On incomparability and related cardinal functions on ultraproducts of Boolean algebras

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ABSTRACT: Let C denote any of the following cardinal characteristics of Boolean algebras: incomparability, spread, character, π-character, hereditary Lindelöf number, hereditary density. It is shown to be consistent that there exists a sequence $\langle B_i : i < \kappa \rangle$ of Boolean algebras and an ultrafilter D on κ such that

$$C(\prod_{i<\kappa} B_i/D) < |\prod_{i<\kappa} C(B_i)/D|.$$

This answers a number of problems posed in [M].

Introduction

For a number of cardinal characteristics C of Boolean algebras it makes sense to ask whether it is consistent to have a sequence $\langle B_i : i < \kappa \rangle$ of Boolean algebras and an ultrafilter D on κ such that

$$C(\prod_{i<\kappa} B_i/D) < |\prod_{i<\kappa} C(B_i)/D|.$$

For C being the length this was proved in [MSh]. The same method of proof can be used to get the analogous thing for C being any one of the following: incomparability (Inc), spread (s), character (χ), π-character ($\pi\chi$), hereditary Lindelöf number (hL), hereditary density (hd). This answers problems 47, 48, 52, 56, 60 of [M]. For irredundancy (Monk's problem 25) this will be done in a subsequent paper of the first author. We won't define these notions here, as they are very clearly defined on pp. 2,3 in [M]. We assume that the reader has a good knowledge of [MSh] and [Mg].

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For C a cardinal function of Boolean algebras which is defined as the supremum of all cardinals which have a certain property, we define C^+ as the least cardinal κ such that this property fails for every cardinal $\lambda \geq \kappa$. Note that all cardinal functions mentioned above are of this form. For χ, note that in [M] $\chi(B)$ has been defined as the minimal κ such that every ultrafilter on B can be generated by κ elements. Clearly, $\chi(B)$ can be equivalently defined as

$$\sup\{\chi(U) : U \text{ is an ultrafilter on } B\},$$

where $\chi(U)$ is the minimal size of a generating subset of U.

In §1 below we deal with incomparability. In §2 we shall show that the results for all the other characteristics can be deduced from this relatively easily.

A key notion for the proofs is that of μ-entangled linear order, μ being a cardinal (see definition 1.1 below). The reason for this is the following observation of Shelah (see [M, p.225]), where $Int(I)$ denotes the interval algebra of some linear order I, i.e. the subalgebra of $\mathcal{P}(I)$ generated by the half-open intervals of the form $[a, b)$, $a, b \in I$.

Fact. *Let μ be a regular uncountable cardinal and let I be a linear order. The following are equivalent:*
(1) I is μ-entangled.
(2) There is no incomparable subset of $Int(I)$ of size μ.

For b a member of some Boolean algebra B, b^1 denotes b and b^0 denotes the complement of b. By $Ult(B)$ we denote the Stone space of B.

1. Incomparability

Definition 1.1. Let $(I, <)$ be a linear order and let $D \subseteq \mathcal{P}(\kappa)$ for some infinite cardinal κ.

(1) $(I, <)$ is called (δ, γ)-entangled, where $\gamma < \delta$ are cardinals, if for every family $\langle t_{\alpha,\varepsilon} : \alpha < \delta, \varepsilon < \varepsilon(*) < \gamma\rangle$ of pairwise distinct members of I and for every $u \subseteq \varepsilon(*)$ there exist $\alpha < \beta < \delta$ such that

$$\forall \varepsilon < \varepsilon(*) \quad t_{\alpha,\varepsilon} < t_{\beta,\varepsilon} \Leftrightarrow \varepsilon \in u.$$

If $\gamma = \omega$ we say that $(I, <)$ is δ-entangled.

(2) $(I,<)$ is called (δ, D)-entangled if for every sequence $\langle t_{\alpha,\varepsilon,l} : \alpha < \delta, \varepsilon \in A, l < n\rangle$ of pairwise distinct members of I, where $A \in D$ and $n < \omega$, and for every $u \subseteq n$ there exist $\alpha < \beta < \delta$ and $B \subseteq A$, $B \in D$, such that

$$\forall \varepsilon \in B \forall l < n \quad t_{\alpha,\varepsilon,l} < t_{\beta,\varepsilon,l} \Leftrightarrow l \in u.$$

Note that if $\kappa < \gamma$ and $(I,<)$ is (δ,γ)-entangled then $(I,<)$ is (δ, D)-entangled for every $D \subseteq \mathcal{P}(\kappa)$.

In the sequel, if $\gamma < \delta$ are regular cardinals, by $C(\gamma,\delta)$ we denote the partial order to add δ Cohen subsets to γ. More precisely,

$$C(\gamma,\delta) = \prod_{i<\delta} Q_i \qquad (< \gamma\text{-support product})$$

where $Q_i = ({}^{<\gamma}2, \supseteq)$. Clearly Q is γ-directedly-closed.

Lemma 1.2. *Let $\gamma < \delta$ be regular cardinals such that $\forall \alpha < \delta \quad \alpha^{<\gamma} < \delta$. Let $L = \{\eta_i : i < \delta\}$ be $C(\gamma,\delta)$-generic. Letting $<_{lex}$ denote the lexicographic order on ${}^{\gamma}2$, we have that $(L, <_{lex})$ is a (δ,γ)-entangled linear order.*

Proof: Suppose $p \Vdash_{C(\gamma,\delta)}$ "$\langle \dot\tau(\alpha,\varepsilon) : \alpha < \delta, \varepsilon < \varepsilon(*) < \gamma\rangle$ is a sequence of pairwise distinct ordinals below δ such that the family $\langle \dot\eta_{\dot\tau(\alpha,\varepsilon)} : \alpha < \delta, \varepsilon < \varepsilon(*) < \gamma\rangle$ contradicts (δ,γ)-entangledness of L, witnessed by set $u \subseteq \varepsilon(*)$". Here $\dot\eta_i$ is the canonical name for the ith Cohen subset of γ.

As $C(\gamma,\delta)$ does not add new ordinal sequences of length $< \gamma$ we may assume that $u \in V$. For the same reason, for each $\alpha < \delta$ we may pick $p_\alpha \leq p$ such that p_α decides the value of $\langle \dot\tau(\alpha,\varepsilon) : \varepsilon < \varepsilon(*)\rangle$, say as $\langle \tau(\alpha,\varepsilon) : \varepsilon < \varepsilon(*)\rangle$. By the Δ-system Lemma and some thinning out there exists $Y \in [\delta]^\delta$ and $r \in [\delta]^{<\gamma}$ such that for all $\alpha,\beta \in Y$, $\alpha \neq \beta$, we have:

(i) $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = r$,

(ii) $\{\tau(\alpha,\varepsilon) : \varepsilon < \varepsilon(*)\} \subseteq \mathrm{dom}(p_\alpha)$,

(iii) $\langle p_\alpha(i) : \alpha \in Y\rangle$ is constant for every $i \in r$,

(iv) $\langle p(\tau(\alpha,\varepsilon)) : \alpha \in Y\rangle$ is constant for every $\varepsilon < \varepsilon(*)$.

