Two consistency results on set mappings

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Abstract. It is consistent that there is a set mapping from the four-tuples of ω_n into the finite subsets with no free subsets of size t_n for some natural number t_n. For any $n < \omega$ it is consistent that there is a set mapping from the pairs of ω_n into the finite subsets with no infinite free sets. For any $n < \omega$ it is consistent that there is a set mapping from the pairs of ω_n into ω_n with no uncountable free sets.

In this paper we consider some problems on *set mappings*, that is, for our current purposes, functions of the type $f : [\kappa]^k \to [\kappa]^{<\mu}$ for some natural number k and cardinals κ, μ, which satisfy $f(x) \cap x = \emptyset$ for $x \in [\kappa]^k$. A subset H of κ is called *free* if $f(x) \cap H = \emptyset$ holds for every $x \in [H]^k$. The most central question of this area of combinatorial set theory is that given k, κ, and μ how large free sets can be guaranteed. The investigation of the case $k = 1$ was started in the thirties by Paul Turán, who asked if there exists an infinite free set if $\mu = \omega$ and κ is the continuum. After G. Grünwald's affirmative answer ([4]) S. Ruziewicz found the right conjecture ([10]); if $\kappa > \mu$ then there is a free set of cardinal κ (remember, $k = 1$ is assumed). Several cases were soon proved, for example S. Piccard solved the case when κ is regular ([9]), but only in 1950 was the full conjecture established by Paul Erdős ([1]) with the assumption of GCH, and ten years later without this assumption, by A. Hajnal ([5]). In the fifties Erdős and Hajnal started the research on the case $k > 1$ following the observation of Kuratowski and Sierpiński (see [4]) that for set mappings on $[\kappa]^k$ there always exists a free set of cardinal $k + 1$ iff $\kappa \geq \mu^{+k}$.

In ZFC alone, Hajnal and Máté extended the Kuratowski-Sierpiński results by showing ([6]) that if $k = 2$ and $\kappa \geq \mu^{+2}$ then there are arbitrarily large finite free sets, and Hajnal proved (see [3]) that a similar result holds for $k = 3$, $\kappa \geq \mu^{+3}$. One of the problems emphasized in [3] is if the result can be extended to $k = 4$, $\kappa \geq \mu^{+4}$. In Theorem 1 we show that it is not

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the case; for every natural number n there exists a natural number t_n such that for any given regular μ it is consistent that there is a set mapping $f : [\mu^{+n}]^4 \to [\mu^{+n}]^{<\mu}$ with no free sets of size t_n. (We assume GCH in the ground model.)

As for the existence of infinite free sets, a special case of a theorem of Erdős and Hajnal states that under CH if $f : [\omega_2]^2 \to [\omega_2]^{<\omega}$ is a set mapping then there is an uncountable free set for f ([2]). Answering a question of [6] the first author proved that without CH even the existence of an infinite free set cannot be guaranteed [7]. Here we extend that result to arbitrary ω_n. Using this result, we answer another question of Hajnal and Máté, by showing that it is consistent that there exists a set mapping from the pairs of ω_n into $[\omega_n]^1$ with no uncountable free sets.

Theorem 1 was proved by S. Shelah; Theorem 2 and Corollary 3 were subsequently proved by P. Komjáth.

Notation and Definitions. We use the standard axiomatic set theory notation. Cardinals are identified with initial ordinals. If S is a set and κ a cardinal, then $[S]^\kappa = \{X \subseteq S : |X| = \kappa\}$, $[S]^{<\kappa} = \{X \subseteq S : |X| < \kappa\}$, $[S]^{\leq\kappa} = \{X \subseteq S : |X| \leq \kappa\}$. For a, b, c and r natural numbers, the Ramsey symbol, $a \longrightarrow (b, c)^r$, means that the following statement is true. Whenever the r-element subsets of an a element set are colored with two colors, say 0 and 1, then either there exists a b-element subset with all its r-tuples colored 0 or there exists a c-element subset with all its r-tuples colored 1. The existence of an appropriate a for any given b, c, r is guaranteed by Ramsey's theorem [3].

