MORE CONSTRUCTIONS FOR BOOLEAN ALGEBRAS SH652

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Abstract. We construct Boolean Algebras with prescribed behaviour concerning depth for the free product of two Boolean Algebras over a third, in ZFC using pcf; assuming squares we get results on ultraproducts. We also deal with the family of cardinalities and topological density of homomorphic images of Boolean Algebras (you can translate it topology - on the cardinalities of closed subspaces); and lastly we deal with inequalities between cardinal invariants, mainly $d(B)^\kappa < |B| \Rightarrow \text{ind}(B) > \kappa \vee \text{ Depth}(B) \geq \log(|B|)$.

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Annotated Content

[We show that if $\square_\lambda$ and $\lambda = \lambda^{\aleph_0}$ then for some Boolean Algebras B_n (for $n < \omega$), we have $\mathrm{Depth}(B_n) \le \lambda$ but for any uniform ultrafilter D on ω, the ultraproduct $\prod_{n<\omega} B_n/D$ has depth $\ge \lambda^+$. The consistency is related to a question on the ideal $I[\mu]$.]

This paper can be translated to compact topologies.
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Notation: For a Boolean Algebra B, we denote by $x \cap y, x \cup y, x - y, -x$ the Boolean operations in B (sometimes B is a Boolean Algebra of sets, sometimes not), and we denote by $0_B, 1_B$ its zero and one, and we denote by $\langle X \rangle_B$ the subalgebra of B generated by $X \subseteq B$. Let B^c be the completion of the Boolean Algebra B.
$B_1 \oplus_A B_2$ is the free product of B_1, B_2 over A (where B_1, B_2 extend A); i.e. without loss of generality $B_1 \cap B_2 = A$ and B is generated by $B_1 \cup B_2$ freely except the equations which hold in B_1 and the equations which hold in B_2.
Let $\mathrm{Ult}(B)$ be the set of ultrafilters on B, and $\mathrm{ult}(B)$ the number of ultrafilters on B. For a fix $B, a \in B, \mathbf{t} \in \{0,1\}$ let $a^{\mathbf{t}}$ be $-a$ if $\mathbf{t} = 0, a$ if $\mathbf{t} = 1$.

§1 On the depth of free products

Monk [M2], Problem 10, 11 ask about the depth of $B \underset{A}{\oplus} C$ (see there for the known results, [M2]), we answer 10 and give some information on 11 here.
We shall define a spectrum $SpDpFP(A)$ for a Boolean Algebra A (in 1.1) and phrase Monk's question with it (1.2(2),(3)). We then phrase a combinatorial statement $\boxtimes_{\lambda,\mu,\theta}$ and prove it gives examples of B, C extending A while $B \underset{A}{\oplus} C$ has depth larger than both (in 1.5), and note that it (provably in ZFC) holds for many cardinals (with $\lambda = \mu$ near a singular) (in 1.7). Later we note some variants of $\boxtimes_{\lambda,\mu,\theta}$ and investigate when the construction in 1.5 works for a Boolean Algebra A, in particular for many infinite Boolean Algebra A, it holds for a class of λ's.

1.1 Definition. 1) For a Boolean Algebra A, we define the spectrum of depth of free products over $A, SpDpFP(A)$ as

$$\big\{(\mu,\kappa) : \text{there are Boolean Algebras } B, C \text{ extending } A \text{ such that :} \\ \text{Depth}(B \underset{A}{\oplus} C) \geq \mu > \kappa \geq \text{ Depth}(B) + \text{ Depth}(A)\big\}.$$

We write κ for (κ^+, κ).
2) Similarly

$$SpDpFP^+(A) = \big\{(\mu,\kappa) : \text{there are Boolean Algebras } B, C \text{ extending } A \text{ such that} \\ \text{Depth}^+(B \underset{A}{\oplus} C) > \mu, \mu \geq \kappa \geq \text{ Depth}^+(B) + \text{ Depth}^+(A)\big\}.$$

We write κ for (κ, κ).

1.2 Remark. 0) Recall that

$$\text{Depth}(B) = \cup\{|X| : X \subseteq B \text{ is well ordered by } <_B\}$$

$$\text{Depth}^+(B) = \cup\{|X|^+ : X \subseteq B \text{ is well ordered by } <_B\}$$

1) Note that

$$\kappa^+ \in SpDpFP^+(A) \text{ iff } \kappa \in SpDpFP(A)$$

so we can deal with 1.1(2) only.
2) So written in our terms, problem 10 of Monk [M2] is:

(∗) for every infinite Boolean Algebra $A, SpDpFP(A)$ is a set of cardinals, i.e. has an upper bound.

3) Written in our terms, problem 11 of Monk is

(∗) for every infinite Boolean Algebra $A, SpDpFP^+(A)$ is non-empty.

4) By 1.5, 1.7 (see $(*)_3$) below, e.g. for some countable Boolean Algebra A, for every strong limit cardinal μ of cofinality $\aleph_0$, we have $\mu^+ \in SpDpFP^+(A)$ (hence $\mu \in SpDPFP(A)$), so Monk's question 10 is answered.
5) The combinatorial property in $\boxtimes_{\lambda,\mu,\theta}$ is close to one considered for investigating the "bad stationary set of a successor of singulars," and more generally the ideal $I[\lambda]$ in [Sh 108], [Sh 88a], [Sh 420, §1].

We then may ask ourselves:
1.3 Question: What occurs to cardinals which are not "near singular" (e.g. $\lambda = \mu = \chi^+, \chi = \chi^{<\chi} > 2^{|A|}$).

1.4 Question: Can you say more on $SpDpFP^+(A)$ when we are given an infinite Boolean Algebra A?

We give some information concerning those problems.

1.5 Claim. *Assume*

$\boxtimes_{\lambda,\mu,\theta}$(a) $\theta < \mathrm{cf}(\mu) \leq \mu \leq \lambda$

(b) $\mathbf{c} : [\lambda]^2 \to \theta$ *satisfies: if* $\zeta_1 < \zeta_2 < \zeta_3 < \lambda$, *then* $\mathbf{c}\{\zeta_1, \zeta_3\} \leq \ Max\{\mathbf{c}\{\zeta_1, \zeta_2\}, \mathbf{c}\{\zeta_2, \zeta_3\}\}$,

(c) *if* $n < \omega, w_\alpha \in [\lambda]^n$ *for* $\alpha < \mu$, *then we can find* α, β *such that* $\alpha < \beta < \mu$ *and letting* $w_\alpha = \{\zeta_\ell : \ell < n\}$ *increasing,* $w_\beta = \{\xi_\ell : \ell < n\}$ *increasing, we have*

(i) $\zeta_\ell = \xi_k \Rightarrow \ell = k$,

(ii) $\mathbf{c}\{\zeta_\ell, \zeta_k\} = \mathbf{c}\{\xi_\ell, \xi_k\}$ *for* $\ell < k < n$ *and*

(iii) *for some* $i < j < \theta$ *satisfying* $i \geq \sup\{\mathbf{c}\{\zeta_\ell, \zeta_k\}, \mathbf{c}\{\xi_\ell, \xi_k\} : \ell, k < n\}$ *we have: for* $\ell, k < n$ *one of the following occurs:*

(α) $\mathbf{c}\{\zeta_\ell, \xi_k\} \geq j \ \ \& \ \ \mathbf{c}\{\zeta_k, \xi_\ell\} \geq j$

(β) $\mathbf{c}\{\zeta_\ell, \xi_k\} = \mathbf{c}\{\zeta_k, \xi_\ell\} < j \ \ \& \ \ [\zeta_\ell < \xi_k \leftrightarrow \xi_\ell < \zeta_k] \ \ \& \ \ \ell \neq k$

(γ) $\zeta_\ell = \xi_k$ *(so* $\ell = k$*)*

(d) $\theta = \ cf(\theta) = \sup(Rang(\mathbf{c}))$
(e) *in clause (c) we can demand* $i \geq i(*)$ *for any pregiven* $i(*) < \theta$.

Then for any $\kappa \in [\theta, \mu)$ *we can find Boolean Algebras* A, B_1, B_2 *such that:*

(α) $|A| = \kappa, A$ *depend just on* θ *and* κ
(β) $|B_1| = |B_2| = \lambda$
(γ) $Depth^+(B_1 \underset{A}{\oplus} B_2) = \lambda^+$ *so* $A \subseteq B_1, A \subseteq B$
(δ) $Depth^+(B_1) \leq \mu$ *and* $Depth^+(B_2) \leq \mu$
$(\delta)^+$ *moreover,* $Length^+(B_i) \leq \mu$ *for* $i = 1, 2$.

1.6 Remark. 1) Let $\boxtimes^-_{\lambda,\mu,\theta}$ just means (a), (b), (c), i.e. omitting clause (d) but weakening (e) to (e)$^-$, which means $i(*) \in$ Rang($\mathbf{c}$). So by 1.9(3) below $\boxtimes^-_{\lambda,\mu,\theta} \Rightarrow \bigvee_{\sigma \leq \theta} \boxtimes_{\lambda,\mu,\sigma}$ and there are obvious monotonicity properties.
2) Let (a)$^-$ mean $\theta < \mu \leq \lambda$. Note that if in the hypothesis of 1.7 below we omit "$\theta <$ cf(μ)", and in the conclusion replace (a) by (a)$^-$ the proof still works.
3) Let $\boxtimes^+_{\lambda,\mu,\theta}$ means that in the definition of $\boxtimes_{\lambda,\mu,\theta}$ in clause (c), when $\mathbf{c}\{\zeta_\ell, \xi_k\} \geq j$ & $\mathbf{c}\{\zeta_k, \xi_\ell\} \geq j$ we add $\mathbf{c}\{\zeta_\ell, \xi_k\} = \mathbf{c}\{\zeta_k, \xi_\ell\}$.
In 1.7 below we can get those versions by working a little more.
4) We could have omitted j in clause (c) of $\boxtimes_{\lambda,\mu,\theta}$, (replacing $\geq j$ by $> i$, so $j = i + 1$), but usually in our proofs we show that j can be arbitrarily larger than i, and in (ii) get $\leq i$ so this formulation makes it clearer.

We quote [Sh:g]

1.7 Observation. The demand on $\theta < \mu \leq \lambda$ is not hard, in fact assuming also $\theta <$ cf(μ) we can find $\mathbf{c}$ such that clauses (a), (b), (c), (d), (e) of $\boxtimes_{\lambda,\mu,\theta}$ hold if the cardinals θ, μ, λ satisfies at least one of the following statements:

$(*)_1$ for some $\chi \in (\theta, \mu)$ we have $\bigwedge_{\alpha < \chi} |\alpha|^{<\theta} < \chi$, cf$(\chi) = \theta$, pp$^+_{J^{bd}_\theta}(\chi) > \lambda$

or

$(*)_2$ $\theta > \aleph_0, \theta =$ cf(χ), $\bigwedge_{\alpha < \chi} |\alpha|^\theta < \chi, \chi < \mu \leq \lambda \leq \chi^\theta$

$(*)_3$ $\lambda = \mathrm{cf}(\lambda) = \mu > \chi > \mathrm{cf}(\chi) = \theta$, $\langle \chi_i : i < \theta \rangle$ is strictly increasing with limit $\chi, i < \theta \Rightarrow \max \mathrm{pcf}\{\chi_j : j < i\} < \chi$ (or just $< \mu$) and $\prod\limits_{i<\theta} \chi_i / J^{bd}_\theta$ has true cofinality $= \lambda$

$(*)_4$ if $\aleph_0 < \mathrm{cf}(\sigma) = \sigma = \mathrm{cf}(\chi) < \chi$, and $(\forall \alpha < \chi)(\mathrm{cf}([\alpha]^{\leq \sigma}, \subseteq) < \chi)$ and $\sigma^{\aleph_0} < \chi$, <u>then</u> for some club C of χ we have:
$\chi_1 \in \{\chi\} \cup C$ & $\lambda = \mathrm{cf}([\chi_1]^{\leq \sigma}, \subseteq)$ & $\mu = \chi_1^+$ & $\theta = \mathrm{cf}(\chi_1) \Rightarrow \boxtimes_{\lambda,\mu,\theta}$

$(*)_5$ like $(*)_3$ replacing $\mathrm{tcf}(\prod\limits_{i<\theta} \chi_i / J^{bd}_\theta) = \lambda$ by: there is a $<_{J^{bd}_\theta}$-increasing sequence $\bar\eta_{\lambda'}$ of length λ' in the product for every regular $\lambda' \in (\chi, \lambda]$ satisfying $|\{\eta_{\lambda',\alpha} \restriction i : \alpha < \lambda', i < \theta\}| \leq \chi$.

Proof. <u>For $(*)_1$</u> If λ is regular by its definition "$pp^+_{J^{bd}_\theta}(\chi) > \lambda$" means that we can find χ_i for $i < \theta$ such that $\chi = \sum\limits_{i<\theta} \chi_i$, with $\theta < \chi_i = \mathrm{cf}(\chi_i) < \chi, \theta = \mathrm{cf}(\chi)$ and $\prod\limits_{i<\theta} \chi_i / J^{\mathrm{bd}}_\theta$ is λ^+-directed; hence in $\prod\limits_{i<\theta} \chi_i$ we can find a $<_{J^{bd}_\theta}$-increasing sequence $\bar\eta = \langle \eta_\alpha : \alpha < \lambda \rangle$; now as $(\forall \alpha < \chi)[|\alpha|^{<\theta} < \chi]$ necessarily $|\{\eta_\alpha \restriction i : \alpha < \lambda\}| \leq \chi$ (really $< \mu$ suffice and $< \chi$ holds). If λ is singular, by combining such examples for regular $\lambda' < \lambda$ we get a sequence as required. Lastly, for $\alpha < \beta$ let $\mathbf{c}\{\alpha, \beta\} = \mathrm{Min}\{i < \theta : (\forall j \in [i, \theta))(\eta_\alpha(i) < \eta_\beta(i)\}$.

Let us check

<u>Clause (a)</u>: By the assumptions on θ, μ, λ clearly $\theta < \chi < \mu \leq \lambda$; note that this is the only place we use the assumption $\theta < \mathrm{cf}(\mu)$.

<u>Clause (b)</u>: Clearly $\mathbf{c}$ is a function from $[\lambda]^2$ to θ. Now suppose $\alpha < \beta < \gamma < \lambda$ and $i = \mathbf{c}\{\alpha, \beta\}$ and $j = \mathbf{c}\{\beta, \gamma\}$ hence: $\max\{i, j\} \leq \zeta < \theta \Rightarrow \eta_\alpha(\zeta) < \eta_\beta(\zeta) < \eta_\gamma(\zeta)$ hence $\mathbf{c}\{\alpha, \gamma\} \leq \max\{i, j\}$ as required.

<u>Clause $(c), (e)$</u>: So suppose $n < \omega$ and $w_\alpha \in [\lambda]^n$ for $\alpha < \mu$. Let $w_\alpha = \{\zeta_{\alpha,\ell} : \ell < n_\alpha\}$ with $\zeta_{\alpha,\ell} < \zeta_{\alpha,\ell+1}$ and $i(*) < \theta$.
Also without loss of generality for some $v \subseteq n$: (just shrink the set for each $\ell < n$)

$\otimes_1$ if $\ell \in v$ then $\langle \zeta_{\alpha,\ell} : \alpha < \chi^+ \rangle$ is strictly increasing with limit ζ^*_ℓ

$\otimes_2$ if $\ell < n, \ell \notin v$ then $\langle \zeta_{\alpha,\ell} : \alpha < \chi^+ \rangle$ is constantly ζ^*_ℓ.

We can also demand

$\otimes_3$ if $\ell \in v, m < n$, <u>then</u> the truth value of $\zeta_{\alpha,\ell} > \zeta^*_m, \zeta_{\alpha,\ell} < \zeta^*_m$ does not depend on $\alpha < \chi^+$

$\otimes_4$ if $\ell \neq m$ are from v and $\zeta^*_\ell = \zeta^*_m$ then $\alpha < \beta < \chi^+ \Rightarrow \zeta_{\alpha,\ell} < \zeta_{\beta,m}$.

For each $\alpha < \chi^+$ let $i_\alpha =$ Max$[\{\mathbf{c}\{\zeta,\xi\} : \zeta \neq \xi \in w_\alpha \cup \{\zeta^*_\ell : \ell < n\} \backslash \{\lambda\}\} \cup \{i(*)\}]$, now clearly $i_\alpha < \theta$ so as $\mathrm{cf}(\chi^+) > \theta$ and $\chi^+ \leq \mu$ (by clause (a)$^-$ or more exactly by our assumptions) without loss of generality $i_\alpha = i^*$ for $\alpha < \chi^+$. As $\langle \eta_{\zeta_{\alpha,\ell}}(i^* + 1) : \ell < n\rangle$ have only $\leq (\chi_{i^*+1})^n < \chi$ possible values, without loss of generality $\alpha < \chi^+ \Rightarrow \eta_{\zeta_{\alpha,\ell}}(i^* + 1) = \gamma_\ell$, moreover $\alpha < \chi^+ \Rightarrow \eta_{\zeta_{\alpha,\ell}} \restriction (i^* + 2) = \nu_\ell$ because $|\{\eta_\alpha \restriction (i^* + 2) : \alpha < \lambda\}| \leq \chi$.

Also without loss of generality

$\otimes_5$ for every $\alpha < \chi^+$ and finite $u \subseteq \theta$, for χ^+ ordinals $\beta < \chi^+$ we have $\bigwedge\limits_{i \in u} \bigwedge\limits_{\ell < n} \eta_{\zeta_{\beta,\ell}}(i) = \eta_{\zeta_{\alpha,\ell}}(i)$.

Note that

$\boxtimes_1$ Assume:

(a) $\alpha, \beta < \chi^+$

(b) $\ell, m < n$

(c) $\zeta_{\alpha,\ell} \neq \zeta_{\beta,m}$

(d) $\zeta_{\alpha,\ell} < \zeta_{\beta,m}$ iff $\zeta_{\beta,\ell} < \zeta_{\alpha,m}$; this holds if $\zeta^*_\ell \neq \zeta^*_m$

(e) $\mathbf{c}\{\zeta_{\alpha,\ell}, \zeta_{\beta,m}\} \leq i^*$

(f) $\mathbf{c}\{\zeta_{\alpha,m}, \zeta_{\beta,\ell}\} \leq i^*$.

<u>Then</u> $\mathbf{c}\{\zeta_{\alpha,\ell}, \zeta_{\beta,m}\} = \mathbf{c}\{\zeta_{\alpha,m}, \zeta_{\beta,\ell}\}$.

Let $\alpha_1 < \alpha_2 < \chi^+$, so we can find $j^* \in (i^* + 2, \theta)$ such that $\{\ell, m\} \subseteq v$ & $\zeta^*_\ell = \zeta^*_m \Rightarrow \eta_{\zeta_{\alpha_1,\ell}}(j^*) < \eta_{\zeta_{\alpha_2,m}}(j^*)$; in fact any large enough $j^* < \theta$ will do. Choose $\alpha_3 \in (\alpha_2, \chi^+)$ such that $\bigwedge\limits_\ell \eta_{\zeta_{\alpha_3,\ell}}(j^*) = \eta_{\zeta_{\alpha_1,\ell}}(j^*)$. Now let $\alpha =: \alpha_2, \beta =: \alpha_3, i = i^*, j = j^*$, they are as required.
Why?

$\otimes_6$ assume $\ell \neq m$ are $< n$ and $\zeta^*_\ell \neq \zeta^*_m$ then we have: $\mathbf{c}\{\zeta_{\alpha,\ell}, \zeta_{\beta,m}\} = \mathbf{c}\{\zeta_{\alpha,m}, \zeta_{\beta,\ell}\}$ is determined by ν_ℓ, ν_m and is $\leq i^* = i$. Also $[\zeta_{\alpha,\ell} < \zeta_{\beta,m} \leftrightarrow \zeta_{\beta,\ell} < \zeta_{\alpha,m}]$.
[Why? E.g. if $\zeta^*_\ell < \zeta^*_m$ then for $\gamma \in [i^*, \theta)$ we have $\eta_{\zeta_{\alpha,\ell}}(\gamma) \leq \eta_{\zeta^*_\ell}(\gamma) <$

$\eta_{\zeta_{\beta,m}}(i)$, so $\mathbf{c}\{\zeta_{\alpha,\ell},\zeta_{\beta,m}\} \le i^*$, noticing $i_\alpha = i_\beta \le i^*$, (and the choice of i_α and the choice of ν_ℓ, ν_m above). Similarly $\mathbf{c}\{\zeta_{\alpha,m},\zeta_{\beta,\ell}\} \le i^*$. Now use $\boxtimes_1$.]

$\otimes_7$ if $\zeta_{\alpha,\ell} \ne \zeta_{\beta,\ell}$ (i.e. $\ell \in v$) then $\mathbf{c}\{\zeta_{\alpha,\ell},\zeta_{\beta,\ell}\} > j^*$
[why? by the choice of j^* we have $\eta_{\beta,\ell}(j^*) = \eta_{\alpha_3,\ell}(j^*) = \eta_{\alpha_1,\ell}(j^*) < \eta_{\alpha_2,\ell}(j^*) = \eta_{\alpha,\ell}(j^*)$ and by the choice of β we have $\alpha < \beta$ hence $\zeta_{\alpha,\ell} < \zeta_{\beta,\ell}$, now use the definition of $\mathbf{c}$.]

$\otimes_8$ if $\ell \ne m < n, \zeta^*_\ell = \zeta^*_n$ and $\{\ell,m\} \subseteq v$ then $\mathbf{c}\{\zeta_{\alpha,\ell},\zeta_{\beta,m}\} > j^* = j$
[why? clearly $\zeta_{\alpha,\ell} < \zeta_{\beta,m}$ as $\alpha < \beta$ (see $\otimes_4$) and
$\eta_{\zeta_{\beta,m}}(j^*) = \eta_{\zeta_{\alpha_3,m}}(j^*) = \eta_{\zeta_{\alpha_1,m}}(j^*) < \eta_{\zeta_{\alpha_2,\ell}}(j^*) = \eta_{\zeta_{\alpha,\ell}}(j^*)$.]

$\otimes_9$ if $\ell \ne m < n, \zeta^*_\ell = \zeta^*_m$ and $\{\ell,m\} \nsubseteq v$, then $\mathbf{c}\{\zeta_{\alpha,\ell},\zeta_{\beta,m}\} \le i^*$. Also $[\zeta_{\alpha,\ell} < \zeta_{\beta,m} \leftrightarrow \zeta_{\beta,\ell} < \zeta_{\alpha,m}]$.
[why? clearly $\{\ell,m\} \cap v = \emptyset$ is impossible (as $\zeta_{\alpha,\ell} \ne \zeta_{\alpha,m}$ for $\alpha < \chi^+, \ell \ne m$), and so $|\{\ell,m\} \cap v| = 1$; now the proof is similar to that of $\otimes_6$.]

Together we are done.

For $(*)_2$:
By [Sh:g, Ch.II,5.4(2),Ch.VIII,§1] the assumptions in $(*)_1$ holds.

For $(*)_3$:
By [Sh:g, Ch.II,3.5] we can choose $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ satisfying $\eta_\alpha \in {}^\theta\chi$ such that $\bar\eta$ is $<_{J^{bd}_\theta}$-increasing, and $|\{\eta_\alpha \restriction i : \alpha < \lambda\}| \le \chi$ for $i < \theta$. Now continue as in the proof of $(*)_1$.

For $(*)_4$:
See [Sh:E12, part(C),17.3].

For $(*)_5$:
Straightforward; but note that λ is regular (for λ singular we only get $|\{\eta_\alpha \restriction i : \alpha < \lambda, i < \theta\}| \le \chi +$ cf(λ) which should be $< \mu$).

Proof of 1.5.

Stage A: Let A be the Boolean Algebra generated by $\{a^t_i : i < \kappa$ and $t \in \{1,2\}\}$ freely except the equations:

$(*)_1$ $a^t_i \le a^t_j$ for $i \le j < \theta$ and $t \in \{1,2\}$

$(*)_2$ $a^1_i \cap a^2_j = 0$ for $i,j < \theta$.