Pick $\alpha,\beta \in Y$ with $\alpha < \beta$. Note that p_α, p_β are compatible, and hence $\{\tau(\alpha,\varepsilon) : \varepsilon < \varepsilon(*)\} \cap \{\tau(\beta,\varepsilon) : \varepsilon < \varepsilon(*)\} = \emptyset$.

Define $q \leq p_\alpha, p_\beta$ by $\mathrm{dom}(q) = \mathrm{dom}(p_\alpha) \cup \mathrm{dom}(p_\beta)$ and

$$q(i) = \begin{cases} p_\alpha(i)\hat{\ }0 & i \in \{\tau(\alpha,\varepsilon) : \varepsilon \in u\} \\ p_\alpha(i)\hat{\ }1 & i \in \{\tau(\alpha,\varepsilon) : \varepsilon \in \varepsilon(*) \setminus u\} \\ p_\beta(i)\hat{\ }0 & i \in \{\tau(\beta,\varepsilon) : \varepsilon \in \varepsilon(*) \setminus u\} \\ p_\beta(i)\hat{\ }1 & i \in \{\tau(\beta,\varepsilon) : \varepsilon \in u\} \\ p_\alpha(i) & i \in \mathrm{dom}(p) \setminus \{\tau(\alpha,\varepsilon) : \varepsilon < \varepsilon(*)\} \\ p_\beta(i) & \text{otherwise.} \end{cases}$$

Then clearly

$$q \Vdash (\forall \varepsilon < \varepsilon(*)) \eta_{\dot\tau(\alpha,\varepsilon)} <_{lex} \eta_{\dot\tau(\beta,\varepsilon)} \Leftrightarrow \varepsilon \in u,$$

a contradiction. $\square$

Assume GCH. Let μ be a supercompact cardinal and let $\kappa < \mu$ be a measurable cardinal. Fix D a normal measure on κ. By [L] we may assume that the supercompactness of μ cannot be destroyed by any μ-directedly-closed forcing.

For any ordinal α let $F(\alpha)$ denote the least inaccessible cardinal above α, if it exists. We assume that $F(\mu)$ exists and denote it with λ.

Let $Q = C(\mu, \lambda)$. Clearly Q is μ-directedly-closed and $V^Q \models 2^\mu = \lambda$.

Work in V^Q. Let U be a normal fine measure on $[H((2^\lambda)^+)]^{<\mu}$. By Lemma 1.2, $H((2^\lambda)^+) \models$ "there exists a (λ, μ)-entangled linear order on λ". Therefore the set A of all $a \in [H((2^\lambda)^+)]^{<\mu}$ such that

(1) $(a, \in) \prec (H((2^\lambda)^+), \in)$,

(2) $a \cap \mu$ is measurable,

(3) the Mostowski collapse of a is $H((2^{F(a\cap\mu)})^+)$,

(4) $H((2^{F(a\cap\mu)})^+) \models$ there is a $(F(a \cap \mu), a \cap \mu)$-entangled linear order on $F(a \cap \mu)$, call it $J^*_{a\cap\mu}$,

(5) $H((2^{F(a\cap\mu)})^+) \models 2^{a\cap\mu} = F(a \cap \mu)$

belongs to U.

By well-known arguments on large cardinals and elementary embeddings we can build a sequence $\bar{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of normal measures on μ such that

(a) $\alpha < \beta \Rightarrow U_\alpha < U_\beta$ (i.e. $U_\alpha \in \mathrm{Ult}(V, U_\beta)$),

(b) $\{a \cap \mu : a \in A\} \in U_\alpha$ for all $\alpha < \kappa$.

The main fact which is used for this is the following lemma which goes back to [SRK]. We thank James Cummings for reconstructing the proof for us.

Lemma 1.3 *For all $a \in V_{\mu+2}$ there exists a normal measure U on μ such that $a \in Ult(V,U)$ and $\{a \cap \mu : a \in A\} \in U$.*

Sketch of proof: Let $j : V \to M$ be the elementary embedding defined by the normal fine measure U above. Fix $<$ a wellordering of V_μ and let

$$<^* = j(<) \restriction V_{\lambda^+}.$$

Assuming that the Lemma is false, let $b \in V_{\mu+2}$ be the $<^*$-minimal counterexample. Let $\bar{U} = \{B \subseteq \mu : \mu \in j(B)\}$ be the normal measure on μ induced by j and let $i : V \to N$ be the corresponding elementary embedding. As usual we have another elementary embedding $k : N \to M$, defined by $k([f]_{\bar{U}}) = j(f)(\mu)$, such that $j = k \circ i$ (see [J, p.312]). As $V_{\mu+2} \subseteq M$ we have that

$$M \models b \text{ is the } j(<)\text{-minimal counterexample.}$$

By elementarity there must exist $\bar{b} \in N$ such that $k(\bar{b}) = b$ and

$$N \models \bar{b} \text{ is the } i(<)\text{-minimal counterexample.}$$

Note that $b \notin N$, as b is a counterexample. Also note that $V_{\mu+1} \subseteq ran(k)$. As k is simply the inverse of the transitive collapse map on $ran(k)$, we conclude $\bar{b} = b$ and hence $b \in N$, a contradiction. □

Let $Q(\bar{U})$ denote Magidor's forcing to change the cofinality of μ to κ by adding a normal sequence $\langle \mu_i : i < \kappa \rangle$ cofinal in μ. Fix such a $Q(\bar{U})$-generic sequence with $\mu_0 > 2^\kappa$. We let

$$\mu'_i = \mu_{\omega i},\ \theta_i = F(\mu_{i+1}),\ \lambda_i = F(\mu_{\omega i}),\ J_i = J^*_{\mu_{i+1}}.$$

Lemma 1.4 *For every $i < \kappa$, $V^{Q * Q(\bar{U})} \models$ "$F(\mu_{i+1}) = \theta_i$ and J_i is (θ_i, D)-entangled".*

Proof: Work in V^Q. Let $\langle \dot{\mu}_i : i < \kappa \rangle$ be a $Q(\bar{U})$-name for the generic sequence. Fix $i < \kappa$. Let $p \in Q(\bar{U})$ such that p decides $\dot{\mu}_j$ as μ_j for $j \in \{i, i+1, i+2\}$. We may assume that the domain of the first coordinate of p is $\{i, i+1, i+2\}$. By the main arguments of [Mg], especially [Mg, Lemma 5.3], it follows that forcing $Q(\bar{U})$ below p factors as $P^i_{\mu_i} * Q^i$, where $P^i_{\mu_i}$ is the union of μ_i many μ_i-directed suborders each of them of size $\leq 2^{\mu_i}$, and Q^i does not add new subsets to μ_{i+2}. Hence clearly $V^{Q * Q(\bar{U})} \models F(\mu_{i+1}) = \theta_i$.

