, \mathbf{w})$ *be a* λ*-candidate. Assume that* $u \subseteq \lambda$ *and* $u = \mathrm{cl}_{(W,\mathbf{w})}(u)$ *(see Definition 1.3(1),(d)) and let* $W^{[u]} = W \cap [u]^{<\aleph_0}$ *and* $\mathbf{w}^{[u]} = \mathbf{w}\restriction W^{[u]}$. *Furthermore suppose that* $(W, \mathbf{w})$ *is non-trivial (which holds in all the cases we construct), i.e.*

$$(*) \qquad i \in v \in W \quad \Rightarrow \quad v \setminus \{i\} \in \mathbf{w}(v).$$

Then:

1. $(W^{[u]}, \mathbf{w}^{[u]})$ *is a* λ*-candidate (here* $u = \mathrm{cl}_{(W,\mathbf{w})}(u)$ *is irrelevant);*
2. $BA(W^{[u]}, \mathbf{w}^{[u]})$ *is a subalgebra of* $BA(W, \mathbf{w})$*, moreover* $BA(W^{[u]}, \mathbf{w}^{[u]}) \lessdot BA(W, \mathbf{w})$*;*
3. *if* $i \in \lambda \setminus u$ *and* $y \in BA(W^{[u]}, \mathbf{w}^{[u]})$ *then*
$$y \neq 0 \quad \Rightarrow \quad y \cap x_i > 0 \ \&\ y - x_i > 0;$$
4. $BA^c(W^{[u]}, \mathbf{w}^{[u]}) \lessdot BA^c(W, \mathbf{w})$.

Proof 1) Trivial.

2) *The first phrase*: if f_0 is a homomorphism from $BA(W^{[u]}, \mathbf{w}^{[u]})$ to the Boolean Algebra $\{0, 1\}$ we define a function f from $\{x_\alpha : \alpha < \lambda\}$ to $\{0, 1\}$ by $f(x_\alpha)$ is $f_0(x_\alpha)$ if $\alpha \in u$ and is zero otherwise. Now

$$v \in W \quad \Rightarrow \quad (\exists \alpha \in v)(f(x_\alpha) = 0).$$

Why? If $v \subseteq u$, then $v \in W^{[u]}$ and "f_0 is a homomorphism", so $f_0(\bigcap_{\alpha\in v} x_\alpha) = 0$. Hence $(\exists\alpha \in v)(f_0(x_\alpha) = 0)$ and hence $(\exists\alpha \in v)(f(x_\alpha) = 0)$. If $v \not\subseteq u$, then choose $\alpha \in v \setminus u$, so $f(x_\alpha) = 0$.

So f respects all the equations involved in the definition of $BA(W, \mathbf{w})$ hence can be extended to a homomorphism $\hat f$ from $BA(W, \mathbf{w})$ to $\{0, 1\}$. Easily $f_0 \subseteq \hat f$ and so we are done.

As for *the second phrase*, let $z \in BA(W, \mathbf{w})$, $z > 0$ and we shall find $y \in BA(W^{[u]}, \mathbf{w}^{[u]})$, $y > 0$ such that

$$(\forall x)[x \in BA(W^{[u]}, \mathbf{w}^{[u]}) \ \&\ 0 < x \leq y \quad \Rightarrow \quad x \cap z \neq 0).$$

We can find disjoint finite subsets s_0, s_1 of λ such that $0 < z' \leq z$ where $z' = \bigcap_{\alpha\in s_1} x_\alpha \cap \bigcap_{\alpha\in s_0} (-x_\alpha)$. Let

$$t = \bigcup\{v : v \in W \text{ a finite subset of } \lambda \text{ and } v \cap s_0 \in \mathbf{w}(v)\} \cup s_0 \cup s_1.$$

We know that t is finite. We can find a partition t_0, t_1 of t (so $t_0 \cap t_1 = \emptyset$, $t_0 \cup t_1 = t$) such that $s_0 \subseteq t_0$ and $s_1 \subseteq t_1$ and $y^* = \bigcap_{\alpha \in t_1} x_\alpha \cap \bigcap_{\alpha \in t_0} (-x_\alpha) > 0$. Note that $y \stackrel{\text{def}}{=} \bigcap_{\alpha \in u \cap t_1} x_\alpha \cap \bigcap_{\alpha \in u \cap t_0} (-x_\alpha)$ is > 0 and, of course, $y \in BA(W^{[u]}, \mathbf{w}^{[u]})$. We shall show that y is as required. So assume $0 < x \leq y$, $x \in BA(W^{[u]}, \mathbf{w}^{[u]})$. As we can shrink x, without loss of generality, for some disjoint finite $r_0, r_1 \subseteq u$ we have $t \cap u \subseteq r_0 \cup r_1$ and $x = \bigcap_{\alpha \in r_1} x_\alpha \cap \bigcap_{\alpha \in r_0} (-x_\alpha)$, so clearly $t_1 \cap u \subseteq r_1$, $t_0 \cap u \subseteq r_0$.