Let I^t be the ideal of A generated by $\{a^t_i : i < \theta\}$ for $t = 1, 2$. Let I be the ideal of A generated by $I^1 \cup I^2$ and I' the ideal generated by $\{a^t_i : i < \kappa, t \in \{1, 2\}\}$ so (if $\kappa = \theta$ then $I = I'$) A/I' is the trivial (= two elements) Boolean Algebra.
Clearly

$$a^{t(1)}_{i(1)} = a^{t(2)}_{i(2)} \Leftrightarrow (t(1), i(1)) = (t(2), i(2))$$

$$t \in \{1, 2\} \ \& \ i < j < \theta \Rightarrow A \models a^t_i < a^t_j$$

[Why? Well, you may check it using $(*)$ of stage D below; for any t, i let $h_{t,i} : \{a^1_j, a^2_j : j < \kappa\} \to \{0, 1\}$ be defined by $h_{t,i}(a^s_j)$ is 1 if $s = t \ \& \ i \le j$ and $h_{t,i}(a^s_j)$ is zero otherwise. Interpreting $\{0, 1\}$ as the trivial Boolean Algebra, $h_{t,i}$ respect the equations is $(*)_1 + (*)_2$ hence extend to a homomorphism $\hat{h}_{t,i}$ from A into $\{0, 1\}$. Now assume $t(1), t(2) \in \{1, 2\}$ and $i(1), i(2) < \kappa$; if $t(1) \ne t(2) \vee i(1) < i(2)$ then $\hat{h}_{t(2),i(2)}$ maps $a_{t(2),i(2)}$ to 1 and $a_{t(1),i(1)}$ to 0 hence $A \models a^{t(1)}_{i(2)} \ne a^{t(2)}_{i(2)}$. Also if $t \in \{1, 2\}$ and $i < j < \theta$ then $a^t_i \le a^t_j$ by $(*)_1$ and $h_{t,j}(a^t_j) = 1, h_{t,j}(a^t_i) = 0$ hence $A \models a^t_i < a^t_j$.]
For $t = 1, 2$ let B_t be the extension of A by $\{x^t_\alpha : \alpha < \lambda\}$ freely except that:

$(*)_3$ $x^t_\alpha - x^t_\beta \le a^t_{\mathbf{c}\{\alpha,\beta\}}$ for $\alpha < \beta < \lambda$

$(*)_4$ $x^t_\alpha \cap a^{3-t}_i = 0$ for $\alpha < \lambda, i < \theta$.

Clearly B_t extends A.
[Why? Well use $(*)$ of stage D, define a function h from $A \cup \{x^t_\alpha : \alpha < \lambda\}$ to A by $h(x) = x$ for $x \in A$ and $h(x^t_\alpha) = 0$. Clearly h respects the equations in the Boolean Algebra A and the equations is $(*)_3 + (*)_4$ hence the homomorphism $\hat{h}$ from B onto $\{0, 1\}$ is well defined and it extends id_A.]

Let $B = B_1 \bigoplus_A B_2$ and let $x_\alpha = x^1_\alpha \cap x^2_\alpha \in B$. Let J^1, J^2, J be the ideals of B which I^1, I^2, I generates (resp.), so $J^1 \cup J^2 \subseteq J$.
Clearly $|B| \le \lambda$.

<u>Stage B</u>: $\mathrm{Depth}^+(B) = \lambda^+$.

Clearly $\mathrm{Depth}^+(B) \le |B|^+ \le \lambda^+$. Hence it suffices to prove that $\langle x_\alpha : \alpha < \lambda \rangle$ is strictly increasing. So let $\alpha < \beta$. First we prove $x_\alpha \le x_\beta$. Let D be an ultrafilter on B (so $D_t = D \cap B_t \in \mathrm{Ult}(B_t)$ and $D_1 \cap A = D_2 \cap A$), and we shall prove $x_\alpha \in D \Rightarrow x_\beta \in D$. This suffices for proving $x_\alpha \le x_\beta$ (as D was any ultrafilter).

<u>Case 1</u>: $D \cap A$ is disjoint to I.

Now modulo J^t in $B^t, x^t_\alpha \le x^t_\beta$ (by $(*)_3$ + "$a^t_i \in I$") hence modulo J, $x^t_\alpha \le x^t_\beta$ hence mod $J, x_\alpha \le x_\beta$ hence $x_\alpha \in D \Rightarrow x_\beta \in D$.

Case 2: $D \cap A$ is not disjoint to I.

So $D \cap I \ne \emptyset$ hence for some $t \in \{1,2\}$ and $i < \theta, a^t_i \in D$, but x^{3-t}_α is disjoint to a^t_i by $(*)_4$, so $x^{3-t}_\alpha \notin D$ hence $x_\alpha = x^1_\alpha \cap x^2_\alpha \notin D$ so trivially $x_\alpha \in D \Rightarrow x_\beta \in D$.

We still have to prove $x_\alpha \ne x_\beta$. Let D^t_β be the ultrafilter on B^t generated by $x^t_\gamma\,(\gamma \ge \beta), -x^t_\gamma\,(\gamma < \beta)$ and the members of $\{-x : x \in I'\}$ (check that it is okay trivially). Now $D^1_\beta \cap A = A \backslash I' = D^2_\beta \cap A$ (check, trivial). So there is an ultrafilter D on B satisfying $D \cap B_t = D^t_\beta$. So $-x^t_\alpha, x^t_\beta \in D^t_\beta$ for $t = 1, 2$ so $-x_\alpha, x_\beta \in D$ as required.

Stage C: $\text{Length}^+(B_t) \le \mu$.

Assume not, so we can find $\langle c_\alpha : \alpha < \mu \rangle$ a chain (so with no repetition). Let $c_\alpha = \tau_\alpha(a^{s_{\alpha,1}}_{i_{\alpha,1}}, \ldots, a^{s_{\alpha,k_\alpha}}_{i_{\alpha,k_\alpha}}, x^t_{\zeta_{\alpha,1}}, \ldots, x^t_{\zeta_{\alpha,n_\alpha}})$ where $i_{\alpha,1} < \ldots < i_{\alpha,k_\alpha} < \kappa$ and $\zeta_{\alpha,1} < \ldots < \zeta_{\alpha,n_\alpha} < \lambda$ and $s_{\alpha,1}, \ldots, s_{\alpha,k_\alpha} \in \{1,2\}$ and τ_α a Boolean term.

As $\text{cf}(\mu) > \theta \ge \aleph_0$, without loss of generality:

$(*)_5$ $k_\alpha = k^*, n_\alpha = n^*, \tau_\alpha = \tau^*$, $s_{\alpha,\ell} = s_\ell$ (and if $|A| <$ cf(μ) then $i_{\alpha,\ell} = i_\ell$).

$(*)_6$ for some $m^* \le k^*$ we have:

- (i) $0 < \ell \le m^* \Rightarrow i_{\alpha,\ell} = i_\ell < \theta$
- (ii) $\ell \in (m^*, k^*] \Rightarrow i_{\alpha,\ell} \ge \theta$
- (iii) for $\alpha < \beta < \mu$, for some $v = v_{\alpha,\beta} \subseteq (m^*, k^*]$ we have
  - (a) $\ell \in v \Rightarrow i_{\alpha,\ell} = i_{\beta,\ell}$ and
  - (b) $\ell \in (m^*, k^*]$ & $\ell \notin v$ implies that $i_{\alpha,\ell} \ne i_{\beta,\ell}$ and both are not in $\{i_{\alpha,k}, i_{\beta,k} : k \ne \ell\}$.

[why $(*)_6$? we can assume clauses (i) + (ii) by the pigeon-hole principle. For clause (iii) if μ is regular, by the $\triangle$-system lemma (so then $v_{\alpha,\beta} = v$) and if singular, apply it twice. More elaborately for singular μ as $\mu > \kappa$, let $\mu = \sum\limits_{\gamma < \text{cf}(\mu)} \mu_\gamma$, where $\gamma_1 < \gamma_2 \to |A| < \mu_{\gamma_1} < \mu_{\gamma_2} < \mu$. For each $\gamma <$ cf(μ) we can find $X'_\gamma \subseteq [\mu_\gamma, \mu^+_\gamma)$ of cardinality μ^+_γ such that $\ell \in [m^*, k^*) \Rightarrow i_{\alpha,\ell} = i^*_{\gamma,\ell}$. Then apply the Δ-system lemma on $\langle < i^*_{\gamma,\ell} : \ell \in [k^*, m^*) >: \gamma <$ cf$(\mu) \rangle$, O.K. as cf(μ) is regular.]

Let $i(*) = \sup\{i_\ell : \ell < m^*\}$ so $i(*) < \theta$.

Let $w_\alpha = \{\zeta_{\alpha,1}, \ldots, \zeta_{\alpha,n^*}\}$. Let $\alpha \neq \beta < \mu$ and $i < j < \theta$ be as guaranteed by clause (c) of the assumption (of 1.5) and $i > i(*)$ (see clause (e)). So c_α, c_β are distinct members of a chain of B_t.
Now read Stage D below.

So let $w =: \{\zeta_{\alpha,\ell}, \zeta_{\beta,\ell} : \ell = 1, \ldots, n^*\} \in [\lambda]^{<\aleph_0}$.
As in Stage D, $B_{t,w}$ is the subalgebra of B_t generated by $A \cup \{x^t_{\beta_\zeta} : \zeta \in w\}$ and clearly $c_\alpha, c_\beta \in B_{t,w}$, hence also in $B_{t,w}, c_\alpha, c_\beta$ are distinct members of a chain.

By symmetry assume $B_{t,w} \models$ "$c_\alpha < c_\beta$", hence there is a homomorphism f from $B_{t,w}$ to (the trivial Boolean algebra) $\{0,1\}$ such that $f(c_\alpha) = 0, f(c_\beta) = 1$ and by Stage D we can extend f to a homomorphism from B_t to $\{0,1\}$; we denote it, too, by f. Let $\gamma(f) = \text{ Min}\{\gamma \leq \theta : \text{if } \gamma < \theta \text{ then } f(a^t_\gamma) = 1\}$.
As in the Boolean Algebra $A \subseteq B^t, \langle a^t_\gamma : \gamma < \theta\rangle$ is increasing, clearly

$(*)_7$ for $\xi < \theta, f(a^t_\xi) = 1 \Leftrightarrow \xi \geq \gamma(f)$

$(*)_8$ for $\xi < \theta, f(a^{3-t}_\xi) = 0$.

[Why does $(*)_8$ hold? Assume that not, so, by $(*)_3$, we have $f(x^t_\gamma) = 0$ for $\gamma < \lambda$. By $(*)_6(iii)(b)$, if $k, \ell \in (m^*, k^*]\backslash v$ then $i_{\alpha,\ell} \neq i_{\beta,k}$. Hence by the definition of A and $(*)$ of Stage D there is an endomorphism g of A such that

$$g(a^s_\varepsilon) = a^s_{i_{\beta,\ell}} \text{ if } s_\ell = s \text{ and } \varepsilon = i_{\alpha,\ell} \text{ for some } \ell \in (m^*, k^*]\backslash v,$$

$$g(a^s_\varepsilon) = a^s_{i_{\alpha,\ell}} \text{ if } s_\ell = s \text{ and } \varepsilon = i_{\beta,\ell} \text{ for some } \ell \in (m^*, k^*]\backslash v,$$

$$g(a^s_\varepsilon) = a^s_\varepsilon \text{ otherwise.}$$

So $s \in \{1,2\}$ & $\varepsilon < \theta \Rightarrow g(a^s_\varepsilon) = a^s_\varepsilon$ and g permutes the $\{a^s_\varepsilon : \varepsilon < \kappa, s \in \{1,2\}\}$. By the form of $(*)_3 + (*)_4$ and the fact that $\mathbf{c}\{\alpha,\beta\} < \theta$, there is a homomorphism g' from B_t into B_t extending g and taking each x^t_γ to x^t_γ for $\gamma < \lambda$. Clearly then $g'(a^{s_\ell}_{i_{\alpha,\ell}}) = a^{s_\ell}_{i_{\beta,\ell}}$ for all $\ell = 1, \ldots, k^*$. Similarly, $g'(a^{s_\ell}_{i_{\beta,\ell}}) = a^{s_\ell}_{i_{\alpha,\ell}}$ for all $\ell = 1, \ldots, k^*$. Hence

$$f(g'(c_\alpha)) = \tau(f(a^{s_1}_{i_{\beta,1}}), \ldots, f(a^{s_{k^*}}_{i_{\beta,k^*}}), 0, 0, \ldots, 0) = f(c_\beta) = 1,$$

and similarly $f(g'(c_\beta)) = 0$, contradicting $c_\alpha < c_\beta$. So $(*)_8$ holds.]
Now

$(*)_9$ the function $g : \{a^s_\varepsilon : \varepsilon < \kappa, s \in \{1,2\}\} \rightarrow \{0,1\}$ induce a homomorphism $\hat{g}$ from A to $\{0,1\}$ where g is defined by (recall i,j were chosen together with α,β):

(i) $g(a^t_\xi) = f(a^t_\xi)$ for $\xi < j$

(ii) $g(a^t_\xi) = 1$ if $\xi \geq j, \xi < \theta$

(iii) $g(a^{3-t}_\xi) = f(a^{3-t}_\xi) = 0$ for $\xi < \theta$

(iv) $g(a^{s_\ell}_{i_{\alpha,\ell}}) = f(a^{s_\ell}_{i_{\beta,\ell}}), g(a^{s_\ell}_{i_{\beta,\ell}}) = f(a^{s_\ell}_{i_{\alpha,\ell}})$ for $\ell = 1,\dots,k^*$

(v) $g(a^s_\varepsilon) = f(a^s_\varepsilon)$ if $a^s_\varepsilon \notin \{a^{s,\ell}_{i_{\alpha,\ell}}, a^{s_\ell}_{i_{\beta,\ell}} : \ell = 1,\dots,k^*\}$ and $\varepsilon \in [\theta,\kappa)$

[why? now g is well defined as, e.g. for contradiction concerns (iv), two instances do not contradict by $(*)_6(iii)$ and they do not contradict others by $(*)_6(i) + (ii)$. By $(*)$ of Stage D below we should check the equations appearing in the definition of A. For those in $(*)_1$, i.e. $a^s_\varepsilon \leq a^s_\xi$ for $\varepsilon < \xi < \theta$, if $s = 3-t$ this is trivial by clause (iii), if $s = t, \theta > \xi \geq j$, this is trivial by clause (ii) and if $s = t, \xi < j$, then $g(a^s_\varepsilon) = f(a^s_\varepsilon) \leq f(a^s_\xi) = g(a^s_\xi)$.

As for the equations in $(*)_2$ that is $a^1_\varepsilon \cap a^2_\xi = 0$ for $\varepsilon,\xi < \theta$ they are preserved trivially by $(*)_7$ and clause (iii).]

Define a function h from $A \cup \{x^t_{\zeta_{\alpha,\ell}}, x^t_{\zeta_{\beta,\ell}} : \ell = 1,\dots,n^*\}$ to $\{0,1\}$ as follows: $h \restriction A$ is the homomorphism $\hat{g}$ to $\{0,1\}$ and define $h(x^t_{\zeta_{\alpha,\ell}}) = f(x^t_{\zeta_{\beta,\ell}})$ and $h(x^t_{\zeta_{\beta,\ell}}) = f(x^t_{\zeta_{\alpha,\ell}})$. Now

$(*)_{10}$ h induces a homomorphism $\hat{h}$ from $B_{t,w}$ to $\{0,1\}$
[why? by clause (c)(i) of $\boxtimes_{\lambda,\mu,\theta}$, i.e. the choice of α,β the function h is well defined; we use $(*)$ of Stage D; now

(a) the equations in A are respected.
By the choice of $h \restriction A$ by $\hat{g}$ being a homomorphism from A to $\{0,1\}$

(b) the equations $x^t_{\zeta_{\gamma,\ell}} \cap a^{3-t}_i = 0$ for $\gamma \in \{\alpha,\beta\}, i < \theta$.
This is respected as $h(a^{3-t}_i) = g(a^{3-t}_i) = 0$.

(c) for $\gamma \in \{\alpha,\beta\}$ the equation $x^t_{\zeta_{\gamma,\ell}} - x^t_{\zeta_{\gamma,m}} \leq a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\gamma,m}\}}$ for $\ell < m$.
Let δ be such that $\{\gamma,\delta\} = \{\alpha,\beta\}$ and remember that $i \geq \mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\gamma,m}\} = \mathbf{c}\{\zeta_{\delta,\ell},\zeta_{\delta,m}\}$ (by the choice of α,β,i,j see $\boxtimes_{\lambda,\mu,\theta}(c)(ii)$ of 1.5) so $h(a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\gamma,m}\}}) = f(a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\gamma,m}\}}) = f(a^t_{\mathbf{c}\{\zeta_{\delta,\ell},\zeta_{\gamma,m}\}})$ by the choice of $h \restriction A$ and of g i.e. $(*)_9(i)$, hence

$$\begin{aligned} h(x^t_{\zeta_{\gamma,\ell}}) - h(x^t_{\zeta_{\gamma,m}}) &= f(x^t_{\zeta_{\delta,\ell}}) - f(x^t_{\zeta_{\delta,m}}) \\ &= f(x^t_{\zeta_{\delta,\ell}} - x^t_{\zeta_{\delta,m}}) \le f(a^t_{\mathbf{c}\{\zeta_{\delta,\ell},\zeta_{\delta,m}\}}) \\ &= h(a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\gamma,m}\}}) \end{aligned}$$

as required.

(d) If $\ell, m \in \{1, \dots, m^*\}$ and $\gamma \neq \delta$ are from $\{\alpha, \beta\}$ and $\zeta_{\gamma,\ell} < \zeta_{\delta,m}$ the equation $x^t_{\zeta_{\gamma,\ell}} - x^t_{\zeta_{\delta,m}} \le a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\delta,m}\}}$.
Now we have to look at clause (c) of $\boxtimes_{\lambda,\mu,\theta}$ of 1.5, by it there are two possibilities

<u>possibility 1</u>: $\mathbf{c}\{\zeta_{\gamma,\ell}, \zeta_{\delta,m}\} \ge j$ (but necessarily $< \theta$), then $g(a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\delta,m}\}}) = 1$ by clause (ii) of $(*)_9$ so the equation is respected.

<u>possibility 2</u>: $\mathbf{c}\{\zeta_{\gamma,\ell}, \zeta_{\delta,m}\} = \mathbf{c}\{\zeta_{\gamma,\ell}, \zeta_{\gamma,m}\} < j$ hence
$[\zeta_{\gamma,\ell} < \zeta_{\delta,m} \Leftrightarrow \zeta_{\delta,\ell} < \zeta_{\gamma,m}]$
(read (c) of $\boxtimes_{\lambda,\mu,\theta}$ so both holds).

So $h(x^t_{\zeta_{\gamma,\ell}}) - h(x^t_{\zeta_{\delta,m}}) = f(x^t_{\zeta_{\delta,\ell}}) - f(x^t_{\zeta_{\gamma,m}}) = f(x^t_{\zeta_{\delta,\ell}} - x^t_{\zeta_{\gamma,m}})$
$\le f(a^t_{\mathbf{c}\{\zeta_{\delta,\ell},\zeta_{\gamma,m}\}}) = f(a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\delta,m}\}}) = g(a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\delta,m}\}}) =$
$h(a^t_{\mathbf{c}\{\zeta_{\gamma,\ell},\zeta_{\delta,m}\}})$.
So we have proved $(*)_{10}$.]

Now we have two homomorphisms $f, \hat{h}$ from $B_{t,w}$ to $\{0, 1\}$ and they satisfy:

$(*)_{11}(a)$ $f(a^{s_\ell}_{i_\alpha,\ell}) = \hat{h}(a^{s_\ell}_{i_\beta,\ell})$

(b) $f(a^{s_\ell}_{i_\beta,\ell}) = \hat{h}(a^{s_\ell}_{i_\alpha,\ell})$

(c) $f(x^t_{\zeta_{\alpha,\ell}}) = \hat{h}(x^t_{\zeta_{\beta,\ell}})$

(d) $f(x^t_{\zeta_{\beta,\ell}}) = \hat{h}(x^t_{\zeta_{\alpha,\ell}})$

[why? for (a) + (b) note that $\hat{h} \restriction A = g \restriction A$ and $(*)_9(iv)$ and for (c) + (d) just see the choice of h.]

So clearly $f(c_\alpha) = \hat{h}(c_\beta), f(c_\beta) = \hat{h}(c_\alpha)$, but $f(c_\alpha) = 0 < f(c_\beta) = 1$ so $\hat{h}(c_\beta) = 0 < \hat{h}(c_\alpha) = 1$ whereas we assume $B_{w,t} \models c_\alpha < c_\beta$, contradiction.

So we have finished the proof of Stage C, hence of 1.5 except a debt: Stage D.

Stage D: First recall

(∗) if a Boolean algebra B is defined by: generated by $\{x_i : i < i^*\}$ freely except the set of equations Γ, and B' is another Boolean Algebra and the function $h : \{x_i : i < i^*\} \to B'$ respect the equations in Γ
(i.e. if $\tau'(x_{i_1}, \dots) = \tau''(x_{j_1}, \dots) \in \Gamma$ then
$B' \models \tau'(h(x_{i_1}), \dots) = \tau''(h(x_{j_1}), \dots))$
then h can be extended to a homomorphism from B to B' (and we call it $\hat{h}$); similarly for "extensions of a Boolean Algebra A".

For $w \subseteq \lambda$ let $B_{t,w}$ be defined just like B_t restricting ourselves to $\alpha \in w$ (so also in the set of equations we consider only the equations such that: x_α appears in the equation $\Rightarrow \alpha \in w$).
A priori it is not guaranteed that $w \subseteq u \subseteq \lambda \Rightarrow B_{t,w} \subseteq B_{t,u}$. Note $B_t = B_{t,\lambda}$.

Fact. For $w \subseteq u \subseteq \lambda, B_{t,w} \subseteq B_{t,u}$ and $B_{t,u}$ is the direct limit of $\{B_{t,w} : w \subseteq u \text{ finite}\}$.

Proof. It is enough to prove this for finite u, so we can ignore the second phrase as it follows. The first phrase we prove by induction on $|u \backslash w|$, so without loss of generality $|u \backslash w| = 1$, let $\zeta \in u \backslash w$.
We define a function $h : A \cup \{x^t_i : i \in u\} \to B_{t,w}$, by

$$h \restriction A = \text{ identity}$$

$$h(x^t_i) = x^t_i \text{ if } i \in w$$

$$h(x^t_\zeta) = \cup\{x^t_\xi - a^t_{\mathbf{c}\{\zeta,\xi\}} : \xi \in w \cap \zeta\}.$$

Now h is as in (∗) (see beginning of stage D, checked below) so there is a homomorphism from $B_{t,u}$ to $B_{t,w}$ which obviously extends the identity so we are done.
Why really is h as required in (∗)? We have to check the "new" equations, i.e. the ones appearing in the definition of $B_{t,u}$ and not in the definition of $B_{t,w}$ (which are satisfied as $h \restriction B_{t,w}$ is the identity):

(*i*) $x^t_\zeta \cap a^{3-t}_j = 0$
[why? obvious by the choice of $h(x^t_\zeta)$ as $h(x^t_\zeta) \cap h(a^{3-t}_j) = h(x^t_\zeta) \cap a^{3-t}_j \le (\bigcup\limits_{\xi \in w \cap \zeta} h(x^t_\xi)) \cap a^{3-t}_j = \bigcup\limits_{\xi \in w \cap \zeta} (h(x^t_\xi) \cap a^{3-t}_j) = \bigcup\limits_{\xi \in w \cap \zeta} 0 = 0$ so (i) holds]

(*ii*) if $\varepsilon < \zeta, \varepsilon \in w$, then the equation $x^t_\varepsilon - x^t_\zeta \le a^t_{\mathbf{c}\{\varepsilon,\zeta\}}$
[why? by the choice of $h(x^t_\zeta)$ clearly $h(x^t_\varepsilon) - h(x^t_\zeta) =$
$x^t_\varepsilon - h(x^t_\zeta) \le x^t_\varepsilon - (x^t_\varepsilon - a^t_{\mathbf{c}\{\zeta,\varepsilon\}}) \le a^t_{\mathbf{c}\{\varepsilon,\zeta\}} = h(a^t_{\mathbf{c}\{\varepsilon,\zeta\}})$ so (ii) holds]

(*iii*) if $\varepsilon > \zeta, \varepsilon \in w$, then the equation $x^t_\zeta - x^t_\varepsilon \le a^t_{\mathbf{c}\{\varepsilon,\zeta\}}$
[why? the meaning of the demand is that we should check

$$h(x^t_\zeta) - h(x^t_\varepsilon) \le h(a^t_{\mathbf{c}\{\varepsilon,\zeta\}})$$

that is

$$h(x^t_\zeta) - x^t_\varepsilon \le a^t_{\mathbf{c}\{\varepsilon,\zeta\}}$$

that is

$$\xi \in w \cap \zeta \Rightarrow (x^t_\xi - a^t_{\mathbf{c}\{\zeta,\xi\}}) - x^t_\varepsilon \le a^t_{\mathbf{c}\{\varepsilon,\zeta\}}$$

for this it suffices to show

$$\xi \in w \cap \zeta \Rightarrow B_{t,w} \models x^t_\xi - x^t_\varepsilon \le a^t_{\mathbf{c}\{\xi,\zeta\}} \cup a^t_{\mathbf{c}\{\zeta,\varepsilon\}}$$

but we know

$$\xi' < \zeta' \ \ \& \ \ \xi' \in w \ \ \& \ \ \zeta' \in w \Rightarrow B_{t,w} \models x^t_{\xi'} - x^t_{\zeta'} \le a^t_{\mathbf{c}\{\xi',\zeta'\}}$$

and apply this to $(\xi', \zeta') = (\xi, \varepsilon)$; so $B_{t,w} \models x^t_\xi - x^t_\varepsilon \le a^t_{\mathbf{c}\{\xi,\varepsilon\}}$. Now as $\langle a^t_i : i < \theta\rangle$ is increasing and as by clause (b) of $\boxtimes_{\lambda,\mu,\theta}$ we have

$$\mathbf{c}\{\xi,\varepsilon\} \le \ \mathrm{Max}\{\mathbf{c}\{\xi,\zeta\}, \mathbf{c}\{\zeta,\varepsilon\}\}$$

we get $B_{t,w} \models a^t_{\mathbf{c}\{\xi,\zeta\}} \cup a^t_{\mathbf{c}\{\zeta,\varepsilon\}} \le a^t_{\mathbf{c}\{\xi,\varepsilon\}}$, so together we are done.] $\square_{1.5}$

1.8 Observation. *1) Assume*

(∗) *$\lambda = \mu$ is weakly compact $> \kappa$.*

If A is a Boolean Algebra, $A \subseteq B_1, A \subseteq B_2, B = B_1 \oplus_A B_1, |A| \le \kappa$ and:

Depth$^+(B) > \lambda$ or just Length$^+(B) > \lambda$, <u>then</u> $\bigvee_{t=1}^{2}$ Depth$^+(B_t) > \mu$.

