KARP COMPLEXITY AND CLASSES WITH THE INDEPENDENCE PROPERTY

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ABSTRACT. A class **K** of structures is *controlled* if for all cardinals λ, the relation of $L_{\infty,\lambda}$-equivalence partitions **K** into a set of equivalence classes (as opposed to a proper class). We prove that no pseudo-elementary class with the independence property is controlled. By contrast, there is a pseudo-elementary class with the strict order property that is controlled (see [4]).

1. INTRODUCTION

It is well known that the class of models of an unstable theory is a rather complicated beast. Perhaps the most familiar statement of this complexity is that every such theory T has 2^κ nonisomorphic models for every $\kappa > |T|$ (see e.g., [7]). In fact, much more is true. For instance, in [6] the second author proves that if **K** is an unsuperstable pseudo-elementary class (for definiteness **K** is the class of L-reducts of an L'-theory T') then for every cardinal $\kappa > |T'|$, **K** contains a family of 2^κ pairwise nonembeddable structures, each of size κ.

Despite these results, our aim is to give some sort of 'classification' to certain unstable classes, or to prove that no such classification is possible. Clearly, because of the results mentioned above, what is meant by a classification in this context is necessarily very weak. Following [4], a class **K** of structures is *controlled* if for every cardinal κ, the relation of $L_{\infty,\kappa}$-equivalence partitions **K** into a *set* of equivalence classes (as opposed to a proper class of classes). In [4] we show that this notion has a number of equivalences. In particular, in [4] we prove the following proposition (see [2] or [4] for definitions of the undefined notions):

Proposition 1.1. *The following are notions are equivalent for any class* **K** *of structures.*

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(1) **K** *is controlled;*
(2) *For any cardinal κ, there is an ordinal bound on the $L_{\infty,\kappa}$-Scott heights of the structures in* **K***;*
(3) *For any cardinal κ, there is an ordinal bound on the κ-Karp complexity of the structures in* **K***;*
(4) *For any cardinal μ, there is a cardinal κ such that for any $M \in$* **K***, there are at most κ distinct L_{∞,μ^+}-types of subsets of M of size at most μ realized in M.*

The whole of this paper is devoted to the proof of the following theorem (see Definition 2.5).

Theorem 1.2. *No pseudo-elementary class with the independence property is controlled.*

To place this result in context, recall that in [7], the second author proves that every unstable theory either has the independence property or has the strict order property. Paradigms for these theories are the theory of the random graph and the theory of dense linear order, respectively. In [4] we prove that the pseudo-elementary class of doubly transitive linear orders (which is a subclass of the class of dense linear orders) is controlled. By contrast, it follows immediately from Theorem 1.2 that every pseudo-elementary subclass of the class of random graphs is uncontrolled. That is, with respect to the relation of $L_{\infty,\kappa}$-equivalence, classes of reducts of extensions of the theory of the random graph are sizably more complicated than certain classes of reducts of extensions of the theory of dense linear order.

The history of this paper is rather lengthy. The statement of Theorem 1.2 was conjectured by the second author almost ten years ago. From the outset it was clear that Theorem 1.2 should be proved by embedding extremely complicated ordered graphs into structures in **K** using the generalization of the Ehrenfeucht-Mostowski construction given in Theorem 2.4. It was also clear (at least to the second author) that the complexity of the ordered graph should come from a complicated coloring of pairs from a relatively small cardinal (see Theorem 2.6). However, the road from these ideas to a formal proof was not smooth. There were a great many false attempts by both authors along the way. The obstruction was not the infinitary combinatorics. Rather, it was the very finitary combinatorics that arose from passing from a well-behaved skeleton to its definable closure that proved difficult.

In Section 2 we develop three notions that arise in the proof of Theorem 1.2. The proof of the theorem is contained in Section 3, with many definitions and easy lemmas relegated to the appendix. As the results in the appendix are wholly self-contained, there is no circularity.

2. The independence property and complicated colorings

We begin this preliminary section by proving a fundamental theorem (Theorem 2.4) about Skolemized theories with the independence property and discussing its consequences for pseudo-elementary classes. Following this, we discuss many complicated colorings of certain uncountable cardinals. We close the section with a short discussion of well-founded trees.

Definition 2.1. A formula $\varphi(\overline{x},\overline{y})$ has the *independence property* with respect to a theory T if for each $n \in \omega$ there is a model M of T and sequences $\langle \overline{b}_i : i < n\rangle$, $\langle \overline{a}_w : w \subseteq n\rangle$ from M such that $M \models \varphi(\overline{a}_w, \overline{b}_i)$ if and only if $i \in w$.

A formula $\psi(\overline{z}_1,\overline{z}_2)$ *codes graphs* if for every (symmetric) graph (G,R) there is a model M_G of T and $\{\overline{c}_g : g \in G\}$ from M_G such that for all $g,h \in G$, $M_G \models \psi(\overline{c}_g,\overline{c}_h)$ if and only if $R(g,h)$.

A theory T has the *independence property* if some formula $\varphi(\overline{x},\overline{y})$ has the independence property with respect to T.

The next lemma tells us that if a theory T has the independence property, then there is a formula that both codes graphs and has the independence property with respect to T.

Lemma 2.2. *Let T be any theory.*

1. *If $\psi(\overline{z}_1,\overline{z}_2)$ codes graphs, then $\psi(\overline{z}_1,\overline{z}_2)$ has the independence property with respect to T.*
2. *If $\varphi(\overline{x},\overline{y})$ has the independence property with respect to T, then the formula*

$$\psi(\overline{x}_1\overline{y}_1,\overline{x}_2\overline{y}_2) := \varphi(\overline{x}_1,\overline{y}_2) \vee \varphi(\overline{x}_2,\overline{y}_1)$$

codes graphs.

Proof. (1) Fix n and let $G = \{g_i : i < n\} \cup \{h_w : w \subseteq n\}$ be any symmetric graph with $n + 2^n$ vertices that satisfies $R(g_i, h_w)$ holds if and only if $i \in w$. Let $\langle \overline{b}_i : i \in n\rangle$, $\langle \overline{a}_w : w \subseteq n\rangle$ be sequences from some model M_G of T that codes G. Then $M_G \models \psi(\overline{a}_w, \overline{b}_i)$ if and only if $i \in w$.

(2) It suffices to show that every finite graph can be coded, so fix a finite (symmetric) graph (G,R) where $G = \{g_i : i < n\}$. For each $i < n$, let $w_i = \{j < n : R(g_i, g_j)\}$. Choose a model M of T and sequences $\langle \overline{b}_i : i < n\rangle$, $\langle \overline{a}_w : w \subseteq n\rangle$ from M exemplifying the independence

property for $\varphi(\overline{x}, \overline{y})$. Let $\overline{c}_i = \overline{a}_{w_i}\overline{b}_i$ for each $i < n$. It is easily verified that $M \models \psi(\overline{c}_i, \overline{c}_j)$ if and only if $R(g_i, g_j)$ holds. ∎

Although coding graphs is a desirable property in its own right, its utility for constructing models is greatly increased when it is combined with an appropriate notion of indiscernibility. With this objective in mind, we generalize the construction of Ehrenfeucht and Mostowski (see e.g., [3]) to admit skeletons that are indexed by structures that are more complicated than linear orderings. We define an *ordered graph* to be a structure $\mathcal{G} = (G, \leq, R)$, where $\leq$ is interpreted as a linear order on G and R is a symmetric, irreflexive binary relation.

