

## ON REFLECTION OF STATIONARY SETS IN $\mathcal{P}_\kappa\lambda$

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ABSTRACT. Let  $\kappa$  be an inaccessible cardinal, and let  $E_0 = \{x \in \mathcal{P}_\kappa\kappa^+ : \text{cf } \lambda_x = \text{cf } \kappa_x\}$  and  $E_1 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x \text{ is regular and } \lambda_x = \kappa_x^+\}$ . It is consistent that the set  $E_1$  is stationary and that every stationary subset of  $E_0$  reflects at almost every  $a \in E_1$ .

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## 1. Introduction.

We study reflection properties of stationary sets in the space  $\mathcal{P}_\kappa\lambda$  where  $\kappa$  is an inaccessible cardinal. Let  $\kappa$  be a regular uncountable cardinal, and let  $A \supseteq \kappa$ . The set  $\mathcal{P}_\kappa A$  consists of all  $x \subset A$  such that  $|x| < \kappa$ . Following [3], a set  $C \subseteq \mathcal{P}_\kappa A$  is *closed unbounded* if it is  $\subseteq$ -cofinal and closed under unions of chains of length  $< \kappa$ ;  $S \subseteq \mathcal{P}_\kappa A$  is *stationary* if it has nonempty intersection with every closed unbounded set. Closed unbounded sets generate a normal  $\kappa$ -complete filter, and we use the phrase “almost all  $x$ ” to mean all  $x \in \mathcal{P}_\kappa A$  except for a nonstationary set.

Almost all  $x \in \mathcal{P}_\kappa A$  have the property that  $x \cap \kappa$  is an ordinal. Throughout this paper we consider only such  $x$ 's, and denote  $x \cap \kappa = \kappa_x$ . If  $\kappa$  is inaccessible then for almost all  $x$ ,  $\kappa_x$  is a limit cardinal (and we consider only such  $x$ 's.) By [5], the closed unbounded filter on  $\mathcal{P}_\kappa A$  is generated by the sets  $C_F = \{x : x \cap \kappa \in \kappa \text{ and } F(x^{<\omega}) \subseteq x\}$  where  $F$  ranges over functions  $F : A^{<\omega} \rightarrow A$ . It follows that a set  $S \subseteq \mathcal{P}_\kappa A$  is stationary if and only if every model  $M$  with universe  $\supseteq A$  has a submodel  $N$  such that  $|N| < \kappa$ ,  $N \cap \kappa \in \kappa$  and  $N \cap A \in S$ . In most applications,  $A$  is identified with  $|A|$ , and so we consider  $\mathcal{P}_\kappa\lambda$  where  $\lambda$  is a cardinal,  $\lambda > \kappa$ . For  $x \in \mathcal{P}_\kappa\lambda$  we denote  $\lambda_x$  the order type of  $x$ .

We are concerned with *reflection* of stationary sets. Reflection properties of stationary sets of ordinals have been extensively studied, starting with [7]. So have been reflection principles for stationary sets in  $\mathcal{P}_{\omega_1}\lambda$ , following [2]. In this paper we concentrate on  $\mathcal{P}_\kappa\lambda$  where  $\kappa$  is inaccessible.

**Definition.** Let  $\kappa$  be an inaccessible and let  $a \in \mathcal{P}_\kappa\lambda$  be such that  $\kappa_a$  is a regular uncountable cardinal. A stationary set  $S \subseteq \mathcal{P}_\kappa\lambda$  *reflects at  $a$*  if the set  $S \cap \mathcal{P}_{\kappa_a}a$  is a stationary set in  $\mathcal{P}_{\kappa_a}a$ .

The question underlying our investigation is to what extent can stationary sets reflect. There are some limitations associated with cofinalities. For instance, let  $S$  and  $T$  be stationary subsets of  $\lambda$  such that every  $\alpha \in S$  has cofinality  $\omega$ , every

$\gamma \in T$  has cofinality  $\omega_1$ , and for each  $\gamma \in T$ ,  $S \cap \gamma$  is a nonstationary subset of  $\gamma$  (cf. [4]). Let  $\widehat{S} = \{x \in \mathcal{P}_\kappa\lambda : \sup x \in S\}$  and  $\widehat{T} = \{a \in \mathcal{P}_\kappa\lambda : \sup a \in T\}$ . Then  $\widehat{S}$  does not reflect at any  $a \in \widehat{T}$ .

Let us consider the case when  $\lambda = \kappa^+$ . As the example presented above indicates, reflection will generally fail when dealing with the  $x$ 's for which  $\text{cf } \lambda_x < \kappa_x$ , and so we restrict ourselves to the (stationary) set

$$\{x \in \mathcal{P}_\kappa\lambda : \text{cf } \kappa_x \leq \text{cf } \lambda_x\}$$

Since  $\lambda = \kappa^+$ , we have  $\lambda_x \leq \kappa_x^+$  for almost all  $x$ .

Let

$$E_0 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x \text{ is a limit cardinal and } \text{cf } \kappa_x = \text{cf } \lambda_x\},$$

$$E_1 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x \text{ is inaccessible and } \lambda_x = \kappa_x^+\}.$$

The set  $E_0$  is stationary, and if  $\kappa$  is a large cardinal (e.g.  $\kappa^+$ -supercompact) then  $E_1$  is stationary; the statement “ $E_1$  is stationary” is itself a large cardinal property (cf. [1]). Moreover,  $E_0$  reflects at almost every  $a \in E_1$  and consequently, reflection of stationary subsets of  $E_0$  at elements of  $E_1$  is a prototype of the phenomena we propose to investigate.

Below we prove the following theorem:

**1.2. Theorem.** *Let  $\kappa$  be a supercompact cardinal. There is a generic extension in which*

- (a) *the set  $E_1 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x \text{ is inaccessible and } \lambda_x = \kappa_x^+\}$  is stationary,*  
*and*
- (b) *for every stationary set  $S \subseteq E_0$ , the set  $\{a \in E_1 : S \cap \mathcal{P}_{\kappa_a} a \text{ is nonstationary in } \mathcal{P}_{\kappa_a} a\}$  is nonstationary.*

A large cardinal assumption in Theorem 1.2 is necessary. As mentioned above, (a) itself has large cardinal consequences. Moreover, (b) implies reflection of stationary subsets of the set  $\{\alpha < \kappa^+ : \text{cf } \alpha < \kappa\}$ , which is also known to be strong (consistency-wise).