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To formulate the next result, set $t_0 = 5$, $t_1 = 7$, in general, t_{n+1} is the least number such that $t_{n+1} \longrightarrow (t_n, 7)^5$.

Theorem 1. (GCH) *Assume that $n < \omega$, $\kappa = \tau^{+n}$ for some regular cardinal τ. Then it is consistent that GCH holds below τ, $2^\tau = \kappa$ if $n > 0$, and there is a set mapping $f : [\kappa]^4 \longrightarrow [\kappa]^{<\tau}$ with no free subset of cardinal t_n.*

Proof. By induction on n. Our set mapping will satisfy the additional condition that $f(\{x_0, x_1, x_2, x_3\}) \subseteq (x_1, x_2)$ (the ordinal interval) for all $x_0 < x_1 < x_2 < x_3$.

The case $n = 0$ is obvious, since we can take $f(\{x_0, x_1, x_2, x_3\}) = (x_1, x_2)$.

Assume that V is a model of set theory satisfying the Theorem for n, and for τ^+ in place of τ. That is, for $\mu \leq \tau$, $2^\mu = \mu^+$ holds, and there is a set mapping $F : [\kappa]^4 \longrightarrow [\kappa]^{\leq\tau}$ satisfying $F(\{x_0, x_1, x_2, x_3\}) \subseteq (x_1, x_2)$ with no free subset of cardinality t_n. We are going to force with a notion of forcing $(P, \leq)$ in which the conditions will be some pairs of the form (s, g) with $s \in [\kappa]^{<\tau}$, $g : [s]^4 \longrightarrow [s]^{<\tau}$ satisfying $g(u) \subseteq F(u)$ for $u \in [s]^4$. Not all pairs as above will be in P but if (s, g), (s', g') are in P then (s', g') will extend (s, g) (in notation $(s', g') \leq (s, g)$) iff $s' \supseteq s$ and $g = g'|[s]^4$.

To describe the condition for $(s, g) \in P$ we introduce two more definitions. If U is a subset of κ then we call U *F-closed*, if $x_2 \in F(x_0, x_1, x_3, x_4)$ holds whenever $x_0 < x_1 < x_2 < x_3 < x_4$ are in U. If U is a subset of s then we call U *g-free*, if $x_2 \notin g(\{x_0, x_1, x_3, x_4\})$ holds for all $x_0 < x_1 < x_2 < x_3 < x_4$ in U. Now put (s, g) into P just in case there is no 7-element subset of s which is F-closed and g-free.

Having defined the notion of forcing $(P, \leq)$, we are going to show some properties of it.

Claim 1. *$(P, \leq)$ is $< \tau$-closed.*

Proof of Claim. Immediate from the finite character of the definition. □

Claim 2. *$(P, \leq)$ is τ^+-c.c.*

Proof of Claim. Assume that $p_\xi = (s_\xi, g_\xi) \in P$ for $\xi < \tau^+$. Using the Δ-system lemma we can assume that $s_\xi = a \cup b_\xi$ for some disjoint sets $\{a\} \cup \{b_\xi : \xi < \tau^+\}$. For $\{x_0, x_1, x_2, x_3\} \in [a]^4$, $g_\xi(\{x_0, x_1, x_2, x_3\})$ is a subset of $F(\{x_0, x_1, x_2, x_3\})$ of cardinal $< \tau$. As $\left|F(\{x_0, x_1, x_2, x_3\})\right| \leq \tau$, and $|a| < \tau$, we can assume, by $\tau^{<\tau} = \tau$, that $g_\xi|[a]^4$ is the same for $\xi < \tau^+$. We show that any two p_ξ, $p_{\xi'}$ are compatible. Set $q = (a \cup b_\xi \cup b_{\xi'}, g)$ where $g \supseteq g_\xi$, $g_{\xi'}$ and if

$$\{x_0, x_1, x_2, x_3\} \in [a \cup b_\xi \cup b_{\xi'}]^4 - [a \cup b_\xi]^4 - [a \cup b_{\xi'}]^4$$

then set

$$g(\{x_0, x_1, x_2, x_3\}) = (a \cup b_\xi \cup b_{\xi'}) \cap F(\{x_0, x_1, x_2, x_3\}).$$

We have to show that $q \in P$, that is, there is no 7-element F-closed, g-free subset of s. Assume that B is such a set. As p_ξ, $p_{\xi'}$ are conditions, $B \not\subseteq a \cup b_\xi$, $B \not\subseteq a \cup b_{\xi'}$. There are, therefore, $\eta_0 \in B \cap b_\xi$, $\eta_1 \in B \cap b_{\xi'}$.