What makes the class of ordered graphs desirable as index structures is the presence of the Nešetřil-Rödl theorem. The version stated below is sufficient for our purposes, but is less general than the statement in either [1] or [5].

Theorem 2.3. *[Nešetřil-Rödl Theorem] For every $e, M \in \omega$ and every finite ordered graph P, there is an ordered graph Q such that for any coloring $F : [Q]^e \to M$ there is an ordered subgraph $Y \subseteq Q$ that is isomorphic to P such that $F(A) = F(B)$ for any $A, B \in [Y]^e$ that are isomorphic as ordered graphs.*

The proof of the theorem below is virtually identical with the proof of the classical Ehrenfeucht-Mostowski theorem, with the Nešetřil-Rödl theorem taking the place of Ramsey's theorem. Recall that a theory T is *Skolemized* if every substructure of every model of T is an elementary substructure.

Theorem 2.4. *Let T be any Skolemized theory with the independence property and suppose that the formula $\varphi(\overline{x}_1, \overline{x}_2)$ codes graphs. For any ordered graph G there is a model M_G of T and $\{\overline{a}_g : g \in G\}$ from M_G such that*

1. *The universe of M_G is the definable closure of $\{\overline{a}_g : g \in G\}$;*
2. *If $f : H_1 \to H_2$ is any ordered graph isomorphism between finite subgraphs of G, then*

$$M_G \models \psi(\overline{a}_g : g \in H_1) \leftrightarrow \psi(\overline{a}_{f(g)} : g \in H_1)$$

   for all formulas ψ; and
3. *For all $g, h \in G$, $M_G \models \varphi(\overline{a}_g, \overline{a}_h)$ if and only if $\mathcal{G} \models R(g, h)$.*

Proof. If we expand the language to $L(G)$ by adding a sequence of new constant symbols $\overline{c}_g$ for every $g \in G$, then Conditions (2) and (3) can be expressed by sets of $L(G)$-sentences. The consistency of these sentences follows immediately from Lemma 2.2, Theorem 2.3 and compactness. ∎

As notation, we call $\{\overline{a}_g : g \in G\}$ the *skeleton* of M_G. Next we extend the notion of independence to pseudo-elementary classes.

Definition 2.5. Fix a language L. A class $\mathbf{K}$ of L-structures is a *pseudo-elementary class* if there is a language $L' \supseteq L$ and an L'-theory T' such that $\mathbf{K}$ is the class of L-reducts of models of T'. Such a class has the *independence property* if some L-formula $\varphi(\overline{x}, \overline{y})$ has the independence property with respect to T'.

Note that as we can always assume that T' is Skolemized, the conclusions of Theorem 2.4 apply to any pseudo-elementary class. The caveat is that in Clause (1), every element of M_G will be in the L'-definable closure of the skeleton, where L' is the language of the Skolemized theory.

Our method of proving Theorem 1.2 will be to use the theorem above to produce a family of elements of $\mathbf{K}$ that code some very complicated ordered graphs. To make this complexity explicit, we discuss some properties of colorings that were developed by the second author. See [9] for a more complete account of these notions. As notation, for x a finite subset of a cardinal μ, let x^m denote the m^{th} element of x in increasing order.

Theorem 2.6. *Suppose that $\mu = \kappa^{++}$ for any infinite cardinal κ. There is a symmetric two-place function $c : \mu \times \mu \to \mu$ such that for every $n \in \omega$, every collection of μ disjoint, n-element subsets $\{x_\alpha : \alpha \in \mu\}$ of μ, and every function $f : n \times n \to \mu$, there are $\alpha < \beta < \mu$ such that*

$$c(x_\alpha^m, x_\beta^{m'}) = f(m, m')$$

for all $m, m' < n$.

The existence of such a coloring c is called $Pr_0(\mu, \mu, \mu, \aleph_0)$ in both [9] and [10]. The same notion is called $Pr^+(\mu)$ in [8]. Theorem 2.6 follows immediately from the results in [8] for all uncountable κ (since the set $S_{\kappa^+} = \{\alpha \in \kappa^{++} : \text{cf}(\alpha) = \kappa^+\}$ is nonreflecting and stationary). The case of $\kappa = \aleph_0$ is somewhat special and is proved in [10] by a separate argument.

We close this section by recalling the definition of a well-founded tree and proving an easy coloring lemma.

Definition 2.7. An *ω-tree* $\mathcal{T}$ is a downward closed subset of ${}^{<\omega}\lambda$ for some ordinal λ. We call $\mathcal{T}$ *well-founded* if it does not have an infinite branch. For a well-founded tree $\mathcal{T}$ and $\eta \in \mathcal{T}$, the *depth of $\mathcal{T}$ above η*, $\mathrm{dp}_{\mathcal{T}}(\eta)$ is defined inductively by

$$\mathrm{dp}_{\mathcal{T}}(\eta) = \begin{cases} \sup\{\mathrm{dp}_{\mathcal{T}}(\nu) + 1 : \eta \lessdot \nu\} & \text{if } \eta \text{ has a successor} \\ 0 & \text{otherwise.} \end{cases}$$

and the depth of $\mathcal{T}$, $\mathrm{dp}(\mathcal{T}) = \mathrm{dp}_{\mathcal{T}}(\langle\rangle)$.

The most insightful example is that for any ordinal δ, the tree $(\mathrm{des}(\delta), \lessdot)$ consisting of all descending sequences of ordinals less than δ ordered by initial segment has depth δ.

Lemma 2.8. *If $\mathcal{T} \subseteq {}^{<\omega}\lambda$ is well-founded and has depth κ^+, then any coloring $f : \mathcal{T} \to \kappa$, there is a sequence $\langle a_n : n \in \omega\rangle$ of elements from $\mathcal{T}$ such that $lg(a_n) = n$ and $f(a_m|_n) = f(a_n)$ for all $n \le m < \omega$.*

Proof. For each $n \in \omega$ we will find a subset $X_n \subseteq \kappa^+$ of size κ^+ and a function $g_n : X_n \to \mathcal{T} \cap {}^n\lambda$ such that $X_{n+1} \subseteq X_n$, every element of $g_{n+1}(X_{n+1})$ is a successor of an element of $g_n(X_n)$, $\mathrm{dp}_{\mathcal{T}}(g_n(\alpha)) \ge \alpha$, and $f|_{g_n(X_n)}$ is constant.

To begin, let $X_0 = \kappa^+$ and let $g_0 : X_0 \to \{\langle\rangle\}$. Given X_n and g_n satisfying our demands, we define X_{n+1} and $g_{n+1} : X_{n+1} \to \mathcal{T} \cap {}^{n+1}\lambda$ as follows: For $\alpha \in X_n$, let β be the least element of X_n greater than α. As $\mathrm{dp}_{\mathcal{T}}(g_n(\beta)) \ge \beta$, we can define $g_{n+1}(\alpha)$ to be a successor of $g_n(\beta)$ of depth at least α. Since X_n has size κ^+, let X_{n+1} be a subset of X_n of size κ^+ such that $f|_{g_{n+1}(X_{n+1})}$ is monochromatic.