## 2. Preliminaries.

We shall first state several results that we shall use in the proof of Theorem 1.2.

We begin with a theorem of Laver that shows that supercompact cardinals have a  $\diamond$ -like property:

**2.1. Theorem.** [6] *If  $\kappa$  is supercompact then there is a function  $f : \kappa \rightarrow V_\kappa$  such that for every  $x$  there exists an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $j$  witnesses a prescribed degree of supercompactness and  $(j(f))(\kappa) = x$ .*

We say that the function  $f$  has *Laver's property*.

**2.2. Definition.** A forcing notion is  $< \kappa$ -strategically closed if for every condition  $p$ , player I has a winning strategy in the following game of length  $\kappa$ : Players I and II take turns to play a descending  $\kappa$ -sequence of conditions  $p_0 > p_1 > \dots > p_\xi > \dots, \xi < \kappa$ , with  $p > p_0$ , such that player I moves at limit stages. Player I wins if for each limit  $\lambda < \kappa$ , the sequence  $\{p_\xi\}_{\xi < \lambda}$  has a lower bound.

It is well known that forcing with a  $< \kappa$ -strategically closed notion of forcing does not add new sequences of length  $< \kappa$ , and that every iteration, with  $< \kappa$ -support, of  $< \kappa$ -strategically closed forcing notions is  $< \kappa$ -strategically closed.

**2.3. Definition.** [8] A forcing notion satisfies the  $< \kappa$ -strategic- $\kappa^+$ -chain condition if for every limit ordinal  $\lambda < \kappa$ , player I has a winning strategy in the following game of length  $\lambda$ :

Players I and II take turns to play, simultaneously for each  $\alpha < \kappa^+$  of cofinality  $\kappa$ , descending  $\lambda$ -sequences of conditions  $p_0^\alpha > p_1^\alpha > \dots > p_\xi^\alpha > \dots, \xi < \lambda$ , with player II moving first and player I moving at limit stages. In addition, player I chooses, at stage  $\xi$ , a closed unbounded set  $E_\xi \subset \kappa^+$  and a function  $f_\xi$  such that for each  $\alpha < \kappa^+$  of cofinality  $\kappa$ ,  $f_\xi(\alpha) < \alpha$ .

Player I wins if for each limit  $\eta < \lambda$ , each sequence  $\langle p_\xi^\alpha : \xi < \eta \rangle$  has a lower bound, and if the following holds: for all  $\alpha, \beta \in \bigcap_{\xi < \lambda} E_\xi$ , if  $f_\xi(\alpha) = f_\xi(\beta)$  for all

$\xi < \lambda$ , then the sequences  $\langle p_\xi^\alpha : \xi < \lambda \rangle$  and  $\langle p_\xi^\beta : \xi < \lambda \rangle$  have a common lower bound.

It is clear that property (2.3) implies the  $\kappa^+$ -chain condition. It is proved in [8] that every iteration with  $< \kappa$ -support, of  $< \kappa$ -strategically  $\kappa^+$ -c.c. forcing notions satisfies the  $< \kappa$ -strategic  $\kappa^+$ -chain condition.

In Lemmas 2.4 and 2.5 below,  $H(\lambda)$  denotes the set of all sets hereditarily of cardinality  $< \lambda$ .

**2.4. Lemma.** *Let  $S$  be a stationary subset of  $E_0$ . For every set  $u$  there exist a regular  $\lambda > \kappa^+$ , an elementary submodel  $N$  of  $\langle H(\lambda), \in, \Delta, u \rangle$  (where  $\Delta$  is a well ordering of  $H(\lambda)$ ) such that  $N \cap \kappa^+ \in S$ , and a sequence  $\langle N_\alpha : \alpha < \delta \rangle$  of submodels of  $N$  such that  $|N_\alpha| < \kappa$  for every  $\alpha$ ,  $N \cap \kappa^+ = \bigcup_{\alpha < \delta} (N_\alpha \cap \kappa^+)$  and for all  $\beta < \delta$ ,  $\langle N_\alpha : \alpha < \beta \rangle \in N$ .*

*Proof.* Let  $\mu > \kappa^+$  be such that  $u \in H(\mu)$ , and let  $\lambda = (2^\mu)^+$ ; let  $\Delta$  be a well ordering of  $H(\lambda)$ . There exists an elementary submodel  $N$  of  $\langle H(\lambda), \in, \Delta \rangle$  containing  $u$ ,  $S$  and  $\langle H(\mu), \in, \Delta \upharpoonright H(\mu) \rangle$  such that  $N \cap \kappa^+ \in S$  and  $N \cap \kappa$  is a strong limit cardinal; let  $a = N \cap \kappa^+$ .

Let  $\delta = \text{cf } \kappa_a$ . As  $a \in S$ , we have  $\text{cf } (\sup a) = \delta$ , and let  $\gamma_\alpha$ ,  $\alpha < \delta$ , be an increasing sequence of ordinals in  $a - \kappa$ , cofinal in  $\sup a$ . Let  $\langle f_\alpha : \kappa \leq \alpha < \kappa^+ \rangle \in N$  be such that each  $f_\alpha$  is a one-to-one function of  $\alpha$  onto  $\kappa$ . (Thus for each  $\alpha \in a$ ,  $f_\alpha$  maps  $a \cap \alpha$  onto  $\kappa_a$ .) There exists an increasing sequence  $\beta_\alpha$ ,  $\alpha < \delta$ , of ordinals cofinal in  $\kappa_a$ , such that for each  $\xi < \alpha$ ,  $f_{\gamma_\alpha}(\gamma_\xi) < \beta_\alpha$ .