An easy calculation shows that no matter what position η_0, η_1 occupy in B, there is a five-tuple $y_0 < y_1 < y_2 < y_3 < y_4$ in B such that η_0, $\eta_1 \in \{y_0, y_1, y_3, y_4\}$. (This is the point where the choice of 7 plays role.) We get, therefore, that $g(\{y_0, y_1, y_3, y_4\}) = F(\{y_0, y_1, y_3, y_4\}) \cap s \ni y_2$ so B cannot be F-closed and g-free. $\square$

Let $G \subseteq P$ be a generic subset of P. Set $S = \bigcup\{s : (s, g) \in G\}$ and $f = \bigcup\{g : (s, g) \in G\}$. Clearly, f is a set mapping of the required type on the set S.

Claim 3. *There is a $p \in P$ forcing that $|S| = \kappa$.*

Proof of Claim. Otherwise, 1 forces that S is bounded in κ, and as $(P, \leq)$ is $< \kappa$-c.c., it forces a bound, say $\xi < \kappa$. But as $\big(\{\xi\}, \emptyset\big) \Vdash \xi \in S$, we get a contradiction. $\square$

Claim 4. *In $V[G]$, f has no free subset of cardinality t_{n+1}.*

Proof of Claim. Assume that $A \subseteq S$ is a free subset of cardinality t_{n+1}. Color the five-tuples of A as follows. If $\{x_0, x_1, x_2, x_3, x_4\} \in [A]^5$, $x_0 < x_1 < x_2 < x_3 < x_4$ and $x_2 \in F(\{x_0, x_1, x_3, x_4\})$ then color $\{x_0, x_1, x_2, x_3, x_4\}$ by 1, otherwise by 0. As $t_{n+1} \longrightarrow (t_n, 7)^5$ either there is a homogeneous subset in color 1 of cardinal 7 or there is a homogeneous subset of color 0 of size t_n. This latter possibility is excluded by the hypothesis on F so we have the former. But that gives a 7-element subset which is F-closed and f-free and this is obviously excluded by the forcing. $\square$

Now Theorem 1 follows from the claims above by induction on n. $\square$

Theorem 2. (GCH) *If τ is a regular cardinal, $\kappa < \tau^{+\omega}$, then it is consistent that there is a set mapping $f : [\kappa]^2 \to [\kappa]^{<\tau}$ with no infinite free sets.*

Proof. For $\kappa \leq \tau$, we can simply take $f(\{x, y\}) = x$.

We are going to show, by induction on positive $n < \omega$ that it is consistent that there exists for $\kappa = \tau^{+n}$ a set mapping f on $[\kappa]^2$ as required. It will also satisfy $f(\{x, y\}) \subseteq x$ for $x < y < \kappa$.

The case $n = 1$ can also be proved in ZFC. If $x < \kappa = \tau^+$, enumerate x as $x = \{\gamma_x(i) : i < \tau\}$. If $x < y$ then let $i(x, y)$ be that index i for which

$x = \gamma_y(i)$ holds. Now set $f\big(\{x,y\}\big) = \big\{\gamma_x(i) : i \leq i(x,y)\big\}$. If $x_0 < x_1 < \cdots$ are the elements of an infinite free set then $i(x_0,x_1) > i(x_1,x_2) > \cdots$ which is impossible.