Now for each $n \in \omega$, simply take $a_n = g_n(\beta_n)$, where β_n is the least element of X_n. ∎

3. Proof of Theorem 1.2

Fix any pseudo-elementary class $\mathbf{K}$ with the independence property. For definiteness, suppose that $L \subseteq L'$ are languages and T' is an L'-theory such that $\mathbf{K}$ is the class of L-reducts of models of T'. Without loss, we may assume that T' is Skolemized. Let $\mu = |T'|^{++}$. Fix an L-formula $\varphi(x_1, x_2)$ that codes graphs (see Lemma 2.2). For notational simplicity we assume that $lg(x_1) = lg(x_2) = 1$.

Now assume by way of contradiction that $\mathbf{K}$ is controlled. It follows from Proposition 1.1(4) that there is a cardinal κ such that for any $M \in \mathbf{K}$ there are fewer than κ distinct L_{∞,μ^+}-types of subsets of size at most μ in M. **Fix, for the whole of the paper, such a κ and put $\delta := \kappa^+$.**

Our strategy for proving Theorem 1.2 is to define one specific structure $M_\delta \in \mathbf{K}$. This M_δ is constructed using Theorem 2.4 and is the reduct the Skolem Hull of an ordered graph I_δ. The ordering on I_δ

is a well-order, but the edge relation on I_δ is extremely complicated as it codes a coloring c given by Theorem 2.6. The definition of I_δ and the construction of M_δ are completed in the paragraph following Definition 3.6. Following our construction of M_δ we use the bound on the number of L_{∞,μ^+}-types to form an ω-sequence $\langle \mathbf{B}_n : n \in \omega \rangle$ of μ-sequences of pairs of elements from M_δ that are reasonably coherent. Then, by combining several of the results from the Appendix with properties of the coloring c we establish three claims whose statements follow Definition 3.7. These claims collectively imply the existence of an infinite, descending sequence of ordinals below δ. This contradiction demonstrates that the class $\mathbf{K}$ is not controlled.

Definition 3.1. The expression $\mathrm{des}(\delta)$ denotes the set of all strictly decreasing sequences of elements from δ. The set of all finite sequences from $\mathrm{des}(\delta)$ is denoted by $\mathrm{des}^{<\omega}(\delta)$.

Every element of $\mathrm{des}(\delta)$ is clearly a finite sequence. It is an easy exercise to show that $\mathrm{des}(\delta)$ is a well-ordering with respect to the lexicographic order $<_{lex}$. As noted in the remarks following Definition 2.7, the ω-tree $(\mathrm{des}(\delta), \lessdot)$ has depth $\delta = \kappa^+$.

Definition 3.2. A function $g : \zeta \to \mathrm{des}^{<\omega}(\delta)$ is *uniform* if $\omega \leq \zeta \leq \mu$; $lg(g(\alpha)) = lg(g(\beta))$ for all $\alpha, \beta \in \zeta$; and (letting $lg(g)$ denote this common length) for all $i < lg(g)$, the sequences $\langle g(\beta)(i) : \beta \in \zeta \rangle$ have constant length and are either constant or $<_{lex}$-strictly increasing. Let $lg(g(-)(i))$ denote the length of $g(\beta)(i)$ for some (every) $\beta \in \zeta$. If the sequence $\langle g(\beta)(i) : \beta \in \zeta \rangle$ is constant we let g_i denote its common value. Let $\mathcal{U}$ denote the set of all uniform functions.

Definition 3.3. The universe of I_δ is the set of all $t = \langle \zeta^t, \eta^t, g^t, p^t \rangle$, where

(1) $\zeta^t \in \mu$;
(2) $\eta^t \in \mathrm{des}(\delta)$;
(3) $g^t : \zeta^t \to \mathrm{des}^{<\omega}(\delta)$ is a uniform function; and
(4) $p^t \in \{0, 1\}$.

We well-order I_δ as follows. First, choose any well-ordering $<_{\mathcal{U}}$ on the set $\mathcal{U}$ of uniform functions. Then, define the ordering on I_δ to be lexicographic i.e., $s <_{I_\delta} t$ if and only if either $\zeta^s < \zeta^t$; or $\zeta^s = \zeta^t$ and $\eta^s <_{\mathrm{lex}} \eta^t$; or $\zeta^s = \zeta^t$ and $\eta^s = \eta^t$ and $g^s <_{\mathcal{U}} g^t$; or $\zeta^s = \zeta^t$ and $\eta^s = \eta^t$ and $g^s = g^t$ and $p^s < p^t$.

In order to define the edge relation on I_δ we require some preparatory definitions.

Definition 3.4. Two uniform functions g and h (possibly with different domains) have the *same shape* if the following four conditions hold:

(1) $lg(g) = lg(h)$;
(2) For each $i < lg(g)$, $lg(g(-)(i)) = lg(h(-)(i))$;
(3) For each $i < lg(g)$, the sequence $\langle g(\beta)(i) : \beta \in \mathrm{dom}(g)\rangle$ is constant if and only if $\langle h(\beta)(i) : \beta \in \mathrm{dom}(h)\rangle)$ is constant;
(4) For all $i, j < lg(g)$ such that $\langle g(\beta)(i) : \beta \in \mathrm{dom}(g)\rangle$ and $\langle g(\beta)(j) : \beta \in \mathrm{dom}(g)\rangle$ are both constant, $g_i = g_j \Leftrightarrow h_i = h_j$ and $g_i \lessdot g_j \Leftrightarrow h_i \lessdot h_j$.

Definition 3.5. Two pairs $(s,t), (s',t') \in (I_\delta)^2$ have the *same type* if the following conditions hold:

(1) $\zeta^s < \zeta^t \Leftrightarrow \zeta^{s'} < \zeta^{t'}$ and $\zeta^s > \zeta^t \Leftrightarrow \zeta^{s'} > \zeta^{t'}$;
(2) $lg(\eta^t) = lg(\eta^{t'})$;
(3) $p^s = p^{s'}$ and $p^t = p^{t'}$;
(4) The uniform functions g^s and $g^{s'}$ have the same shape;
(5) For all $i < lg(g^s)$, $g^s(\zeta^t)(i) = \eta^t \Leftrightarrow g^{s'}(\zeta^{t'})(i) = \eta^{t'}$ and $g^s(\zeta^t)(i) \lessdot \eta^t \Leftrightarrow g^{s'}(\zeta^{t'})(i) \lessdot \eta^{t'}$.

Evidently, having the same type induces an equivalence relation on pairs from I_δ with countably many classes. We let $\mathrm{tp}(s,t)$ denote the class of pairs that have the same type as (s,t) and let $\mathcal{E}$ denote the set of equivalence classes. Let $\mathcal{H}$ denote any countable collection of (total) functions from $\mathcal{E}$ to $\{0,1\}$ such that for any partial function $f : \mathcal{E} \to \{0,1\}$ whose domain is finite there is an $h \in \mathcal{H}$ extending f.