For each  $\alpha < \delta$ , let  $N_\alpha$  be the Skolem hull of  $\beta_\alpha \cup \{\gamma_\alpha\}$  in  $\langle H(\mu), \in, \Delta \upharpoonright H(\mu), \langle f_\alpha \rangle \rangle$ .  $N_\alpha$  is an elementary submodel of  $H(\mu)$  of cardinality  $< \kappa_a$ , and  $N_\alpha \in N$ . Also, if  $\xi < \alpha$  then  $\gamma_\xi \in N_\alpha$  (because  $f_{\gamma_\alpha}(\gamma_\xi) < \beta_\alpha$ ) and so  $N_\xi \subseteq N_\alpha$ .

As  $N \cap \kappa$  is a strong limit cardinal, it follows that for all  $\beta < \delta$ ,  $\langle N_\alpha : \alpha < \beta \rangle \in N$ . Also,  $N_\alpha \subseteq N$  for all  $\alpha < \delta$ , and it remains to prove that  $a \subseteq \bigcup_{\alpha < \delta} N_\alpha$ .

As  $\sup\{\beta_\alpha : \alpha < \delta\} = \kappa_a$ , we have  $\kappa_a \subseteq \bigcup_{\alpha < \delta} N_\alpha$ . If  $\gamma \in a$ , there exists a  $\xi < \alpha < \delta$  such that  $\gamma < \gamma_\xi$  and  $f_{\gamma_\xi}(\gamma) < \beta_\alpha$ . Then  $\gamma_\xi \in N_\alpha$  and so  $\gamma \in N_\alpha$ .

**2.5. Lemma.** *Let  $S$  be a stationary subset of  $E_0$  and let  $P$  be a  $< \kappa$ -strategically closed notion of forcing. Then  $S$  remains stationary in  $V^P$ .*

*Proof.* Let  $\dot{C}$  be a  $P$ -name for a club set in  $\mathcal{P}_\kappa \kappa^+$ , and let  $p_0 \in P$ . We look for a  $p \leq p_0$  that forces  $S \cap \dot{C} \neq \emptyset$ .

Let  $\sigma$  be a winning strategy for I in the game (2.2). By Lemma 2.4 there exist a regular  $\lambda > \kappa^+$ , an elementary submodel  $N$  of  $\langle H(\lambda), \epsilon, \Delta, P, p_0, \sigma, S, \dot{C} \rangle$  (where  $\Delta$  is a well-ordering) such that  $|N| < \kappa$  and  $N \cap \kappa^+ \in S$ , and a sequence  $\langle N_\alpha : \alpha < \delta \rangle$  of submodels of  $N$  such that  $|N_\alpha| < \kappa$  for every  $\alpha$ ,  $N \cap \kappa^+ = \bigcup_{\alpha < \delta} (N_\alpha \cap \kappa^+)$  and for all  $\beta < \delta$ ,  $\langle N_\alpha : \alpha < \beta \rangle \in N$ .

We construct a descending sequence of conditions  $\langle p_\alpha : \alpha < \delta \rangle$  below  $p_0$  such that for all  $\beta < \delta$ ,  $\langle p_\alpha : \alpha < \beta \rangle \in N$ : at each limit stage  $\alpha$  we apply the strategy  $\sigma$  to get  $p_\alpha$ ; at each  $\alpha + 1$  let  $q \leq p_\alpha$  be the  $\Delta$ -least condition such that for some  $M_\alpha \in \mathcal{P}_\kappa \kappa^+ \cap N$ ,  $M_\alpha \supseteq N_\alpha \cap \kappa^+$ ,  $M_\alpha \supseteq \bigcup_{\beta < \alpha} M_\beta$  and  $q \Vdash M_\alpha \in \dot{C}$  (and let  $M_\alpha$  be the  $\Delta$ -least such  $M_\alpha$ ), and then apply  $\sigma$  to get  $p_{\alpha+1}$ . Since  $M_\alpha \in N$ ,  $N \models |M_\alpha| < \kappa$  and so  $M_\alpha \subseteq N$ ; hence  $M_\alpha \subseteq N \cap \kappa^+$ . Since for all  $\beta < \delta$ ,  $\langle N_\alpha : \alpha < \beta \rangle \in N$ , the construction can be carried out inside  $N$  so that for each  $\beta < \delta$ ,  $\langle p_\alpha : \alpha < \beta \rangle \in N$ .

As I wins the game, let  $p$  be a lower bound for  $\langle p_\alpha : \alpha < \delta \rangle$ ;  $p$  forces that  $\dot{C} \cap (N \cap \kappa^+)$  is unbounded in  $N \cap \kappa^+$  and hence  $N \cap \kappa^+ \in \dot{C}$ . Hence  $p \Vdash S \cap \dot{C} \neq \emptyset$ .

□

### 3. The forcing.

We shall now describe the forcing construction that yields Theorem 1.2. Let  $\kappa$  be a supercompact cardinal.

The forcing  $P$  has two parts,  $P = P_\kappa * \dot{P}^\kappa$ , where  $P_\kappa$  is the *preparation forcing* and  $\dot{P}^\kappa$  is the *main iteration*. The preparation forcing is an iteration of length  $\kappa$ , with Easton support, defined as follows: Let  $f : \kappa \rightarrow V_\kappa$  be a function with Laver's

property. If  $\gamma < \kappa$  and if  $P_\kappa \upharpoonright \gamma$  is the iteration up to  $\gamma$ , then the  $\gamma^{\text{th}}$  iterand  $\dot{Q}_\gamma$  is trivial unless  $\gamma$  is inaccessible and  $f(\gamma)$  is a  $P_\kappa \upharpoonright \gamma$ -name for a  $< \gamma$ -strategically closed forcing notion, in which case  $\dot{Q}_\gamma = f(\gamma)$  and  $P_{\gamma+1} = P_\gamma * \dot{Q}_\gamma$ . Standard forcing arguments show that  $\kappa$  remains inaccessible in  $V^{P_\kappa}$  and all cardinals and cofinalities above  $\kappa$  are preserved.