Now suppose $p \Vdash$ "$\langle \dot{t}_{\alpha,\varepsilon,l} : \alpha < \theta_i, \varepsilon \in A, l < n \rangle$ is a one-to-one family of elements of J_i". By [Mg, Lemma 4.6], for each $\alpha < \theta_i$ we can find $p_\alpha \leq p$ such that p_α and p have the same first coordinate and for all $\varepsilon \in A$ and $l < n$ there exists $w_{\alpha,\varepsilon,l} \in [i]^{<\omega}$ such that below p_α, the value of $\dot{t}_{\alpha,\varepsilon,l}$ depends only on the value of $\langle \dot{\mu}_j : j \in w_{\alpha,\varepsilon,l}\rangle$. As D is κ-complete and $i < \kappa$, there exists $B_{\alpha,l} \in D$ and $w_{\alpha,l}$ such that $w_{\alpha,\varepsilon,l} = w_{\alpha,l}$ for all $\varepsilon \in B_{\alpha,l}$ and $l < n$. Let $w_\alpha = \bigcup_{l<n} w_{\alpha,l}$, $B_\alpha = \bigcap_{l<n} B_{\alpha,l}$. As $2^\kappa < \theta_i$ and $i < \theta_i$ we can find $Y \subseteq \theta_i$ of size θ_i, $w^* \in [i]^{<\omega}$ and $B \in D$ such that $B_\alpha = B$ and $w^* = w_\alpha$ for all $\alpha \in Y$. Let v be the domain of the first coordinate of any p_α. By [Mg, Lemma 3.3], for each $\alpha \in Y$ we can find $p'_\alpha \le p_\alpha$ such that the domain of the first coordinate of p'_α is $w^* \cup v$. Then p'_α decides $\langle \dot{\mu}_j : j \in w^*\rangle$, say as $\langle \mu^\alpha_j : j \in w^*\rangle$, and hence p'_α decides $\langle \dot{t}_{\alpha,\varepsilon,l} : \varepsilon \in B, l < n\rangle$, say as $\langle t_{\alpha,\varepsilon,l} : \varepsilon \in B, l < n\rangle$. Note that this sequence is one-to-one. As $\mu_i < \theta_i$ we can find $Y' \subseteq Y$ of size θ_i and $\langle \mu_j : j \in w^*\rangle$ such that $\langle \mu^\alpha_j : j \in w^*\rangle = \langle \mu_j : j \in w^*\rangle$ and $(p_\alpha)_{i+2} = (p_\beta)_{i+2}$ (we use the notation of [Mg, p.67]), for all $\alpha, \beta \in Y'$. By [Mg, Lemma 4.1] it follows that p_α and p_β are compatible for all $\alpha, \beta \in Y'$. It follows that $\langle t_{\alpha,\varepsilon,l} : \alpha \in Y', \varepsilon \in B, l < n\rangle$ is a one-to-one family. Applying (θ_i, D)-entangledness of J_i in V^Q, for any $u \subseteq n$ we obtain $B' \in D$, $B' \subseteq B$ and $\alpha < \beta$, $\alpha, \beta \in Y'$, such that for all $\varepsilon \in B'$, for all $l < n$, $t_{\alpha,\varepsilon,l} < t_{\beta,\varepsilon,l} \Leftrightarrow l \in u$. What we have shown suffices to prove the Lemma. □

For every $i < \kappa$ we define a linear order $I_i \subseteq \prod_{j<\omega i} \theta_j$ as follows: For every $i' < \omega i$ fix a family $\langle A^\rho : \rho \in \prod_{j<i'} \theta_j\rangle$ of pairwise disjoint subsets of $\theta_{i'} \cap Card$, each of them of cardinality $\theta_{i'}$. This is possible as $|\prod_{j<i'} \theta_j| < \theta_{i'}$ and $\theta_{i'}$ is a regular limit cardinal. Let I_i be the set of all $\eta \in \prod_{j<\omega i} \theta_j$ such that for all $j < \omega i$, $\eta(j) \in A^{\eta \restriction j}$. Define a linear order $<_i$ on I_i as follows: For distinct $\eta, \nu \in I_i$ let $\varepsilon = \min\{j < \omega i : \eta(j) \ne \nu(j)\}$. Now let

$$\eta <_i \nu \Leftrightarrow \begin{cases} \varepsilon \text{ is even and } \eta(\varepsilon) <_{J_\varepsilon} \nu(\varepsilon), \text{ or} \\ \varepsilon \text{ is odd and } \eta(\varepsilon) < \nu(\varepsilon). \end{cases}$$

We claim that in I_i we can choose a one-to-one family $\langle \eta^i_{\zeta,\varepsilon} : \varepsilon \le \zeta < \lambda_i\rangle$ such that the following hold:

(a) $\forall \zeta_1 < \zeta_2 \forall \varepsilon_1 \le \zeta_1 \forall \varepsilon_2 \le \zeta_2 \quad \eta^i_{\zeta_1,\varepsilon_1} <_{J^{bd}_{\omega i}} \eta^i_{\zeta_2,\varepsilon_2}$,

(b) $\langle \eta^i_{\zeta,0} : \zeta < \lambda_i\rangle$ is cofinal in $\prod_{j<\omega i} \theta_j / J^{bd}_{\omega i}$,

(c) the mapping $\langle (\eta^i_{\zeta,2\varepsilon}, \eta^i_{\zeta,2\varepsilon+1}) : \varepsilon < \zeta\rangle$ is $<_i$-preserving.

