A RESULT RELATED TO THE PROBLEM CN OF FREMLIN

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Abstract. We show that the set of injective functions from any uncountable cardinal less than the continuum into the real numbers is of second category in the box product topology.

In this paper, we use a definability argument to resolve under mild set–theoretic assumptions a problem about injective functions in the box product topology. Suppose κ is a cardinal; let $\mathbf{S}_\kappa$ be the set of injective functions from κ into the closed unit interval $[0,1]$, and let $\Box^\kappa[0,1]$ be the product of κ copies of $[0,1]$ equipped with the box product topology (basic open sets are just products of basic open subsets of $[0,1]$: is $\mathbf{S}_\kappa$ of first category in $\Box^\kappa[0,1]$. We shall prove:

Theorem 1. *Suppose that* $\aleph_1 \leq \kappa < 2^{\aleph_0}$*. Then* $\mathbf{S}_\kappa$ *is of second category in* $\Box^\kappa[0,1]$.

David Fremlin asks [2, Problem CN] whether $\mathbf{S}_{\aleph_1}$ is co-meagre and whether $\mathbf{S}_{2^{\aleph_0}}$ is of the second category. We do not know the answer to these questions, but feel that Theorem 1 represents some progress on the former. As regards the latter question, we note that it is easy to prove that if MA holds, then every countable intersection of dense open sets in $\Box^{2^{\aleph_0}}[0,1]$ contains functions which are injective on a closed unbounded subset of $2^{\aleph_0}$ (see Corollary 5).

In proving Theorem 1, we make use (often tacitly) of some standard results about elementary submodels of $\mathcal{H}(\chi)$, the set of all sets hereditarily of cardinality less than χ. For reader's convenience we record these next.

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Fact 2. (1) *The cardinals ω and ω_1 belong to $\mathcal{H}(\chi)$ for $\chi > \omega_1$.*
(2) *If $\chi > \omega_1$ is regular, then $\mathcal{H}(\chi)$ is a model of all the axioms of ZFC except possibly the power set axiom.*
(3) *If $\chi > \omega_1$ is singular, then $\mathcal{H}(\chi)$ is a model of all the axioms of* ZFC *except possibly the power set, union and replacement axioms.* ■

Lemma 3. *Suppose that $\chi > \omega_1$ is regular and N is an elementary submodel of $\mathcal{H}(\chi)$. Then:*
(1) *If $a \in \mathcal{H}(\chi)$ is definable with parameters from N (i.e., there is a formula $\varphi(x)$ with one free variable x and possibly parameters in N such that a is the unique element which satisfies $\varphi(x)$ in $(\mathcal{H}(\chi), \in)$), then $a \in N$.*
(2) *The ordinals ω and ω_1 belong to N and $\omega \subseteq N$.*
(3) *If $a, A, B, f \in N$ and (in $\mathbf{V}$) f is a function from A to B then $f(a) \in N$.*
(4) *For every $\alpha \in \omega_1 \cup \{\omega_1\}$, $\alpha \cap N$ is an ordinal.*
(5) *If $\kappa \in N$ and $\{A_\alpha : \alpha < \kappa\} \in N$, then $(\forall \alpha \in \kappa \cap N)(A_\alpha \in N)$.* ■

The proofs of these well-known facts can be found in many places, for example, in [1] or in the appendix of [3].

Since the proof of Theorem 1 involves some notation, we describe in intuitive terms how it works. First we shall fix a family of "rational boxes" which are products of open subintervals of $[0,1]$ having rational endpoints. For a given countable family of dense open subsets G_n of $\Box^\kappa[0,1]$, for each n, we define a function f_n on rational boxes such that $f_n(B) \subseteq G_n$ and the diameters of open sets used to define the box $f_n(B)$ decrease as n increases. For each real τ, we define using the f_n's an element $x_\tau \in \bigcap_{n<\omega} G_n$. We then take an elementary submodel N of cardinality κ (of $\mathcal{H}(\chi)$ for a large enough χ) containing the f_n's, G_n's, all the ordinals up to $\kappa+1$, and whatever else is necessary. Since $\kappa < 2^{\aleph_0}$, there is a real $\tau \in {}^\omega 2 \setminus N$. We complete the proof by showing that $x_\tau \in \mathbf{S}_\kappa$, and this is done by demonstrating that if x_τ is not injective, then x_τ is definable in N and hence belongs to N – a contradiction. So the essence of the argument lies in connecting non-injectivity and definability.

Proof of Theorem 1 Let $\{G_n : n \in \omega\}$ be a countable family of dense open sets in $\Box^\kappa[0,1]$. We must show that $\mathbf{S}_\kappa \cap \bigcap_{n<\omega} G_n \neq \emptyset$.

First we set up some notation.