Using Theorem 2.6 choose a symmetric, binary function

$$c : \mu \times \mu \to \mathcal{H}$$

such that for every $k \in \omega$, for every collection of μ disjoint, k-element subsets $\{x_\alpha : \alpha \in \mu\}$ of μ, and for every function $f : k \times k \to \mathcal{H}$, there are $\alpha < \beta < \mu$ such that $c(x_\alpha^m, x_\beta^{m'}) = f(m,m')$ for all $m, m' < k$. (Here, x_α^m denotes the m^{th} element of x_α.)

We are now able to complete our description of the ordered graph I_δ by defining the edge relation $R(x,y)$ on I_δ.

Definition 3.6. For $s,t \in I_\delta$, $R_0(s,t)$ holds if and only if the following conditions are satisfied:

(1) $\zeta^s > \zeta^t$;
(2) $lg(\eta^s) < lg(\eta^t)$;
(3) $p^s = 1$; $p^t = 0$; and
(4) $c(\zeta^s, \zeta^t)[\mathrm{tp}(s,t)] = 1$.

We say $R(s,t)$ holds if and only if $R_0(s,t)$ or $R_0(t,s)$ holds.

Let $M'_\delta \models T'$ be a model satisfying the conclusions of Theorem 2.4 with respect to the ordered graph $(I_\delta, \leq_{I_\delta}, R)$ defined above. To ease notation, we identify the I_δ with the skeleton $\{a_g : g \in I_\delta\}$ of M'_δ. In particular, every element of M'_δ is an L'-term applied to a finite sequence from I_δ. Let $M_\delta \in \mathbf{K}$ be the L-reduct of M'_δ.

As notation, let $g^\alpha_{\langle\rangle}$ denote the function whose domain is α and $g(\beta) = \langle\rangle$ for all $\beta \in \alpha$. For $\nu \in \text{des}(\delta)$, let $A_{\nu,\alpha} \in M^2_\delta$ denote the pair of elements $\langle(\alpha, \nu, g^\alpha_{\langle\rangle}, 0), (\alpha, \nu, g^\alpha_{\langle\rangle}, 1)\rangle$ from I_δ (recall that we are identifying I_δ with the skeleton) and let $\mathbf{A}_\nu$ denote the sequence $\langle A_{\nu,\alpha} : \alpha \in \mu\rangle$.

As the number of L_{∞,μ^+}-types of subsets of M_δ of size at most μ is bounded by κ and $lg(\mathbf{A}_\nu) \leq \mu$ for all $\nu \in \text{des}(\delta)$, there is a function $f : \text{des}(\delta) \to \kappa$ such that $f(\nu) = f(\nu')$ if and only if $lg(\nu) = lg(\nu')$ and $\text{tp}_{\infty,\mu^+}(\langle \mathbf{A}_{\nu|l} : l \leq lg(\nu)\rangle) = \text{tp}_{\infty,\mu^+}(\langle \mathbf{A}_{\nu'|l} : l \leq lg(\nu)\rangle)$. Since the depth of the ω-tree $(\text{des}(\delta), \lessdot)$ is $\delta = \kappa^+$, it follows from Lemma 2.8 applied to this function f that there is a sequence $\langle \nu^*_n : n \in \omega\rangle$ of elements from $\text{des}(\delta)$ such that for all $n \in \omega$, $lg(\nu^*_n) = n$ and the sequences $\langle \mathbf{A}_{\nu^*_n|l} : l \leq n\rangle$ and $\langle \mathbf{A}_{\nu^*_m|l} : l \leq n\rangle$ have the same L_{∞,μ^+}-type in M_δ for all $m \geq n$.

Thus, one can construct by induction on n an ω-sequence $\langle \mathbf{B}_n : n \in \omega\rangle$ in M_δ such that:

- Each $\mathbf{B}_n$ is a sequence $\langle B_{n,\alpha} : \alpha \in \mu\rangle$, where each $B_{n,\alpha}$ is a pair of elements from M_δ;
- $\mathbf{B}_0 = \mathbf{A}_{\langle\rangle}$; and
- The sequences $\langle \mathbf{B}_l : l \leq n\rangle$ and $\langle \mathbf{A}_{\nu^*_n|l} : l \leq n\rangle$ have the same L_{∞,μ^+}-type for every $n \in \omega$.

Fix sequences $\langle \nu^*_n : n \in \omega\rangle$ and $\langle \mathbf{B}_n : n \in \omega\rangle$ satisfying the properties described above. As notation, we write a_α for the element $(\alpha, \langle\rangle, g^\alpha_{\langle\rangle}, 1) \in I_\delta$ (i.e., the second coordinate of $A_{\langle\rangle,\alpha}$). For $n > 0$ we write $a_{n,\alpha}$ for $(\alpha, \nu^*_n, g^\alpha_{\langle\rangle}, 0)$ (the first coordinate of $A_{\nu^*_n,\alpha}$) and write $b_{n,\alpha}$ for the first coordinate of $B_{n,\alpha}$. We let $\Gamma_n = \text{tp}(a_\alpha, a_{n,\beta})$ for all $\alpha > \beta$ from μ. So, for example, when $\alpha > \beta$ then

$$M_\delta \models \varphi(a_\alpha, b_{n,\beta}) \Leftrightarrow M_\delta \models \varphi(a_\alpha, a_{n,\beta}) \Leftrightarrow c(\alpha, \beta)[\Gamma_n] = 1.$$

Next we use results from the Appendix to obtain subsequences of the sequences $\langle a_\alpha : \alpha \in \mu\rangle$ and $\langle b_{n,\alpha} : \alpha \in \mu\rangle$ with desirable regularity properties. It may be helpful to the reader to skip ahead to the Appendix at this point in order to become familiar with the definitions therein. Specifically, iterating Lemma A.9 yields a descending sequence $Y_1 \supseteq Y_2 \supseteq \dots$ of stationary subsets of μ such that for all $n > 0$ the sequences $\langle a_\alpha : \alpha \in Y_n\rangle$ and $\langle b_{n,\alpha} : \alpha \in Y_n\rangle$ form a clean pair (see Definition A.8). As notation, for each $n > 0$ fix a number $m(n)$, an L'-term τ_n, and for each $l < m(n)$, tidy sequences $\langle b^l_{n,\alpha} : \alpha \in Y_n\rangle$ of elements from the skeleton I_δ such that for all $\alpha \in Y_n$,

$$b_{n,\alpha} = \tau_n(b^l_{n,\alpha} : l < m(n)).$$

Let $\zeta^l_{n,\alpha}$, $\eta^l_{n,\alpha}$, $g^l_{n,\alpha}$, and $p^l_{n,\alpha}$ denote the four components of $b^l_{n,\alpha}$.