2) Similarly if $\lambda \to (\mu)^2_\kappa$ and cf$(\lambda) > 2^\kappa$.
3) We can above replace $\lambda =$ cf$(\lambda) > 2^\kappa$ by $\lambda > \kappa$.

Remark. Clearly part (2) generalizes part (1).

Proof. 1), 2) We can find pairwise distinct $c_\alpha \in B$ (non-zero) for $\alpha < \lambda$ such that $\alpha < \beta < \lambda \Rightarrow c_\alpha, c_\beta$ are comparable in B.

For each α we can find $b^t_{\alpha,\ell} \in B_t$ ($\ell < n_\alpha, t = 1, 2$) such that $c_\alpha = \bigcup\limits_{\ell=0}^{n_\alpha - 1} (b^1_{\alpha,\ell} \cap b^2_{\alpha,\ell})$.

Without loss of generality $\langle b^2_{\alpha,\ell} : \ell < n_\alpha \rangle$ are pairwise disjoint (in B_2). Without loss of generality $n_\alpha = n(*)$.

For each $\alpha < \lambda$ there is an ultrafilter D_α of B such that

$$c_\alpha \in D_\alpha, \bigwedge_{\beta < \lambda} [c_\beta <_B c_\alpha \Rightarrow c_\beta \notin D_\alpha]$$

(remember: $\{c_\alpha : \alpha < \lambda\}$ is a chain in B).

As cf$(\lambda) = \lambda > 2^\kappa \geq 2^{|A|}$, without loss of generality $\bigwedge\limits_{\alpha < \lambda} [D_\alpha \cap A = D^*]$. Let $I^* = \{a \in A : 1_A - a \in D^*\}$, it is a maximal ideal of A. Let I_t be the ideal which I^* generates in B_t and I be the ideal which I^* generates in B. So easily $\langle c_\alpha / I : \alpha < \lambda \rangle$ is a chain with no repetition. Now easily $B/I = (B_1/I_1) \oplus (B_2/I_2)$; as in B/I there is a chain of cardinality λ, by Monk McKenzie [McMo82] this holds in B_1/I_1 or in B_2/I_2, so without loss of generality in B_1/I_1 say it is $\langle b_\alpha : \alpha < \lambda \rangle$. Now for some $a_{\{\alpha,\beta\}} \in I^*$, if $b_\alpha / I_1 < b_\beta / I_1$ then

$$B_1 \models b_\alpha - a_{\{\alpha,\beta\}} < b_\beta - a_{\{\alpha,\beta\}}.$$

So it is enough to find $X \in [\lambda]^\mu$ and $a \in A$ and truth value $\mathbf{t}$ such that $a_{\{\alpha,\beta\}} = a$ & $[B \models c_\alpha < c_\beta \Leftrightarrow \mathbf{t} =$ truth$]$ for $\alpha < \beta \in X$ which we can. So $\langle b_\alpha - a : \alpha \in X \rangle$ or $\langle 1 - (b_\alpha - a) : \alpha \in X \rangle$ is a strictly increasing sequence of order type λ. (Of course, for the version with depth not length, $\mathbf{t}$ is redundant). In more details for each $\alpha < \beta$ choose $a = a_{\alpha,\beta} \in A \backslash \{0\}$ such that $b_\alpha \cap a, b_\beta \cap a$ are distinct but comparable and $\mathbf{t}_{\alpha,\beta} \in \{0, 1\}$ be such that $(b_\alpha < b_\alpha) \equiv (\mathbf{t}_{\alpha,\beta} = 1)$.
Define a colouring $\mathbf{c} : [\lambda]^2 \to A \times 2$ by: if $\alpha \neq \beta < \lambda$ let $\mathbf{c}\{\alpha, \beta\} = (a_{\{\alpha,\beta\}}, \mathbf{t}_{\alpha,\beta}) \in A \times 2$. Now apply the partition property.
3) We start as above. Let D^0_α be the filter of B generated by $\{c_\alpha - c_\beta : \beta < \alpha\}$, so $\langle D^0_\alpha \cap A : \alpha < \lambda \rangle$ is an increasing sequence of subsets of A hence is eventually

constant so without loss of generality $\alpha < \lambda \Rightarrow D^0_\alpha \cap A = D^0$. Let D^* be an ultrafilter of A extending D^0 and let D_α be an ultrafilter of B extending $D^0_\alpha \cup D^*$. Now continue as above. $\square_{1.8}$

Remark. If we deal only with 1.8(1), we could use the partition property twice: first for the colouring $\{\alpha,\beta\} \mapsto \mathbf{c}_{\{\alpha,\beta\}}$ and second for the colouring $\langle \alpha,\beta\rangle \mapsto \mathbf{t}_{\{\alpha,\beta\}}$.

1.9 Claim. *1) In 1.5 we may replace in clauses (b), (c) of the assumption the usual (well) ordering of the ordinals $< \lambda$ and of the ordinals $< \theta$ by linear orders $<^*_\lambda, <^*_\theta$, but can retain the order on μ, this does not make a difference (call this assumption $\boxtimes'_{\lambda,\mu,\theta}$), provided that we weaken clause (γ) of the conclusion by*

$(\gamma)^-$ *Length*$^+(B_1 \underset{A}{\oplus} B_2) = \lambda^+$

2) In $\boxtimes_{\lambda,\mu,\theta}$ of 1.5 if we add $\theta =$ sup Rang$(\mathbf{c})$, then we can omit clause (e) as it follows.
3) If $\mathbf{c},\lambda,\mu,\theta$ satisfies (a),(b),(c) of $\boxtimes_{\lambda,\mu,\theta}$, <u>then</u> Rang$(\mathbf{c})$ has no last element.
4) If $\boxtimes_{\lambda,\mu,\theta}$ and $\mu \le \mu_1 \le \lambda_1 \le \lambda$, <u>then</u> $\boxtimes_{\lambda_1,\mu_1,\theta}$.

Proof. 1) Same proof as in 1.5.
2) Add to w_α dummy members to increase i, possible as Rang$(\mathbf{c})$ is unbounded in θ (and included in the next proof).
3) Let $\delta^* = \{\mathbf{c}\{\alpha,\beta\}+1 : \alpha,\beta < \lambda\}$, and let $\sigma =$ cf(δ^*), now $\delta^* > 0$ as Rang$(\mathbf{c}) \ne \emptyset$, if δ^* is a successor ordinal, say α^*+1 let $\zeta_0 \ne \zeta_1$ be such that $\mathbf{c}(\zeta_0,\zeta_1) = \alpha^*$ and let $\zeta_{\alpha,\ell} = \zeta_\ell$ for $\ell < 2, \alpha < \mu$; applying clause (c) we get a contradiction. So δ^* is a limit ordinal, and let $\langle \gamma_\varepsilon : \varepsilon < \sigma\rangle$ be increasing continuous with limit δ^*. Define $\mathbf{c}' : [\lambda]^2 \to \sigma$ by $\mathbf{c}'\{\alpha,\beta\} =$ Min$\{\varepsilon < \sigma : \mathbf{c}\{\alpha,\beta\} < \gamma_\varepsilon\}$ so $\mathbf{c}'\{\alpha,\beta\}$ is always a successor ordinal.

Let us prove (c) + (e). So let $\varepsilon(*) < \sigma$ and let $w_\alpha = \{\zeta_{\alpha,\ell} : \ell < n\}, \zeta_{\alpha,0} < \zeta_{\alpha,1} < \ldots < \zeta_{\alpha,n-1} < \lambda$ for $\alpha < \mu$. Let $\varepsilon_\alpha =$ max$\{\mathbf{c}'\{\zeta_{\alpha,\ell},\zeta_{\alpha,m}\} : \ell < m < n\}$ and as cf$(\mu) > \sigma$ (so cf$(\mu) > \theta$ is an overkill) without loss of generality ε_α is constant so $\varepsilon^* =$ max$\{\varepsilon(*)+1, \varepsilon_\alpha + 1 : \alpha < \lambda\} < \sigma$, so as $\delta^* =$ sup(Rang $\mathbf{c}$), for some $\alpha^* < \beta^* < \lambda$ we have $\mathbf{c}\{\alpha^*,\beta^*\} > \gamma_{\varepsilon^*}$, hence $\mathbf{c}'\{\alpha^*,\beta^*\} > \varepsilon^*$. Without loss of generality for all α's the truth value of $\zeta_{\alpha,\ell} < \alpha^*, \zeta_{\alpha,\ell} > \alpha^*, \zeta_{\alpha,\ell} < \beta^*, \zeta_{\alpha,\ell} > \beta^*$ are the same. Now apply the "old clause (c)" to $w'_\alpha = w_\alpha \cup \{\alpha^*,\beta^*\}$ and we can find $\alpha \ne \beta$ and i,j as there. Now $\alpha,\beta,i' =$ Min$\{\varepsilon : j < \gamma_\varepsilon\}, j' =: i'+1$ are as required.
4) Trivial. $\square_{1.9}$

1.10 Definition. 1) $Qr_2(\lambda,\mu,\theta)$ means:

(∗) if $f : [\lambda]^2 \to \mathscr{P}(\theta)\backslash\{\theta\}$ satisfies $\alpha < \beta < \gamma \Rightarrow \emptyset \ne f\{\alpha,\beta\} \cap f\{\beta,\gamma\} \subseteq f\{\alpha,\gamma\}$, then for some $X \in [\lambda]^\mu$ we have $(\bigcap_{\alpha\ne\beta\in X} f\{\alpha,\beta\}) \ne \emptyset$; note that by assumption $f\{\alpha,\beta\} \ne \emptyset$.

2) $NQs_2(\lambda,\mu,A,I)$ where $\lambda \ge \mu$ are (infinite) cardinals and A is a Boolean algebra and I is an ideal of A (possibly $I = A$) means that there is a function $f : [\lambda]^2 \to \mathscr{J}(I) =: \{J \subseteq I : J$ is non-empty closed upward and closed under intersection but $0_A \notin I\}$ such that

(a) if $\alpha < \beta < \gamma$, then $f\{\alpha,\gamma\} \supseteq f\{\alpha,\beta\} \cap f\{\beta,\gamma\}$

(b) for no $X \in [\lambda]^\mu$ and $b \in I$ do we have

$$\alpha \ne \beta \in X \Rightarrow b \in f(\{\alpha,\beta\}).$$

We say in this case "f witnesses $NQs(\lambda,\mu,A,I)$".

3) $NQs^+_2(\lambda,\mu,A,I)$ means that some $f : [\lambda]^2 \to I$ witnesses it which means that $f' : [\lambda]^2 \to \mathscr{J}(I)$ which is defined by $f'\{\alpha,\beta\} =: \{b \in I : f\{\alpha,\beta\} \le b\}$ witnesses $NQs_2(\lambda,\mu,A,I)$.
4) $NQs^*_2(\lambda,\mu,A,I)$ means $NQs_2(\lambda,\mu,A,I)$ is witnessed by some f which satisfies

(c) if $n < \omega, \zeta_{\alpha,0} < \ldots < \zeta_{\alpha,n-1}$ for $\alpha < \mu$, then for some $\alpha < \beta$ and b satisfying: $0 < b \in I$ we have:

  (i) $\zeta_{\alpha,\ell} \le \zeta_{\beta,\ell}$ and $\zeta_{\alpha,\ell} = \zeta_{\beta,m} \to \ell = m$

  (ii) if $\ell < m < n$ then
  $b \in f\{\zeta_{\alpha,\ell},\zeta_{\alpha,m}\} \cap f\{\zeta_{\beta,\ell},\zeta_{\beta,m}\}$

  (iii) if $\ell, m < n$ then
  $b \in f\{\zeta_{\alpha,\ell},\zeta_{\beta,m}\} \cap f(\zeta_{\beta,\ell},\zeta_{\beta,m})$ or b does not belong to
  $f\{\zeta_{\alpha,\ell},\zeta_{\beta,m}\} \cup f\{\zeta_{\beta,\ell},\zeta_{\alpha,m}\}$.

5) In part (1) addition of the letter N (that is $NQr_2(\lambda,\mu,\theta)$) means the negation; similarly in parts (2), (3) omitting N means the negation. We can replace μ by D, a filter on λ meaning replacing "there is $X \in [\lambda]^\mu$ such that ..." by "$\{\mathrm{otp} X : X \subseteq \lambda$ is such that $\ldots\} \in D$".
6) If we omit A we mean I is a Boolean ring, A the Boolean Algebra it generates. Omitting I in the N-version means "for some I". If we omit A and I and write θ we mean in the N-version, "for some $A, I, |A| \le \theta$" (so without N for every such A, I).

Among obvious implications are

1.11 Claim. *1) $NQr_2(\lambda,\mu,\theta)$ if $NQs_2(\lambda,\mu,\mathscr{P}(\theta),[\theta]^{<\aleph_0})$.*
2) $NQs_2^+(\lambda,\mu,A,I)$ implies $NQs_2(\lambda,\mu,A,I)$; also $NQs_2^*(\lambda,\mu,A,I)$ implies $NQs_2(\lambda,\mu,A,I)$.
3) In $NQs_2(\lambda,\mu,A,I)$ we can replace I by $A \restriction \{a : a \in I \text{ or } 1_A - a \in I\}$.
4) If $\lambda = \mu$ is weakly compact $> \theta$ then $Qr_2(\lambda,\mu,\theta)$.

Proof. Easy, e.g. for (1), if f exemplifies $NQs_2(\lambda,\mu,\mathscr{P}(\theta),[\theta]^{<\aleph_0}])$ let $f' : [\lambda]^2 \to \mathscr{P}(\theta)\backslash\{\theta\}$ be $f\{\alpha,\beta\} =: \{h(\gamma) : \gamma \in f'\{\alpha,\beta\}\}$ where h is one to one from $[\theta]^{<\aleph_0}$ to θ; now f exemplifies $NQr_2(\lambda,\mu,\theta)$. $\square_{1.11}$

1.12 Claim. *Assume $\lambda =$ cf$(\lambda) > 2^\theta$ and $Qs_2(\lambda,\mu,A)$.*
1) If A, B_1, B_2 are Boolean Algebras, $A \subseteq B_1, A \subseteq B_2, |A| \le \theta$ and
Depth$^+(B_1 \underset{A}{\oplus} B_2) > \lambda$ then $\bigvee\limits_{t=1}^{2}$ Depth$^+(B_t) > \mu$.
2) Similarly for length, if in Definition 1.10(2) we add to f a linear order $<^*$ of λ and use it in clause (a) there.

Proof. 1) We start as in the proof of 1.8(1), getting I^*, I, I_1, I_2 and find $t \in \{1,2\}$ and $b_\alpha \in B_t$ such that:

- $(*)_1$ $\langle b_\alpha/I_t : \alpha < \lambda\rangle$ is strictly increasing in B_t/I_t
- $(*)_2$ I^* is an ideal of A (in fact, a maximal one), and I_t is the ideal of B_t which I^* generates.

Let $f : [\lambda]^2 \to \mathscr{I}(I^*)$ be defined as follows: for $\alpha < \beta < \lambda$ we let $f\{\alpha,\beta\} = \{d \in I^* : b_\alpha - d \le b_\beta - d\}$. So $f\{\alpha,\beta\}$ is a non-empty subset of I^*, upward closed and closed under intersection (remember $\alpha < \beta < \lambda \Rightarrow b_\alpha/I_t < b_\beta/I_t$ and $(\forall c \in I_t)(\exists d \in I^*)[B_t \models c \le d]$). Now the preliminary demands in Definition 1.10(2) hold; also clause (a) holds as $d \in f\{\alpha,\beta\} \cap f\{\beta,\gamma\}, \alpha < \beta < \gamma$ implies $B_t \models b_\alpha - d < b_\beta - d < b_\gamma - d$ so $B_t \models b_\alpha - d < b_\gamma - d$ hence $d \in f\{\alpha,\gamma\}$, thus proving the $\subseteq$. Hence by the assumptions of our claim 1.12, clause (b) of Definition 1.10(2) fail, so for some $X \in [\lambda]^\mu$ and $b \in I^*$ we have $A \models 0 < b$ and $\alpha \ne \beta \in X \Rightarrow b \in f(\{\alpha,\beta\})$. So $\langle b_\alpha - b : \alpha < \lambda\rangle$ is strictly increasing in B_t.
2) Similar only now $\{b_\alpha/I_t : \alpha < \lambda\}$ is a chain with no repetitions. $\square_{1.12}$

1.13 Claim. *If $\lambda \to [\mu]^2_{\theta,<\aleph_0}$ then $Qs_2(\lambda,\mu,\theta)$.*

Proof. Straight.

By Definition 1.10(6), $Qs_2(\lambda,\mu,\theta)$ mean that we are given a Boolean algebra A, of cardinality $\leq \theta$, an ideal I of A and a function $f:[\lambda]^2 \rightarrow \mathscr{J}(I)$.

Let $f':[\lambda]^2 \rightarrow I$ be such that $\alpha < \beta < \lambda \Rightarrow f'\{\alpha,\beta\} \in f(\alpha,\beta)$. So f' is a function with domain $[\lambda]^2$ and with range of cardinality $\leq |I| \leq |A| \leq \theta$. By the definition of $\lambda \rightarrow [\mu]^2_{\theta,<\aleph_0}$ there is $X \in [\lambda]^\mu$ and finite $w \subseteq \theta$ such that $\alpha < \beta \in X \Rightarrow f'(\alpha,\beta) \in w$. Now let $b = \cup\{a : a \in w\}$, as a finite union of members of I clearly b is in I. Also for $\alpha < \beta$ from X, $f'(\alpha,\beta)$ is a member of w hence is $\leq b$, but is also a member of $f(\alpha,\beta)$ which is a subset of I upward closed, so $b \in f(\alpha,\beta)$, is as required (we did not use "$f\{\alpha,\beta\}$ is closed under intersection"). $\square_{1.13}$

1.14 Remark. 1) So we have consistency results by [Sh 276], [Sh 546].
2) In 1.15 below we can moreover get $Qr_2(\lambda,\lambda,\mu)$.
3) When does $\boxtimes$ of 1.5 fail? See below (and 5.5).

1.15 Claim. *1) Assume $\theta = cf(\theta) < \chi = \chi^{<\chi} < \lambda, \lambda$ measurable, then in $\mathbf{V}_1 = \mathbf{V}^{\mathrm{Levy}(\chi,<\lambda)}$ we have $\neg\boxtimes_{\lambda,\lambda,\theta}$.*
2) If in $\mathbf{V}_1, \lambda = \chi^+, \chi > \theta$ and

- $(*)$ *D is a normal filter on λ for which in the game $\partial = \partial(\lambda,D,\theta)$ of length $\theta+1$ between the even and odd players choosing $A_i \in D^+$ for $i \leq \theta$ decreasing (of course, for i even, the even player chooses A_i, for i odd, the odd player chooses A_i), the even player can guarantee that for limit $\delta \leq \theta$, $\bigcap_{i<\delta} A_i \in D^+$.*

Then $\mathbf{V}_1$ satisfies the conclusion in part (1).

Proof. 1) By part (2) relying on [JMMP] (see on the subject [Sh:b] or [Sh:f]).
2) Toward contradiction assume $\mathbf{c}:[\lambda]^2 \rightarrow \theta$ exemplifies $\boxtimes_{\lambda,\lambda,\theta}$. Let St be a winning strategy of the even player. We chose by induction on $i < \theta$ sets A_i, B_i such that

- (a) $\langle A_j : j \leq i\rangle$ is a play of $\partial(\lambda,D,\theta)$ in which even use his winning strategy St
- (b) $\langle B_j : j \leq i\rangle$ is a play of $\partial(\lambda,D,\theta)$ in which even use his winning strategy St
- (c) for i odd for some $\gamma_i < \lambda$ and $j_i \in (i,\theta)$ we have

$$\gamma_i < \mathrm{Min}(A_i)$$

$$\gamma_i < \mathrm{Min}(B_i)$$

$$(\forall \alpha \in A_i)[\mathbf{c}\{\gamma_i, \alpha\} < j_i]$$

$$(\forall \beta \in B_i)[\mathbf{c}\{\gamma_i, \beta\} \geq j_i]$$

For i even we have no free choice.

For i odd we ask

$(*)_i$ is there $\gamma < \lambda$ such that
$j < \theta \Rightarrow B_{i,j} = \{\beta \in B_{i-1} : \beta > \gamma \text{ and } \mathbf{c}\{\gamma, \beta\} \geq j\} \in D^+$?

If yes, choose such $\gamma = \gamma_i$, so $A_{i-1} \backslash (\gamma_i + 1) = \bigcup\limits_{j<\theta} \{\alpha \in A_{i-1} : \mathbf{c}\{\gamma_i, \alpha\} = j\}$. Hence for some $j = j_i$

$$\{\alpha : \alpha \in A_{i-1}, \alpha > \gamma_i, \mathbf{c}\{\gamma_i, \alpha\} < j\} \in D^+$$

and without loss of generality$j_i > i$. Choose this set as A_i and B_i as $B_{i,j}$.

If no, let for $\gamma < \lambda, j(\gamma) < \theta$ be a counterexample. So by the normality of the filter, $B' =: \{\beta \in B_{i-1} :$ for every $\gamma < \beta$ we have $\mathbf{c}\{\gamma, \beta\} \geq j(\gamma)\} = \emptyset \bmod D$. So $B_{i-1} \backslash B' \in D^+$ but normality implies "θ^+-completeness" hence for some $j^* < \theta$ we have $B^* = \{\gamma \in B_{i-1} : \gamma \notin B' \text{ and } j(\gamma) = j^*\} \in D^+$.

So $B^* \in D^+$ and for $\gamma_1 < \gamma_2$ in B^* we have $\mathbf{c}\{\gamma_1, \gamma_2\} < j(\gamma_1) = j^*$; but $B^* \in D^+ \Rightarrow |B^*| = \lambda$ (again normality).

This contradicts the choice of $\mathbf{c}$, that is clause (c) of 1.5 so clearly $(*)_i$ holds for our odd i.

So we succeed to choose $\langle A_i : i \leq \theta \rangle, \langle B_i : i \leq \theta \rangle$, so we can find $\alpha \in A_\theta$ and $\beta \in B_\theta \backslash (\alpha + 1)$ (as $A_\theta, B_\theta \in D^+$).

So for each $i < \theta$

$(**)_i$ $\gamma_i < \alpha < \beta, \mathbf{c}\{\gamma_i, \alpha\} < j_i, \mathbf{c}\{\gamma_i, \beta\} \geq j_i$ so $\mathbf{c}\{\gamma_i, \alpha\} < \mathbf{c}\{\gamma_i, \beta\}$.