Fix a canonical family of non-empty open intervals I_ρ for $\rho \in {}^{\omega>}\omega$ as follows. Let $I_{\langle\rangle} = (0,1)$. Suppose that I_ρ has been defined, I_ρ a non-empty open subinterval of $[0,1]$, and the length $\ell g(\rho)$ of the sequence ρ is n. Choose 2^{2^n} disjoint open subintervals of I_ρ of equal length, say $I_{\rho\frown\langle k\rangle}$, $0 \le k < 2^{2^n}$ such that $I_\rho \setminus \bigcup_{k<2^{2^n}} I_{\rho\frown\langle k\rangle}$ is finite. This completes the definition of the family $\{I_\rho : \rho \in {}^{\omega >}\omega\}$.

Next for each $n \in \omega$, we choose a function f_n defined on the family of non-empty open boxes $B = \prod_{\iota<\kappa} B_\iota$, where each B_ι is a non-empty open subinterval of $[0,1]$, as follows: $f_n(B) = \prod_{\iota<\kappa} A_{\iota,n}$, where for all $\iota < \kappa$:

(1) $A_{\iota,n} \in \{I_\rho : \ell g(\rho) > n\}$,
(2) $\mathrm{cl}(A_{\iota,n}) \subseteq B_n$,
(3) $f_n(B) \subseteq G_n$.

There is no problem in choosing f_n as above, since G_n is dense open and for each $\iota < \kappa$ we can take $\ell g(\rho)$ large enough to ensure that $\mathrm{cl}(I_\rho) \subseteq B_\iota$.

Finally, fix κ reals $\{\eta_i : i < \kappa\}$, $\eta_i \in {}^\omega 2$, $\eta_i \neq \eta_j$ for $i < j < \kappa$.

We associate with each real τ an element $x_\tau \in \bigcap_{n<\omega} G_n$. Define by induction on n, for every $i < \kappa$, a non-empty open subinterval $C_{i,\tau\restriction n} \subseteq [0,1]$ and an open box $C_{\tau\restriction n} = \prod_{i<\kappa} C_{i,\tau\restriction n}$ as follows. Let $C_{i,\tau\restriction 0} = (0,1)$ (for $i < \kappa$). Suppose that $C_{i,\tau\restriction n}$ has been defined (for $i < \kappa$) and is a non-empty open subinterval of $[0,1]$. By (1), $f_n(C_{\tau\restriction n}) = \prod_{i<\kappa} I_{\rho_i}$, for some $\rho_i \in {}^{\omega >}\omega$ such that $\ell g(\rho_i) > n$. Let

$$k_i = 2 \cdot |\{\eta \in {}^n 2 : \eta \le_{\ell ex} \eta_i \restriction n\}| + \tau(n).$$

Note that (trivially) $k_i < 2^{2^{\ell g(\rho_1)}}$, so $I_{\rho_i\frown\langle k_i\rangle}$ is a non-empty open subinterval. Put $C_{i,\tau\restriction n+1} = I_{\rho_i\frown\langle k_i\rangle}$. By (1), (2) and (3), there exists a unique element

$$x_\tau \in \bigcap_{n<\omega} C_{\tau\restriction n} \subseteq \bigcap_{n<\omega} G_n.$$

So to complete the proof, it will suffice to choose a real τ such that $x_\tau \in \mathbf{S}_\kappa$.

Let $N \prec \mathcal{H}(\chi)$ be an elementary submodel for χ regular large enough $((\beth_\omega)^+$ will do) such that

(1) $\{f_n : n \in \omega\} \subseteq N$, $\kappa + 1 \subseteq N$, $\{\eta_i : \imath < \kappa\} \subseteq N$, $\{G_n : n \in \omega\} \subseteq N$, $\{I_\rho : \rho \in {}^{\omega >}\omega\} \subseteq N$,
(2) $|N| = \kappa$.

Since $|N| = \kappa < 2^{\aleph_0}$, there exists a real $\tau \in {}^\omega 2 \setminus N$. We complete the proof by proving that x_τ is an injective function from κ into $[0,1]$.

Suppose that x_τ is not injective: so there are $i < j < \kappa$ such that $x_\tau(i) = x_\tau(j)$. We derive a contradiction by showing that τ is definable using parameters in N and hence $\tau \in N$ by Lemma 3.