We need to show $x \cap z \neq 0$, and for this it is enough to show that $x \cap z' \neq 0$. Now, it is enough to find a function $f : \{x_\alpha : \alpha < \lambda\} \longrightarrow \{0, 1\}$ respecting all the equations in the definition of $BA(W, \mathbf{w})$ such that $\hat{f}$ maps $x \cap z'$ to 1. So let $f(x_\alpha) = 1$ for $\alpha \in r_1 \cup s_1$ and $f(x_\alpha) = 0$ otherwise. If this is O.K., fine as $f \restriction r_0$, $f \restriction s_0$ are identically zero and $f \restriction r_1$, $f \restriction s_1$ are identically one. If this fails, then for some $v \in \mathbf{w}$ we have $v \subseteq r_1 \cup s_1$. But then $v \cap r_1 \in \mathbf{w}(v)$ or $v \cap s_1 \in \mathbf{w}(v)$. Now if $v \cap r_1 \in \mathbf{w}(w)$ as $r_1 \subseteq u$ necessarily $v \subseteq u$, but $v \subseteq r_1 \cup s_1$ and $s_1 \cap u \subseteq t_1 \subseteq r_1$, so $v \subseteq r_1$ is a contradiction to $x > 0$. Lastly, if $v \cap s_1 \in \mathbf{w}(v)$, then $v \subseteq t$ so as $v \subseteq r_1 \cup s_1$ we have $v \subseteq s_1 \cup (t \cap r_1)$ and so $v \subseteq s_1 \cup t_1$ and hence $v \subseteq t_1$ — a contradiction to $y^* > 0$. So f is O.K. and we are done.

3) Let f_0 be a homomorphism from $BA(W^{[u]}, \mathbf{w}^{[u]})$ to the trivial Boolean Algebra $\{0, 1\}$. For $t \in \{0, 1\}$ we define a function f from $\{x_\alpha : \alpha < \lambda\}$ to $\{0, 1\}$ by

$$f(x_\alpha) = \begin{cases} f_0(x_\alpha) & \text{if} \quad \alpha \in u \\ t & \text{if} \quad \alpha = i \\ 0 & \text{if} \quad \alpha \in \lambda \setminus u \setminus \{i\}. \end{cases}$$

Now f respects the equations in the definition of $BA(W, \mathbf{w})$. Why? Let $v \in W$. We should prove that $(\exists \alpha \in v)(f(\alpha) = 0)$. If $v \subseteq u$, then

$$f \restriction \{x_\alpha : \alpha \in v\} = f_0 \restriction \{x_\alpha : \alpha \in v\} \quad \text{and}$$
$$0 = f_0(0_{BA(W^{[u]}, \mathbf{w}^{[u]})}) = f_0(\bigcap_{\alpha \in v} x_\alpha) = \bigcap_{\alpha \in v} f_0(x_\alpha),$$

so $(\exists \alpha \in v)(f_0(x_\alpha) = 0)$. If $v \not\subseteq u \cup \{i\}$ let $\alpha \in v \setminus u \setminus \{i\}$, so $f(x_\alpha) = 0$ as required.

So we are left with the case $v \subseteq u \cup \{i\}$, $v \not\subseteq u$. Then by the assumption $(*)$, $v \cap u = v \setminus \{i\} \in \mathbf{w}(v)$ and $v \subseteq u$, a contradiction.

4) Follows. $\blacksquare_{2.3}$

Remark 2.4. We can replace $\aleph_0$ by say $\kappa = \mathrm{cf}(\kappa)$ (so in 2.2, $\mu = \kappa^{++}$, in 1.7, $(\forall \alpha < \mu)(|\alpha|^{<\kappa} < \mu = \mathrm{cf}(\mu))$.

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