But $\mathbf{c}\{\gamma_i, \beta\} \leq \text{Max}\{\mathbf{c}\{\gamma_i, \alpha\}, \mathbf{c}\{\alpha, \beta\}\}$. Hence $\mathbf{c}\{\alpha, \beta\} \geq \mathbf{c}\{\gamma_i, \beta\}$ but the latter is $\geq j_i$ and $j_i \geq i$, so $\mathbf{c}\{\alpha, \beta\} \geq i$. As this holds for any $i < \theta$ we have gotten a contradiction. $\square_{1.15}$

1.16 Remark. 1) We can weaken the demand on the even player in the game $\Game(\lambda, D, \theta)$ for $\delta = \theta$ to the demand $\bigcap\limits_{i<\delta} A_i \neq \emptyset$ is enough, as we can:

(α) make $A_1 \cap B_1 = \emptyset$

(β) if $i = 4j+1$ retain demand (c) in the proof but if $i = 4j+3$ uses a similar demand interchanging A_i and B_i.

2) Moreover, instead "even has a winning strategy" it is enough, that "odd has no winning strategy for winning at least one of two plays, played simultaneously". Now we can deal with other variants.

We may wonder what is required from A in Claim 1.5.

1.17 Claim. *Assume* $\boxdot_{\lambda,\mu,\theta}$ *(see below) and* A *is a Boolean Algebra of cardinality* $< cf(\mu)$ *(for simplicity) and*

(∗) *there are* $a^t_i \in A$ *for* $i < \theta, t \in \{1,2\}$ *such that:*
- (i) $i < j < \theta \Rightarrow A \models a^t_i \le a^t_j$
- (ii) $a^1_i \cap a^2_i = 0$ *for* $i < \theta$
- (iii) *there are no disjoint* $a_1, a_2 \in B$ *such that* $t \in \{1,2\}$ & $i < \theta \Rightarrow a^t_i \le a_t$.

<u>*Then*</u> $(\lambda,\mu) \in SpDpFP(A)$ *where*

$\boxdot_{\lambda,\mu,\theta}$ *is like* $\boxtimes_{\lambda,\mu,\theta}$ *but we replace clause (c) by (c)′ below where*
(c)′ *like clause (c) but we are given also an unbounded* $Y \subseteq \theta$ *and demand* $j \in Y$.

Proof. Similar to the proof of 1.5.

We mark the changes
<u>Stage A</u>:
We let I_t be the ideal of A generated by $\{a^t_i : i < \theta\}$ and let $I^\perp_t = \{a \in A$: we have $a^t_i \cap a = 0_A$ for every $i < \theta\}$. Clearly $I_{3-t} \subseteq I^\perp_t$. Let I' be a maximal ideal of B which include $I^\perp_1 \cup I^\perp_2$ (it exists by clause (iii) of the assumptions).
We let B_t be the extension of A by $\{x^t_\alpha : \alpha < \lambda\}$ freely except the equations:

$(*)_3$ $x^t_\alpha - x^t_\beta \le a^t_{\mathbf{c}\{\alpha,\beta\}}$ for $\alpha < \beta < \lambda$

$(*)_4$ $x^t_\alpha \cap a = 0$ for $a \in I^\perp_t$.

Stage C:

Toward contradiction, let $\{c_\alpha : \alpha < \mu\}$ be a chain in B_t (with no repetitions), and let $c_\alpha = \tau_\alpha(a_{\alpha,1}, \ldots, a_{\alpha,k_\alpha}, x^t_{\zeta_{\alpha,1}}, \ldots, x^t_{\zeta_{\alpha,n_\alpha}})$ where $\zeta_{\alpha,1} < \ldots < \zeta_{\alpha,n_\alpha} < \lambda$ and $a_{\alpha,k} \in A$. Without loss of generality

$(*)_5$ $\tau_\alpha = \tau, k_\alpha = k^*, a_{\alpha,\ell} = a_\ell, n_\alpha = n^*$ (recall $\mathrm{cf}(\mu) > |A|$).

For each α let D_α be an ultrafilter on B_t such that $c_\alpha \in D_\alpha$ and $\beta < \mu$ & $A \models c_\beta < c_\alpha \Rightarrow c_\beta \notin D_\alpha$. We can find $a^*_\alpha \in D_\alpha \cap A$ such that $(\forall \ell \in \{1, \ldots, k^*\})(a^*_\alpha \leq a_{\alpha,\ell} \vee a^*_\alpha \cap a_{\alpha,\ell} = 0_A)$ and $a_\alpha \leq \mathbf{a}^t_{\mathbf{c}\{\zeta_{\alpha,\ell},\zeta_{\alpha,m}\}} \vee a_\alpha \cap a^t_{\mathbf{c}\{\zeta_{\alpha,\ell},\zeta_{\alpha,m}\}} = 0_{B_t}$ for $\ell, m \in \{1, \ldots, n^*\}$.

Without loss of generality (as $\mathrm{cf}(\mu) > |A|$, if μ regular we can ask for more)

$(*)_6$ $a_\alpha = a^*$ and the $\mathbf{c}\{\zeta_{\alpha,\ell}, \zeta_{\alpha,m}\} = i^*_{\ell,m}$ and $\mathrm{T.V.}(a_\alpha \leq a^t_{\mathbf{c}\{\zeta_{\alpha,\ell},\zeta_{\alpha,m}\}}) = \mathbf{t}_{\ell,m}$.

So changing names

$(*)'_6$ $c_\alpha = \tau(x_{\zeta_{\alpha,1}}, \ldots, x_{\zeta_{\alpha,n^*}}) \cap a^*$.

Now

$(*)_7$ $a^* \notin I^\perp_t$, moreover $\forall i < \theta \exists j < \theta [a^* \cap a^t_j - a^t_i \notin I^\perp_t]$
[why? otherwise let ε be a counterexample so $a^* - a^t_\varepsilon \in I^\perp_t$, by the definition of $I^\perp_t$ clearly $\alpha < \mu$ & $\ell \in [1, n^*] \Rightarrow x_{\zeta_{\alpha,\ell}} \cap (a^* - a^t_\varepsilon) = 0_{B_t}$ (by $(*)_4$) hence $c_\alpha \cap (a^* - a^t_\varepsilon)$ does not depend on α so call it c^*, so also $\{c_\alpha - c^* : \alpha < \mu\}$ is a chain of B_t with no repetitions but $c_\alpha \leq a^*$ by $(*)'_6$ so $c_\alpha - c^* \leq a^t_i$ (for our fixed i). We apply clause (c) + (e) of the assumption $\boxtimes_{\lambda,\mu,\theta}$ to $\langle w_\alpha : \alpha < \mu \rangle$ with $w_\alpha = \{\zeta_{\alpha,1}, \ldots, \zeta_{\alpha,n^*}\}$ and $i(*) = \varepsilon + 1$, and get $\alpha \neq \beta, i < j$ as there, so ζ_ℓ, ξ_ℓ there stands for $\zeta_{\alpha,\ell+1}, \zeta_{\beta,\ell+1}$ here.
Without loss of generality $\alpha < \beta$ and let h be a homomorphism from B_t to $\{0,1\}$ such that $h(c_\alpha - c^*) = 0, h(c_\beta - c^*) = 1$; hence $h(a^t_i) = 1$. Let g be the following function from $A \cup \{x_{\zeta_{\alpha,\ell}}, x_{\zeta_{\beta,\ell}} : \ell = 1, \ldots, n^*\}, g \restriction A = h \restriction A$ and $g(x_{\zeta_{\alpha,\ell}}) = h(x_{\zeta_{\beta,\ell}})$ and $g(x_{\zeta_{\beta,\ell}}) = h(x_{\zeta_{\alpha,\ell}})$. We would like to apply $(*)$ of Stage D of the proof of 1.5 + the fact there to extend g to a homomorphism from B_t to $\{0,1\}$; this suffices, as then $\hat{g}(c_\alpha - c^*) = h(c_\beta - c^*) = 1, \hat{g}(c_\beta - c^*) = h(c_\alpha - c^*) = 0$ contradicting $B_t \models c_\alpha - c^* < c_\beta - c^*$.
Does the condition of (*) hold? As in the proof of $(*)_9$ in the proof of 1.5, noting that $\mathbf{c}\{\zeta, \xi\} \geq \varepsilon$ & $(h(x_\zeta, x_\xi)$ well defined$) \Rightarrow h$ respects $x_\zeta - x_\xi \leq a^t_{\mathbf{c}\{\zeta,\xi\}}$ as $h(a^t_\varepsilon) = g(a^t_\varepsilon) = 1$.]

Let $Y = \{j < \theta : (\forall \varepsilon < j)(A \models (a_j - a_\varepsilon) \cap a^* > 0)\}$, it is unbounded in θ by $(*)_7$ as if $j_0 < \theta, j_1 = \mathrm{Min}\{j : a^* \cap b_j - b_{j_0} \notin I^\perp_t\}$ then $j_1 \in Y$.
We choose α, β, i, j as in clause $(c)'$ + (e) of $\boxdot_{\lambda,\mu,\theta}$ (see 1.5) but $j \in Y$ is O.K. as

we use $\boxdot_{\lambda,\mu,\theta}$. Let $\alpha < \beta$ and f be a homomorphism from B_t to $\{0,1\}$ such that $f(x_\alpha) = 0, f(x_\beta) = 1$. Now

$(*)'_8$ $a \in I_t^\perp \Rightarrow f(a) = 0$
[why? if $a' \in I_t^\perp$ is a counterexample then $\alpha < \lambda \in w \Rightarrow f(x^t_\alpha) = f(x^t_\alpha - a') = f(x^\ell_\alpha) - 1 = 0$ so clearly $f(c_\alpha) = f(c_\beta)$, contradiction]

$(*)_9$ $f(a^*) = 1$
[why? otherwise $f(c_\alpha) = 0 = f(c_\beta)$, contradiction]

$(*)_{10}$ there is a homomorphism $g : A \to \{0,1\}$ such that

- (i) $g(a^t_\xi) = f(a^t_\xi)$ for $\xi < j$
- (ii) $g(a^t_\xi) = 1$ if $\xi \geq j, \xi < \theta$
- (iii) $g(a^*) = 1$
- (iv) $b \in I_t^\perp \Rightarrow g(b) = 0$.

[Why? If $(\forall \varepsilon)(j \leq \varepsilon < \theta \to f(a^t_\varepsilon) = 1)$, then $g = f$ is O.K. If not, then clause (i) means $g(a^t_\xi) = 0$ for $\xi < j$; hence for proving the existence of q it is enough to show

$$j' < j \Rightarrow a^* \cap (a^t_j - a^t_{j'}) \notin I_t^\perp$$

this holds by the choice of (α, β, i, j), i.e. $j \in Y$.]

We continue defining h as in the proof of $(*)_9$ in the proof of 1.5. $\square_{1.17}$

1.18 Claim. *In 1.7 we can also get $\boxdot_{\lambda,\mu,\theta}$.*

Proof. We repeat the proof but: proving clauses (c) + (e) we are also given an unbounded set $Y \subseteq \{i < \theta : i \text{ odd}\}$, and choose j^* there. $\square_{1.18}$

1.19 Claim. *Assume*

- (a) *A is a Boolean Algebra and I_1, I_2 are ideals of A and $I_1 \cap I_2 = \{0\}$*
- (b) *for $\ell = 1, 2$ we have $NQs^*_2(\lambda, \mu, A, I_\ell)$ and $|A| <$ cf(μ)*
- (c) *$I_1 \cup I_2$ generates A (or less).*

<u>Then</u> there are Boolean Algebras B_1, B_2 extending A such that Depth$^+(B_1 \underset{A}{\oplus} B_2) = \lambda^+$, Depth$^+(B_1) \leq \mu$, Depth$^+(B_2) \leq \mu$.

Remark. We can weaken clause (c). We may wonder on using more ideals.

Proof. Like the proofs of 1.5, 1.17 but: A is given let $\mathbf{c}_\ell : [\lambda]^2 \to \mathscr{J}(I)$ exemplifies $NQs_2(\lambda, \mu, A, I_\ell)$ and $(*)_3, (*)_4$ are now

$(*)_3$ $x^t_\alpha - x^t_\beta \le a$ when $a \in f_t\{\alpha, \beta\}, t \in \{1, 2\}$

$(*)_4$ $x^t_\alpha \cap a = 0$ if $a \in I^\perp_t, \alpha < \lambda$.

$\square_{1.19}$

Note (not used here, just for background see more on clause (c) in §5)

1.20 Claim. *1) Assume*

(a) $\theta = cf(\theta) < \mu$, *and*

(b) μ *is strong limit singular,* $cf(\mu) = \kappa < \theta$

(c) *for any* $\mathbf{c} : [\theta]^2 \to \kappa$ *satisfying* $\alpha < \beta < \gamma \Rightarrow \mathbf{c}\{\alpha, \gamma\} \subseteq Max\{\mathbf{c}\{\alpha, \beta\}, \mathbf{c}\{\beta, \gamma\}\}$, *there is* $X \in [\theta]^\theta$ *such that* $Rang(\mathbf{c} \restriction [X]^2)$ *is a bounded subset of* κ.

then $\{\delta < \mu^+ : cf(\delta) = \theta\} \in I[\lambda]$.
2) We can replace (b) by

$(b)^-$ $\mu > \theta > \kappa = cf(\mu)$ *and* $(\forall \sigma < \mu)[\sigma^{<\theta} < \mu]$.

Proof. See [Sh 108], [Sh 88a].

§2 On the family of homomorphic images of a Boolean Algebra

Our best result is 2.6(2) but we first deal with more specific cases. On background see [JuSh 612].

2.1 Lemma. *Assume $\kappa < \mu < \lambda$ and λ is a strong limit singular of cofinality κ and $cf(\mu) = \kappa$.*

Then every Boolean Algebra of cardinality λ has a homomorphic image of cardinality $\in [\mu, 2^{<\mu}]$.

First we prove

2.2 Claim. *Assume $\kappa = cf(\mu) < \mu < 2^{<\mu}$. Any Boolean Algebra B of cardinality $\geq \chi =: \sum\limits_{\theta<\mu}(2^\theta)^+$ has a homomorphic image of cardinality $\in [\mu, \chi)$; note that $(2^{<\mu})^+ \geq \chi$.*

Proof. Let $\mu = \sum\limits_{i<\kappa} \mu_i$ be such that $i < j \Rightarrow \mu_i < \mu_j$ and moreover $i < j \Rightarrow \mu < 2^{\mu_i} \leq 2^{\mu_j} \leq 2^{<\mu}$ and $\mu_i > \kappa$ and if $\langle 2^{\mu_i} : i < \kappa\rangle$ is not eventually constant then it is strictly increasing.

If B has an independent subset of cardinality μ, then it has a homomorphic image of cardinality $\in [\mu, \mu^{\aleph_0}]$ but $\mu^{\aleph_0} < \chi$ so without loss of generality there is no independent $X \subseteq B$ of cardinality μ hence for no $\theta < \mu$ does B satisfy the θ^+-c.c., hence $c(B) > \theta$. So $c(B) \geq \mu$ but μ is singular hence by a theorem of Erdös and Tarski, B has an antichain $\{a_\alpha : \alpha < \mu\}$. For each $i < \mu$, let B_i be the subalgebra of B generated by $\{a_\alpha : \alpha < \mu_i\}$ and let B^c_i be the completion of B_i. So id_{B_i} is a homomorphism from B_i into B^c_i hence it can be extended to a homomorphism f_i from B into B^c_i.

Clearly $B_i \subseteq \mathrm{Rang}(f_i) \subseteq B^c_i$ so $\mu_i \leq (\mathrm{Rang}(f_i)) \leq |B^c_i| \leq 2^{\mu_i}$.

If for some $i, |\mathrm{Rang}(f_i)| \geq \mu$ we are done: $B'_i =: \mathrm{Rang}(f_i)$ is a homomorphic image of $B, \mu \leq |B'_i| \leq |B^c_i| \leq 2^{\mu_i} < \chi$. Otherwise, we have $i < \kappa \Rightarrow |B'_i| < \mu$, let $B^* = \prod\limits_{i<\kappa} B'_i$, and we define a homomorphism f from B into $B^*, f(x) = \langle f_i(x) : i < \kappa\rangle$. Clearly B^* is a Boolean Algebra and f a homomorphism from B into B^*. Now let $B' = \mathrm{Rang}(f)$, clearly B' is a homomorphic image of B and $|B'| = |\mathrm{Rang}(f)| \leq |B^*| \leq \prod\limits_{i<\kappa} |B'_i| \leq \prod\limits_{i<\kappa} \mu = \mu^\kappa \leq 2^{\mu_0} < \chi$. On the other hand f is one to one on each B_i (as f_i is) hence is one to one on $\bigcup\limits_{i<\kappa} B_i$ which has cardinality

μ, so $|B'| \geq \mu$.
So we are done. $\square_{2.2}$

2.3 Remark. If μ is regular, $|B| \geq \mu$, then B has a homomorphic image of cardinality $\in [\mu, 2^{<\mu}]$, this follows by Juhasz [Ju1].

2.4 Claim. *Assume (δ a limit ordinal)*

- (a) *for $\ell = 0,1,2,3$ we have $\langle B^\ell_i : i \leq \delta \rangle$ is an increasing continuous sequence of Boolean Algebras and for simplicity B^ℓ_0 is trivial*
- (b) *$B^2_i \subseteq B^3_i$ and $B^0_i \subseteq B^1_i$ for $i \leq \delta$*
- (c) *for $i < \delta$ non-limit, B^1_i is complete*
- (d) *h_i is a homomorphism from B^2_i into B^0_i, increasing with $i \leq \delta$*
- (e) *if $x \in B^2_{i+1}, y \in B^3_i$ and $B^3_{i+1} \models$ "$x \cap y = 0$", then for some $z \in B^2_i$ we have $B^3_i \models z \cap y = 0$ and $B^2_{i+1} \models x \leq z$.*

Then we can find a homomorphism h from B^3_δ into B^1_δ extending h_δ.

Proof. We choose by induction on i a homomorphism f_i from B^3_i into B^1_i increasing with i such that: f_i extends h_i.

For $i = 0$: Trivial as the B^ℓ_i are trivial.

For i limit: Let $f_i = \bigcup_{j<i} f_i$.

For $i = j+1$: Let $H_{j+1} = \{f : f$ is a homomorphism from some subalgebra B_f of B^3_{j+1} into B^1_{j+1} extending f_j and $h_{j+1}\}$. As B^1_{j+1} is complete, it is enough to prove that $H_{j+1} \neq \emptyset$. Let B'_{j+1} be the subalgebra of B^3_{j+1} generated by $B^2_{j+1} \cup B^3_j$, so it is enough to prove that $f_j \cup h_{j+1}$ induce a homomorphism from B'_{j+1} into B^1_{j+1}. Easily for this it suffices to prove:

- $(*)$ if $x \in B^2_{j+1}, y \in B^3_j$ are disjoint (in B^3_{j+1}), then $h_{j+1}(x), f_j(y)$ are disjoint (in B^1_{j+1}).

By the assumption (e) there is $z \in B^2_j$ such that $B^2_{j+1} \models x \leq z$ and $B^3_j \models y \cap z = 0$ hence $B^0_{j+1} \models h_{j+1}(x) \leq h_{j+1}(z)$ and $B^1_j \models f_j(y) \cap f_j(z) = 0$. As $f_j(z) = h_j(z) = h_{j+1}(z)$ we are done. $\square_{2.4}$

Proof of 2.1. So by 2.2 without loss of generality $\mu = 2^{<\mu}$ hence is strong limit (as $\mu = 2^{<\mu}$ & μ not strong limit $\Rightarrow \mu = \mathrm{cf}(\mu)$), so let $\mu = \sum\limits_{i<\kappa} \mu_i, \kappa < \mu_i$ and $\prod\limits_{i<j} 2^{\mu_i^+} < \mu_j$. Similarly, let $\lambda = \sum\limits_{i<\kappa} \lambda_i$ be such that $i < j \Rightarrow \kappa < \mu_i < \lambda_i < \lambda_j$, moreover, $\lambda_i = \mathrm{cf}(\lambda_i) > \prod\limits_{j<i} 2^{\lambda_j}$. As in the proof of 2.2, it suffices to deal with the following three cases.

Case A: There is an antichain $\{a_\alpha : \alpha < \lambda\}$ of B. As we can replace B by any homomorphic image of cardinality $\geq \lambda$ without loss of generality $\{a_\alpha : \alpha < \lambda\}$ is a maximal antichain of B and each a_α an atom. So without loss of generality B is a subalgebra of $\mathscr{P}(\lambda)$ and $a_\alpha = \{\alpha\}$. Let $B = \bigcup\limits_{i<\kappa} B_i, B_i$ increasing continuous, $|B_i| \leq 2 + \sum\limits_{j<i} \lambda_j$.

We can find $X_i \subseteq \lambda_i$ of cardinality λ_i such that:

$(*)_1$ for each $a \in B_i$ either $(\forall \alpha \in X_i)(a_\alpha \leq b)$ or $(\forall \alpha \in X_i)(a_\alpha \cap b = 0)$

$(*)_2$ $\{a_\alpha : \alpha \in X_i\} \subseteq B_{j_i}$ where $i < j_i < \kappa$.
[Why? Define a two place relation E_i on λ_i by $\alpha E_i \beta$ iff $\alpha, \beta < \lambda_i$ and $(\forall a \in B_i)[x \in a \equiv \beta \in a]$. Clearly this is an equivalence relation and the number of E_i-equivalence classes is $\leq 2^{|B_i|} \leq 2^{\Sigma\{\lambda_j, j<i\}} = \prod\limits_{j<i} 2^{\lambda_j} < \lambda_i$. As λ_i is regular necessarily some E_i-equivalence class X'_i has cardinality λ_i. As $X'_i = \cup\{X'_{i,j} : j \in (i,\kappa)\}$ where $X'_{i,j} = \{\alpha \in X_i : \{\alpha\} \in B_j\}$ and $\kappa < \lambda_i = \mathrm{cf}(\lambda_i)$ for some $j_i, |X'_{i,j_i}| = \lambda_i$, and let $X_i = X'_{i,j_i}$.]

Choose $Y_i \subseteq X_i$ of cardinality μ_i, recalling $\mu_i < \lambda_i$.

Let $Y = \bigcup\limits_{i<\kappa} Y_i$ and let $f : B \to \mathscr{P}(Y)$ be the following homomorphism: $f(a) = a \cap Y$ and let B' be Rang(f), so $B' \subseteq \mathscr{P}(Y)$ is a homomorphic image of B and

$$|B'| \geq |\{a_\alpha : \alpha \in Y\}| = |Y| = \sum_{i<\kappa} |Y_i| = \sum_{i<\kappa} \mu_i = \mu$$

$$\begin{aligned}|B'| \le |\{a \cap Y : a \in B\}| &\le \sum_{i<\kappa} |\{a \cap Y : a \in B_i\}| \\ &= \sum_{i<\kappa} |\{a : a \subseteq Y \text{ and } (\forall j)(i \le j < \kappa \to a \cap Y_i \in \{\emptyset, Y_i\})| \\ &\le \sum_{i<\kappa} (2^\kappa \times 2^{\mu_i}) = \sum_{i<\kappa} 2^{\mu_i} = 2^{<\mu} = \mu.\end{aligned}$$

Case B: B satisfies the θ-c.c., $\theta < \lambda, \kappa > \aleph_0$. Then B has an independent subset of cardinality χ for each $\chi < \lambda$, in particular $\chi = \mu$, say $\{a_\alpha : \alpha < \mu\}$. Let B'_0 be the subalgebra of B which $\{a_\alpha : \alpha < \mu\}$ generates, and B^c_0 be its completion, so id_{B_0} can be extended to a homomorphism f from B into B^c_0, let $B_1 = \mathrm{Rang}(f)$ so $\mu = |B_0| \le |B_1| \le |B^c_0| \le \mu^{\aleph_0} = \mu$ so B_1 is as required.