The main iteration  $\dot{P}^\kappa$  is an iteration in  $V^{P_\kappa}$ , of length  $2^{(\kappa^+)}$ , with  $< \kappa$ -support. We will show that each iterand  $\dot{Q}_\gamma$  is  $< \kappa$ -strategically closed and satisfies the  $< \kappa$ -strategic  $\kappa^+$ -chain condition. This guarantees that  $\dot{P}^\kappa$  is (in  $V^{P_\kappa}$ )  $< \kappa$ -strategically closed and satisfies the  $\kappa^+$ -chain condition, therefore adds no bounded subsets of  $\kappa$  and preserves all cardinals and cofinalities.

Each iterand of  $\dot{P}^\kappa$  is a forcing notion  $\dot{Q}_\gamma = Q(\dot{S})$  associated with a stationary set  $\dot{S} \subseteq \mathcal{P}_{\kappa\kappa^+}$  in  $V^{P_\kappa * \dot{P}^\kappa \upharpoonright \gamma}$ , to be defined below. By the usual bookkeeping method we ensure that for every  $P$ -name  $\dot{S}$  for a stationary set, some  $\dot{Q}_\gamma$  is  $Q(\dot{S})$ .

Below we define the forcing notion  $Q(S)$  for every stationary set  $S \subseteq E_0$ ; if  $S$  is not a stationary subset of  $E_0$  then  $Q(S)$  is the trivial forcing. If  $S$  is a stationary subset of  $E_0$  then a generic for  $Q(S)$  produces a closed unbounded set  $C \subseteq \mathcal{P}_{\kappa\kappa^+}$  such that for every  $a \in E_1 \cap C$ ,  $S \cap \mathcal{P}_{\kappa_a} a$  is stationary in  $\mathcal{P}_{\kappa_a} a$ . Since  $\dot{P}^\kappa$  does not add bounded subsets of  $\kappa$ , the forcing  $Q(\dot{S})$  guarantees that in  $V^P$ ,  $\dot{S}$  reflects at almost every  $a \in E_1$ . The crucial step in the proof will be to show that the set  $E_1$  remains stationary in  $V^P$ .

To define the forcing notion  $Q(S)$  we use certain models with universe in  $\mathcal{P}_{\kappa\kappa^+}$ . We first specify what models we use:

**3.1. Definition.** A *model* is a structure  $\langle M, \pi, \rho \rangle$  such that

- (i)  $M \in \mathcal{P}_{\kappa\kappa^+}$ ;  $M \cap \kappa = \kappa_M$  is an ordinal and  $\lambda_M =$  the order type of  $M$  is at most  $|\kappa_M|^+$
- (ii)  $\pi$  is a two-place function;  $\pi(\alpha, \beta)$  is defined for all  $\alpha \in M - \kappa$  and  $\beta \in M \cap \alpha$ .  
For each  $\alpha \in M - \kappa$ ,  $\pi_\alpha$  is the function  $\pi_\alpha(\beta) = \pi(\alpha, \beta)$  from  $M \cap \alpha$  onto

$M \cap \alpha$ , and moreover,  $\pi_\alpha$  maps  $\kappa_M$  onto  $M \cap \alpha$ .

(iii)  $\rho$  is a two-place function;  $\rho(\alpha, \beta)$  is defined for all  $\alpha \in M - \kappa$  and  $\beta < \kappa_M$ .

For each  $\alpha \in M - \kappa$ ,  $\rho_\alpha$  is the function  $\rho_\alpha(\beta) = \rho(\alpha, \beta)$  from  $\kappa_M$  into  $\kappa_M$ , and  $\beta \leq \rho_\alpha(\beta) < \kappa_M$  for all  $\beta < \kappa_M$ .

Two models  $\langle M, \pi^M, \rho^M \rangle$  and  $\langle N, \pi^N, \rho^N \rangle$  are *coherent* if  $\pi^M(\alpha, \beta) = \pi^N(\alpha, \beta)$  and  $\rho^M(\alpha, \beta) = \rho^N(\alpha, \beta)$  for all  $\alpha, \beta \in M \cap N$ .  $M$  is a *submodel* of  $N$  if  $M \subseteq N$ , and  $\pi^M \subseteq \pi^N$  and  $\rho^M \subseteq \rho^N$ .

**3.2. Lemma.** *Let  $M$  and  $N$  be coherent models with  $\kappa_M \leq \kappa_N$ . If  $M \cap N$  is cofinal in  $M$  (i.e. if for all  $\alpha \in M$  there is a  $\gamma \in M \cap N$  such that  $\alpha < \gamma$ ), then  $M \subseteq N$ .*

*Proof.* Let  $\alpha \in M$ ; let  $\gamma \in M \cap N$  be such that  $\alpha < \gamma$ . As  $\pi_\gamma^M$  maps  $\kappa_M$  onto  $M \cap \gamma$ , there is a  $\beta < \kappa_M$  such that  $\pi_\gamma^M(\beta) = \alpha$ . Since both  $\beta$  and  $\gamma$  are in  $N$ , we have  $\alpha = \pi^M(\gamma, \beta) = \pi^N(\gamma, \beta) \in N$ .

We shall now define the forcing notion  $Q(S)$ :

**3.3 Definition.** Let  $S$  be a stationary subset of the set  $E_0 = \{x \in \mathcal{P}_\kappa \kappa^+ : \kappa_x$  is a limit cardinal and  $\text{cf } \lambda_x = \text{cf } \kappa_x\}$ . A *forcing condition* in  $Q(S)$  is a model  $M = \langle M, \pi^M, \rho^M \rangle$  such that

- (i)  $M$  is  $\omega$ -closed, i.e. for every ordinal  $\gamma$ , if  $\text{cf } \gamma = \omega$  and  $\sup(M \cap \gamma) = \gamma$ , then  $\gamma \in M$ ;
- (ii) For every  $\alpha \in M - \kappa$  and  $\beta < \kappa_M$ , if  $\kappa_M \leq \gamma < \alpha$ , and if  $\{\beta_n : n < \omega\}$  is a countable subset of  $\beta$  such that  $\gamma = \sup\{\pi_\alpha^M(\beta_n) : n < \omega\}$ , then there is some  $\zeta < \rho_\alpha^M(\beta)$  such that  $\gamma = \pi_\alpha^M(\zeta)$ .
- (iii) For every submodel  $a \subseteq M$ , if

$$a \in E_1 = \{x \in \mathcal{P}_\kappa \kappa^+ : \kappa_x \text{ is inaccessible and } \lambda_x = \kappa_x^+\},$$

then  $S \cap \mathcal{P}_{\kappa_a} a$  is stationary in  $\mathcal{P}_{\kappa_a} a$ .