Assume now that $\tau < \kappa$, GCH holds up to and including τ and there is a set mapping $F : [\kappa]^2 \to [\kappa]^{\leq\tau}$ with no infinite free sets and with $F\big(\{x,y\}\big) \subseteq x$ for $x < y < \kappa$. We are going to define a $< \tau$-closed partial ordering $(P,\leq)$ which adds a set mapping $f : [S]^2 \to [S]^{<\tau}$ for some $S \in [\kappa]^\kappa$ and with no infinite free sets. It will also satisfy $f(\{x,y\}) \subseteq F(\{x,y\})$ for all $\{x,y\} \subseteq S$.

An element of P will be a triplet of the form $p = (s,g,r)$ where $s \in [\kappa]^{<\tau}$, $g : [s]^2 \to [s]^{<\tau}$ is a set mapping with $g \subseteq F$. If U is a subset of κ then we call U *F-closed*, if $x \in F(y,z)$ holds if $x < y < z$ are in U. If U is a subset of s then we call U *g-free*, if $x \notin g(\{y,z\})$ holds for $x < y < z$ in U. We require that there be no infinite g-free, F-closed subsets of s and r will be a rank function witnessing this. For this, we call a finite subset $u \in [s]^{<\omega}$ *secured* if $|u| \geq 3$, u is g-free and F-closed. What we assume on r is that it is a function from the secured subsets to τ with $r(u) > r(v)$ if v properly end-extends u. $p' = (s',g',r')$ extends $p = (s,g,r)$ if $s' \subseteq s$, $g' \subseteq g$, $r' \subseteq r$.

It is obvious that $(P,\leq)$ is transitive and $< \tau$-closed.

Claim 1. *$(P,\leq)$ is τ^+-c.c.*

Proof of Claim. Assume, for a contradiction, that we are given τ^+ conditions, $p_\xi = (s_\xi, g_\xi, r_\xi) \in P$ for $\xi < \tau^+$. By the Δ-system lemma we can assume that there are disjoint sets $\{a\} \cup \{b_\xi : \xi < \tau^+\}$ such that $s_\xi = a \cup b_\xi$. As for x, $y \in a$, since $g_\xi(\{x,y\}) \in [F(\{x,y\})]^{<\tau}$, by removing at most τ members from the family we can assume that $F\big(\{x,y\}\big) \cap b_\xi = \emptyset$ holds for x, $y \in a$. Then, $g_\xi\big(\{x,y\}\big) \subseteq a$, and with one more shrinking, we can assume that $g_\xi\big(\{x,y\}\big)$ is independent of ξ. We can also assume that the functions r_ξ are identical on the secured subsets of a.

Assume now that $\xi < \xi' < \tau^+$, we want to find a common extension of p_ξ and $p_{\xi'}$. Set $q = (a \cup b_\xi \cup b_{\xi'}, g, r)$ where $g \supseteq g_\xi \cup g_{\xi'}$ is the maximal extension, that is, $g\big(\{x,y\}\big) = (a \cup b_\xi \cup b_{\xi'}) \cap F\big(\{x,y\}\big)$ if $\{x,y\} \cap b_\xi \neq \emptyset$ and $\{x,y\} \cap b_{\xi'} \neq \emptyset$.

We now consider if we can define r. As q is the union of two conditions both omitting infinite g-free, F-closed sets, q won't have such sets, either. So *some* rank function r can be defined; the question is, if one extending

r_ξ, $r_{\xi'}$ can be given. To show this, it suffices to prove, that if u is a g-free, F-closed set, which is new, that is, has points in b_ξ, as well as in $b_{\xi'}$, then it cannot end extend an "old" secured set (one in p_ξ or in $p_{\xi'}$). Assume that $x_0 < x_1 < x_2 < \cdots$ are the elements of u. If $x_i \in b_\xi$, $x_j \in b_{\xi'}$, and i, $j \neq 0$, then $x_0 \in F\big(\{x_i, x_j\}\big)$, so $x_0 \in g\big(\{x_i, x_j\}\big)$ by the definition of g and so our set is not g-free. We get, therefore, that x_0 is the *only* element of $u \cap b_\xi$ (say). The possibility that both x_1 and x_2 are in a is ruled out by our above condition that $b_\xi \cap F\big(\{x_1, x_2\}\big) = \emptyset$. This means that $\{x_0, x_1, x_2\}$ is a "new" set, so u is indeed not an end extension of an old secured set as we assumed that secured sets have at least three elements. □

If $G \subseteq P$ is a generic subset, then define $S = \bigcup\{s : (s, g, r) \in G\}$, $f = \bigcup\{g : (s, g, r) \in G\}$, $R = \bigcup\{r : (s, g, r) \in G\}$, .