Case C: B satisfies the θ-c.c., $\theta < \lambda, \kappa = \aleph_0$.
Let $\lambda = \sum_{n<\omega} \lambda_n, \theta < \lambda_n, 2^{\lambda_n} < \lambda_{n+1}, \lambda_n = \mathrm{cf}(\lambda_n), \lambda_n^\theta = \lambda_n$. Easily we can find pairwise disjoint $\langle b_n : n < \omega\rangle$ such that $|B \restriction b_n| \ge \lambda_n^+$.
[Why? Choose by induction on n, b_n such that $\ell < n \Rightarrow b_\ell \cap b_n = 0, (B \restriction b_n) \ge \lambda_n^+$ and $|B \restriction (1_B - \bigcup_{\ell \le n} b_\ell)| = \lambda$; if we are stuck in n, then let $I =: \{x \in B : x \cap \bigcup_{\ell<n} b_\ell = 0$ and $|B \restriction x| \le \lambda_n\}$ and let $B' = B \restriction (1_B - \bigcup_{\ell<n} b_\ell)$, so $|B'| = \lambda$ and I is a maximal ideal of B' (as we are stuck), hence I has cardinality λ. Try to choose by induction on $\alpha < \lambda, b_\alpha \in I$ such that $\beta < \alpha \Rightarrow b_\beta \cap b_\alpha = 0_B$; if we succeed we contradict our being in case C, if we are stuck in stage $\alpha < \lambda$, then $\{x \in B' : (\exists \beta < \alpha) x \le b_\alpha$ and $x > 0\}$ is a dense subset of $B' \backslash \{0_B\}$ of cardinality $\le |\alpha| + 2^{\lambda_n} < \lambda$, contradiction to $|B'| = \lambda$.]

We can find for each $n, \langle a^n_\alpha : \alpha < \lambda_n^+\rangle$ such that $a^n_\alpha \le b_n$ and $\langle a^n_\alpha : \alpha < \lambda_n^+\rangle$ is independent in $B \restriction b_n$.
[Why? As $B \restriction b_n$ is a Boolean Algebra of cardinality $\ge \lambda_n^+$ and $\lambda_n = \lambda_n^\theta$ and $B \restriction b_n$ satisfies the θ-c.c.] So some homomorphic image of B has λ atoms and we get Case A. $\square_{2.1}$

2.5 Observation. If μ is strong limit of cofinality $\aleph_0$ and $\mu \le \lambda < 2^\mu, B$ a Boolean Algebra of cardinality λ, then $\mathrm{ult}(B) > \lambda \Rightarrow \mathrm{ult}(B) \ge 2^\mu$.

Proof. Straight, or by [Sh 454a] (for even a more general setting: a topology, not necessarily Hausdorff).
Or directedly, let

$$Q = \{X : X \text{ is a subset of } B \text{ of cardinality } < \lambda \text{ with the FIP and } \{D \in \text{Ult}(B) : X \subseteq D\}| > \lambda\}.$$

Now

(a) Q is not empty (as $\emptyset \in Q$)
(b) for $X \in Q$ define the two place relation E_X on B : $aE_\lambda b$ iff the set $\{D \in \text{Ult}(B) : X \subseteq D \text{ and } a \in D \leftrightarrow b \in D\}$ has cardinality $\leq \lambda$.

Now E_X is an equivalence relation; easily

$$|\{D \in \text{Ult}(B) : X \subseteq D\}| \leq 2^{|B/E_X|} + \lambda$$

hence necessarily $|B/E_X| \geq \mu$. So by the Erdos Rado theorem, we have

(c) for any $\chi < \mu$ there is a sequence $\langle a_\alpha : \alpha < \chi \rangle$ such that $\alpha < \beta \Rightarrow X \cup \{-a_\alpha, a_\beta\} \in Q$ moreover by the proof of Erdos-Rado theorem $X \cup \{-a_\beta : \beta < \alpha\} \cup \{a_\alpha\} \in Q$.

We can let $\mu = \sum\limits_n \chi_n$ with $\chi_n < \chi_{n+1}$. We then can easily get a tree $\langle a_\eta : \eta \in \prod\limits_{\ell < n} \chi_\ell$ for $n < \omega\}$ and for each $\eta \in \prod\limits_{\ell < \omega} \chi_\ell$ the set $\{a_{\eta \restriction \ell} : \ell < \omega\} \cup \{-a_{\eta \restriction \ell \frown <\beta>} : \beta < \eta(\ell)\}$ has the FIP, so we get $\mu^{\aleph_0} = 2^\mu$ filters, extending each to an ultrafilter we are done. $\square_{2.5}$

We can add

2.6 Claim. *1) If λ is a strong limit singular and B a Boolean Algebra of cardinality λ, then for some homomorphic image B' of B we have $|B'| = |Ult(B')| = \lambda$.*
2) If we further assume $\lambda \geq \mu >$ cf$(\mu) =$ cf(λ), then for some homomorphic image B' of B we have:

$$\mu \leq |B'| \leq 2^{<\mu}, \mu \leq |Ult(B')| \leq \sum_{\theta < \mu} 2^{2^\theta}.$$

2.7 Remark. The free Boolean Algebra generated by $\{x_\alpha : \alpha < \lambda\}, B_\lambda$, has as homomorphic image any Boolean Algebra (in particular, any σ-complete Boolean

Algebra of cardinality $\leq \lambda$.) The completion of B_λ has as homomorphic images many σ-complete Boolean Algebras of cardinality $\leq \lambda$ (though consistency not all by a result of Dow and Vermeer).

Proof. 1) Without loss of generality B is a Boolean Algebra of subsets of λ and let $\lambda = \sum_{i<\theta} \mu_\zeta, \theta = \mathrm{cf}(\lambda), \mu_\zeta > \theta + \prod_{\xi<\zeta} 2^{\mu_\xi^+}$ and let $B = \bigcup_{i<\theta} B_i, B_i$ increasing continuous, $|B_\zeta| \leq 2 + \sum_{\xi<\zeta} \mu_\xi^+$. We know that there are $(a_i^\zeta, \alpha_i^\zeta)$ for $i < \mu_\zeta^+$ such that $a_i^\zeta \in B, \alpha_i^\zeta < \lambda, [\alpha_j^\zeta \in a_i^\zeta \Leftrightarrow j = i]$ (e.g. by the first construction in case A of the proof of 2.2). So without loss of generality $\alpha_i^\zeta < \mu_\zeta^+, a_i^\zeta \in B_{\zeta+1}$. As we can replace B by a homomorphic image without loss of generality $\alpha_i^\zeta = \mu_\zeta + i$. We can find $X_\zeta \in [\{\alpha : \mu_\zeta \leq \alpha < \mu_\zeta^+\}]^{\mu_\zeta^+}$ for $\zeta < \theta$ such that $(\forall a \in B_\zeta)((\forall \alpha \in X_\zeta)(\alpha \in a) \vee (\forall \alpha \in X_\zeta)(\alpha \notin a))$ (like case A of the proof of 2.1, so actually there it suffices). Let $X = \bigcup_{\zeta<\theta} X_\zeta$ and let $Y_\zeta = \bigcup_{\varepsilon<\zeta} X_\varepsilon$.
Now let $h : B \to \mathscr{P}(X)$ be $h(a) = a \cap X$ and let $B' = \mathrm{Rang}(h)$. Let $B'' = \{a \subseteq X :$ for every $i < \theta, X_i \subseteq a$ or $X_i \cap a = \emptyset\}$.
Let B^+ be the Boolean Algebra of subsets of X generated by $B'' \cup B'$. Now clearly

$$\begin{aligned}
|\mathrm{Ult}(B')| &\leq |\mathrm{Ult}(B^+)| \\
&\leq |\{p \in \mathrm{Ult}(B^+) : \text{for every } i < \theta, \text{ we have } Y_i \notin p\}| + \sum_{i<\theta} |\mathrm{Ult}(B^+ \restriction Y_i)| \\
&\leq |\mathrm{Ult}(B'')| + \sum_{i<\theta} 2^{2^{|Y_i|}} \leq 2^{2^\theta} + \sum_{i<\theta} 2^{2^{|Y_i|}} \leq \lambda.
\end{aligned}$$

2) As in the proof of part (1) without loss of generality B is a subalgebra of $\mathscr{P}(\lambda)$. Let $\theta, \langle \mu_\zeta : \zeta < \theta \rangle, \langle B_i : i < \theta \rangle$ be as we have gotten in the proof of part (1); without loss of generality $\mu < \mu_\zeta$. Choose $\mu'_\zeta \in (\theta, \mu)$ increasing with ζ such that $\mu = \sum_{\zeta<\theta} \mu'_\zeta$. Let $Y'_\zeta \subseteq [\mu_\zeta, \mu_\zeta^+)$ be of cardinality μ'_ζ and let $Y' = \bigcup_{\zeta<\theta} Y'_\zeta$ and $B'_\zeta = \{a \cap \bigcup_{\varepsilon<\zeta} Y'_\varepsilon : a \in B\}$ and $B' = \{a \cap Y' : a \in B\}$. So B', B'_ζ are homomorphic images of B. If some B'_ζ has cardinality $\geq \mu$ then $|B'_\zeta| \in [\mu, 2^{\mu'_\zeta}) \in [\mu, \sum_{\kappa<\mu} (2^\kappa)^+)$ and $|\mathrm{Ult}(B'_\zeta)| \leq 2^{|B'_\zeta|} \leq 2^{2^{\mu'_\zeta}}$ so B'_ζ is as required. Otherwise B' is of cardinality

$\in [\mu, \prod_{\zeta<\theta} B'_\zeta] \subseteq [\mu, \mu^\theta) \subseteq [\mu, \sum_{\kappa<\mu}(2^\kappa)^+)$ moreover as above $\geq \mu$ by the present assumption and $\leq \sup\{|B'_\zeta| : \zeta < \theta\}$ as in part (1) hence $\leq \mu$ so $= \mu$. Lastly we can bound the number of ultrafilters as in the proof of part (1). $\square_{2.6}$

2.8 Observation: 1) Let B_σ be the Boolean Algebra generated freely by $\{x_\alpha : \alpha < \sigma\}$ and B^c_σ its completion. Then there is a homomorphic image B'_σ of B^c_σ say by f such that B'_σ is a Boolean Algebra of subsets of σ and $[\sigma]^{<\aleph_0} \subseteq f(B^c_\sigma) = B'_\sigma \subseteq B^*_\sigma = \{X \subseteq \sigma : |X| \leq \aleph_0 \text{ or } |\sigma \backslash X| \leq \aleph_0\}$.
2) It follows that B'_σ has $\sigma^{\aleph_0}$ elements and $\sigma^{\aleph_0} + \beth_2$ ultrafilters.
3) We can arrange that $B'_\sigma = B^*_\sigma$.

2.9 Remark. Note that though the homomorphic image of a complete Boolean algebra is not necessarily complete (e.g. $\mathscr{P}(\omega)$ finite), it satisfies the countable separation principle.

Proof. 1) Let for $\alpha < \sigma, B^c_{\sigma,\alpha}$ be the complete subalgebra of B^c_σ which $\{x_\beta : \beta < \sigma$ and $\beta \neq \alpha\}$ generated, so for every $b \in B^c_{\sigma,\alpha} \backslash \{0,1\}$ we have $b \cap a_\alpha > 0$ & $b - a_\alpha > 0$. Let D be an ultrafilter on B^c_σ such that $\{-x_\alpha : \alpha < \sigma\} \subseteq D$ and for $\alpha < \sigma$ let D_α be an ultrafilter on B^c_σ such that: $D_\alpha \cap B^c_{\sigma,\alpha} = D \cap B^c_{\sigma,\alpha}$ and $a_\alpha \in D_\alpha$. We define a homomorphism $f : B^c_\sigma \to \mathscr{P}(\sigma)$ by $f(b) = \{\alpha < \sigma : b \in D_\alpha\}$. Clearly f is a homomorphism. Let B'_σ be the range of f. Now $f(x_\alpha) = \{\alpha\}$ for $\alpha < \sigma$ so B'_σ contains all singletons from σ. Next suppose $b \in B^c_\sigma$. Hence for some $c_n \in B_\sigma$ for $n < \omega$ we have $b = \sup\{b_n : n < \omega\}$. Hence for some $\alpha_n < \sigma$ (for $n < \omega$) and (infinite) Boolean term $\tau, b = \tau(\ldots, x_{\alpha_n}, \ldots)_{n<\omega}$, so $\alpha \in \sigma \backslash \{\alpha_n : n < \omega\} \Rightarrow [\alpha \in f(b) \leftrightarrow b \in D_\alpha \leftrightarrow b \in D]$, hence $f(b)$ contains $\sigma \backslash \{\alpha_n : n < \omega\}$ or is disjoint to it hence $f(b) \in B^*_\sigma$, so we are done.
2) Now B'_σ as a homomorphic image of a complete Boolean Algebra satisfies the countable separation principle hence satisfies $|B'_\sigma| = |B'_\sigma|^{\aleph_0}$, but $\sigma = |[\sigma]^{<\aleph_0}| \leq |B'_\sigma| \leq |[\sigma]^{\aleph_0}| \times 2 = \sigma^{\aleph_0}$ so necessarily $|B'_\sigma| = \sigma^{\aleph_0}$. Also $\sigma^{\aleph_0} \leq |B'_\sigma| \leq |\text{Ult}(B'_\sigma)| \leq 1 + \Sigma\{|\{D : D \text{ an ultrafilter on } B'_\sigma \text{ to which } a \text{ belongs}\}| : a \in [\sigma]^{\leq \aleph_0}\}| \leq 2^{2^{\aleph_0}} \times |B'_\sigma| = \sigma^{\aleph_0} + 2^{2^{\aleph_0}}$ and for any $a \in [\sigma]^{\aleph_0}$ we have $\{b \cap a : b \in B'_\sigma\} = \mathscr{P}(a)$ and necessarily there is such a hence $|\text{Ult}(B'_\sigma)| \geq |\text{Ult}(\mathscr{P}(a))| = 2^{2^{\aleph_0}}$, so together B'_σ has $\sigma^{\aleph_0} + \beth_2$ ultrafilters.
3) Clearly $a \in B'_\sigma \cap [\sigma]^{\leq \aleph_0} \Rightarrow \mathscr{P}(a) \subseteq B'_\sigma$, hence there are no two disjoint members of $[\sigma]^{\leq \aleph_0} \backslash B'_\sigma$ (as we can find $b \in B'_\sigma$ separating them and b or $\sigma \backslash b$ is in $[\sigma]^{\aleph_0}$ and contains one of them). If $[\sigma]^{\leq \aleph_0} \subseteq B'_\sigma$ we are done, if not let $X^* \in [\sigma]^{\leq \aleph_0} \backslash B'_\sigma$ and replace σ by $\sigma \backslash X^*$. $\square_{2.8}$

§3 If $d(B)$ is small, then depth or ind are not tiny

The following definition 3.1 inspires the main result of this section 3.3 though is not used. The result says that e.g. $d(B) \leq \lambda = \lambda^\kappa < |B|$ & $\kappa \geq \text{ind}^+(B) \Rightarrow \kappa \leq \text{Depth}(B)$.

3.1 Definition. 1) We say $\bar{a} = \langle a_\beta : \beta < \beta^* \rangle$ is semi-independent in B if: it is a sequence of distinct elements in a Boolean Algebra B and some ideal I on B witness the semi-independence of $\bar{a}$ in B, which means:

$\boxdot_{B,I,\bar{a}}$ for any $\alpha < \gamma < \beta^*$ and $b \in \langle a_\beta : \beta < \alpha \rangle_B$ we have

$(*)_1$ $b \in I \Rightarrow b \cap a_\gamma = b \cap a_\alpha$

$(*)_2$ $b \notin I \Rightarrow \{b \cap a_\alpha, b \cap a_\gamma\}$ is an independent set in $B \restriction b$, (so e.g. $b \cap a_\alpha > 0$)

$(*)_3$ $b \notin I, b \cap a_\alpha \in I \Rightarrow b \cap a_\gamma \in I$

$(*)_4$ $b \notin I, b - a_\alpha \in I \Rightarrow b - a_\gamma \in I$.

2) $si^+(B) = \text{Min}\{\lambda :$ there is no $\langle a_\beta : \beta < \lambda \rangle$ in B which is semi-independent$\}$ and we say $\bar{a} = \langle a_\beta : \beta < \lambda \rangle$ and I witness $\lambda < si^+(B)$ if I witnesses the semi-independence of $\bar{a}$ in B. Let $si(B) = \sup\{\lambda :$ there is a semi-independent sequence $\langle a_\beta : \beta < \lambda \rangle$ in $B\}$.
3) $si^{1+}(B)$ is defined similarly to $si^+(B)$ except that we use $\langle a_\beta : \beta < \lambda + 1 \rangle$. We say $I, \langle a_\beta : \beta \leq \lambda \rangle$ witness $\lambda < si^{1+}(B)$.

Remark. In fact as it is known that $t^+(B) \leq \text{ind}^+(B) +$ Depth$^+(B)$, below part (2) of 3.2 follows from part (3) of 3.2; proof included for familiarity with si.

3.2 Fact. 0) $si(B), si^{1+}(B) \leq si^+(B) \leq (si(B))^+$.
1) $\text{ind}^+(B) \leq si^+(B)$.
2) $si^+(B) \leq t^+(B)$.
3) $si^+(B) \leq \text{ind}^+(B) + \text{Depth}^+(B)$.

Proof. 0) Read the definition.
1) $\text{ind}^+(B) \leq si^+(B)$ holds as independent implies semi-independent for the ideal $\{0_B\}$.
2) Let $\lambda < si^+(B)$ and $\langle a_\beta : \beta < \lambda \rangle, I$ witness it.

Let D be an ultrafilter on B disjoint to I. Without loss of generality $(\forall\alpha)(a_\alpha \in D)$ or $(\forall\alpha)(a_\alpha \notin D)$. As we can replace $\langle a_\beta : \beta < \lambda\rangle$ by $\langle -a_\beta : \beta < \lambda\rangle$, without loss of generality $\bigwedge\limits_{\beta<\lambda} a_\beta \in D$. Let $\beta_0 < \ldots < \beta_{m-1} < \beta_m < \ldots < \beta_{n-1}$. Let for $k \in [m,n], b_k = \bigcap\limits_{\ell<m} a_{\beta_\ell} \cap \bigcap\limits_{\ell\in[m,k)} (-a_{\beta_\ell})$. So $b_m \in D$ as $b_{\beta_0}, \ldots, b_{\beta_{m-1}} \in D$. So $b_m \notin I$.
We should prove that $b_n > 0$, and let $\beta_{n+\ell} =: \beta_{n-1} + 1 + \ell$.
Let $k \in [m,n]$ be maximal such that $b_k \notin I$.

So k is well defined, if $k = n$ we are done as then $b_n \notin I \Rightarrow b_n > 0$, so we can assume $k < n$. Clearly $b_k \in \langle b_\beta : \beta < \beta_k\rangle_B$ holds and $\beta_k < \beta_{k+1}$ so in Definition 3.1(1) with $(b_k, \beta_k, \beta_{k+1})$ here standing for (b,α,γ) there the demands hence the conclusions of $(*)_2 - (*)_4$ holds. Now by trivial reasons we have $b_{k+1} \cap (-a_{\beta_{k+1}}) = b_k \cap (-a_{\beta_k}) \cap (-a_{\beta_{k+1}}) = (b_k \cap (-a_{\beta_k})) \cap (b_k \cap (-a_{\beta_{k+1}}))$; now by 3.1(1) $(*)_2$ this is > 0 and by the maximality of $k, b_{k+1} \in I$, so by 3.1(1) $(*)_1$ we have $b_{k+1} \cap (-a_{\beta_{k+1}}) = b_{k+1} \cap (-a_{\beta_{k+2}}) = \ldots$ hence $b_{k+1} \cap (-a_{\beta_k}) = b_{k+2} = \ldots = b_n$ so $b_n \neq 0$.
3) Let $\langle a_\beta : \beta < \lambda\rangle, I$ witness $\lambda < si^+(B)$.

If $\langle a_\beta : \beta < \lambda\rangle$ is independent we are done, so assume not, so let $\beta^* < \lambda$ be minimal such that $\langle a_\beta : \beta \leq \beta^*\rangle$ is not independent modulo I; so $\langle a_\beta : \beta < \beta^*\rangle$ is independent modulo I and for some $b \in \langle a_\beta : \beta < \beta^*\rangle_B$ satisfying $b > 0$, (so $b \notin I$ by the assumption on β^*), we have $b \cap a_{\beta^*} \in I$ or $b - a_{\beta^*} \in I$. Now by symmetry (as we can replace $\langle a_\alpha : \alpha < \lambda\rangle$ by $\langle -a_\alpha : \alpha < \lambda\rangle$) without loss of generality the former holds.

Hence $\beta \in [\beta^*, \lambda) \Rightarrow b \cap a_\beta \in I$ (by $(*)_3$ of $\boxdot_{B,I,\bar a}$ from Definition 3.1(1)) hence $\beta^* \leq \beta < \gamma_1 < \gamma_2 < \lambda \Rightarrow b \cap a_\beta \cap a_{\gamma_1} = b \cap a_\beta \cap a_{\gamma_2}$ (by $(*)_1$ of $\boxdot_{B,I,\bar a}$ from Definition 3.1(1) applied to $b \cap b_\beta$) hence $\beta(*) \leq \beta < \gamma < \lambda \Rightarrow b \cap a_\beta \cap a_{\beta+1} = b \cap a_\beta \cap a_{\beta+1} \cap a_\gamma = b \cap a_\beta \cap a_{\beta+1} \cap a_\gamma \cap a_{\gamma+1} \leq b \cap a_\gamma \cap a_{\gamma+1}$.
So $\langle b \cap a_\beta \cap a_{\beta+1} : \beta \in [\beta^*, \lambda)\rangle$ is increasing. Now for β from $[\beta^*, \lambda), b \notin I, b \cap a_\beta \in I$ hence $b - (b \cap a_\beta \cap a_{\beta+1})$ belongs to $\langle a_j : j < \beta + 2\rangle_B$ but does not belong to I hence by $\boxdot_{B,I,\bar a}(*)_2$ we have $(b - (b \cap a_\beta \cap a_{\beta+1})) \cap a_{\beta+2} \cap a_{\beta+3} > 0$ hence $b \cap a_\beta \cap a_{\beta+1} \neq b \cap a_{\beta+2} \cap a_{\beta+3}$ hence $\langle b \cap a_{2\beta} \cap b_{2\beta+1} : 2\beta \in [\beta^*, \lambda)\rangle$ is strictly increasing in B, a required. $\square_{3.2}$

3.3 Claim. *Assume B is an infinite Boolean Algebra satisfying $\kappa \geq$ ind$^+(B)$ and $d(B) \leq \lambda = \lambda^{<\kappa} < |B|$. Then Depth$^+(B) > \kappa$.*

Remark. I think that:

(∗) if in addition $\lambda^+ \to (\mu+1)^3_\sigma$ for every $\sigma < \kappa$, then Depth$^+(B) > \mu$.

Proof. Let $\langle a_\alpha : \alpha < \lambda^+ \rangle$ be a list of pairwise distinct elements of B. As $\lambda \geq d(B)$ without loss of generality B is a subalgebra of $\mathscr{P}(\lambda)$. Let $B_\alpha = \langle \{a_\beta : \beta < \alpha\} \rangle_B$ and

$$E =: \{\delta < \lambda^+ : B_\delta \cap \{a_\alpha : \alpha < \lambda^+\} = \{a_\alpha : \alpha < \delta\}\}$$

clearly it is a club of λ^+.

For every $\delta \in S_0 =: \{\delta \in E : \mathrm{cf}(\delta) \geq \kappa\}$ we let $I_\delta =: \{b \in B_\delta : a_\delta \cap b \in B_\delta\}$, so I_δ is an ideal of the subalgebra B_δ of B. Let $\delta \in S_0$, and J be an ideal on B_δ and now we try to choose by induction on $i < \kappa$, an ordinal $\alpha_{\delta,J,i}$ such that:

(a) $\alpha_{\delta,J,j} < \alpha_{\delta,J,i} < \delta$ for $j < i$

(b) $\langle a_{\alpha_{\delta,J,j}}/J : j < i \rangle$ is independent in the Boolean Algebra B_δ/J.