A forcing condition  $N$  is *stronger* than  $M$  if  $M$  is a submodel of  $N$  and  $|M| < |\kappa_N|$ .

The following lemma guarantees that the generic for  $Q_S$  is unbounded in  $\mathcal{P}_{\kappa\kappa^+}$ .

**3.4. Lemma.** *Let  $M$  be a condition and let  $\delta < \kappa$  and  $\kappa \leq \varepsilon < \kappa^+$ . Then there is a condition  $N$  stronger than  $M$  such that  $\delta \in N$  and  $\varepsilon \in N$ .*

*Proof.* Let  $\lambda < \kappa$  be an inaccessible cardinal, such that  $\lambda \geq \delta$  and  $\lambda > |M|$ . We let  $N = M \cup \lambda \cup \{\lambda\} \cup \{\varepsilon\}$ ; thus  $\kappa_N = \lambda + 1$ , and  $N$  is  $\omega$ -closed. We extend  $\pi^M$  and  $\rho^M$  to  $\pi^N$  and  $\rho^N$  as follows:

If  $\kappa \leq \alpha < \varepsilon$  and  $\alpha \in M$ , we let  $\pi_\alpha^N(\beta) = \beta$  for all  $\beta \in N$  such that  $\kappa_M \leq \beta \leq \lambda$ . If  $\alpha \in M$  and  $\varepsilon < \alpha$ , we define  $\pi_\alpha^N$  so that  $\pi_\alpha^N$  maps  $\kappa_N - \kappa_M$  onto  $(\kappa_N - \kappa_M) \cup \{\varepsilon\}$ . For  $\alpha = \varepsilon$ , we define  $\pi_\varepsilon^N$  in such a way that  $\pi_\varepsilon^N$  maps  $\lambda$  onto  $N \cap \varepsilon$ .

Finally, if  $\alpha, \beta \in N$ ,  $\beta < \kappa \leq \alpha$ , and if either  $\alpha = \varepsilon$  or  $\beta \geq \kappa_M$  we let  $\rho_\alpha^N(\beta) = \lambda$ .

Clearly,  $N$  is a model,  $M$  is a submodel of  $N$ , and  $|M| < |\kappa_N|$ . Let us verify (3.3.ii). This holds if  $\alpha \in M$ , so let  $\alpha = \varepsilon$ . Let  $\beta \leq \lambda$ , let  $\{\beta_n : n < \omega\} \subseteq \beta$  and let  $\gamma = \sup\{\pi_\varepsilon^N(\beta_n) : n < \omega\}$  be such that  $\kappa \leq \gamma < \varepsilon$ . There is a  $\zeta < \lambda = \rho_\varepsilon^N(\beta)$  such that  $\pi_\varepsilon^N(\zeta) = \gamma$ , and so (3.3.ii) holds.

To complete the proof that  $N$  is a forcing condition, we verify (3.3. iii). This we do by showing that if  $a \in E_1$  is a submodel of  $N$  then  $a \subseteq M$ .

Assume that  $a \in E_1$  is a submodel of  $N$  but  $a \not\subseteq M$ . Thus there are  $\alpha, \beta \in a$ ,  $\beta < \kappa \leq \alpha$  such that either  $\alpha = \varepsilon$  or  $\beta \geq \kappa_M$ . Then  $\rho_\alpha^a(\beta) = \rho_\alpha^N(\beta) = \lambda$  and so  $\lambda \in a$ , and  $\kappa_a = \lambda + 1$ . This contradicts the assumption that  $\kappa_a$  is an inaccessible cardinal.  $\square$

Thus if  $G$  is a generic for  $Q_S$ , let  $\langle M_G, \pi_G, \rho_G \rangle$  be the union of all conditions in  $G$ . Then for every  $a \in E_1$ , that is a submodel of  $M_G$ ,  $S \cap \mathcal{P}_{\kappa_a} a$  is stationary in  $\mathcal{P}_{\kappa_a} a$ . Thus  $Q_S$  forces that  $S$  reflects at all but nonstationary many  $a \in E_1$ .

We will now prove that the forcing  $Q_S$  is  $< \kappa$ -strategically closed. The key

technical devices are the two following lemmas.

**Lemma 3.5.** *Let  $M_0 > M_1 > \dots > M_n > \dots$  be an  $\omega$ -sequence of conditions. There exists a condition  $M$  stronger than all the  $M_n$ , with the following property:*

(3.6)

*If  $N$  is any model coherent with  $M$  such that there exists some  $\gamma \in N \cap M$*

$$\text{but } \gamma \notin \bigcup_{n=0}^{\infty} M_n, \text{ then } \kappa_N > \lim_n \kappa_{M_n}.$$

*Proof.* Let  $A = \bigcup_{n=0}^{\infty} M_n$  and  $\delta = A \cap \kappa = \lim_n \kappa_{M_n}$ , and let  $\pi^A = \bigcup_{n=0}^{\infty} \pi^{M_n}$  and  $\rho^A = \bigcup_{n=0}^{\infty} \rho^{M_n}$ . We let  $M$  be the  $\omega$ -closure of  $(\delta + \delta) \cup A$ ; hence  $\kappa_M = \delta + \delta + 1$ . To define  $\pi^M$ , we first define  $\pi_\alpha^M \supset \pi_\alpha^A$  for  $\alpha \in A$  in such a way that  $\pi_\alpha^M$  maps  $\delta + \delta$  onto  $M \cap \alpha$ . When  $\alpha \in M - A$  and  $\alpha \geq \kappa$ , we have  $|M \cap \alpha| = |\delta|$  and so there exists a function  $\pi_\alpha^M$  on  $M \cap \alpha$  that maps  $\delta + \delta$  onto  $M \cap \alpha$ ; we let  $\pi_\alpha^M$  be such, with the additional requirement that  $\pi_\alpha^M(0) = \delta$ . To define  $\rho^M$ , we let  $\rho^M \supset \rho^A$  be such that  $\rho^M(\alpha, \beta) = \delta + \delta$  whenever either  $\alpha \notin A$  or  $\beta \notin A$ .