Fix an $n > 0$. Let $Y^*_n = \{(\alpha, \beta) \in (Y_n)^2 : \alpha > \beta\}$. It follows from the remarks following Definition A.6 that

$$b^0_{n,\beta} <_{I_\delta} b^1_{n,\beta} <_{I_\delta} \cdots <_{I_\delta} b^{m(n)-1}_{n,\beta} <_{I_\delta} a_\alpha$$

for all $(\alpha, \beta) \in Y^*_n$ and that $R(b^l_{n,\beta}, b^{l'}_{n,\beta}) \leftrightarrow R(b^l_{n,\beta'}, b^{l'}_{n,\beta'})$ for all $\beta, \beta' \in Y_n$ and for all $l, l' < m(n)$. Thus, the only freedom we have in determining whether $\varphi(a_\alpha, b_{n,\beta})$ holds or fails for various $(\alpha, \beta) \in Y^*_n$ is whether or not $R(a_\alpha, b^l_{n,\beta})$ holds or fails for various $l < m(n)$. Accordingly, we call a subset $Z \subseteq m(n)$ *true for* n if

$$M_\delta \models \varphi(y, \tau_n(x^l : l < m(n)))$$

for all $<_{I_\delta}$-increasing sequences $x^0, \dots, x^{m(n)-1}, y$ from I_δ such that $R(x^l, x^{l'})$ holds if and only if $R(b^l_{n,\beta}, b^{l'}_{n,\beta})$ holds for $\beta \in Y_n$ and $R(y, x^l)$ holds if and only if $l \in Z$. A subset $Z \subseteq m(n)$ is *false for* n if it is not true for n.

We call an index $l \in m(n)$ n-*constant* if $\zeta^l_\beta = \zeta^l_{\beta'}$ for all $\beta, \beta' \in Y_n$. Let β^*_l denote this common value. As $\langle a_\alpha : \alpha \in Y_n\rangle$ and $\langle b_{n,\alpha} : \alpha \in Y_n\rangle$ form a clean pair, it follows that for every n-constant l, the values of both $c(\alpha, \beta^*_l)$ and $\mathrm{tp}(a_\alpha, b^l_{n,\beta})$ are constant for all $(\alpha, \beta) \in Y^*_n$. Thus, for all n-constant l,

$$R(a_\alpha, b^l_{n,\beta}) \leftrightarrow R(a_{\alpha'}, b^l_{n,\beta})$$

for all $(\alpha, \beta), (\alpha', \beta') \in Y^*_n$. Let P_n denote the set of all n-constant l's such that $R(a_\alpha, b^l_{n,\beta})$ holds for all $(\alpha, \beta) \in Y^*_n$.

Switching our attention to the non-constants, let J_n denote the set of non-constant $l \in m(n)$. Let

$$V_n = \{l \in J_n : \zeta^l_\beta = \beta \text{ and } \mathrm{tp}(a_\alpha, b^l_{n,\beta}) = \Gamma_n \text{ for all (some) } (\alpha, \beta) \in Y^*_n\}.$$

There is a natural equivalence relation E_n on V_n defined by $E_n(l, l')$ if and only if $\eta^l_\beta = \eta^{l'}_\beta$ for all $\beta \in Y_n$. (It follows from Condition 4 of Definition A.6 that whether or not $\eta^l_\beta = \eta^{l'}_\beta$ is independent of β.) We are now able to state the crucial definition for the argument that follows.

Definition 3.7. An E_n-class C is n-*interesting* if there is a union of E_n-classes $X \subseteq V_n$ such that $P_n \cup X$ is false for n, while $P_n \cup X \cup C$ is true for n.

In what follows, we will prove the following three claims.

Claim 1. For every $n > 0$ there is an n-interesting E_n-class C.

Claim 2. For every $n > 0$ and for every n-interesting E_n-class C there is an $\eta^C \in \mathrm{des}(\delta)$ of length n such that $\eta^l_\beta = \eta^C$ for all $l \in C$ and all $\beta \in Y_n$.

Claim 3. For every $n' > n > 0$ and for every n-interesting E_n-class C there is an n'-interesting $E_{n'}$-class C' such that $\eta^C \lessdot \eta^{C'}$.

Clearly, one can deduce a contradiction from the three claims by building an infinite, descending sequence of ordinals. Thus, to complete the proof of Theorem 1.2 it suffices to prove the claims. The proofs of all three appeal to the complexity of the coloring c. The first application is direct, but the other two involve constructing appropriate surrogates to the a_α's before invoking the properties of the coloring.

Proof of Claim 1. Fix $n > 0$. Let $\bar{\alpha} = \{\alpha\}$, let $\bar{\beta} = \{\beta\} \cup \{\zeta^l_\beta : l \in J_n\}$, and choose $k \geq |\bar{\beta}|$. Note that by Condition 5 of Definition A.6, $\alpha > \zeta^l_\beta$ for all $(\alpha, \beta) \in Y^*_n$. Let $\pi_1 : J_n \to k$ be the function defined by $\pi_1(l) = t$ if and only if ζ^l_β is the t^{th} element of $\bar{\beta}$. For each $l \in J_n$ let $\Gamma_l = \mathrm{tp}(a_\alpha, b^l_{n,\beta})$ for all $(\alpha, \beta) \in Y^*_n$. (As g^{a_α} is the trivial function and as $\langle b_{n,\alpha} : \alpha \in Y_n\rangle$ is clean, it is easily verified that there is only one such type for each $l \in J_n$.)

Let $h, h' : k \times k \to \mathcal{H}$ be any functions that satisfy:

(1) $h(0, \pi_1(l))[\Gamma_l] = 0$ for all $l \in J_n$;
(2) $h(0, 0)[\Gamma_n] = 0$; and
(3) $h' = h$ EXCEPT that $h'(0, 0)[\Gamma_n] = 1$.

It follows easily from the properties of the coloring c that there is $(\alpha, \beta) \in Y^*_n$ such that $c(\alpha, \zeta^{\pi_1(l)}_\beta) = h(0, \pi_1(l))$ for all $l \in J(n)$. Fix such a pair (α, β) and choose $(\alpha', \beta') \in Y^*_n$ such that $c(\alpha', \zeta^{\pi(l)}_{\beta'}) = h'(0, \pi_1(l))$ for all $l \in J(n)$. It is readily verified that

$$\{l \in m(n) : R(a_\alpha, b^l_{n,\beta}) \text{ holds}\} = P_n,$$

while

$$\{l \in m(n) : R(a_{\alpha'}, b^l_{n,\beta'}) \text{ holds}\} = P_n \cup V_n.$$

But, as $c(\alpha, \beta)[\Gamma_n] = 0$ and $c(\alpha', \beta')[\Gamma_n] = 1$,

$$M_\delta \models \neg\varphi(a_\alpha, b_{n,\beta}) \wedge \varphi(a_{\alpha'}, b_{n,\beta'}),$$

so P_n is false for n, while $P_n \cup V_n$ is true for n.

Let $\langle C_j : j < s\rangle$ be an enumeration of the E_n-classes of V_n. Choose $j < s$ such that $P_n \cup \bigcup_{i<j} C_i$ is false for n, while $P_n \cup \bigcup_{i\leq j} C_i$ is true for n. Then C_j is n-interesting. ∎

Proof of Claim 2. Fix $n > 0$ and an n-interesting E_n-class C. Choose $X \subseteq V_n$, X a union of E_n-classes, such that $P_n \cup X$ is false for n, while $P_n \cup X \cup C$ is true for n.