If we succeed, then $\langle a_{\alpha_{\delta,J,i}} : i < \kappa \rangle$ contradict the assumption $\kappa \geq \mathrm{ind}^+(B)$, so for some $i(\delta, J) < \kappa$ we have: $\alpha_{\delta,J,i}$ is defined iff $i < i(\delta, J)$. This is true in particular for $J = I_\delta$. So for some stationary $S_1 \subseteq S_0$ and $i(*) < \delta$ and $\langle \alpha_i : i < i(*) \rangle$, an increasing sequence of ordinals $< \lambda^+$, the set $S_1 = \{\delta \in S_0 : i(\delta, I_\delta) = i(*)$ and $i < i(*) \Rightarrow \alpha_{\delta,I_\delta,i} = \alpha_i\}$ is stationary.
Let $\langle b_\gamma : \gamma < \gamma(*) \rangle$ list the non-zero Boolean combinations of $\{a_{\alpha_i} : i < i(*)\}$ so $\gamma(*) < \kappa$ (as $\kappa \geq \mathrm{ind}^+(B) \geq \aleph_1$) and for $\delta \in S_1$ let $Y_\delta \subseteq \gamma(*)$ be such that $\gamma < \gamma(*)$ & $\delta \in S_1 \Rightarrow [b_\gamma \in I_\delta \equiv \gamma \in Y_\delta]$. As B is a subalgebra of $\mathscr{P}(\lambda)$ we can choose a function H such that $\mathrm{Dom}(H) = B \backslash \{\emptyset\}, H(c) \in c$ and define, for $\mathbf{s} \in \{0,1\}$ the function $H^{\mathbf{s}}_\delta$ with domain $\{c \in B_\delta : c \cap a^{\mathbf{s}}_\delta > 0_B\}$ by $H^{\mathbf{s}}_\delta(c) = H(c \cap a^{\mathbf{s}}_\delta)$. Choose a function F_δ such that $\mathrm{Dom}(F_\delta) = I_\delta$ and $c \in I_\delta \Rightarrow F_\delta(c) = c \cap a_\delta \in B_\delta$. Again for some $\langle x^{\mathbf{s}}_\gamma : \gamma < \gamma(*)$ and $\mathbf{s} = 0, 1 \rangle$ and Y we have

$$\begin{aligned} S_2 = \{\delta \in S_1 : & Y_\delta = Y \text{ and for } \gamma \in \gamma(*) \backslash Y \text{ we have} \\ & H(b_\gamma \cap a^{\mathbf{s}}_\delta) = x^{\mathbf{s}}_\gamma \text{ for } \mathbf{s} = 0, 1\} \end{aligned}$$

is a stationary subset of λ^+.

For each $\delta \in S_2$ and $\mathbf{t} \in \{0,1\}$ and $\gamma \in Y$ we try to choose by induction on $i < \kappa, B_{\delta,\gamma,\mathbf{t},i}$ and an ordinal $\beta_{\delta,\gamma,\mathbf{t},i}$ such that:

(a)′ $\beta_{\delta,\gamma,\mathbf{t},j} < \beta_{\delta,\gamma,\mathbf{t},i} < \delta$ for $j < i$

(b)′ $\beta_{\delta,\gamma,\mathbf{t},i} > \alpha_j$ for $j < i(*)$

(c)′ $a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}} \cap b_\gamma \in I_\delta$
(remember $a^{\mathbf{t}}$ is a if $\mathbf{t} = 1$ and is $-a$ if $\mathbf{t} = 0$)

$(d)'$ $B_{\delta,\gamma,\mathbf{t},i}$ is the smallest subalgebra of B_δ containing $\{b_\gamma\} \cup \{a_{\alpha_j} : j < i(*)\} \cup \{a_{\beta_{\delta,\gamma,\mathbf{t},j}} : j < i\}$

$(e)'$ if $c \in I_\delta \cap B_{\delta,\gamma,\mathbf{t},i}$ then $a_{\beta_{\delta,\gamma,\mathbf{t},i}} \cap c = F_\delta(c) = a_\delta \cap c$ (in fact just $c \in \{b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} : j < i\}$ suffice)

$(f)'$ if $c \in B_{\delta,\gamma,\mathbf{t},i}$ and $\mathbf{s} \in \{0,1\}$ then $c \cap a^{\mathbf{s}}_\delta \neq 0 \Rightarrow H(c \cap a^{\mathbf{s}}_\delta) \in a^{\mathbf{s}}_{\beta_{\delta,\gamma,\mathbf{t},i}}$ (in fact just $c \in \{b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} : j < i\}$ suffice).

Now

$(*)$ If for some $\delta \in S_2$ and $\gamma \in \gamma(*) \backslash Y$ and $\mathbf{t} \in \{0,1\}$ we succeed, <u>then</u> we can prove $\text{Depth}^+(B) > \kappa$.

[Why? We just prove that $\langle a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}} : i < \kappa \rangle$ is strictly increasing. Let $j < i < \kappa$, so by clause $(c)'$ we know that $b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \in I_\delta$ but $b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \in B_{\delta,\gamma,\mathbf{t},i}$ by clause $(d)'$ hence $a_{\beta_{\delta,\gamma,\mathbf{t},i}} \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} = F_\delta(b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}) = a_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}$ by clause $(e)'$ and it follows that $(-a_{\beta_{\delta,\gamma,\mathbf{t},i}}) \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}} = (-a_\delta) \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}$.
So as $\mathbf{t} \in \{0,1\}$ we always have $a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}} \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} = a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}$.
So

$$x \in a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}} \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \Leftrightarrow x \in a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}.$$

So if $x \in a^{\mathbf{t}}_\delta \cap b_\gamma$ then

$$x \in a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}} \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \Leftrightarrow x \in a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}.$$

hence

$$x \in a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \Rightarrow x \in a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}}.$$

The above statement means

$$x \in a^{\mathbf{t}}_\delta \cap b_\gamma \Rightarrow [x \in a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \Rightarrow x \in a^{\mathbf{t}}_{\beta,\delta,\mathbf{t},i}]$$

hence $\langle a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}} : i < \kappa \rangle$ is $\leq$-increasing. But $b_\gamma \notin I_\delta$ hence $a^{\mathbf{t}}_\delta \cap b_\gamma \notin B_\delta$, hence for $j < i$, $a^{\mathbf{t}}_\delta \cap b_\gamma \neq a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}$ (as by clause $(c)'$ we know that $b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \in I_\delta$ so $a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \in B_\delta$).
So $0 < a^{\mathbf{t}}_\delta \cap b_\gamma - a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} = (b_\gamma - a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}) \cap a^{\mathbf{t}}_\delta$ hence $x =: H((b_\gamma - a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}) \cap a^{\mathbf{t}}_\delta) \in a^{\mathbf{t}}_\delta$ is well defined and belongs to $a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}}$ by clause $(f)'$. So x belongs to

$a^{\mathbf{t}}_\delta, b_\gamma - a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}}, a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}}$ so it exemplifies $a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},j}} \neq a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}}$. So $\langle a^{\mathbf{t}}_\delta \cap b_\gamma \cap a^{\mathbf{t}}_{\beta_{\delta,\gamma,\mathbf{t},i}} : i < \kappa\rangle$ is strictly increasing.]

We have proved $(*)$, so assume toward contradiction that for $\delta \in S_2, \gamma \in \gamma(*)\backslash Y$ and $\mathbf{t} \in \{0,1\}$ the ordinal $\beta_{\delta,\gamma,\mathbf{t},i}$ is well defined iff $i < j(\delta,\gamma,\mathbf{t})$ where $j(\delta,\gamma,\mathbf{t}) < \kappa$. So again for some ordinals $j(\gamma,\mathbf{t})$ and $\beta_{\gamma,\mathbf{t},i}$ (for $\gamma \in \gamma(*)\backslash Y, \mathbf{t} \in \{0,1\}$ and $i < j(\gamma,\mathbf{t})$) and $B_{\gamma,\mathbf{t},i}$ we have

$$\begin{aligned} S_3 = \Big\{\delta \in S_2 :& \text{for every } \gamma \in \gamma(*)\backslash Y \text{ we have} \\ & j(\delta,\gamma,\mathbf{t}) = j(\gamma,\mathbf{t}), [i < j(\gamma,\mathbf{t}) \Rightarrow \beta_{\delta,\gamma,\mathbf{t},i} = \beta_{\gamma,\mathbf{t},i}] \\ & \text{and } [i \le j(\gamma,\mathbf{t}) \Rightarrow B_{\delta,\gamma,\mathbf{t},i} = B_{\gamma,\mathbf{t},i}]\Big\} \end{aligned}$$

is stationary.
So for some stationary $S_4 \subseteq S_3$ we have: $\mathbf{s} \in \{0,1\}$ & $\delta_1,\delta_2 \in S_4 \Rightarrow H^{\mathbf{s}}_{\delta_1} \restriction \bigcup\limits_{\gamma,\mathbf{t},i} B_{\gamma,\mathbf{t},i} = H^{\mathbf{s}}_{\delta_2} \restriction \bigcup\limits_{\gamma,\mathbf{t},i} B_{\gamma,\mathbf{t},i}$ and $F_{\delta_1}(y \cap a^{\mathbf{t}}_{\delta_1}) = F_{\delta_2}(y \cap a^{\mathbf{t}}_{\delta_1})$ for any $y \in \bigcup\limits_{\gamma,\mathbf{t},i} B_{\gamma,\mathbf{t},i}$ belonging to I_{δ_2} (equivalently I_{δ_1}) and $I_{\delta_1} \cap \bigcup\limits_{\gamma,\mathbf{t},i} B_{\gamma,\mathbf{t},i} = I_{\delta_2} \cap \bigcup\limits_{\gamma,\mathbf{t},i} B_{\gamma,\mathbf{t},i}$. Let $\delta_1 < \delta_2$ be in S_4 and we shall get a contradiction as follows.

Recalling the choice of $i(\delta_2, I_{\delta_2}), \langle \alpha_{\delta_2,I_{\delta_2},i} : i < i(\delta_2,I_{\delta_2})\rangle$, clearly δ_1 cannot serve as $\alpha_{\delta_2,I_{\delta_2},i(\delta_2,I_{\delta_2})}$, which means that for some $\mathbf{t} \in \{0,1\}$ and $b \in \langle a_{\alpha_{\delta_2,I_{\delta_2},i}} : i < i(\delta_2,I_{\delta_2}\rangle_B \backslash \{0_B\}$ we have $a^{\mathbf{t}}_{\delta_1} \cap b \in I_{\delta_2}$, note that necessarily $b \notin I_{\delta_2}$ by clause (b) for δ_2. However, as $\delta_2 \in S_1$ clearly $i(\delta_2, I_{\delta_2}) = i(*), \alpha_{\delta_2,I_{\delta_2},i} = \alpha_i$ for $i < i(*)$, so $b \in \langle a_{\alpha_i} : i < i(*)\rangle_B \backslash \{0_B\}$ and $b \notin I_{\delta_2}$ hence $b = b_\gamma$ for some $\gamma \in \gamma(*)\backslash Y_{\delta_2} = \gamma(*)\backslash Y$. So as $\delta_2 \in S_2$, we have defined $\langle \beta_{\delta_2,\gamma,\mathbf{t},i} : i < j(\delta_2,\gamma,\mathbf{t})\rangle$ but $\delta_2 \in S_3$ hence $\beta_{\delta_2,\gamma,\mathbf{t},i} = \beta_{\gamma,\mathbf{t},i}$ for $i < j(\delta_2,\gamma,\mathbf{t}) = j(\gamma,\mathbf{t})$. Also $\delta_1 \in S_3$ hence $\beta_{\delta_1,\gamma,\mathbf{t},i} = \beta_{\gamma,\mathbf{t},i}$ for $i < j(\delta_1,\gamma,\mathbf{t}) = j(\gamma,\mathbf{t})$. Can δ_1 serve as $\beta = \beta_{\delta_2,\gamma,\mathbf{t},j(\gamma,\mathbf{t})}$? Now clauses $(a)', (b)'$ are trivial[1], clause $(c)'$ was proved above (as b_{δ_1} cannot serve as $\alpha_{\delta_2,I_{\delta_2},i(\delta_2,I_{\delta_2})}$ and the choice of γ and $\mathbf{t}$) and clause $(d)'$ is a definition. Now clause $(e)'$ holds as by the choice of S_4 above for $c \in I_{\delta_2} \cap B_{\delta_2,\gamma,\mathbf{t},j(\delta_2,\gamma,\mathbf{t})} = I_{\delta_2} \cap B_{\gamma,\mathbf{t},j(\gamma,\mathbf{t})} = I_{\delta_1} \cap B_{\delta_1,\gamma,j(\delta_1,\gamma,\mathbf{t})}$ we have $a_{\delta_1} \cap c = F_{\delta_1}(c) = F_{\delta_2}(c) = a_{\delta_2} \cap c$ as required. Lastly for clause $(f)'$, let $c \in B_{\delta_2,\gamma,\mathbf{t},i}$ and $\mathbf{s} \in \{0,1\}$ and $c \cap a^{\mathbf{s}}_{\delta_2} \neq 0_B$ then, remembering $\delta_1,\delta_2 \in S_4$, we have $H(c \cap a^{\mathbf{s}}_{\delta_2}) = H^{\mathbf{s}}_{\delta_2}(c) = H^{\mathbf{s}}_{\delta_1}(c) = H(c \cap a^{\mathbf{s}}_{\delta_1})$ hence $(c \cap a^{\mathbf{s}}_{\delta_1}) \cap a^{\mathbf{s}}_{\delta_2} \neq \emptyset, H(c \cap a^{\mathbf{s}}_{\delta_2}) \in a^{\mathbf{s}}_{\delta_1}$ as required.

[1]in particular, $\delta_1 > \alpha_i$ for $i < i(*)$ because $\delta_1 \in S_1$ and by clause (a), $\alpha_i = \alpha_{\delta_1,I_\delta,i} < \delta_1$; similarly $\delta_1 > \beta_{\gamma,\mathbf{t},i}$

So actually δ_1 is a good candidate for $\beta_{\delta,\gamma,\mathbf{t},j(\gamma,\mathbf{t})}$, a contradiction to the choice of $j(\gamma,\mathbf{t}) = j(\delta_2,\gamma,\mathbf{t})$, contradiction. $\square_{3.3}$

3.4 Conclusion. 1) If $\kappa =$ Depth$^+(B) +$ ind$^+(B)$, then $|B| \le d(B)^{<\kappa}$.

3.5 Remark. 1) Instead $\lambda = d(B)^{<\kappa}$ we can use $\lambda = \lambda^{<\kappa}$ such that:

(∗) there are no $\gamma^* < \kappa$ and $b_\gamma > 0$ for $\gamma < \gamma^*$ and $\langle a_\alpha : \alpha < \lambda^+\rangle$ such that for every $\alpha < \beta < \lambda^+$, for some $\gamma < \gamma^*$ in the Boolean Algebra $B \restriction b_\gamma$ we have: $a_\gamma \cap a_\alpha \ne b_\gamma \cap a_\beta$ are disjoint or have disjoint compliment.

2) Can think of parallel replacing in $(*)_2$ of 3.1(1), 2 by n, that is: let $2 \le n < \omega$. We say $\langle a_\beta : \beta < \beta^*\rangle$ is n-semi-independent (in the Boolean Algebra B) if some I witnesses it which means $a_\beta \in B$ are pairwise distinct and if for $\alpha \le \gamma_0 < \ldots < \gamma_{n-1}$ and $b \in \langle a_\beta : \beta < \alpha\rangle_B$ we have $(*)_1, (*)_3, (*)_4$ and

$(*)_{2,n}$ in $\{a_{\gamma_\ell} \cap b : \ell < n\}$ is independent in $B \restriction b$.

There are some variants and I have not tried if this gives something interesting.

§4 On omitting cardinals by compact spaces

We continue Juhasz Shelah [JuSh 612]. We investigate what homomorphic images some Boolean Algebras may have, and (in 4.16) prove the topological analog of §2, showing the existence of some subspaces for Hausdorff spaces (not, necessarily compact).
Recall

4.1 Definition. 1) $\mathbf{U}_\theta(\mu) = \text{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\mu]^\theta \text{ and } (\forall X \in [\mu]^\theta)(\exists a \in \mathscr{P}) (|a \cap X| = \theta)\}$.
2) Let $\mathfrak{a}_\theta(\mu) = \text{Min}\{|\mathscr{A}| : \mathscr{A} \subseteq [\mu]^\theta$ is θ-MAD$\}$ where $\mathscr{A} \subseteq [\mu]^\theta$ is called θ-AD if $A \neq B \in \mathscr{A} \Rightarrow |A \cap B| < \theta$ and we have $\mathscr{A}$ is θ-MAD when in addition $\mathscr{A}$ is maximal under those restrictions.

4.2 Remark. 1) In the case $\mu \geq 2^\theta$, in which we are interested, $\mathbf{U}_\theta(\mu) = \mathfrak{a}_\theta(\mu) = |\mathscr{A}|$ whenever $\mathscr{A}$ is θ-MAD for μ. (See more and connection of pcf theory [Sh 506], [Sh 589], but we do not use any non-trivial fact.)
2) We could have used other variants like $\mu^{[\theta,\sigma]} = \text{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\mu]^\theta$, and every $A \in [\mu]^\theta$ is the union of $< \sigma$ members of $\mathscr{P}\}$ <u>or</u> $\text{cov}(\mu, \theta^+, \theta, \sigma) = \mu$.
3) Our main interest below is in regular θ.
4) Recall that $B \models$ "$b = \bigcup\limits_{i \in \mathscr{U}} a_i$" if b is the lub of $\{a_i : i \in \mathscr{U}\}$ in B and then we say "$\bigcup\limits_{i \in \mathscr{U}} a_i$ exists (in the sense of $\mathscr{B}$)".

4.3 Definition. 1) For $J_1 \subseteq J_2$ ideals of a Boolean Algebra $\mathscr{B}$, we say J_2 is θ-full over J_1 inside $\mathscr{B}$, <u>if</u> (α) or at least (β) where

- (α) θ is regular and for every $X \in [J_1]^\theta$ there is $b \in J_2$ such that $|\{x \in X : \mathscr{B} \models x \leq b\}| = \theta$
- (β) θ not necessarily regular and for every $x_i \in J_1$ for $i < \theta$ there is $b \in J_2$ such that $|\{i < \theta : x_i \leq b\}| = \theta$.

(When θ is regular, they are equivalent).
2) The ideal J of $\mathscr{B}$ is θ-full if: J is θ-full over J inside $\mathscr{B}$.
3) J is τ-local inside $\mathscr{B}$, <u>if</u> $|\{b \in \mathscr{B} : \mathscr{B} \models b \leq x\}| \leq \tau$ for $x \in J$.
4) In part (1), J_2 is strongly θ-full over J_1 inside $\mathscr{B}$ if (α) or just (β) where

- (α) θ is regular and for every $X \in [J_1]^\theta$ for some $Y \in [X]^\theta$, for every $Z \subseteq Y$ we have $\bigcup\limits_{b \in Z} a$ exists in the sense of $\mathscr{B}$ and belongs to J_2

(β) θ is not necessarily regular and for every $x_i \in J$ for $i < \theta$, for some $Y \in [\theta]^\theta$, for every $Z \subseteq Y$, the union $\bigcup_{i \in Z} x_i$ in the sense of $\mathscr{B}$ exists and belongs to J_2.

(For θ regular (α), (β) are equivalent).
4A) Similarly in part (2).

<u>4.4 Fact</u>: 1) If J_2 is θ-full over J_1 inside $\mathscr{B}, h$ is a homomorphism from $\mathscr{B}$ onto $\mathscr{B}^*$ and $J^*_\ell = h(J_\ell)$ for $\ell = 1, 2$, <u>then</u> J^*_2 is θ-full over J^*_1 inside $\mathscr{B}^*$.
2) If J_2 is θ-full over J_1 inside $\mathscr{B}$ and J_2 is τ-local inside $\mathscr{B}$, <u>then</u> $\mathbf{U}_\theta(|J_1|) \leq \tau^\theta + |J_2|$.
3) If the ideal J of $\mathscr{B}$ is θ-full and τ-local inside $\mathscr{B}$, then $\mathbf{U}_\theta(|J|) \leq \tau^\theta + |J|$.
4) In parts 2) and 3), if $\tau = \theta$ then we get $\mathbf{U}_\theta(|J_1|) \leq |J_2|, \mathbf{U}_\theta(|J|) \leq |J|$ respectively.

Proof. 1) Trivial.
2) Let $J_1 = \{a_i : i < |J_1|\}$, let $\mathscr{P}_y = \{X \subseteq |J_1| : |X| = \theta$ and $(\forall i \in X)(\mathscr{B} \models a_i \leq y)\}$ for each $y \in J_2$, so $|\mathscr{P}_y| \leq \tau^\theta$. Lastly, let $\mathscr{P} = \cup\{\mathscr{P}_y : y \in J_2\}$ so $|\mathscr{P}| \leq |J_2| \times \sup_{y \in J_2} |\mathscr{P}_y| \leq \tau^\theta + |J_2|$. Easily $\mathscr{P}$ is as required in Definition 4.1.
3) Follows by part (2).
4) Similar to the proof of part (2) only now $\mathscr{P}_y = \{\{i < |J_1| : \mathscr{B} \models a_i \leq y\}\}$.

$\square_{4.4}$

<u>4.5 Fact</u>: 1) Assume

($*$) $\lambda < \kappa \leq \mu \leq \kappa^\lambda$ and $\Theta \subseteq \Theta_{\mu,\lambda} =: \{\theta \leq \lambda : \mathbf{U}_\theta(\mu) = \mu\}$ and $\sigma = \text{cf}(\sigma) \leq \lambda^+$ and for every $\theta \in \Theta$ we have $\text{cf}(\theta) \neq \sigma$, (clearly there is such cardinal: $\sigma = \lambda^+$).

<u>Then</u> there is a Boolean algebra $\mathscr{B}$ such that

(a) $|\mathscr{B}| = \mu$
(b) $\mathscr{B}$ is atomic with exactly κ atoms, say $[\kappa]^{<\aleph_0} \subseteq \mathscr{B} \subseteq \mathscr{P}(\kappa)$
(c) $\mathscr{B}$ has a maximal ideal J which is 2^λ-local, moreover $x \in J \Rightarrow |\{y : \mathscr{B} \models$ "$y \leq x$ & y an atom (of $\mathscr{B}$)"$\}| \leq \lambda$; in other words, $J \subseteq [\kappa]^{\leq\lambda}$
(d) J is θ-full (inside $\mathscr{B}$) for every $\theta \in \Theta$
(e) (α) if $2^\lambda \leq \mu$ then $\mathscr{P}(\lambda)$ is isomorphic to some $\mathscr{B} \restriction \{x : x \leq a\}$
(β) If $2^\lambda > \mu, \mathscr{B}_0 \subseteq \mathscr{P}(\lambda)$ has cardinality $\leq \mu$ then in (α) we can replace $\mathscr{P}(\lambda)$ by $\mathscr{B}_0$.

2) We can add in part (1)

- (f) if $2^\lambda \le \mu, x \in J$ and $y \subseteq x$, then $y \in \mathscr{B}$
- (g) $2^\lambda \le \mu \ \ \& \ \ \theta \in \Theta \Rightarrow J$ is strongly θ-full inside $\mathscr{B}$
- (h) if $2^\lambda \ge \mu$ and we fix $\mathscr{B}_0 \subseteq \mathscr{P}(\lambda)$ which has cardinality $\le \mu$ then for some subalgebra $\mathscr{B}_1$ of $\mathscr{P}(\lambda)$ extending $\mathscr{B}_0$ we have
  - (α) for every $x \in J \subseteq \mathscr{B}$ satisfying $|x| = \lambda$ we have $\mathscr{B} \restriction x = \mathscr{B} \cap \mathscr{P}(x)$ is isomorphic to $\mathscr{B}_1$
  - (β) $\forall x \in J \exists y \in I(x \subseteq y \ \ \& \ \ |y| = \lambda)$
  - (γ) if $x, y \in \mathscr{B}_1$ and $|x| = |y|$ then $\mathscr{B}_1 \restriction x \cong \mathscr{B}_1 \restriction y$.

Proof. 1), 2).

Case A: $2^\lambda \le \mu$.

As $\mu \le \kappa^\lambda$ we can find pairwise distinct $x_i \in [\kappa]^\lambda$ for $i < \mu, x_0 = \{i : i < \lambda\}$. Let J_0 be the ideal of $\mathscr{P}(\kappa)$ generated by $\{x_i : i < \mu\} \cup \big\{\{\alpha\} : \alpha < \kappa\big\}$, so $J_0 \subseteq [\kappa]^{\le \lambda}$ and $|J_0| = \mu$ as $\mu = |\{x_i :< \mu\}| \le |J_0| \le \sum\limits_{i<\mu} 2^{|x_i|} \le \mu + 2^\lambda = \mu$. We choose by induction on $\zeta \le \sigma$, an ideal J_ζ of $\mathscr{P}(\kappa), J_\zeta \subseteq [\kappa]^{\le\lambda}, |J_\zeta| = \mu, J_\zeta$ is increasing continuous in ζ. For $\zeta = 0, J_0$ was defined, for ζ limit let $J_\zeta = \bigcup\limits_{\varepsilon<\zeta} J_\varepsilon$. For $\zeta = \varepsilon + 1$, let $J_\varepsilon = \{a^\varepsilon_i : i < \mu\}$, for $\theta \in \Theta$ let $\mathscr{P}^\varepsilon_\theta \subseteq [\mu]^{\le\theta}$ exemplifies $\mathbf{U}_\theta(\mu) = \mu$ (which follows from $\theta \in \Theta$), so $|\mathscr{P}^\varepsilon_\theta| \le \mu$, and let J_ζ be the ideal of $\mathscr{P}(\kappa)$ generated by $J_\varepsilon \cup \{\bigcup\limits_{i\in x} a^\varepsilon_i : x \in \mathscr{P}^\varepsilon_\theta$ for some $\theta \in \Theta\}$, easy to check the inductive demand.