Here $J^{bd}_{\omega i}$ denotes the ideal of bounded subsets of ωi. For the construction of such a family remember from [MSh] that $\prod_{j<\omega i} \theta_j / J^{bd}_{\omega i}$ has true cofinality λ_i. Clearly, in I_i we can find a family $\langle \eta^i_\zeta : \zeta < \lambda_i\rangle$ which is increasing and cofinal in $\prod_{j<\omega i} \theta_j / J^{bd}_{\omega i}$ and satisfies

$\eta^i_\zeta(j)\cdot 3<\eta^i_{\zeta+1}(j)$ for almost all $j<\omega i$. Now let $\zeta<\lambda_i$ and $2\varepsilon<\zeta$. Define $\eta^i_{\zeta,2\varepsilon}$ and $\eta^i_{\zeta,2\varepsilon+1}$ by letting

$$\eta^i_{\zeta,2\varepsilon+l}(j)=\begin{cases}\eta_\zeta(j) & \text{if } j \text{ is even,}\\ \eta_\zeta(j)+\eta_\varepsilon(j) & \text{if } j \text{ is odd and } l=0,\\ \eta_\zeta(j)+\eta_\varepsilon(j)\cdot 2 & \text{if } j \text{ is odd and } l=1.\end{cases}$$

It is easy to see that this definition works.

Lemma 1.5. *In $V^{Q*Q(\bar U)}$ the following holds: Whenever $\langle J_i : i<\kappa\rangle$ is a family such that for every i, J_i is a (θ_i,D)-entangled linear order on θ_i and I_i is defined as above, then $(I_i,<_i)$ is (λ_i,D)-entangled but not $(\lambda',\aleph_0)$-entangled for any $\lambda'<\lambda_i$.*

Proof: The last statement easily follows from the existence of the family $\langle\eta^i_{\zeta,\varepsilon}:\varepsilon\le\zeta<\lambda_i\rangle$. Let $\langle t_{\alpha,\varepsilon,l}:\alpha<\lambda_i,\varepsilon\in A,l<n\rangle$ be a family of pairwise distinct members of I_i, where $A\in D$ and $n<\omega$. Hence

$$t_{\alpha,\varepsilon,l}=\eta^i_{\zeta(\alpha,\varepsilon,l),\nu(\alpha,\varepsilon,l)}$$

for some $\nu(\alpha,\varepsilon,l)\le\zeta(\alpha,\varepsilon,l)<\lambda_i$. Fix $\alpha<\lambda_i$. As $i<\kappa$ there is $A'_\alpha\in D$ and $i^*_\alpha<\omega i$ such that for all distinct $\varepsilon,\varepsilon'\in A'_\alpha$ and $l,m<n$ do we have $t_{\alpha,\varepsilon,l}\restriction i^*_\alpha\neq t_{\alpha,\varepsilon',l}\restriction i^*_\alpha$, $t_{\alpha,\varepsilon,l}\restriction i^*_\alpha\neq t_{\alpha,\varepsilon,m}\restriction i^*_\alpha$ and $t_{\alpha,\varepsilon,l}\restriction i^*_\alpha\neq t_{\alpha,\varepsilon',m}\restriction i^*_\alpha$. As $2^\kappa<\lambda_i$ we may assume that $\langle A'_\alpha:\alpha<\lambda_i\rangle$ and $\langle i^*_\alpha:\alpha<\lambda_i\rangle$ are constant, say with values A^*, i^*. As in [Sh462, Claim 3.1.1] one shows that there must exist cofinally many even $j\in(i^*,\omega i)$ such that for every $\xi<\theta_j$ there is $\alpha<\lambda_i$ with the property $\forall\varepsilon\in A^*\forall l<n\quad t_{\alpha,\varepsilon,l}(j)>\xi$. Fix such j. Construct an increasing sequence $\langle\alpha(\nu):\nu<\theta_j\rangle$ such that

$$\forall\nu<\rho<\theta_j\forall\varepsilon,\varepsilon'\in A'\forall l,m<n\quad t_{\alpha(\nu),\varepsilon,l}(j)<t_{\alpha(\rho),\varepsilon',m}(j).$$

As $(\prod_{l<j}\theta_l)^\kappa<\theta_j$, we may assume that the sequence $\langle\langle t_{\alpha(\nu),\varepsilon,l}\restriction j:\varepsilon\in A^*,l<n\rangle:\nu<\theta_j\rangle$ is constant. Note that by construction,

$$\langle t_{\alpha(\nu),\varepsilon,l}(j):\nu<\theta_j,\varepsilon\in A^*,l<n\rangle$$

is a sequence of pairwise distinct members. We can apply (θ_j,D)-entangledness of J_j and, for given $u\subseteq n$, we get $B\in D$, $B\subseteq A^*$, and $\nu<\rho<\theta_j$ such that

$$\forall\varepsilon\in B\forall l<n\quad t_{\alpha(\nu),\varepsilon,l}(j)<_{J_i}t_{\alpha(\rho),\varepsilon,l}(j)\Leftrightarrow l\in u.$$

By construction we conclude that

$$\forall \varepsilon \in B \forall l < n \quad t_{\alpha(\nu),\varepsilon,l} <_i t_{\alpha(\rho),\varepsilon,l} \Leftrightarrow l \in u.$$

$\square$

Lemma 1.6. *Letting $I = \prod_{i<\kappa} I_i/D$, I is $(\lambda, \aleph_0)$-entangled in $V^{Q*Q(\bar{U})*Coll(\mu^+,<\lambda)}$.*

Proof: By Lemmas 1.4 and 1.5 and as $Coll(\mu^+, <\lambda)$ does not add new subsets to μ, in $V^{Q*Q(\bar{U})*Coll(\mu^+,<\lambda)}$ it is true that I_i is (λ_i, D)-entangled and D is a normal fine measure on κ. Moreover, note that $\prod_{i<\kappa} \lambda_i/D$ has order-type λ in $V^{Q*Q(\bar{U})*Coll(\mu^+,<\lambda)}$. This is true because it holds in $V^{Q*Q(\bar{U})}$ by [MSh] and because $Coll(\mu^+, <\lambda)$ does not add new functions to $\prod_{i<\kappa} \lambda_i$. As $V^{Q*Q(\bar{U})*Coll(\mu^+,<\lambda)} \models \lambda = \mu^{++}$ we have that the cofinality of $\prod_{i<\kappa} \lambda_i/D$ is λ.

Let $\langle t^l_\alpha : \alpha < \lambda, l < n\rangle$, $n < \omega$, be a family of pairwise distinct elements of I. So t^l_α is of the form

$$t^l_\alpha = \langle \eta^i_{\zeta_i(\alpha,l),\varepsilon_i(\alpha,l)} : i < \kappa\rangle/D,$$

where $\eta^i_{\zeta_i(\alpha,l),\varepsilon_i(\alpha,l)} \in I_i$. By the above observations, wlog we may assume that

($*_1$) $\langle\langle\zeta_i(\alpha,l) : i < \kappa\rangle/D : \alpha < \lambda\rangle$ is increasing and cofinal in $\prod_{i<\kappa} \lambda_i/D$, for every $l < n$.