Using Lemma A.13, choose a stationary subset $W \subseteq Y_n$ and a uniform function $g : \mu \to \mathrm{des}^{<\omega}(\delta)$ that satisfies

$$g(\beta) = \langle \eta^l_{n,\beta} : l < m(n) \rangle \quad \text{for all } \beta \in W.$$

For all $\alpha \in W$, let e_α denote the element $(\alpha, \langle\rangle, g|_\alpha, 1)$ from the skeleton I_δ of M_δ. By applying Lemma A.9 and possibly shrinking W, we may additionally assume that the sequences $\langle e_\alpha : \alpha \in W \rangle$ and $\langle b_{n,\alpha} : \alpha \in W \rangle$ form a clean pair. The e_α's should be thought of as being a surrogate for the a_α's that carry just enough data from the $b_{n,\beta}$'s.

Let $\Gamma^*_l = \mathrm{tp}(e_\alpha, b^l_{n,\beta})$ for all $\alpha > \beta$ from W. The fact that the values of these types does not depend on our choice of (α, β) follows from our choice of the functions g^{e_α} and Condition (4) of Definition A.6 applied to $\langle b_{n,\alpha} : \alpha \in W \rangle$. To elaborate, the crucial point is that from our definition of $g^{e_\alpha}|_W$, relations such as '$g^{e_\alpha}(\zeta^l_\beta)(l') = \eta^l_\beta$' are essentially unary (depending only on β) when restricted to pairs $\alpha > \beta$ from W. Note that for each $l < m(n)$ the type $\Gamma^*_l(x, y)$ contains the relation

$$\eta^y = g^x(\zeta^y)(l). \tag{1}$$

Let $\Gamma^*_C = \Gamma^*_l$ for any $l \in C$.

As $\langle a_{n,\alpha} : \alpha \in W \rangle$ realizes the same L_{∞,μ^+}-type as $\langle b_{n,\alpha} : \alpha \in W \rangle$, we can choose $\langle d_\alpha : \alpha \in W \rangle$ such that the sequences

$$\langle d_\alpha : \alpha \in W \rangle^\frown \langle b_{n,\alpha} : \alpha \in W \rangle \text{ and } \langle e_\alpha : \alpha \in W \rangle^\frown \langle a_{n,\alpha} : \alpha \in W \rangle. \tag{2}$$

have the same L_{∞,μ^+}-type.

By applying Lemma A.9 we can find a stationary subset $Z \subseteq W$ such that the sequences $\langle d_\alpha : \alpha \in Z \rangle$ and $\langle a_{n,\alpha} : \alpha \in Z \rangle$ form a clean pair. Let $Z^* = \{(\alpha, \beta) \in Z^2 : \alpha > \beta\}$. For each $\alpha \in Z$ say

$$d_\alpha = \theta(d^r_\alpha : r < r(d)),$$

where θ is an L'-term and $\langle d^r_\alpha : r < r(d) \rangle$ is a strictly $<_{I_\delta}$-increasing sequence from I_δ. As notation, let $\hat{\zeta}^r_\alpha$ denote the ζ-component of d^r_α. Let $J_d = \{r \in r(d) : \hat{\zeta}^r_\alpha \text{ is not constant}\}$. For each $r < r(d)$, let $\Phi_r = \mathrm{tp}(d^r_\alpha, a_{n,\beta})$ for any $(\alpha, \beta) \in Z^*$. Note that since $g^{d_\alpha}(-)(r)$ is strictly increasing or constant for each $\alpha \in Z$ and $\eta^{a_{n,\beta}} = \nu^*_n$ for all $\beta \in Z$, both of the relations

$$g^{d_\alpha}(\beta) = \eta^{a_{n,\beta}} \quad \text{and} \quad g^{d_\alpha}(\beta) \lessdot \eta^{a_{n,\beta}}$$

concentrate on tails for all $r < r(d)$ (see Definition A.10). Thus, it follows from Lemma A.11 that by possibly trimming Z further, we may assume that for each $r < r(d)$, the value of Φ_r is independent of our choice of $(\alpha, \beta) \in Z^*$.

Subclaim. There is an $r \in J_d$ such that $\hat{\zeta}^r_\alpha = \alpha$ for $\alpha \in Z$ and $\Phi_r = \Gamma^*_C$

Proof. Let $\bar{\alpha} = \{\alpha\} \cup \{\hat{\zeta}^r_\alpha : r \in J_d\}$, let $\bar{\beta} = \{\beta\} \cup \{\zeta^l_{n,\alpha} : l \in J_n\}$ and choose $k \geq |\bar{\alpha}|, |\bar{\beta}|$. Let $\pi_0 : J_d \to k$ be the function that satisfies $\pi_0(r) = s$ if and only if $\hat{\zeta}^r_\alpha$ is the s^{th} element of $\bar{\alpha}$ and let $\pi_1 : J_n \to k$ be the function that satisfies $\pi_1(l) = t$ if and only if $\zeta^l_{n,\alpha}$ is the t^{th} element of $\bar{\beta}$. Since the sequences $\langle d_\alpha : \alpha \in Z\rangle$ and $\langle b_{n,\alpha} : \alpha \in Z\rangle$ are clean, the lengths of $\bar{\alpha}$, $\bar{\beta}$ and the values of π_0 and π_1 do not depend on our choice of $\alpha \in Z$.

Now, if the subclaim were false we could find two functions $h, h' : k \times k \to \mathcal{H}$ that satisfy the following conditions:

(1) $h(\pi_0(r), \pi_1(l))[\Phi_r] = h'(\pi_0(r), \pi_1(l))[\Phi_r]$ for $r \in J_d$ and $l \in J_n$;
(2) $h(\pi_0(r), 0)[\Gamma^*_l] = h'(\pi_0(r), 0)[\Gamma^*_l] = 1$ for $r \in J_d$, $l \in X$;
(3) $h(\pi_0(r), 0)[\Gamma^*_l] = h'(\pi_0(r), 0)[\Gamma^*_l] = 0$ for $r \in J_d$, $l \in V_n \setminus X \setminus C$;
(4) $h(0,0)[\Gamma^*_C] = 0$; $h'(0,0)[\Gamma^*_C] = 1$.