We shall now verify that  $M$  satisfies (3.3. ii). Let  $\alpha, \beta \in M$  be such that  $\alpha \geq \kappa$  and  $\beta < \kappa$  and let  $\gamma \in M$ ,  $\kappa \leq \gamma < \alpha$ , be an  $\omega$ -limit point of the set  $\{\pi_\alpha^M(\xi) : \xi < \beta\}$ . We want to show that  $\gamma = \pi_\alpha^M(\eta)$  for some  $\eta < \rho_\alpha^M(\beta)$ . If both  $\alpha$  and  $\beta$  are in  $A$  then this is true, because  $\alpha, \beta \in M_n$  for some  $n$ , and  $M_n$  satisfies (3.3 ii). If either  $\alpha \notin A$  or  $\beta \notin A$  then  $\rho_\alpha^M(\beta) = \delta + \delta$ , and since  $\pi_\alpha^M$  maps  $\delta + \delta$  onto  $M \cap \alpha$ , we are done.

Next we verify that  $M$  satisfies (3.6). Let  $N$  be any model coherent with  $M$ , and let  $\gamma \in M \cap N$  be such that  $\gamma \notin A$ . If  $\gamma < \kappa$  then  $\gamma \geq \delta$  and so  $\kappa_N > \delta$ . If  $\gamma \geq \kappa$  then  $\pi_\gamma^M(0) = \delta$ , and so  $\delta = \pi_\gamma^N(0) \in N$ , and again we have  $\kappa_N > \delta$ .

Finally, we show that for every  $a \in E_1$ , if  $a \subseteq M$  then  $S \cap \mathcal{P}_{\kappa_a} a$  is stationary. We do this by showing that for every  $a \in E_1$ , if  $a \subseteq M$  then  $a \subseteq M_n$  for some  $M_n$ .

Thus let  $a \subseteq M$  be such that  $\kappa_a$  is regular and  $\lambda_a = \kappa_a^+$ . As  $\kappa_a \leq \kappa_M = \delta + \delta + 1$ , it follows that  $\kappa_a < \delta$  and so  $\kappa_a < \kappa_{M_{n_0}}$  for some  $n_0$ . Now by (3.6) we have

$a \subseteq \bigcup_{n=0}^{\infty} M_n$ , and since  $\lambda_a$  is regular uncountable, there exists some  $n \geq n_0$  such that  $M_n \cap a$  is cofinal in  $a$ . It follows from Lemma 3.2 that  $a \subseteq M_n$ .

**Lemma 3.7.** *Let  $\lambda < \kappa$  be a regular uncountable cardinal and let  $M_0 > M_1 > \dots > M_\xi > \dots$ ,  $\xi < \lambda$ , be a  $\lambda$ -sequence of conditions with the property that for every  $\eta < \lambda$  of cofinality  $\omega$ ,*

(3.8)

*If  $N$  is any model coherent with  $M_\eta$  such that there exists some  $\gamma \in N \cap M_\eta$*

*but  $\gamma \notin \bigcup_{\xi < \eta} M_\xi$ , then  $\kappa_N > \lim_{\xi \rightarrow \eta} \kappa_{M_\xi}$ .*

*Then  $M = \bigcup_{\xi < \lambda} M_\xi$  is a condition.*

*Proof.* It is clear that  $M$  satisfies all the requirements for a condition, except perhaps (3.3 iii). ( $M$  is  $\omega$ -closed because  $\lambda$  is regular uncountable.) Note that because  $|M_\xi| < \kappa_{M_{\xi+1}}$  for all  $\xi < \lambda$ , we have  $|M| = |\kappa_M|$ .

We shall prove (3.3 iii) by showing that for every  $a \in E_1$ , if  $a \subseteq M$ , then  $a \subseteq M_\xi$  for some  $\xi < \lambda$ . Thus let  $a \subseteq M$  be such that  $\kappa_a$  is regular and  $\lambda_a = \kappa_a^+$ .

As  $\lambda_a = |a| \leq |M| = |\kappa_M|$ , it follows that  $\kappa_a < \kappa_M$  and so  $\kappa_a < \kappa_{M_{\xi_0}}$  for some  $\xi_0 < \lambda$ . We shall prove that there exists some  $\xi \geq \xi_0$  such that  $M_\xi \cap a$  is cofinal in  $a$ ; then by Lemma 3.2,  $a \subseteq M_\xi$ .

We prove this by contradiction. Assume that no  $M_\xi \cap a$  is cofinal in  $a$ . We construct sequences  $\xi_0 < \xi_1 < \dots < \xi_n < \dots$  and  $\gamma_1 < \gamma_2 < \dots < \gamma_n < \dots$  such that for each  $n$ ,

$$\gamma_n \in a, \quad \gamma_n > \sup(M_{\xi_n} \cap a), \quad \text{and} \quad \gamma_n \in M_{\xi_{n+1}}$$

Let  $\eta = \lim_n \xi_n$  and  $\gamma = \lim_n \gamma_n$ . We claim that  $\gamma \in a$ .

As  $\lambda_a$  is regular uncountable, there exists an  $\alpha \in a$  such that  $\alpha > \gamma$ . Let  $\beta_n$ ,  $n \in \omega$ , be such that  $\pi_\alpha^a(\beta_n) = \gamma_n$ , and let  $\beta < \kappa_a$  be such that  $\beta > \beta_n$  for all  $n$ . As  $M$  satisfies (3.3. ii), and  $\gamma = \sup\{\pi_\alpha^M(\beta_n) : n < \omega\}$ , there is some  $\zeta < \rho_\alpha^M(\beta)$  such that  $\gamma = \pi_\alpha^M(\zeta)$ . Since  $\zeta < \rho_\alpha^M(\beta) = \rho_\alpha^a(\beta) < \kappa_a$ , we have  $\zeta \in a$ , and  $\gamma = \pi_\alpha^a(\zeta) \in a$ .