Let $m_0 = \min\{n : \eta_i(n) \neq \eta_j(n)\}$ (we can calculate m_0 in N since $\{\eta_i : i < \kappa\} \subseteq N$ and $i, j \in N$). It suffices to show that we can define $\tau \restriction n$ in N for every $n > m_0$. We prove this by induction on n. Suppose that we have defined $\tau \restriction n$, $n > m_0$. We show how to calculate $\tau(n)$ in N, thereby defining $\tau \restriction (n+1)$. Note that $C_{i,\sigma}$ and C_σ are definable in N (for $i < \kappa$ and $\sigma \in {}^{\omega >}2$) and hence $f_n(C_{\tau\restriction n}) = \prod_{\alpha<\kappa} I_{\rho_\alpha} \in N$. Consider I_{ρ_i} and I_{ρ_j}: in N, $I_{\rho_i} \cap I_{\rho_j} \neq \emptyset$ since $x_\tau(i) = x_\tau(i)$ and $N \prec \mathcal{H}(\chi)$. Also in N, $\rho_i \leq_{\ell ex} \rho_j$ or $\rho_j \leq_{\ell ex} \rho_i$, so without loss of generality, $\rho_i \leq_{\ell ex} \rho_j$. In fact, $\rho_i <_{\ell ex} \rho_j$, for if $\rho_i = \rho_j$, then $\eta_i \restriction n = \eta_j \restriction n$, contradicting $n > m_0$. Since $I_{\rho_i} \cap I_{\rho_j} \neq \emptyset$, it follows that $\ell g(\rho_i) < \ell g(\rho_j)$ and $\rho_j \restriction \ell g(\rho_i) = \rho_i$, and so there is a unique natural number k^* such that $\rho_j \restriction (\ell g(\rho_i)+1) = \rho_i{}^\frown\langle k^*\rangle$. As we can compute k^* and $2 \cdot |\{\eta \in {}^n2 : \eta \leq_{\ell ex} \eta_i \restriction n\}|$ inside N, it will suffice to show (in $\mathcal{H}(\chi)$) that:

$(*)$ $\quad \tau(n) = k^* - 2 \cdot |\{\eta \in {}^n2 : \eta \leq_{\ell ex} \eta_i \restriction n\}|.$

To see this, notice that $C_{i,\tau\restriction(n+1)} = I_{\rho_i{}^\frown\langle k^*\rangle}$. Why? If not, then $C_{i,\tau\restriction(n+1)} = I_{\rho_i{}^\frown\langle \ell\rangle}$ for some $\ell \neq k^*$, and so $I_{\rho_i{}^\frown\langle k^*\rangle} \cap I_{\rho_i{}^\frown\langle \ell\rangle} = \emptyset$, and hence

$(**)$ $\quad I_{\rho_j} \cap I_{\rho_i{}^\frown\langle \ell\rangle} = \emptyset$

(as $I_{\rho_j} \subseteq I_{\rho_i{}^\frown\langle k^*\rangle}$). However, $x_\tau(i) \in C_{i,\tau\restriction(n+1)} = I_{\rho_i{}^\frown\langle \ell\rangle}$, and $x_\tau(i) = x_\tau(j) \in I_{\rho_j}$, so $I_{\rho_j} \cap I_{\rho_i{}^\frown\langle \ell\rangle} \neq \emptyset$, contradicting $(**)$. By the definition of $C_{i,\tau\restriction(n+1)} = I_{\rho_i{}^\frown\langle k_i\rangle}$, it is now immediate that $k_i = k^*$. Recalling that

$$k_i = 2 \cdot |\{\eta \in {}^n2 : \eta \leq_{\ell ex} \eta_i \restriction n\}| + \tau(n),$$

we obtain $(*)$, and so $\tau \restriction (n+1)$ is definable in N.

Thus, $\tau = \bigcup_{n<\omega} (\tau \restriction n)$ is definable using parameters in N and hence belongs to N – a contradiction. It follows that x_τ is injective. This completes the proof of the theorem. ■

Proposition 4. *Suppose that $2^{\aleph_0}$ is regular and there is an enumeration $\{\mathfrak{r}_\alpha : \alpha < 2^{\aleph_0}\}$ of the real numbers such that for every β, $\{\mathfrak{r}_\alpha : \alpha < \beta\}$ is of the first category. Then every countable intersection of dense open sets in $\square^{2^{\aleph_0}}[0,1]$ contains functions which are injective on a closed unbounded set of $2^{\aleph_0}$.*

Proof Let $X_\beta = \{\mathfrak{r}_\alpha : \alpha < \beta\} \subseteq \bigcup_{n<\omega} F_{\beta,n}$, where $F_{\beta,n}$ is closed nowhere dense in $[0,1]$, and let $G_{\beta,n} = {}^\omega 2 \setminus F_{\beta,n}$. So $G_n = \prod_{\alpha<2^{\aleph_0}} G_{\alpha,n}$ is dense and open. It is enough to show that every $f \in \bigcap_{n<\omega} G_n$ is injective on some club. Suppose that $f \in \bigcap_{n<\omega} G_n$. For each $\alpha < 2^{\aleph_0}$, let

$$\beta_\alpha \stackrel{\text{def}}{=} \min\{\beta : f(\alpha) = \mathfrak{r}_\beta\}.$$

The interval of ordinals $(\beta_\alpha, 2^{\aleph_0})$ is a club, and hence the diagonal intersection

$$C = \mathop{\triangle}_{\alpha}(\beta_\alpha, 2^{\aleph_0}) = \{\beta < 2^{\aleph_0} : (\forall \alpha < \beta)(\beta \in (\beta_\alpha, 2^{\aleph_0}))\}$$

is a club. It is trivial to verify that if $\beta \in C$ and $\alpha < \beta$ then $f(\alpha) \neq f(\beta)$. ■

Corollary 5. *Martin's Axiom implies that every countable intersection of dense open subsets in* $\square^{2^{\aleph_0}}[0,1]$ *contains functions which are injective on some club.* ■

References

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