From the properties of the coloring c, choose (α, β) and (α', β') from Z^* such that

$$c(\hat{\zeta}^{\pi_0(r)}_\alpha, \zeta^{\pi_1(l)}_\beta) = h(\pi_0(r), \pi_1(l)) \quad \text{and} \quad c(\hat{\zeta}^{\pi_0(r)}_{\alpha'}, \zeta^{\pi_1(l)}_{\beta'}) = h'(\pi_0(r), \pi_1(l))$$

for all $r \in J_d$ and all $l \in J_n$. Thus,

$$\{l \in m(n) : R(e_\alpha, b^l_{n,\beta}) \text{ holds}\} = P_n \cup X,$$

which is false for n, while

$$\{l \in m(n) : R(e_{\alpha'}, b^l_{n,\beta'}) \text{ holds}\} = P_n \cup X \cup C,$$

which is true for n. Hence

$$M_\delta \models \neg\varphi(e_\alpha, b_{n,\beta}) \wedge \varphi(e_{\alpha'}, b_{n,\beta'}),$$

so it follows from Equation (2) that

$$M_\delta \models \neg\varphi(d_\alpha, a_{n,\beta}) \wedge \varphi(d_{\alpha'}, a_{n,\beta'}). \tag{3}$$

However, as $\langle d_\alpha : \alpha \in Z\rangle$ and $\langle a_{n,\alpha} : \alpha \in Z\rangle$ form a clean pair, the sequences $\langle a_{n,\beta}\rangle^\frown\langle d^r_\alpha : r < r(d)\rangle$ and $\langle a_{n,\beta'}\rangle^\frown\langle d^r_{\alpha'} : r < r(d)\rangle$ are both $<_{I_\delta}$-strictly increasing. As well, $R(d^r_\alpha, d^{r'}_\alpha) \leftrightarrow R(d^r_{\alpha'}, d^{r'}_{\alpha'})$ holds for all $r, r' < r(d)$ by the remark following Definition A.6. Since $c(\hat{\zeta}^r_\alpha, \beta)[\Phi_r] = c(\hat{\zeta}^r_{\alpha'}, \beta')[\Phi_r]$ for all $r \in J_d$, $R(d^r_\alpha, a_{n,\beta}) \leftrightarrow R(d^r_{\alpha'}, a_{n,\beta'})$ holds for all $r <$

$r(d)$ as well. That is, the pairs (α, β) and (α', β') generate isomorphic ordered subgraphs of I_δ. Hence

$$M_\delta \models \varphi(d_\alpha, a_{n,\beta}) \leftrightarrow \varphi(d_{\alpha'}, a_{n,\beta'}),$$

which contradicts Equation (3). ∎

To complete the proof of Claim 2 choose any $r < r(d)$ such that $\Phi_r = \Gamma^*_C$ and $\hat{\zeta}^r_\alpha = \alpha$ for all $\alpha \in Z$. As well, fix $\alpha > \beta > \beta'$ from Z, let $\overline{g}$ denote the g-component from d^r_α, and choose any $l^* \in C$. Since $\mathrm{tp}(d^r_\alpha, a_{n,\beta}) = \mathrm{tp}(d^r_\alpha, a_{n,\beta'}) = \Gamma^*_C$, it follows from Equation (1) that

$$\overline{g}(\beta)(l^*) = \nu^*_n = \overline{g}(\beta')(l^*).$$

Since $\overline{g}$ is uniform, the function $\overline{g}(-)(l^*)$ must be constant. As well, this information is part of the shape of $\overline{g}$. However, since $\mathrm{tp}(e_\alpha, b^{l^*}_{n,\beta}) = \mathrm{tp}(d^r_\alpha, a_{n,\beta})$, the function g^{e_α} has the same shape as $\overline{g}$, so the function $g^{e_\alpha}(-)(l^*)$ must be constant as well. But the l^*-th coordinate of $g^{e_\alpha}(\beta)$ was chosen to be $\eta^{l^*}_{n,\beta}$ for all $\beta \in W$. That is, $\langle \eta^{l^*}_{n\beta} : \beta \in W\rangle$ is constant. But, as the sequence $\langle \eta^{l^*}_{n,\beta} : \beta \in Y_n\rangle$ forms a Δ-system, it too must be constant. Let η^C denote the common value of $\eta^{l^*}_{n,\beta}$. That $\eta^l_{n,\beta} = \eta^C$ for all $l \in C$ and all $\beta \in Y_n$ follows immediately from Condition (4) of Definition A.6 and the definition of E_n.

Finally, since $\mathrm{tp}(a_\alpha, b^l_{n,\beta}) = \Gamma_n$ for all $l \in V_n$ and all $(\alpha, \beta) \in Y_n$, $lg(\eta^{l^*}_{n,\beta}) = n$ as required. ∎

Proof of Claim 3. Fix $n' > n > 0$ and an n-interesting E_n-class C. By reindexing, we may assume that the index sets J_n and $J_{n'}$ are disjoint. Choose $X \subseteq V_n$, X a union of E_n-classes, such that $P_n \cup X$ is false for n, while $P_n \cup X \cup C$ is true for n.

As we are choosing between finitely many possibilities, by shrinking $Y_{n'}$ further, we may assume that for all $l, l' \in J_n \cup J_{n'}$ the truth values of the relations

$$\text{`}\eta^l_\alpha = \eta^{l'}_{\zeta^l_\alpha}\text{,'} \quad \text{`}\eta^l_\alpha \lessdot \eta^{l'}_{\zeta^l_\alpha}\text{,'} \quad \text{and} \quad \text{`}\eta^{l'}_{\zeta^l_\alpha} \lessdot \eta^l_\alpha\text{,'}$$

are invariant among all $\alpha \in Y_{n'}$. By analogy with the argument in Claim 2, use Lemma A.13 to find a stationary subset $W \subseteq Y_{n'}$ and a uniform function $g : \mu \to \mathrm{des}^{<\omega}(\delta)$ that satisfies

$$g(\beta) = \langle \eta^l_{n,\beta} : l < m(n)\rangle^\frown\langle \eta^{l'}_{n',\beta} : l' < m(n')\rangle \quad \text{for all } \beta \in W.$$

For all $\alpha \in W$, let e_α denote the element $(\alpha, \langle\rangle, g|_\alpha, 1)$ from the skeleton of M_δ. (These e_α's are not the same as in the proof of Claim 2 as the function g is different.)

Let $\Gamma^*_{n,l} = \mathrm{tp}(e_\alpha, b^l_{n,\beta})$ and let $\Gamma^*_{n',l'} = \mathrm{tp}(e_\alpha, b^{l'}_{n',\beta})$ for all $\alpha > \beta$ from W. As was the case in the proof of Claim 2, the values of $\Gamma^*_{n,l}$ and $\Gamma_{n',l'}$ do not depend on our choice of (α, β). The verification of this depends on Condition (4) of Definition A.6 and the further reduction performed above. Note that for each $l < m(n)$ the type $\Gamma^*_{n,l}(x, y)$ contains the relation '$\eta^y = g^x(\zeta^y)(l)$,' while the type $\Gamma^*_{n',l'}(x, y)$ contains the relation '$\eta^y = g^x(\zeta^y)(m(n) + l')$,' for all $l' < m(n')$. As well, note that if $E_n(l_1, l_2)$, then $\Gamma^*_{n,l_1} = \Gamma^*_{n,l_2}$. Let $\Gamma^*_C = \Gamma^*_l$ for any $l \in C$.