Now since  $\gamma \in a$  and  $\gamma > \sup(M_{\xi_n} \cap a)$  we have  $\gamma \notin M_{\xi_n}$ , for all  $n$ . As  $M_\eta$  is  $\omega$ -closed, and  $\gamma_n \in M_\eta$  for each  $n$ , we have  $\gamma \in M_\eta$ . Thus by (3.8) it follows that  $\kappa_a > \lim_n \kappa_{M_{\xi_n}}$ , a contradiction.  $\square$

**Lemma 3.9.**  *$Q_S$  is  $< \kappa$ -strategically closed.*

*Proof.* In the game, player I moves at limit stages. In order to win the game, it suffices to choose at every limit ordinal  $\eta$  of cofinality  $\omega$ , a condition  $M_\eta$  that satisfies (3.8). This is possible by Lemma 3.5.  $\square$

We shall now prove that  $Q_S$  satisfies the  $< \kappa$ -strategic  $\kappa^+$ -chain condition. First a lemma:

**Lemma 3.10.** *Let  $\langle M_1, \pi_1, \rho_1 \rangle$  and  $\langle M_2, \pi_2, \rho_2 \rangle$  be forcing conditions such that  $\kappa_{M_1} = \kappa_{M_2}$  and that the models  $M_1$  and  $M_2$  are coherent. Then the conditions are compatible.*

*Proof.* Let  $\lambda < \kappa$  be an inaccessible cardinal such that  $\lambda > |M_1 \cup M_2|$  and let  $M = M_1 \cup M_2 \cup \lambda \cup \{\lambda\}$ . We shall extend  $\pi_1 \cup \pi_2$  and  $\rho_1 \cup \rho_2$  to  $\pi^M$  and  $\rho^M$  so that  $\langle M, \pi^M, \rho^M \rangle$  is a condition.

If  $\alpha \in M_i - \kappa$ , we define  $\pi_\alpha^M \supset \pi_i$  so that  $\pi_\alpha^M$  maps  $\lambda - \kappa_{M_1}$  onto  $M \cap \alpha$ , and such that  $\pi_\alpha^M(\beta) = \lambda$  whenever  $\kappa \leq \beta < \alpha$ ,  $\alpha \in M_1 - M_2$  and  $\beta \in M_2 - M_1$  (or vice versa). We define  $\rho_\alpha^M \supset \rho_i$  by  $\rho_\alpha^M(\beta) = \lambda$  for  $\kappa_{M_1} \leq \beta \leq \lambda$ . It is easy to see that  $M$  is an  $\omega$ -closed model that satisfies (3.3 ii).

To verify (3.3 iii), we show that every  $a \in E_1$  that is a submodel of  $M$  is either  $a \subseteq M_1$  or  $a \subseteq M_2$ . Thus let  $a$  be a submodel of  $M$ ,  $a \in E_1$ , such that neither  $a \subseteq M_1$  nor  $a \subseteq M_2$ . First assume that  $\kappa_a \leq \kappa_{M_1}$ . Then there are  $\alpha, \beta \in a$  such that  $\kappa \leq \beta < \alpha$  and  $\alpha \in M_1 - M_2$  while  $\beta \in M_2 - M_1$  (or vice versa). But then  $\pi^a(\alpha, \beta) = \pi^M(\alpha, \beta) = \lambda$  which implies  $\lambda \in a$ , or  $\kappa_a = \lambda + 1$ , contradicting the inaccessibility of  $\kappa_a$ .

Thus assume that  $\kappa_a > \kappa_{M_1}$ . Let  $\alpha \in a$  be such that  $\alpha \geq \kappa$ , and then we have  $\rho^a(\alpha, \kappa_{M_1}) = \rho^M(\alpha, \kappa_{M_1}) = \lambda$ , giving again  $\lambda \in a$ , a contradiction.  $\square$

**Lemma 3.11.**  *$Q_S$  satisfies the  $< \kappa$ -strategic  $\kappa^+$ -chain condition.*

*Proof.* Let  $\lambda$  be a limit ordinal  $< \kappa$  and consider the game (2.3) of length  $\lambda$ . We may assume that  $\text{cf } \lambda > \omega$ . In the game, player I moves at limit stages, and the key to winning is again to make right moves at limit stages of cofinality  $\omega$ . Thus let  $\eta$  be a limit ordinal  $< \lambda$ , and let  $\{M_\xi^\alpha : \alpha < \kappa^+, \text{ cf } \alpha = \kappa\}$  be the set of conditions played at stage  $\xi$ .

By Lemma 3.5, player I can choose, for each  $\alpha$ , a condition  $M_\eta^\alpha$  stronger than each  $M_\xi^\alpha$ ,  $\xi < \eta$ , such that  $M_\eta^\alpha$  satisfies (3.8). Then let  $E_\eta$  be the closed unbounded subset of  $\kappa^+$

$$E_\eta = \{\gamma < \kappa^+ : M_\eta^\alpha \subset \gamma \text{ for all } \alpha < \gamma\},$$

and let  $f_\eta$  be the function  $f_\eta(\alpha) = M_\eta^\alpha \upharpoonright \alpha$ , this being the restriction of the model  $M_\eta^\alpha$  to  $\alpha$ .