It is also easy to check that then $J_{\zeta+1}$ is θ-full over J_ζ (inside $\mathscr{P}(\kappa)$), when $\theta \in \Theta$. Let $J = J_\sigma, \mathscr{B} = J \cup \{\kappa \backslash x : x \in J\}$ = the Boolean subalgebra of $\mathscr{P}(\kappa)$ which J generates. Clearly J is θ-full over J inside $\mathscr{B}$. (If $\sigma = \lambda^+, X \subseteq J, |X| \le \lambda$, as $\mathrm{cf}(\theta) < \sigma = \mathrm{cf}(\sigma)$ and $\langle J_\zeta : \zeta \le \sigma\rangle$ is increasing continuous clearly for some $\zeta < \lambda^+$ we have $X \subseteq J_\zeta$, so $|X| \in \Theta \Rightarrow (\exists y \in J_{\zeta+1})[|X| = |\{x \in X : B \models x \in y\}|]$. If $\sigma < \lambda^+$ and $|X| \in \Theta$, then $\mathrm{cf}(\theta) \ne \sigma = \mathrm{cf}(\sigma)$ and $\langle J_\zeta : \zeta \le \sigma\rangle$ is increasing continuous hence for some $\zeta < \sigma, |X \cap J_\zeta| = \theta$ and proved as above.) So easily $\mathscr{B}, J$ are as required.

Case B: $2^\lambda > \mu$.

First, add $\mathscr{B}_0$ to $\bar{J}_0$. Second, we should replace above everywhere "J_ζ is the ideal of $\mathscr{P}(\kappa)$ such that ..." "J_ζ is the Boolean subring of $\mathscr{P}(\kappa)$ such that ..."; and also choose $\mathfrak{B}_\zeta \prec (\mathscr{H}(\chi), \in)$ of cardinality μ increasing continuous, $\langle \mathfrak{B}_\varepsilon : \varepsilon \le \xi\rangle \in \mathfrak{B}_{\xi+1}, \mu + 1 \subseteq \mathfrak{B}_\zeta$ and use $J_\zeta = \mathfrak{B}_\zeta \cap [\mu]^{\le\lambda}$. $\square_{4.5}$

4.6 Claim. *1) If $\mathscr{B}, J$ (and $\lambda, \kappa, \mu, \Theta$) are as in fact 4.5(1),(2), $2^\lambda \le \mu$ and $\mathscr{B}^*$ is a homomorphic image of $\mathscr{B}$ and $\|\mathscr{B}^*\| \ge 2^\lambda$, <u>then</u> $\theta \in \Theta \Rightarrow \mathbf{U}_\theta(\|\mathscr{B}^*\|) = \|\mathscr{B}^*\|$.*
2) Hence if $\theta \in \Theta, 2^\lambda \le \chi < \kappa, (\forall \alpha < \chi)(|\alpha|^{<\theta} < \chi \ \ \& \ \ \mathrm{cf}(\chi) = \theta)$, <u>then</u> $\|\mathscr{B}^*\| \notin [\chi, \chi^\theta)$.
3) Also it follows that the number of ultrafilters of $\mathscr{B}^*$ is $\le 2^{2^\lambda} + \|\mathscr{B}^*\|$. Moreover, if $\|\mathscr{B}^*\| > 2^\lambda$ & $\lambda \in \Theta$ equality holds (in any case $|Ult(\mathscr{B}^*)| \ge \|\mathscr{B}^*\|$).

4.7 Remark. 1) Note that in 4.6(2), if $\chi^\theta = \chi$ then the conclusion says nothing as $[\chi, \chi^\theta) = \emptyset$ (this occurs e.g. if $\chi = 2^\lambda$).
2) In second clause of 4.6(3), the assumption "$\lambda \in \Theta$" can be weaken to "if $X \subseteq J, |X| = (2^\lambda)^+$ then for some $y \in I$ we have $\lambda = |\{x \in X : x \le y\}|$", to get this we need much less than $\mathbf{U}_\lambda(\mu) = \lambda$.

Proof. 1) 2), 3) Let $h^* : \mathscr{B} \to \mathscr{B}^*$ be a homomorphism from $\mathscr{B}$ onto $\mathscr{B}^*$, let $J^* = \{h^*(x) : x \in J\}$. First assume $1_{\mathscr{B}^*} \in J^*$, this means that for some $x \in J, h^*(x) = 1_{\mathscr{B}^*}$, so $\mathscr{B} \restriction x =: \mathscr{B} \restriction \{y : \mathscr{B} \models y \le x\}$ has $\mathscr{B}^*$ as a homomorphic image so $\|\mathscr{B}^*\| \le |\mathscr{P}(x)| \le 2^\lambda$, and the number of ultrafilters of $\mathscr{B}^*$ is $\le 2^{2^\lambda}$; also if $\|\mathscr{B}^*\| \ge 2^\lambda$ we get $\|\mathscr{B}^*\| = 2^\lambda$ hence $\mathbf{U}_\theta(\|\mathscr{B}^*\|) = \mathbf{U}_\theta(2^\lambda) = 2^\lambda$ as $(2^\lambda)^\theta = 2^\lambda$; this finishes. So assume $1_{\mathscr{B}^*} \notin J^*$, hence J^* is a maximal ideal of $\mathscr{B}^*$, also clearly J^* is θ-full inside $\mathscr{B}^*$ for every $\theta \in \Theta$ (by Fact 4.4(1)). As J is 2^λ-local (see Fact 4.5(c)), clearly J^* is 2^λ-local, hence by 4.4(2), (letting $\chi =: |J^*|$), we have $\mathbf{U}_\theta(\chi) \le (2^\lambda)^\theta + \chi$, but $\mathbf{U}_\theta(\chi) \ge \chi \ge 2^\lambda = (2^\lambda)^\theta$ so $\mathbf{U}_\theta(\chi) = \chi$. Also $\|\mathscr{B}^*\| = \chi + \chi = \chi$. So we have gotten the conclusion of 4.6(1). Now 4.6(2) follows easily as $\mathbf{U}_\theta(\chi) \ge \chi^\theta$ as $\mathscr{P} =: \big\{\{\eta \restriction \alpha : \alpha < \theta\} : \eta \in \chi^\theta\big\}$ is a θ-AD family of subsets of $X =: \{\eta : \eta \in {}^{\theta >}\chi\}$ now $|X| = \chi^{<\theta} = \chi$ while the family $\mathscr{P}$ has cardinality χ^θ, see 4.2(1).

[In details: first note that $\chi = |{}^{\theta>}\chi|$. Hence it suffices to assume that $\mathscr{A} \subseteq [{}^{\theta>}\chi]^\theta$ and $\forall X \in [{}^{\theta>}\chi]^\theta \exists a \in \mathscr{A}[|a \cap X| = \theta]$, and show that $|\mathscr{A}| = \chi^\theta$. Let $\mathscr{P} = \{\{\eta \restriction \alpha : \alpha < \theta\} : \eta \in {}^\theta\chi\}$. Thus $\forall \eta \in {}^\theta\chi \exists a \in \mathscr{A}[|\{\eta \restriction \alpha : \eta \restriction \alpha \in a\}| = \theta]$. So

$$(*) \qquad {}^\theta\chi = \bigcup_{a \in \mathscr{A}} \{\eta \in {}^\theta\chi : |\{\alpha < \theta : \eta \restriction \alpha \in a\}| = \theta\}.$$

Now fix $a \in \mathscr{A}$ and let $T_a = \{\eta \in {}^\theta\chi : |\{\alpha < \theta : \eta \restriction \alpha \in a\}| = \theta\}$. For each $\eta \in T_a$ let $f(\eta) = \{\eta \restriction \alpha : \eta \restriction \alpha \in a\}$. Clearly f is a one-to-one function. Hence $|T_a| \le 2^\theta \le 2^\lambda \le \chi$. Hence by $(*), \chi^\theta \le |\mathscr{A}| \cdot 2^\lambda$. Since $2^\lambda < \chi^\theta$, it follows that $|\mathscr{A}| = \chi^\theta$.]

Lastly, for 4.6(3), if D is an ultrafilter of $\mathscr{B}^*$ then either $D = \mathscr{B}^* \backslash J^*$ or for some $x \in J^*, x \in D$ but for each $x \in J^*, \mathscr{B}^* \restriction \{y \in \mathscr{B}^* : y \le x\}$ has $\le 2^\lambda$ members so the number of ultrafilters of $\mathscr{B}$ to which x belongs is $\le 2^{2^\lambda}$, that means

4.8 Fact: If the Boolean Algebra $\mathscr{B}^*$ has a τ-local maximal ideal, then $|\mathrm{Ult}(\mathscr{B}^*)| \le 2^\tau + \|\mathscr{B}^*\|$.

Last point is the second sentence in part (3); that is we assume $\chi =: \|\mathscr{B}^*\| > 2^\lambda$; we have to prove that $\mathscr{B}^*$ has $\ge 2^{2^\lambda}$ ultrafilters. Let $x_i \in J$ for $i < \chi$ be such that $i < j < \chi \Rightarrow h^*(x_i) \ne h^*(x_j)$ (possible as $\|\mathscr{B}^*\| = |J^*|$). So by the $\triangle$-system argument without loss of generality for some $x, i < j < (2^\lambda)^+ \Rightarrow x_i \cap x_j = x$. Also $i < j < \chi \Rightarrow x_i \ne x_j$ hence $\langle x_i - x : i < (2^\lambda)^+\rangle$ is a sequence of $(2^\lambda)^+$ pairwise disjoint non-zero (in the sense of $\mathscr{B}$) members of $\mathscr{B}$. Now we know that J is λ-full (ideal of $\mathscr{B}$, by 4.5(1), clause (d), J is θ-full for every $\theta \in \Theta$). Apply the definition of "J is λ-full" to the set $\langle x_i - x : i < \lambda\rangle$ so we can find $y \in \mathscr{B}$ such that $w = \{i < \lambda : x_i - x^* \le y\}$ has cardinality λ (remember $\lambda \in \Theta$ by assumption of the second inequality in 4.6(3)). Hence $\mathscr{P}(\cup\{x_i - x : i \in w\}) \subseteq \mathscr{B}$. Now for every set $u \subseteq w$, $z_u = \bigcup\limits_{i\in u}(x_i - x)$ by clause (f) of 4.5(2) belongs to $\mathscr{B}$ and $[i \in u \Rightarrow x_i - x \le z_u]$ and $[i \in w\backslash u \rightarrow (x_i - x) \cap z_u = 0]$. But h^* is a homomorphism so in $\mathscr{B}^*$ we have $(\forall u \subseteq w)(\exists z \in \mathscr{B}^*)([i \in u \rightarrow h^*(x_i - x) \le z]\ \&\ [i \in w\backslash u \Rightarrow h^*(x_i - x) \cap z = 0])$. Hence $\mathscr{B}^*$ has a homomorphic image isomorphic to $\mathscr{P}(\lambda)$, just let D_i be an ultrafilter of $\mathscr{B}^*$ to which $h(x_i - x)$ belongs and $g : \mathscr{B}^* \rightarrow \mathscr{P}(\lambda)$ by $g(a) = \{i : a \in D_i\}$. Hence $\mathscr{B}$ has $\ge 2^{2^\lambda}$ ultrafilters.
Together we finish. $\square_{4.6}$

* * *

By claims 4.5, 4.6 we really finish. Let me point some specific conclusions: conclusion 4.9 is the theorem of Juhasz Shelah [JuSh 612].

4.9 Conclusion. For every $\kappa > 2^\lambda$, there is a Boolean algebra $\mathscr{B}_\kappa$ such that: $\mathscr{B}_\kappa$ is atomic with κ atoms, $\|\mathscr{B}\| = \kappa^\lambda$ and for every homomorphic image $\mathscr{B}^*$ of $\mathscr{B}$ of cardinality $\chi > 2^\lambda$ we have $\chi = \chi^\lambda$ and the number of ultrafilters of $\mathscr{B}^*$ is $2^{2^\lambda} + \chi$ in particular $\chi \in [\kappa, \kappa^\lambda)$ is impossible.

Proof. We apply fact 4.5 + claim 4.6 to $\mu = \kappa^\lambda$ and our κ, so $\Theta = \{\theta : \theta \le \lambda\}$. So for Θ we get $\mathscr{B}$. Let $\mathscr{B}^*$ be a homomorphic image of $\mathscr{B}$ (equivalently, a quotient of $\mathscr{B}$) and $\chi = \|\mathscr{B}^*\| > 2^\lambda$. So $\theta \in \Theta \Rightarrow \mathbf{U}_\theta(\chi) = \chi$, now if $\chi < \chi^\lambda$ let $\sigma = \mathrm{Min}\{\theta : \chi^\theta > \chi\}$, so $\sigma = \mathrm{cf}(\sigma) \in \Theta$ and $\chi^{<\sigma} = \chi < \chi^\sigma$ and (as in the proof of 4.6) we get a contradiction to $\mathbf{U}_\sigma(\chi) = \chi$. For the number of ultrafilters use 4.6(3). $\square_{4.9}$

4.10 Conclusion. If λ is strong limit singular (e.g. $\beth_\omega$) and $2^\lambda < \kappa \le \mu \le \kappa^\lambda$, then there is an atomic Boolean Algebra $\mathscr{B}$ with κ atoms, $|\mathscr{B}| = \mu$ such that: for every large enough regular $\theta < \lambda$ we have:

> every homomorphic image $\mathscr{B}^*$ of $\mathscr{B}$ of cardinality $> (2^\lambda)^+$ satisfies
> $\mathbf{U}_\theta[\|\mathscr{B}^*\|] = \|\mathscr{B}^*\|$
> (so for any cardinality τ, we have $\|\mathscr{B}^*\| \in [\tau^{<\theta}, \tau^\theta)$ is impossible).

Proof. By conclusion 4.5 + 4.6 as by [Sh 460] for every regular large enough $\theta < \lambda$ we have $\mathbf{U}_\theta(\mu) = \mu$. $\square_{4.10}$

4.11 Remark. In addition to $\mathbf{U}_\theta(-)$ we can use other functions (e.g. as in [Sh 589, §1], even if their number is $> \mu$ it does not matter as for each $\chi \in (2^\lambda, \mu)$ we can choose one) but does not seem worth elaborating.

4.12 Remark. Assume κ is strong limit singular of cofinality $\theta^* < \lambda < \kappa$. There are many μ above κ (and if 2^κ large enough, below 2^κ) such that $\Theta_{\mu,\lambda} = \{\theta \le \lambda : \mathrm{cf}(\theta) \ne \theta^*\}$. E.g. $\mu \in \{\kappa^{+n} : n < \omega\}$, also (see [Sh:g, Ch.IX]) for a club E of $\lambda^{+4}, \delta \in E$ & $\mathrm{cf}(\delta) \ge \lambda^+$ & $n < \omega \Rightarrow \Theta_{\kappa^{+\delta+n},\lambda} = \{\theta \le \lambda : \mathrm{cf}(\theta) \ne \theta^*\}$.

4.13 Claim. *In Claim 4.6 we can add (see [Sh 460] or below)*

(a) *if $\theta \le \lambda$ and $\mu = \mu^{[\theta]}$ then $\|\mathscr{B}^*\|^{[\theta]} = \|\mathscr{B}^*\|$*
(b) *similar more general condition (see [Sh 589, §1,end]).*

Remark. Recall $\mu^{[\theta]} = \mathrm{Min}\{|\mathscr{P} : \mathscr{P} \subseteq [\mu]^\theta$ and every $x \in [\mu]^\theta$ is included in the union of $< \theta$ members of $\mathscr{P}\}$.

4.14 Conclusion. If μ is strongly limit singular and $\mathrm{cf}(\mu) \le \theta < \mu$, then for some atomic Boolean Algebra $\mathscr{B}$ we have:

(a) $\mathscr{B}$ has cardinality μ
(b) $\mathscr{B}$ has μ ultrafilters
(c) if $\mathscr{B}^*$ is a homomorphic image of $\mathscr{B}$ of singular strong limit cardinality $\chi > \theta$, then $\mathrm{cf}(\chi) = \mathrm{cf}(\mu)$ and $\mathscr{B}^*$ has χ ultrafilters
(d) if $\mathscr{B}^*$ is a homomorphic image of $\mathscr{B}, \chi = \|\mathscr{B}^*\| > 2^\lambda$, then $\chi^{<\mathrm{cf}(\mu)} = \chi$ and $\chi = \mathrm{cov}(\chi, \theta^+, \theta^+, (\mathrm{cf}(\mu))^+)$.

Proof. By 4.6 and 4.13.

Another example

4.15 Conclusion. If μ is strong limit singular, $\theta = \mathrm{cf}(\delta) < \mathrm{cf}(\mu), \mu^{+\delta} < 2^\mu, (\mu^{+\delta})^{<\theta} = \mu^{+\delta}$ and $\mathrm{cf}([\mu^{+\delta}]^{<\mathrm{cf}(\mu)}, \subseteq) = \mu^{+\delta}$, <u>then</u> for some Boolean Algebra $\mathscr{B}$ of cardinality $\mu^{+\delta}$ it has $\mu^{+\delta}$ ultrafilters, and for every homomorphic image $\mathscr{B}^*$ of $\mathscr{B}$ of cardinality $\chi, 2^{\mathrm{cf}(\mu)} < \chi < \mu$ we have:

(∗) if χ is strong limit then $\mathrm{cf}(\chi) \geq \mathrm{cf}(\mu)$ or $\mathrm{cf}(\chi) = \mathrm{cf}(\delta)$.

We now turn to topological version of the results of §2.

4.16 Claim. *Assume*

(a) λ *is strong limit singular,* $\kappa =$ *cf*(λ) *and* $\kappa = \mathrm{cf}(\mu) < \mu \leq \lambda$
(b) X *is a Hausdorf topological space with* $w(X) = \lambda$.

<u>*Then*</u> *X has a closed subset Y such that:*

(α) $\mu \leq w(Y) \leq 2^{<\mu}$
(β) $\mu \leq |Y| \leq \sum_{\theta<\mu} 2^{2^\theta}$.

Remark. 1) If we speak on Boolean Algebras $\mathscr{B}$, let $X = \mathrm{Ult}(\mathscr{B})$, so $w(X) = |\mathscr{B}|$ and $\{Y : Y \subseteq X \text{ closed}\} = \{\mathrm{Ult}(\mathscr{B}/I) : I \text{ and ideal of } \mathscr{B}\}$ essentially.
2) So this result reasonably compliments Juhasz [Ju1], Juhasz Shelah [JuSh 612].

Proof. <u>Case 1</u>: $\mu = \lambda$.

Let $W = \{\mathscr{U}_i : i < \lambda\}$ be a basis of X. Choose $\langle \lambda_i : i < \kappa \rangle$ an increasing continuous sequence with limit $\lambda, \lambda_0 = 0, (\forall \sigma < \lambda_{i+1})[\sigma^{\lambda_i} < \lambda_{i+1} = \mathrm{cf}(\lambda_{i+1})]$. As $|X| \geq \lambda$ (as $w(X) = \lambda$ and λ is strong limit) necessarily $s^+(X) > \lambda$, see Juhasz [Ju], so there is $\{y_\alpha : \alpha < \lambda\} \subseteq X$, (with no repetitions) which is discrete. We can choose $Z_i \in [\lambda_{i+1}]^{\lambda_{i+1}}$ such that

$$(\forall \alpha < \lambda_i)(\bigwedge_{\zeta \in Z_i} y_\zeta \in \mathscr{U}_\alpha \vee \bigwedge_{\zeta \in Z_i} y_\zeta \notin \mathscr{U}_\alpha)$$

(as the λ_i sets $\langle \mathscr{U}_\alpha : \alpha < \lambda_i \rangle$ divides λ_{i+1} to $\leq 2^{\lambda_i} < \lambda_{i+1} = \mathrm{cf}(\lambda_{i+1})$ parts; of course possibly $[\zeta \in Z_i \Rightarrow y_\zeta \in \mathscr{U}_0]$ while $[\zeta \in Z_i \Rightarrow y_\zeta \notin \mathscr{U}_1]$).

By renaming without loss of generality $Z_i = [\lambda_i, \lambda_{i+1})$. Let $Y^* = c\ell\{y_\alpha : \alpha < \lambda\}$. It suffices to prove that $|Y^*| = \lambda$ (i.e. clause (β), noting that clause (α) holds trivially), for this it suffices to prove

$(*)$ if $x \in Y^*$, then for some $i < \kappa$ we have

$$x \in c\ell(\{y_\alpha : \alpha < \lambda_i\} \cup \{y_{\lambda_j} : j < \kappa\}).$$

If x contradicts $(*)$, then for every $i < \lambda$ there is $\alpha_i < \lambda$ such that $x \in \mathscr{U}_{\alpha_i}, \mathscr{U}_{\alpha_i} \cap (\{y_\alpha : \alpha < \lambda_i\} \cup \{y_{\lambda_j} : j < \kappa\}) = \emptyset$.
Now $\alpha_i < \lambda = \bigcup\limits_{j<\kappa} \lambda_j$ so for some $j, \alpha_i < \lambda_j$, so $\mathscr{U}_{\alpha_i} \cap \mathscr{U}_{\alpha_j}$ is an open neighborhood of x (as $\mathscr{U}_{\alpha_i}, \mathscr{U}_{\alpha_j}$ are) disjoint to $\{y_\alpha : \alpha < \lambda_j\}$ as $\mathscr{U}_{\alpha_j}$ is, and for each $\beta \in [\lambda_j, \lambda)$ we have

$(**)$ for some $\zeta \in [j, \kappa)$ we have $\beta \in [\lambda_\zeta, \lambda_{\zeta+1})$ and so (as $\alpha_i < \lambda_j, j \leq \zeta$) we have $y_\beta \in \mathscr{U}_{\alpha_i} \Leftrightarrow y_{\lambda_\zeta} \in \mathscr{U}_{\alpha_i}$; but $y_{\lambda_\zeta} \notin \mathscr{U}_{\alpha_i}$ by the choice of $\mathscr{U}_{\alpha_i}$, hence

$$\beta \in [\lambda_j, \lambda] \Rightarrow y_\beta \notin \mathscr{U}_{\alpha_i} \Rightarrow y_\beta \notin \mathscr{U}_{\alpha_i} \cap \mathscr{U}_{\alpha_j}.$$

So $\mathscr{U}_{\alpha_i} \cap \mathscr{U}_{\alpha_j}$ is an open neighborhood of x disjoint to $\{y_\alpha : \alpha < \lambda\}$ so $x \notin Y^*$, contradiction so $(*)$ holds, hence we are done.

<u>Case 2</u>: $\mu < \lambda$.
Let $\mu = \sum\limits_{i<\kappa} \mu_i, \mu_i$ strictly increasing with i. Repeat the above and let $Y' = c\ell\{y_\alpha : \alpha \in [\lambda_i, \lambda_i + \mu_i)$ for some $i\}$.
So clearly

$$\begin{aligned} y \in Y' &\Rightarrow y \in c\ell\{y_\alpha : \alpha \in \bigcup_{i<\kappa}[\lambda_i, \lambda_i + \mu_i)\} \\ &\Rightarrow \bigvee_{j<\kappa} y \in c\ell(\{y_\alpha : \alpha \in \bigcup_{i<j}[\lambda_i, \lambda_i + \mu_i)\} \cup \{y_{\lambda_i} : i < \kappa\}). \end{aligned}$$

So clearly $\mu = \sum\limits_{i<\kappa} \mu_i \leq |Y'| \leq \sum\limits_{i<\kappa} 2^{2^{\mu_i}}$.
Consider the family $W' = \bigcup\limits_{\substack{j<\kappa \\ A\subseteq\kappa}} W_{j,A}$ where

$$W_{j,A} = \Big\{ \mathscr{U}_\alpha \cap Y' : \alpha < \lambda_j \text{ and } [i \in [j,\kappa) \cap A \Rightarrow \bigwedge_{\alpha \in [\lambda_i, \lambda_i + \mu_i)} y_\alpha \in \mathscr{U}_\alpha] \text{ and } [i \in [j,\kappa) \backslash A \Rightarrow \bigvee_{\alpha \in [\lambda_i, \lambda_i + \mu_i)} y_\alpha \notin \mathscr{U}_\alpha] \Big\}.$$

$\boxdot_1$ W' is a family of open subsets of Y'.
[Why? As each $\mathscr{U}_\alpha \in W'$ is open subset of X.]