Claim 2. $|S| = \kappa$.

Proof of Claim. As in the corresponding proof in Theorem 1. □

Claim 3. *F has no infinite free set in $V[G]$.*

Proof of Claim. This is a well-known fact. It follows from the rank characterization of the nonexistence of free sets. □

Claim 4. *f has no infinite free set.*

Proof of Claim. Assume that $x_0 < x_1 < \cdots$ form an infinite f-free set. By Ramsey's theorem we can assume that either for every triplet $i < j < k < \omega$, $x_i \in F(x_j, x_k)$ holds or for every triplet $i < j < k < \omega$, $x_i \notin F(x_j, x_k)$ holds. The latter is impossible by Claim 3. Therefore $\{x_0, x_1, \ldots\}$ is f-free, F-closed, but then $R\big(\{x_0, x_1, x_2\}\big) > R\big(\{x_0, x_1, x_2, x_3\}\big) > \cdots$, which is impossible. □

An easy application of Theorem 2 solves another problem of [6].

Corollary 3. *For every $n < \omega$ it is consistent that there exists a set mapping $f : [\omega_n]^2 \to [\omega_n]^1$ with no uncountable free set.*

Proof. Applying Theorem 2 assume that $F : [\omega_n]^2 \to [\omega_n]^{\aleph_0}$ is a set mapping with no infinite free sets so that $F(\{x, y\}) \subseteq x$ for all $x < y < \omega_n$. Define the notion of forcing as follows, $(s, g) \in P$ iff $s \in [\omega_n]^{<\omega}$, $g : [s]^2 \to [s]^1$, and $g(u) \subseteq F(u)$ for all $u \in [s]^2$. Set $(s', g') \leq (s, g)$ iff $s' \supseteq s$, $g' \supseteq g$.

Claim 1. *If $\alpha < \omega_n$ then the set $D_\alpha = \{(s, g) : \alpha \in s\}$ is dense in $(P, \leq)$.*

Proof of Claim. Straightforward. □

Claim 2. $(P, \leq)$ *is c.c.c.*

Proof of Claim. Assume that $p_\xi \in P$ for $\xi < \omega_1$. By the usual thinning out procedure we can assume that $p_\xi = (s \cup s_\xi, g_\xi)$ where $s_\xi \cap F(x,y) = \emptyset$ holds for x, $y \in s$, and the functions $g_\xi|[s]^2$ are identical. Now any two p_ξ-s are compatible. □

If $G \subseteq P$ is a generic set, put $f = \bigcup\{g : (s,g) \in G\}$.

Claim 3. *f has no uncountable free set.*

Proof of Claim. Assume that $p \Vdash X$ is an uncountable free set. There are, for $\xi < \omega_1$, conditions $p_\xi \leq p$ and ordinals α_ξ with $p_\xi \Vdash \alpha_\xi \in X$. Again, we can assume, that $p_\xi = (s \cup s_\xi, g_\xi)$, $\alpha_\xi \in s_\xi$, and the functions $g_\xi \cap [s]^2$ are identical. As F has no infinite free sets ("no uncountable" suffices) there are ordinals ξ_0, ξ_1, $\xi_2 < \omega_1$ such that $\alpha_{\xi_0} \in F\big(\alpha_{\xi_1}, \alpha_{\xi_2}\big)$. We can now extend p to a condition $p' = (s', g')$ where

$$s' = s \cup s_{\xi_0} \cup s_{\xi_1} \cup s_{\xi_2},$$

g' extends g_{ξ_0}, g_{ξ_1}, g_{ξ_2} and $g'\big(\{\alpha_{\xi_1}, \alpha_{\xi_2}\}\big) = \alpha_{\xi_0}$. □

Now Corollary 3 follows from the claims above. □

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