For every $\alpha < \lambda$ and $i < \kappa$ there is $j < \omega i$ such that for every $l < m < n$, if $\langle\zeta_i(\alpha,l), \varepsilon_i(\alpha,l)\rangle \neq \langle\zeta_i(\alpha,m), \varepsilon_i(\alpha,m)\rangle$ then

$$\eta^i_{\zeta_i(\alpha,l),\varepsilon_i(\alpha,l)} \restriction j \neq \eta^i_{\zeta_i(\alpha,m),\varepsilon_i(\alpha,m)} \restriction j.$$

By Los' Theorem and since D is normal, there exist $B'_\alpha \in D$ and $j'_\alpha < \kappa$ such that for all $i \in B'_\alpha$ and $l < m < n$ we have

$$\eta^i_{\zeta_i(\alpha,l),\varepsilon_i(\alpha,l)} \restriction j'_\alpha \neq \eta^i_{\zeta_i(\alpha,m),\varepsilon_i(\alpha,m)} \restriction j'_\alpha.$$

As $2^\kappa < \lambda$, wlog we may assume that

($*_2$) there are $B^1 \in D$ and $j' < \kappa$ such that for all $\alpha < \lambda$, $i \in B^1$ and $l < m < n$

$$\eta^i_{\zeta_i(\alpha,l),\varepsilon_i(\alpha,l)} \restriction j' \neq \eta^i_{\zeta_i(\alpha,m),\varepsilon_i(\alpha,m)} \restriction j'.$$

Moreover we have

$(*_3)$ there exist $B^2 \in D$, $B^2 \subseteq B^1$, and $\langle j_i^2 : i \in B^2 \rangle$ such that $j_i^2 < \omega i$ and for every $g \in \prod_{i \in B^2} \lambda_i$, $f_i \in \prod\{\theta_j : j_i^2 \leq j < \omega i\}$ and $\alpha < \lambda$ we can find $\beta \in (\alpha, \lambda)$ such that for every $i \in B^2$, $j_i^2 \leq j < \omega i$ and $l < n$ we have $g(i) < \zeta_i(\beta, l)$ and

$$f_i(j) < \eta^i_{\zeta_i(\beta,l),\varepsilon_i(\beta,l)}(j).$$

If $(*_3)$ failed, for every candidate $y = \langle B^y, \langle j_i^y : i \in B^y \rangle\rangle$ to satisfy $(*_3)$ we had g^y, $\langle f_i^y : i \in B^y \rangle$, α^y which witness that y does not satisfy $(*_3)$. Note that there are only 2^κ candidates. Let

$$\alpha = \sup\{\alpha^y : y \text{ is a candidate}\}$$

and

$$f_i(j) = \sup\{f_i^y(j) : y \text{ is a candidate and } j \in \mathit{dom}(f_i^y)\}.$$

As there are only 2^κ candidates we have $\alpha < \lambda$ and $f_i(j) < \theta_j$. We can choose $\beta_i < \lambda_i$ such that $\beta_i > \zeta_i(\alpha, l)$ for every $l < n$ and

$$f_i <_{J^{bd}_{\omega i}} \eta^i_{\beta_i,\varepsilon}$$

for every $\varepsilon \leq \beta_i$. Finally we define $g \in \prod_{i<\kappa} \lambda_i$ by letting

$$g(i) = \sup\{g^y(i) : y \text{ is a candidate and } i \in B\} \cup \{\beta_i + 1\}.$$

By $(*_1)$ we can find $\gamma \in (\alpha, \lambda)$ and $B \in D$ such that $g(i) < \langle \zeta_i(\gamma, l) : i < \kappa \rangle$ for all $i \in B$ and $l < n$. By construction, for every $i \in B$ there is $j_i < \omega i$ such that for all $j_i \leq j < \omega i$ and $l < n$

$$f_j(j) < \eta^i_{\zeta_i(\gamma,l)\varepsilon_i(\gamma,l)}(j).$$

Then $y = \langle B, \langle j_i : i \in B \rangle\rangle$ is a candidate which contradicts the definition of α, $\langle f_i : i < \kappa \rangle$, g. This finishes the proof ok $(*_3)$.

As D is normal, wlog we may assume that in $(*)_3$, $\langle j_i^2 : i \in B^3 \rangle$ is constant with value $j^2 < \kappa$. Now choose $i^* \in B^3$ even with $\max\{j^1, j^2\} < i^*$. Using $(*_3)$ it is straightforward to find an increasing sequence $\langle \alpha(\nu) : \nu < \theta_{i^*} \rangle$ in λ such that for all $i \in B^3 \setminus i^* + 1$ and $l, m < n$ we have

$$\eta^i_{\zeta_i(\alpha(\nu),l)\varepsilon_i(\alpha(\nu),l)}(i^*) < \eta^i_{\zeta_i(\alpha(\nu+1),l)\varepsilon_i(\alpha(\nu+1),l)}.$$

As $(\prod_{j<i^*}\theta_j)^\kappa < \theta_{i^*}$, wlog we may assume that

$$\langle\langle \eta^i_{\zeta_i(\alpha(\nu),l)\varepsilon_i(\alpha(\nu),l)} \restriction i^* : i \in B^3 \setminus (i^*+1), l<n\rangle : \nu < \theta_{i^*}\rangle$$

is constant. By construction we have that, letting

$$s_{\nu,i,l} = \eta^i_{\zeta_i(\alpha(\nu),l)\varepsilon_i(\alpha(\nu),l)}(i^*),$$

$$\langle s_{\nu,i,l} : \nu < \theta_{i^*}, i \in B^3 \setminus (i^*+1), l<n\rangle$$

is a sequence of pairwise distinct members of I_{i^*}. Hence by Lemma 1.5, for every $u \subseteq n$ we can find $\nu < \xi < \theta_{i^*}$ and $A \in D$, $A \subseteq B^3 \setminus (i^*+1)$ such that for all $i \in A$ and $l < n$ we have

$$s_{\nu,i,l} < s_{\xi,i,l} \Leftrightarrow l \in u.$$

This implies

$$t^l_{\alpha(\nu)} < t^l_{\alpha(\xi)} \Leftrightarrow l \in u,$$

which finishes the proof. $\square$

As a corollary we obtain the following:

Theorem 1.7 *For $i<\kappa$ let I_i be the linear order defined above and let $B_i = Int(I_i)$. In the model $V^{Q*Q(\bar{U})*Coll(\mu^+,<\lambda)}$ the following hold:*

(i) $Inc(B_i) = Inc^+(B_i) = \lambda_i$ for all $i<\kappa$, and hence $\prod_{i<\kappa} Inc(B_i)/D = \lambda = \mu^{++}$,

(ii) $Inc^+(\prod_{i<\kappa} B_i/D) \leq \lambda$ and hence $Inc(\prod_{i<\kappa} B_i/D) \leq \mu^+$.