$\boxdot_2$ W' is a basis of Y'.
[Why? Let $z \in Y', z \in \mathscr{U}, \mathscr{U}$ an open subset of X. As $W = \{\mathscr{U}_\alpha : \alpha < \lambda\}$ is a basis of X, for some $\alpha < \lambda$ we have $z \in \mathscr{U}_\alpha \subseteq \mathscr{U}$. As $z \in Y'$, for some $i(0) < \kappa$ and $\varepsilon < \mu_{i(0)}$ we have $z = y_{\lambda_{i(0)}+\varepsilon}$ and as $\alpha < \lambda$ for some $i(1) < \kappa$ we have $\alpha < \lambda_{i(1)}$. Lastly, let $j < \kappa$ be $> i(0)$ and $> i(1)$ and $A = \{i < \kappa : y_{\lambda_i} \in \mathscr{U}_\alpha\}$. Clearly α satisfies the requirements in the definition of $W_{j,A}$ so $\mathscr{U}_\alpha \cap Y' \in W_{j,A}$, as $W_{j,A} \subseteq W'$ we are done.]

$\boxdot_3$ W' has cardinality $\le \sum_{\theta<\mu} 2^{2^\theta}$.
[Why? $|W'| \le \sum\{|W_{j,A}| : j < \kappa, A \subseteq \kappa\} \le \sum\{2^{\lambda_j} : j < \kappa, A \subseteq \kappa\} \le \sum\{2^{\lambda_j}, 2^\kappa : j < \kappa\} = \sum\{2^{\lambda_j} : j < \kappa\} = \sum_{\theta<\mu} 2^{2^\theta}$ (the second inequality by the definition of $W_{j,A}$ the only freedom left is about $U_\alpha \cap [0, \lambda_j)$.)]

So clearly we are done. $\square_{4.16}$

We may consider this in the following framework (see [M2] for such functions)

4.17 Definition. 1) For a Boolean Algebra $\mathscr{B}$ let

$$\mathrm{wcSp}(B) = \{(\|\mathscr{B}'\|, \mathrm{ult}(\mathscr{B}')) : \mathscr{B}' \text{ is an infinite homomorphic image of } \mathscr{B}\}$$

(remember

$$\mathrm{ult}(\mathscr{B}') = |\mathrm{Ult}(\mathscr{B}')|, \mathrm{Ult}(B') = \{D : D \text{ an ultrafilter of } \mathscr{B}'\}).$$

2) For a topological space X let

$$\mathrm{wcSp}(X) = \{(|Y|, w(Y)) : Y \text{ is a closed subspace of } X\}.$$

(remember $w(X)$ is the weight of the topological space X).

4.18 Remark. Of course, we can use disjoint sums of Boolean Algebras to get more examples (and similarly for topological spaces) as

$$\begin{aligned}\mathrm{wcSp}(\sum_{i<\alpha} B_i) = \Big\{ (\sum_{i<\alpha} \lambda_i, \sum_{i<\alpha} \mu_i) :& (\lambda_i, \mu_i) \in \mathrm{wcSp}(B_i) \\ & \cup \{(2^n, 2^{2^n}) : n < \omega\} \cup \{(0,0)\} \\ & \text{for } i < \alpha \text{ and } \sum_{i<\alpha} \lambda_i \text{ is infinite} \Big\}.\end{aligned}$$

In this way we can get more examples from the ones from [JuSh 612], but this does not cover all the above.

§5 Depth of Ultraproducts of Boolean Algebras

5.1 Claim. *Assume* $\square_\lambda$ *(i.e. there is* $\langle C_\delta : \delta < \lambda^+ \text{ limit}\rangle$ *such that* C_δ *is a club of* δ, C_δ *of order type* $< \delta$ *if* $\text{cf}(\delta) < \delta$ *and* $\delta_1 \in \text{acc}(C_{\delta_2}) \Rightarrow C_{\delta_1} = C_{\delta_2} \cap \delta_1$*).*
Let $\kappa = \text{cf}(\kappa) < \lambda$*. Then there are Boolean Algebras* B_ε *for* $\varepsilon < \kappa$ *such that:*

- (a) *Depth*$(B_\varepsilon) \leq \lambda$
- (b) *for any uniform ultrafilter* D *on* κ *we have* $\lambda^+ \leq$ *Depth*$(\prod\limits_{\varepsilon<\kappa} B_\varepsilon/D)$.

5.2 Remark. 1) This can be expressed through §1, see later.
2) If $\lambda = \lambda^\kappa$ we get Depth$(\prod\limits_{\varepsilon<\kappa} B_\varepsilon/D) > \prod\limits_{\varepsilon<\kappa}$ Depth$(B_\varepsilon)/A$.

Proof. Let $\langle C_\delta : \delta < \lambda^+ \text{ limit}\rangle$ exemplify $\square_\lambda$. Clearly there are an ordinal γ^* and a stationary $S \subseteq \lambda^+$ such that $(\forall \alpha \in S)[\text{otp}(C_\alpha) = \gamma^*]$ (so $\alpha \in S \Rightarrow \alpha$ limit) and, $\text{cf}(\gamma^*) = \kappa$ and γ^* divisible by ω^2.
[Why? If $\kappa > \aleph_0$, for $\gamma < \lambda^+$ let us define $S_\gamma =: \{\delta < \lambda^+ : \text{cf}(\delta) = \kappa$ and $\text{otp}(C_\delta)$ is $\gamma\}$, if so by Fodor's lemma for some γ the set S_γ is a stationary subset of λ^+; as C_δ is unbound in δ, clearly $\text{cf}(\gamma) = \kappa$, hence γ is divisible by ω^2. If $\lambda > \aleph_1, \kappa = \aleph_0$ the set $S' = \{\delta < \lambda^+ : \text{cf}(\delta) = \aleph_0$, and $\text{otp}(C_\delta)$ is divisible by $\omega^2\}$ is stationary (if E is a club disjoint to S', choose $\delta \in \text{acc}(E)$ of cofinality $\aleph_1$ and a club of $\delta' \in C_\delta$ are as required). If $\lambda = \aleph_1, \kappa = \aleph_0$, use the construction in [Sh 351, §4].]
So without loss of generality for every limit $\delta < \lambda^+$ we have $C_\delta \cap S = \emptyset$ (why? by deleting the first $\gamma^* + 1$ elements from any C_δ of greater order type) also without loss of generality $[\alpha \in C_\alpha \setminus \text{acc}(C_\alpha) \Rightarrow \alpha$ non-limit$]$. Without loss of generality $S \cap (\lambda + 1) = \emptyset$ and let $\mu = \lambda^+$.
So

$\boxtimes$(a) $\kappa < \mu$ are regular
- (b) $\langle C_\delta : \delta < \mu \text{ limit}\rangle$ is a square sequence
- (c) $S \subseteq \{\delta < \mu : \text{cf}(\delta) = \kappa\}$ is stationary
- (d) $\delta \in S \Rightarrow \text{otp}(C_\delta) = \gamma^*$ so $\text{cf}(\gamma^*) = \kappa$
- (e) $C_\delta \cap S = \emptyset$
- (f) γ^* is divisible by ω^2.

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<u>5.3 Fact</u>: Assume $\mu, \kappa, S, \langle C_\delta : \delta < \mu \text{ limit}\rangle$ satisfy $\boxtimes$ above. <u>Then</u> there are sets $A_{\alpha,\varepsilon}$ (for $\alpha < \mu, \varepsilon < \kappa$) such that:

$\boxtimes^{\mu,\kappa,S}_{\langle A_{\alpha,\varepsilon}:\alpha<\mu,\varepsilon<\kappa\rangle}$ ($\kappa < \mu$ are regular, $S \subseteq \{\delta < \mu : \text{cf}(\delta) = \kappa\}$ is stationary and)

(i) $\varepsilon < \zeta < \kappa \Rightarrow A_{\alpha,\varepsilon} \subseteq A_{\alpha,\zeta}$

(ii) $\bigcup\limits_{\varepsilon<\kappa} A_{\alpha,\varepsilon} = \alpha$

(iii) $\beta \in A_{\alpha,\varepsilon} \Rightarrow A_{\beta,\varepsilon} = A_{\alpha,\varepsilon} \cap \beta$

(iv) $\alpha \in S \;\; \& \;\; \varepsilon < \kappa \Rightarrow \sup(A_{\alpha,\varepsilon}) < \alpha$

(v) $A_{\alpha,\varepsilon}$ is a closed subset of α

(vi) if $\beta \in \text{acc}(C_\alpha)$ (hence α is a limit ordinal) then $\beta \in A_{\alpha,0}$

(vii) $\beta \in A_{\beta+1,0}$.

Proof of 5.3. We shall choose by induction $\alpha < \mu$ a sequence $\langle A_{\alpha,\varepsilon} : \varepsilon < \kappa\rangle$ such that clauses (i)-(vii) hold.
Let $\langle \gamma_\varepsilon : \varepsilon < \kappa\rangle$ be increasing continuous in ε sequence of ordinals with limit γ^*, each γ_ε a limit ordinal.
How do we carry the definition?

<u>Case 1</u>: $\alpha < \text{Min}(S)$.
Let $A_{\alpha,\varepsilon} = \alpha$.

<u>Case 2</u>: $\alpha = \beta + 1 \geq \text{Min}(S)$.
Let $A_{\alpha,\varepsilon} = A_{\beta,\varepsilon} \cup \{\beta\}$; note that $A_{\alpha,\varepsilon}$ is closed in α as $A_{\beta,\varepsilon}$ is closed in β.

<u>Case 3</u>: α limit, $\alpha > \sup(\text{acc}(C_\alpha))$ and $\alpha \geq \text{Min}(S)$.
So necessarily $\text{cf}(\alpha) = \aleph_0$ and $\text{otp}(C_\alpha)$ is not divisible by ω^2 hence $\alpha \notin S$. Let $\beta_0 \in C_\alpha$ and $\beta_0 = \max(\text{acc}(C_\alpha))$ if $\text{acc}(C_\alpha) \neq \emptyset$. Choose β_n (for $n \in [1,\omega)$) such that $n \geq 0 \Rightarrow \beta_n < \beta_{n+1}, \beta_n \in C_\alpha, \alpha = \bigcup\limits_{n<\omega} \beta_n$. Choose $\varepsilon_n < \kappa$ such that $\varepsilon_0 = 0, \varepsilon_n \leq \varepsilon_{n+1}, \beta_n \in A_{\beta_{n+1},\varepsilon_{n+1}}$. Lastly, for $\varepsilon < \kappa$ we let $A_{\alpha,\varepsilon}$ be: $\bigcup\{\{\beta_n\} \cup A_{\beta_n,\varepsilon} : n \text{ satisfies } \varepsilon_n \leq \varepsilon\}$.
Now check.

<u>Case 4</u>: α limit, $\alpha = \sup(\text{acc } C_\alpha), \alpha \notin S, \alpha \geq \text{Min}(S)$.
Let $A_{\alpha,\varepsilon} = \cup\{A_{\beta,\varepsilon} : \beta \in \text{acc}(C_\alpha)\}$.
Remember that $C_\delta \cap S = \emptyset$ for every limit $\delta < \lambda^+$.

<u>Case 5</u>: α limit, $\alpha = \sup(\text{acc}(C_\alpha)), \alpha \in S, \alpha \geq \text{Min}(S)$.

Let $A_{\alpha,\varepsilon} = \{\beta_\varepsilon\} \cup A_{\beta_\varepsilon,\varepsilon}$ where β_ε is the γ_ε-th member of $\mathrm{acc}(C_\alpha)$ (so necessarily $\beta_\varepsilon \in \mathrm{acc}(C_\alpha)$ and $\xi < \zeta \Rightarrow C_{\beta_\xi} = C_{\beta_\zeta} \cap \beta_\xi \Rightarrow \beta_\xi \in A_{\beta_\zeta,0} \Rightarrow (\forall \varepsilon)[(A_{\beta_\xi,\varepsilon} = A_{\beta_\zeta,\varepsilon} \cap \beta_\varepsilon)]$.
Check. $\square_{5.3}$

5.4 Remark. This may be relevant to a problem from [Sh 108], see 1.20.
Continuation of the proof of 5.1. Let $\bar{A} = \langle A_{\alpha,\varepsilon} : \alpha < \lambda^+, \varepsilon < \kappa \rangle$ be as in 5.3. Let $<_\varepsilon$ be the following two place relation on $\lambda^+ : \alpha <_\varepsilon \beta \Rightarrow \alpha \in A_{\beta,\varepsilon}$. It is a partial order (by clause (iii) of 5.3). Also

$(*)_1$ $\alpha < \beta \Rightarrow \bigvee\limits_{\zeta < \kappa} \bigwedge\limits_{\varepsilon \in [\zeta,\kappa)} \alpha <_\varepsilon \beta$

by clauses (i) + (ii) of 5.3. Let B_ε be $BA[(\lambda^+, <_\varepsilon)]$, i.e. it is a Boolean Algebra generated by $\langle x^\varepsilon_\alpha : \alpha < \lambda^+ \rangle$ freely except

$\bigotimes$ $x^\varepsilon_\alpha \le x^\varepsilon_\beta$ when $\alpha <_\varepsilon \beta$.

Clearly if D is a filter on κ containing the co-bounded subsets of κ then $\mathrm{Depth}^+(\prod\limits_{i<\kappa} B_i/D) > \lambda^+$ as $\langle < x_\alpha : \varepsilon < \kappa > /D : \alpha < \lambda^+ \rangle$ exemplifies this by $(*)_1$ above. Assume toward contradiction that $\varepsilon < \kappa$ and $\mathrm{Depth}^+(B_\varepsilon) > \lambda^+$ so assume $\bar{b} = \langle b_\gamma : \gamma < \lambda^+ \rangle$ is (strictly) increasing in B_ε. Choose by induction on $\gamma < \lambda^+$ a model $M_\gamma \prec (\mathscr{H}(\lambda^{++}), \in, <^*_{\lambda^{++}})$ of cardinality λ, increasing continuous with γ such that $\{\bar{C}, <_\varepsilon, \bar{b}\} \in M_0$ and $\langle M_\beta : \beta \le \gamma \rangle \in M_{\gamma+1}$. So $C^* = \{\delta < \lambda^+ : M_\delta \cap \lambda^+ = \delta\}$ is a club of λ^+ so choose $\delta(*) \in S \cap \mathrm{acc}(C^*)$.
Note

$(*)_2$ $b_\delta \notin M_\delta$ for $\delta \in C^*$ (in particular $\delta = \delta(*)$).
[Why? $\bar{b} = \langle b_\gamma : \gamma < \lambda^+ \rangle$ belongs to M_0 so for every $b \in M_\delta$, $\mathrm{Min}\{\gamma : b_\gamma = b \vee \gamma = \lambda^+\}$ necessarily belongs to M_δ. But $\bar{b}$ is without repetition so $b = b_\beta \Rightarrow \mathrm{Min}\{\gamma : b_\gamma = b \vee \gamma = \lambda^+\} = \beta$ so apply this to $\beta = \delta$.]

For $\gamma < \lambda^+$ let $b_\gamma = \tau_\gamma(x^\varepsilon_{\alpha(\gamma,0)}, \ldots, x^\varepsilon_{\alpha(\gamma,n_\gamma-1)})$ with τ_γ a Boolean term and $\alpha(\gamma,0) < \alpha(\gamma,1) < \ldots \alpha(\gamma,n_\gamma-1) < \lambda^+$, and for $Y \subseteq \lambda^+$ let $B_{\varepsilon,Y}$ be the subalgebra of B_ε generated by $\{x^\varepsilon_\alpha : \alpha \in Y\}$. Easily

$(*)_3$ $B_{\varepsilon,Y}$ is the algebra generated by $\{x^\varepsilon_\alpha : \alpha \in Y\}$ freely except the equations $x^\varepsilon_\alpha \le x^\varepsilon_\beta$ when $\alpha <_\varepsilon \beta$ & $\alpha \in Y$ & $\beta \in Y$, i.e. $B_{\varepsilon,Y} = BA[(Y, <_\varepsilon \restriction Y)]$.

[Why? As in the proof of 1.5, stage D.]
Without loss of generality $\langle \tau_\gamma, \langle \alpha_{\alpha(\gamma,\ell)} : \ell < n_\gamma \rangle : \gamma < \lambda^+ \rangle$ belongs to M_0. Clearly

for each $\ell < n_{\delta(*)}$ by clause (ii) of 5.3 for some $\zeta_\ell < \kappa$ we have $\alpha(\delta(*),\ell) > \delta(*) \Rightarrow \delta(*) \in A_{\alpha(\delta(*),\ell),\zeta_\ell}$ and $\alpha(\delta(*),\ell) < \delta(*) \Rightarrow \alpha(\delta(*),\ell) \in A_{\delta(*),\zeta_\ell}$ and let $\xi = \max(\{\varepsilon\} \cup \{\zeta_\ell : \ell < n_{\delta(*)}\})$, so $[\alpha(\delta(*),\ell) \geq \delta(*) \Rightarrow A_{\alpha(\delta(*),\ell),\xi} \cap \delta(*) = A_{\delta(*),\xi}]$ and $\delta(*) \cap \bigcup\limits_{\ell < n_{\delta(*)}} A_{\alpha(\delta(*),\ell),\xi} \subseteq A_{\delta(*),\xi}$.

Let $\alpha_0(*) = \sup(\delta(*) \cap \bigcup\limits_{\ell < n_{\delta(*)}} A_{\alpha(\delta(*),\ell),\xi})$, now by clause (v) and clause (iv) of 5.3 we have $\delta(*) > \sup(A_{\delta(*),\xi})$ hence clearly $\alpha_0(*) < \delta(*)$. Let $\alpha(*) = \text{Min}(C^* \backslash \alpha_0(*))$ so $\alpha(*) < \delta(*)$ (as $\delta(*) \in \text{acc}(C^*)$). Let $Y_0 = \alpha(*), Y_1 = \delta(*), Y_2 = \alpha(*) \cup \{\alpha(\delta(*),0),\dots,\alpha(\delta(*),n_{\delta(*)}-1)\}$.

Easily: $Y_0 = Y_1 \cap Y_2, Y_1 \cup Y_2 \subseteq \lambda^+$ and

$$\beta_1 \in Y_1 \backslash Y_2 \;\&\; \beta_2 \in Y_2 \backslash Y_1 \Rightarrow (\beta_1,\beta_2 \text{ are } <_\xi\text{-incomparable}) \Rightarrow (\beta_1,\beta_2 \text{ are } <_\varepsilon\text{-incomparable}).$$

By $(*)_3$ above and the definition of C^* clearly:

(a) $B_{\varepsilon,Y_1 \cup Y_2}$ is the free product of B_{ε,Y_1} and B_{ε,Y_2} over B_{ε,Y_0}
(b) $b_{\delta(*)} \notin M_{\delta(*)}$ (by $(*)_2$),
(c) $B_\varepsilon \restriction M_{\delta(*)} = B_{\varepsilon,Y_1}$ so $b_{\delta(*)} \notin B_{\varepsilon,Y_1}$ hence $b_{\delta(*)} \notin B_{\varepsilon,Y_0}$
(d) $b_{\alpha(*)} \notin M_{\alpha(*)}$ [again by $(*)_2$],
(e) $B_\varepsilon \restriction M_{\alpha(*)} = B_{\xi,Y_0}$ so $b_{\alpha(*)} \notin B_{\varepsilon,Y_0}$.

But by the choice of $\bar b, B_\varepsilon \models b_{\alpha(*)} < b_{\delta(*)}$ hence $B_{\varepsilon,Y_1 \cup Y_2} \models b_{\alpha(*)} < b_{\delta(*)}$ by $(*)_3$ hence by clause (a) for some $c \in B_{\varepsilon,Y_0}$ we have $B_{\varepsilon,Y_1} \models b_{\alpha(*)} \leq c$ and $B_{\varepsilon,Y_2} \models c \leq b_{\delta(*)}$. So $c \in M_{\alpha(*)}$ and $\alpha(*) \in Z =: \{\alpha : b_\alpha \leq c\}$, this last set Z is necessarily an initial segment of λ^+, it belongs to $M_{\alpha(*)}$, hence its supremum belongs to $M_{\alpha(*)}$, but the supremum $\in \{\alpha : \alpha \leq \lambda^+\}$, and is not in λ^+ (as $M_{\alpha(*)} \cap \lambda^+ = \alpha(*)$ and $\alpha(*)$ belongs to the set), so $Z = \lambda^+$. So $B_\xi \models b_{\delta(*)} \leq c$ but (by c's choice) $B_\varepsilon \models c \leq b_{\delta(*)}$ so $c = b_{\delta(*)}$, but $c \in M_{\alpha(*)} \prec M_{\delta(*)}$ where $b_{\delta(*)} \notin M_{\delta(*)}$ by $(*)_2$, contradiction. $\square_{5.1}$

<u>5.5 Conclusion</u>: Under the assumption of claim 5.1

(a) we can find $\mathbf{c} : [\lambda]^2 \to \kappa$ such that $\boxtimes_{\lambda,\lambda,\theta}$ (from 1.5)
(b) $NQs_2(\lambda,\mu,\kappa)$ (see Definition 1.10(2), in fact $NQs_2(\lambda,\mu,A,I)$ where A is the interval Boolean Algebra of $\kappa, I = \{a \in A : \sup(a) < \kappa\}$.

Proof. Easy, e.g.

(a) let $\langle\langle A_{\alpha,\varepsilon} : \varepsilon < \kappa\rangle : \alpha < \lambda^+\rangle$ be as in the proof of 5.1. Let for $\alpha < \beta$

$$\mathbf{c}\{\alpha, \beta\} =: \text{Min}\{\varepsilon : \alpha \in A_{\beta,\varepsilon}\}.$$

Now to prove clause (c) of $\boxtimes_{\lambda,\lambda,\theta}$ from 1.5, use "stationary" and get possibility (i) there. $\square_{5.5}$

5.6 Remark. 1) If μ is (weakly) inaccessible $> \kappa = \text{cf}(\kappa), S \subseteq \{\delta < \kappa : \text{cf}(\delta) = \kappa\}$ is stationary non-reflecting and we have a square as required in the beginning of the proof of 5.1 (e.g. if $\mathbf{V} = L$, by Beller and Litman [BeLi80]) then for some $\bar{A}$, the statement $\boxtimes^{\mu,\kappa,S}_{\bar{A}}$ is proved by 5.3. Then repeating the proof of 5.1 we get Boolean Algebras B_ε for $\varepsilon < \kappa$ such that $\text{Depth}^+(B_\varepsilon) = \mu$, and for every uniform ultrafilter D on $\kappa, \mu < \text{Depth}^+(\prod\limits_{\varepsilon<\kappa} B_\varepsilon/D)$.

2) In the proof of 5.1 we can with minor changes use a weaker version of $\boxtimes^{\mu,\kappa,S}_{\bar{A}}$, which is

$\boxdot^{\mu,\kappa,S}_{\bar{A}}$ $\mu > \kappa$ are regular, $S \subseteq \{\delta < \mu : \text{cf}(\delta) = \kappa\}$ is stationary, clause (i) and (ii) of 5.3 hold and

$(iii)^-$ $\beta \in A_{\alpha,\varepsilon} \Rightarrow A_{\beta,\varepsilon} \subseteq A_{\alpha,\varepsilon}$

$(iv)^+$ $\alpha \in S \ \& \ \varepsilon < \kappa \ \& \ \beta < \kappa \Rightarrow \sup(A_{\beta,\varepsilon} \cap \alpha) < \alpha$.

3) If $\mu = \lambda^+, \kappa = \text{cf}(\lambda) < \lambda$ and $S = \{\delta < \mu : \text{cf}(\delta) = \kappa\}$ then for some $\bar{A}, \boxdot^{\mu,\kappa,S}_{\bar{A}}$ (as in [Sh 108], [Sh 88a]) but $\mu \leq \lambda^\kappa/D$ so this does not give an example about the depth of the ultraproduct above the ultraproducts of the depths).

Clearly by 5.1:

<u>5.7 Conclusion</u>: If e.g. $\mathbf{V} = L, \lambda > \kappa = \text{cf}(\kappa)$ and $\lambda^\kappa = \lambda$ <u>then</u> for some sequence $\langle B_\varepsilon : \varepsilon < \kappa\rangle$ of Boolean Algebras for every uniform ultrafilter D on κ we have $\text{Depth}(\prod\limits_{\varepsilon<\kappa} B_\varepsilon/D) \geq \lambda^+ > \lambda \geq \prod\limits_{\varepsilon<\kappa} \text{Depth}(B_\varepsilon)/D$.

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