We claim that player I wins following this strategy: By Lemma 3.7, player I can make a legal move at every limit ordinal  $\xi < \lambda$ , and for each  $\alpha$  (of cofinality  $\kappa$ ),  $M^\alpha = \bigcup_{\xi < \lambda} M_\xi^\alpha$  is a condition. Let  $\alpha < \beta$  be ordinals of cofinality  $\kappa$  in  $\bigcap_{\xi < \lambda} E_\xi$  such that  $f_\xi(\alpha) = f_\xi(\beta)$  for all  $\xi < \lambda$ . Then  $M^\alpha \subset \beta$  and  $M^\beta \upharpoonright \beta = M^\alpha \upharpoonright \alpha$ , and because the functions  $\pi$  and  $\rho$  have the property that  $\pi(\gamma, \delta) < \gamma$  and  $\rho(\gamma, \delta) < \gamma$  for every  $\gamma$  and  $\delta$ , it follows that  $M^\alpha$  and  $M^\beta$  are coherent models with  $\kappa_{M^\alpha} = \kappa_{M^\beta}$ . By Lemma 3.10,  $M^\alpha$  and  $M^\beta$  are compatible conditions.  $\square$

#### 4. Preservation of the set $E_1$ .

We shall complete the proof by showing that the set

$$E_1 = \{x \in \mathcal{P}_{\kappa\kappa^+} : \kappa_x \text{ is inaccessible and } \lambda_x = \kappa_x^+\}$$

remains stationary after forcing with  $P = P_\kappa * \dot{P}^\kappa$ .

Let us reformulate the problem as follows: Let us show, working in  $V^{P_\kappa}$ , that for every condition  $p \in \dot{P}^\kappa$  and every  $\dot{P}^\kappa$ -name  $\dot{F}$  for an operation  $\dot{F} : (\kappa^+)^{<\omega} \rightarrow \kappa^+$  there exists a condition  $\bar{p} \leq p$  and a set  $x \in E_1$  such that  $\bar{p}$  forces that  $x$  is closed under  $\dot{F}$ .

As  $\kappa$  is supercompact, there exists by the construction of  $P_\kappa$  and by Laver's Theorem 2.1, an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  that witnesses that  $\kappa$  is  $\kappa^+$ -supercompact and such that the  $\kappa^{\text{th}}$  iterand of the iteration  $j(P_\kappa)$  in  $M$  is (the name for) the forcing  $\dot{P}^\kappa$ . The elementary embedding  $j$  can be extended, by a standard argument, to an elementary embedding  $j : V^{P_\kappa} \rightarrow M^{j(P_\kappa)}$ . Since  $j$  is elementary, we can achieve our stated goal by finding, in  $M^{j(P_\kappa)}$ , a condition  $\bar{p} \leq j(p)$  and a set  $x \in j(E_1)$  such that  $\bar{p}$  forces that  $x$  is closed under  $j(\dot{F})$ .

The forcing  $j(P_\kappa)$  decomposes into a three step iteration  $j(P_\kappa) = P_\kappa * \dot{P}^\kappa * \dot{R}$  where  $\dot{R}$  is, in  $M^{P_\kappa * \dot{P}^\kappa}$ , a  $< j(\kappa)$ -strategically closed forcing.

Let  $G$  be an  $M$ -generic filter on  $j(P_\kappa)$ , such that  $p \in G$ . The filter  $G$  decomposes into  $G = G_\kappa * H * K$  where  $H$  and  $K$  are generics on  $\dot{P}^\kappa$  and  $\dot{R}$  respectively, and  $p \in H$ . We shall find  $\bar{p}$  that extends not just  $j(p)$  but each member of  $j''H$  ( $\bar{p}$  is a *master condition*). That will guarantee that when we let  $x = j''\mathcal{P}_{\kappa^+}$  (which is in  $j(E_1)$ ) then  $\bar{p}$  forces that  $x$  is closed under  $j(\dot{F})$ : this is because  $\bar{p} \Vdash j(\dot{F}) \upharpoonright x = j''F_H$ , where  $F_H$  is the  $H$ -interpretation of  $\dot{F}$ .

We construct  $\bar{p}$ , a sequence  $\langle p_\xi : \xi < j(2^{\kappa^+}) \rangle$ , by induction. When  $\xi$  is not in the range of  $j$ , we let  $p_\xi$  be the trivial condition; that guarantees that the support of  $\bar{p}$  has size  $< j(\kappa)$ . So let  $\xi = j(\gamma)$  be such that  $\bar{p} \upharpoonright \xi$  has been constructed.

Let  $M$  the model  $\bigcup \{j(N) : N \in H_\gamma\}$  where  $H_\gamma$  is the  $\gamma^{\text{th}}$  coordinate of  $H$ . The  $\gamma^{\text{th}}$  iterand of  $\dot{P}^\kappa$  is the forcing  $Q(S)$  where  $S$  is a stationary subset of  $E_0$ . In order for  $M$  to be a condition in  $Q(j(S))$ , we have to verify that for every submodel  $a \subseteq M$ , if  $a \in j(E_1)$  then  $j(S)$  reflects at  $a$ .

Let  $a \in j(E_1)$  be a submodel of  $M$ . If  $\kappa_a < \kappa_M = \kappa$ , then  $a = j''\bar{a} = j(\bar{a})$  for some  $\bar{a} \in E_1$ , and  $\bar{a}$  is a submodel of some  $N \in H_\gamma$ . As  $S$  reflects at  $\bar{a}$  it follows that  $j(S)$  reflects at  $a$ .

If  $\kappa_a = \kappa$ , then  $\lambda_a = \kappa^+$ , and  $a$  is necessarily cofinal in the universe of  $M$ , which is  $j''\kappa^+$ . By Lemma 3.2, we have  $a = M$ , and we have to show that  $j(S)$  reflects at  $j''\kappa^+$ . This means that  $j''S$  is stationary in  $\mathcal{P}_\kappa(j''\kappa^+)$ , or equivalently, that  $S$  is stationary in  $\mathcal{P}_\kappa\kappa^+$ .

We need to verify that  $S$  is a stationary set, in the model  $M^{j(P_\kappa)*j(\dot{P}_\kappa)\upharpoonright j(\gamma)}$ , while we know that  $S$  is stationary in the model  $V^{P_\kappa*\dot{P}_\kappa\upharpoonright\gamma}$ . However, the former model is a forcing extension of the latter by a  $< \kappa$ -strategically closed forcing, and the result follows by Lemma 2.5.

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