Proof: (i) follows from the fact mentioned in the introduction and Lemma 1.5. Note that Lemma 1.5 holds also in $V^{Q*Q(\bar{U})*Coll(\mu^+,<\lambda)}$ as $Coll(\mu^+,<\lambda)$ does not add new subset of μ.

(ii) follows from the same fact, by Lemma 1.6 and by the fact that $\prod_{i<\kappa} B_i/D$ is isomorphic to Int $\prod_{i<\kappa} I_i/D$. This last fact holds by Los' Theorem and as D is $\aleph_1$-complete. $\square$

2. Other characteristics

Definition 2.1. If $(I,<)$ is a linear order, by $Sq(I)$ we denote the Boolean subalgebra of $(\mathcal{P}(I^2),\subseteq)$ generated by sets of the form

$$X_{a,b} = \{(a',b') \in I^2 : a' < a \text{ and } b' < b\},$$

for $a, b \in I$.

Recall that a sequence $\langle y_\alpha : \alpha < \lambda \rangle$ of elements of some Boolean algebra is *left-separated* iff for every $\alpha < \lambda$, y_α does not belong to $Id\langle y_\beta : \beta > \alpha \rangle$, the ideal generated by $\langle y_\beta : \beta > \alpha \rangle$. Similary, $\langle y_\alpha : \alpha < \lambda \rangle$ is *right-separated* if for every $\alpha < \lambda$, y_α does not belong to $Id\langle y_\beta : \beta < \alpha \rangle$, the ideal generated by $\langle y_\beta : \beta < \alpha \rangle$.

Lemma 2.2. *Suppose $(I, <)$ is a λ-entangled linear order, where $\lambda > \omega$ is regular. Then $Sq(I)$ has neither a left-separated nor a right-separated sequence of length λ.*

Proof: We prove the Lemma only for right-separated sequences. The proof for left-separated sequences is similar. Suppose $\langle y_\alpha : \alpha < \lambda \rangle$ is a right-separated sequence in $\mathrm{Sq}(I)$. We shall obtain a contradiction. Each y_α is a finite union of finite intersections of sets of the form $X_{a,b}$ or $-X_{a,b}$. One of these finite intersections does not belong to $Id\langle y_\beta : \beta < \alpha \rangle$. Hence wlog we may assume that each y_α is such a finite intersection. As $\mathrm{cf}(\lambda) > \omega$, wlog there exist $n < \omega$ and $\eta : n \to 2$ such that

$$y_\alpha = \bigcap_{l<n} X^{\eta(l)}_{a(\alpha,l),b(\alpha,l)}$$

for some $a(\alpha, l), b(\alpha, l) \in I$, for all $\alpha < \lambda$.

Case I: $\exists l < n \quad \eta(l) = 1$.

As the intersection of any two sets of the form $X_{a,b}$ has the same form, wlog we may assume that $\eta(0) = 1$ and $\eta(l) = 0$ for all $0 < l < n$. We may also assume that $0 < l < l' < n$ implies $a(\alpha, l) \neq a(\alpha, l')$, $b(\alpha, l) \neq b(\alpha, l')$ and $a(\alpha, l) < a(\alpha, l') \Leftrightarrow b(\alpha, l) > b(\alpha, l')$, for all $\alpha < \lambda$. Otherwise we could choose a smaller n. Hence we have two subcases according to whether $a(\alpha, 1) < \ldots < a(\alpha, n-1)$ and $b(\alpha, 1) > \ldots > b(\alpha, n-1)$ or $a(\alpha, 1) > \ldots > a(\alpha, n-1)$ and $b(\alpha, 1) < \ldots < b(\alpha, n-1)$ holds. We assume the first alternative holds. The second one is symmetric.

For fixed $\alpha < \lambda$ define the following sets:

$$\begin{aligned}
z_0 &= X_{a(\alpha,0),b(\alpha,0)} - X_{a(\alpha,n-1),b(\alpha,0)},\\
z_1 &= X_{a(\alpha,n-1),b(\alpha,0)} - X_{a(\alpha,n-1),b(\alpha,n-1)} - X_{a(\alpha,n-2),b(\alpha,0)},\\
&\ \ldots\\
z_{n-2} &= X_{a(\alpha,2),b(\alpha,0)} - X_{a(\alpha,2),b(\alpha,2)} - X_{a(\alpha,1),b(\alpha,0)},\\
z_{n-1} &= X_{a(\alpha,1),b(\alpha,0)} - X_{a(\alpha,1),b(\alpha,1)}.
\end{aligned}$$

Note that $y_\alpha = \bigcup_{j<n} z_j$. Hence there exists $j < n$ such that $z_j \notin Id\langle y_\beta : \beta < \alpha\rangle$. Wlog we may assume that j is the same for all $\alpha < \lambda$ and that $y_\alpha = z_j$ for all $\alpha < \lambda$. Then y_α has the form $X_{a,b} - X_{a',b} - X_{a,b'}$ or $X_{a,b} - X_{a',b}$ or $X_{a,b} - X_{a,b'}$ for some $a' < a$ and $b' < b$. Let us assume y_α is of the first form. The others are even easier to handle. Hence we have

$$y_\alpha = X_{c(\alpha,0),d(\alpha,0)} - X_{c(\alpha,0),d(\alpha,1)} - X_{c(\alpha,1),d(\alpha,0)},$$

where $c(\alpha,1) < c(\alpha,0)$ and $d(\alpha,1) < d(\alpha,0)$.

Choose $F \subseteq 2 \times 2$ maximal such that there exist $\sigma : F \to I$ and cofinally many $\alpha \in \lambda$ with the property that $(0,j) \in F$ implies $c(\alpha,j) = \sigma(0,j)$ and $(1,j) \in F$ implies $d(\alpha,j) = \sigma(1,j)$ for all $j < 2$. Wlog we may assume that the above holds for all $\alpha < \lambda$ and that for all $\alpha < \beta < \lambda$ and $(i,j) \in 2 \times 2 \setminus F$, if $i = 0$ then $c(\alpha,j) \neq c(\beta,j)$ and if $i = 1$ then $d(\alpha,j) \neq d(\beta,j)$. Depending on F we have 16 cases to consider. However we consider only the case $F = \emptyset$, as the others are similar.

We have more subcases to consider according to the order-type of the sequence $\langle c(\alpha,0), c(\alpha,1), d(\alpha,0), d(\alpha,1)\rangle$. Wlog we may assume that it does not depend on α. We only work through two typical examples. Let us first assume that this sequence consists of pairwise distinct elements. As we assumed $F = \emptyset$ we conclude that $\langle c(\alpha,j), d(\alpha,j) : \alpha < \lambda, j < 2\rangle$ is a family of pairwise distinct elements. By λ-entangledness of I we get $\alpha > \beta$ such that $c(\alpha,0) < c(\beta,0)$, $d(\alpha,0) < d(\beta,0)$, $c(\alpha,1) > c(\beta,1)$ and $d(\alpha,1) > d(\beta,1)$. We conclude $y_\alpha \leq y_\beta$, a contradiction. Now suppose $c(\alpha,0) = d(\alpha,0) < c(\alpha,1) < d(\alpha,1)$. In this case the family $\langle c(\alpha,j), d(\alpha,1) : \alpha < \lambda, j < 2\rangle$ consists of pairwise distinct elements. By λ-entangledness we obtain $\alpha > \beta$ such that $c(\alpha,0) < c(\beta,0)$, $c(\alpha,1) > c(\beta,1)$ and $d(\alpha,1) > d(\beta,1)$. Again we conclude $y_\alpha \leq y_\beta$, a contradiction. The other cases are similar.

Case II: $\forall l < n \quad \eta(l) = 0$.

Again we may assume that $a(\alpha,0) < a(\alpha,1) < \ldots < a(\alpha,n-1)$ and $b(\alpha,0) > b(\alpha,1) > \ldots > b(\alpha,n-1)$ for all $\alpha < \lambda$. Notice that wlog we may assume that

$$X^0_{a(\alpha,n-1),b(\alpha,0)} \notin Id\langle y_\beta : \beta < \alpha\rangle$$

for all $\alpha < \lambda$, as otherwise we may replace y_α by $y_\alpha \cap X_{a(\alpha,n-1),b(\alpha,0)}$ and proceed as in Case I. Hence wlog

$$y_\alpha = X^0_{a(\alpha,n-1),b(\alpha,0)}$$

for all $\alpha < \lambda$. Let $a_\alpha = a(\alpha, n-1), b_\alpha = b(\alpha, 0)$. Clearly, if $a_\alpha = a_\beta$ for some $\alpha < \beta$ then $b_\alpha < b_\beta$, as otherwise $y_\alpha \leq y_\beta$. Similarly, $b_\alpha = b_\beta$ implies $a_\alpha < a_\beta$. As a λ-entangled linear order does not have any increasing or decreasing sequences of length λ, wlog we may assume that both families $\langle a_\alpha : \alpha < \lambda \rangle$ and $\langle b_\alpha : \alpha < \lambda \rangle$ are one-to-one. By a similar argument we may assume that $a_\alpha \neq b_\alpha$ for all $\alpha < \lambda$ and also that $a_\alpha \neq b_\beta$ for all $\alpha \neq \beta$. We can apply λ-entangledness of I to the family $\langle a_\alpha, b_\alpha : \alpha < \lambda \rangle$ and get some $\alpha > \beta$ such that $a_\alpha > a_\beta$ and $b_\alpha > b_\beta$. Hence $y_\alpha \leq y_\beta$, a contradiction. □

Lemma 2.3. *Let $(I, <)$ be a linear order and μ a cardinal such that there exist $\{(a_\alpha, b_\alpha) : \alpha < \mu\} \subseteq I^2$ and $c \in I$ with the property that $a_\alpha \neq a_\beta$, $b_\alpha < c$ and that $a_\alpha < a_\beta$ implies $b_\alpha < b_\beta$ for all $\alpha, \beta < \mu$, $\alpha \neq \beta$. Then $s^+(Sq(I)) > \mu$ holds.*

Proof: Let $y_\alpha = X_{a_\alpha, c} - X_{a_\alpha, b_\alpha}$, for $\alpha < \mu$. Note that

$$y_\alpha \not\leq \bigcup_{\beta \in F} y_\beta$$

for all $\alpha < \mu$ and finite $F \subseteq \mu$ with $\alpha \notin F$. Indeed, let $F_0 = \{\beta \in F : a_\alpha < a_\beta\}$, $F_1 = F \setminus F_0$, let β_0 be the subscript of the smallest a_β, $\beta \in F_0$ and let β_1 be the subscript of the largest a_β, $\beta \in F_1$. Then $y_\alpha \setminus \bigcup_{\beta \in F} y_\beta = y_\alpha \setminus (y_{\beta_0} \cup y_{\beta_1})$. As $(a_\alpha, b_\alpha) \in y_\alpha \setminus (y_{\beta_0} \cup y_{\beta_1})$ we are done. Hence there exists a family of ultrafilters $\langle U_\alpha : \alpha < \mu \rangle$ with $y_\alpha \in U_\alpha$ and $-y_\beta \in U_\alpha$ for all $\alpha \neq \beta$. Then $\langle U_\alpha : \alpha < \mu \rangle$ is a discrete set of cardinality μ in the Stone space of $Sq(I)$. □

Corollary 2.4. *Using the notation of §1, letting $B_i = Sq(I_i)$ for $i < \kappa$, in the model $V^{Q * Q(\bar{U}) * Coll(\mu^+, <\lambda)}$ the following hold:*

(i) $s(B_i) = s^+(B_i) = hL(B_i) = hL^+(B_i) = hd(B_i) = hd^+(B_i) = \lambda_i$ for all $i < \kappa$, and hence $|\prod_{i<\kappa} s(B_i)/D| = |\prod_{i<\kappa} hL(B_i)/D| = |\prod_{i<\kappa} hd(B_i)/D| = \lambda = \mu^{++}$,

(ii) $hL^+(\prod_{i<\kappa} B_i/D) = hd^+(\prod_{i<\kappa} B_i/D) \leq \lambda$ and hence $s(\prod_{i<\kappa} B_i/D)$, $hL(\prod_{i<\kappa} B_i/D)$ and $hd(\prod_{i<\kappa} B_i/D)$ are all at most μ^+. □

Proof: We first prove (i). The proofs of Theorem 6.7 and Lemma 6.8 in [M] show that for every Boolean algebra B, if $hd(B) = \kappa$, κ being regular and infinite, then $hd(B)$ is attained (i.e. there exists a subspace $X \subseteq Ult(B)$ with $d(X) = \kappa$) iff B has a left-separated sequence of length κ. Similarly, the proof of Theorem 15.1 in [M] shows that if $hL(B)$ is regular and infinite, then $hL(B) = \kappa$ is attained iff B has a right-separated sequence of length κ. As trivially $s^+(B) \leq \min\{hL^+(B),\ hd^+(B)\}$ and hence $s(B) \leq$

$\min\{hL(B),\ hd(B)\}$ holds, we conclude that all cardinal coefficients of B_i mentioned in (i) are at most λ_i. That they are at least λ_i follows from Lemma 2.3, the construction of I_i and the trivial fact that every linear order of cardinality μ^+, for some cardinal μ, has a subset of size μ which has an upper bound.

In order to prove (ii) note that by Los' Theorem and $\aleph_1$-completeness of D we have that $\prod_{i<\kappa} B_i/D$ is isomorphic to $Sq(\prod_{i<\kappa} I_i/D)$. By Lemmas 1.6 and 2.2 and the previous argument we get (ii). $\square$

Definition 2.5. Let $\langle y_\alpha : \alpha < \lambda\rangle$ be a one-to-one enumeration of some infinite linear order $(J, <_J)$. Define a linear order $(L(J), <)$ by letting $L(J) = \{(y_\alpha, \beta) : \alpha < \lambda,\ \beta < \alpha\}$ and

$$(y_\alpha, \beta) < (y'_\alpha, \beta') \Leftrightarrow (y_\alpha <_J y_{\alpha'}) \vee (y_\alpha = y_{\alpha'} \wedge \beta < \beta').$$

Lemma 2.6. *Let σ be an infinite, regular cardinal which is not the successor of a singular cardinal. Let $(J, <_J)$ be a linear order of size λ which does not have any increasing or decreasing chain of length λ. Then*

$$\chi^+(Int(L(J))) = \pi\chi^+(Int(L(J))) = \lambda$$

holds.

Proof: As trivially $\pi\chi^+(B) \leq \chi^+(B)$ holds for every Boolean algebra B, it suffices to show $\chi^+(Int(L(J))) \leq \lambda$ and $\pi\chi^+(Int(L(J))) \geq \lambda$. Let U be an ultrafilter on $Int(L(J))$. Let

$$L_U = \{z \in L(J) : (-\infty, z) \in U\}.$$

Clearly L_U is a (possibly empty) end-segment of $L(J)$. It is straightforward to see that

$$\chi(U) \leq cf(L \setminus L_U) + cf(L_U^*),$$

where the cofinality of a linear order is the minimal length of a well-ordered cofinal subset, and L_U^* is the inverse order of L_U. We claim that $cf(L \setminus L_U) + cf(L_U^*) < \lambda$. Let us first consider $cf(L \setminus L_U)$. If $(L \setminus L_U) \cap J \times \{0\}$ is unbounded in $L \setminus L_U$ then $cf(L \setminus L_U)$ equals the cofinality of some well-ordered increasing chain in J, which is assumed to be $< \lambda$. Otherwise $L \setminus L_U \subseteq \{(y_\beta, \gamma) : \beta \leq \alpha,\ \gamma < \beta\}$ for some $\alpha < \lambda$. Then $cf(L \setminus L_U) \leq |\alpha| < \lambda$. We conclude $\chi^+(Int(L(J))) \leq \lambda$.

In order to prove $\pi\chi^+(Int(L(J))) \geq \lambda$ let $\sigma < \lambda$ be regular. Let U be the ultrafilter on $Int(L(J))$ generated by the intervals

$$[(y_{\sigma+1}, \alpha), (y_{\sigma+1}, \sigma)), \quad \alpha < \sigma.$$

Now let $Y \subseteq Int(L(J)) \setminus \{0\}$ be dense in U. If $|U| < \sigma$ there exists $y \in Y$ such that $y \subseteq [(y_{\sigma+1}, \alpha), (y_{\sigma+1}, \sigma))$ holds for cofinally many $\alpha < \sigma$. This is clearly impossible. $\square$

Corollary 2.7. *Using the notation of* §1 *and definition 2.5, letting* $B_i = Int(L(I_i))$ *for* $i < \kappa$*, in the model* $V^{Q*Q(\bar{U})*Coll(\mu^+,<\lambda)}$ *the following hold:*

(i) $\pi(B_i) = \pi^+(B_i) = \pi\chi(B_i) = \pi\chi^+(B_i) = \lambda_i$ *for all* $i < \kappa$*, and hence* $|\prod_{i<\kappa} \chi(B_i)/D| = |\prod_{i<\kappa} \chi^+(B_i)/D| = \lambda = \mu^{++}$,

(ii) $\chi^+(\prod_{i<\kappa} B_i/D) = \pi\chi^+(\prod_{i<\kappa} B_i/D) = \lambda$ *and hence* $\chi(\prod_{i<\kappa} B_i/D) = \pi\chi(\prod_{i<\kappa} B_i/D) = \mu^+$. $\square$

Proof: First note that I_i, $i < \kappa$, has a dense subset of size $\mu_{\omega i}$. Indeed, for each $s \in \bigcup_{j'<\omega i} \prod_{j<j'} \theta_j$ choose $\eta_s \in I_i$ with $s \subseteq \eta_s$ if this is possible. It is easy to see that the collection of all these η_s is dense in I_i. As there are only $\mu_{\omega i}$ many s we are done. Hence clearly I_i does not have a well-ordered increasing or decreasing chain of length λ_i. Hence by Lemma 2.6 we have (i). By Los' Theorem and $\aleph_1$-completeness of D we have that $\prod_{i<\kappa} B_i/D$ is isomorphic to $Int(L(\prod_{i<\kappa} I_i/D))$. By Los' Theorem again and as $\prod_{i<\kappa} \lambda_i = \lambda$, it follows that $\prod_{i<\kappa} I_i/D$ does not have a well-ordered increasing or decreasing chain of length λ. By Lemma 2.6 we conclude (ii). $\square$

References

[J] T. Jech, **Set theory,** Academic Press, New York (1978).

[L] R. Laver, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, **Israel J. Math. 29** (1978), no. 4, 385–388.

[Mg] M. Magidor, *Changing cofinality of cardinals*, **Fund. Math. 99** (1978), 61-71.

[MSh] M. Magidor and S. Shelah, *Length of Boolean algebras and ultraproducts*, **Math. Japonica**, to appear.

[M] J.D. Monk, **Cardinal Invariants on Boolean Algebras**, Birkhäuser Verlag, Basel, 1996.

[SRK] R. Solovay, W. Reinhardt and A. Kanamori, *Strong axioms of infinity and elementary embeddings* **Ann. Math. Logic 13** (1978), no. 1, 73–116.

[Sh462] S. Shelah, **Cardinal arithmetic**, Oxford Logic Guides, 29. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.

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