As $\langle a_{n,\alpha} : \alpha \in W\rangle^\frown\langle a_{n',\alpha} : \alpha \in W\rangle$ realizes the same L_{∞,μ^+}-type as $\langle b_{n,\alpha} : \alpha \in W\rangle^\frown\langle b_{n',\alpha} : \alpha \in W\rangle$, we can choose $\langle d_\alpha : \alpha \in W\rangle$ from M_δ such that

$$\langle d_\alpha : \alpha \in W\rangle^\frown\langle b_{n,\alpha} : \alpha \in W\rangle^\frown\langle b_{n',\alpha} : \alpha \in W\rangle$$

has the same L_{∞,μ^+}-type as

$$\langle e_\alpha : \alpha \in W\rangle^\frown\langle a_{n,\alpha} : \alpha \in W\rangle^\frown\langle a_{n',\alpha} : \alpha \in W\rangle \tag{4}$$

Using Lemma A.9, choose a stationary subset $Z \subseteq W$ such that both pairs of sequences $\{d_\alpha : \alpha \in Z\rangle$, $\{a_{n,\alpha} : \alpha \in Z\rangle$ and $\{d_\alpha : \alpha \in Z\rangle$, $\{a_{n',\alpha} : \alpha \in Z\rangle$ are clean pairs. Let $Z^* = \{(\alpha, \beta) \in Z^2 : \alpha > \beta\}$. For each $\alpha \in Z$ say

$$d_\alpha = \theta(d^r_\alpha : r < r(d)),$$

where θ is an L'-term and $\langle d^r_\alpha : r < r(d)\rangle$ is a strictly $<_{I_\delta}$-increasing sequence from I_δ. As notation, let $\hat\zeta^r_\alpha$ denote the ζ-component of d^r_α. Let $J_d = \{r \in r(d) : \hat\zeta^r_\alpha \text{ is not constant}\}$. As in the proof of Claim 2, we can use Lemma A.11 to shrink Z so that the values of $\mathrm{tp}(d^r_\alpha, a_{k,\beta})$ is independent of the choice of $(\alpha, \beta) \in Z^*$ for all $r < r(d)$ and all $k \in \{n, n'\}$. Let $\Phi_r = \mathrm{tp}(d^r_\alpha, a_{n,\beta})$ for all $(\alpha, \beta) \in Z^*$.

Let

$$\bar\alpha = \{\alpha\} \cup \{\hat\zeta^r_\alpha : r \in J_d\}, \bar\beta = \{\beta\} \cup \{\zeta^l_{n,\alpha} : l \in J_n\} \cup \{\zeta^{l'}_{n',\alpha} : l' \in J_{n'}\}$$

and choose $k \geq |\bar\alpha|, |\bar\beta|$. (Recall that we chose the index sets J_n and $J_{n'}$ to be disjoint.) Let $\pi_0 : J_d \to k$ be the function that satisfies $\pi_0(r) = s$ if and only if $\hat\zeta^r_\alpha$ is the s^{th} element of $\bar\alpha$ and let $\pi_1 : J_n \cup J_{n'} \to k$ be the function that satisfies $\pi_1(l) = t$ if and only if $l \in J_n$ and $\zeta^l_{n,\alpha}$ is the t^{th} element of $\bar\beta$ **OR** $l \in J_{n'}$ and $\zeta^l_{n',\alpha}$ is the t^{th} element of $\bar\beta$. As was the case in the proof of Claim 2, the lengths of $\bar\alpha$ and $\bar\beta$ and the functions π_0 and π_1 do not depend on $\alpha \in Z$.

Suppose that $\Phi(x, y)$ is any type that satisfies $lg(\eta^y) = n$. We call a type Ψ an *extension of* Φ if there are s, t, t' from I_δ such that $lg(\eta^t) = n$, $lg(\eta^{t'}) = n'$, $\mathrm{tp}(s, t) = \Phi$, $\mathrm{tp}(s, t') = \Psi$, and $\eta^t \lessdot \eta^{t'}$. Note that any type Φ has only finitely many extensions. As well, note that one of the types $\Gamma^*_{n',l'}$ is an extension of Γ^*_C, then necessarily $l' \in V_{n'}$.

We call a function $h : k \times k \to \mathcal{H}$ *closed under r-extensions* if

$$h(\pi_0(r), 0)[\Phi_r] = h(\pi_0(r), 0)[\Psi]$$

for all $r \in J_d$ and all of the (finitely many) types Ψ extending Φ_r.

Since $\mathrm{tp}(x, a_{n',\beta})$ is an extension of $\mathrm{tp}(x, a_{n,\beta})$ for any β and any x from I_δ, it follows easily that if h is closed under r-extensions and some $(\alpha, \beta) \in Z^*$ satisfies

$$c(\hat{\zeta}^{\pi_0(r)}_\alpha, \zeta^{\pi_1(l)}_\beta) = h(\pi_0(r), \pi_1(l)) \quad \text{and} \quad c(\hat{\zeta}^{\pi_0(r)}_{\alpha'}, \zeta^{\pi_1(l)}_{\beta'}) = h'(\pi_0(r), \pi_1(l))$$

for all $r \in J_d$ and all $l \in J_n \cup J_{n'}$, then

$$R(d^r_\alpha, a_{n,\beta}) \leftrightarrow R(d^r_\alpha, a_{n',\beta}) \quad \text{for all } r < r(d),$$

(recall that if $\hat{\zeta}^r_\alpha$ is constant then it follows from cleaning that $\beta > \hat{\zeta}^r_\alpha$ for all $\beta \in Z$, so $R(d^r_\alpha, a_{n,\beta})$ and $R(d^r_\alpha, a_{n',\beta})$ both fail). So, the ordered graph with universe $\{d^r_\alpha : r < r(d)\} \cup \{a_{n,\beta}\}$ is isomorphic to the ordered graph with universe $\{d^r_\alpha : r < r(d)\} \cup \{a_{n',\beta}\}$, hence

$$M_\delta \models \varphi(d_\alpha, a_{n,\beta}) \leftrightarrow \varphi(d_\alpha, a_{n',\beta}) \tag{5}$$

for any such $(\alpha, \beta) \in Z^*$. Let

$$D' = \{l' \in J_{n'} : \pi_1(l') = 0 \text{ and } \Gamma^*_{n',l'} \text{ extends } \Gamma^*_C\}$$

and let

$$X' = \{l' \in J_{n'} : \pi_1(l') = 0 \text{ and } \Gamma^*_{n',l'} \text{ extends } \Gamma^*_{n,l} \text{ for some } l \in X\}.$$

Clearly, both D' and X' are subsets of $V_{n'}$ and are unions of $E_{n'}$-classes.

Now fix any function $h : k \times k \to \mathcal{H}$ that is closed under r-extensions and satisfies the following conditions:

(1) For all $l \in J_n$

$$h(0, \pi_1(l))[\Gamma^*_{n,l}] = \begin{cases} 1 & \text{if } l \in X \\ 0 & \text{otherwise;} \end{cases}$$

(2) For all $l' \in J_{n'}$

$$h(0, \pi_1(l'))[\Gamma^*_{n',l'}] = \begin{cases} 1 & \text{if } l' \in X' \\ 0 & \text{otherwise;} \end{cases}$$

(3) For all $r \in J_d$

$$h(\pi_0(r), 0)[\Phi_r] = \begin{cases} 1 & \text{if } \Phi_r = \Gamma^*_{n,l} \text{ for some } l \in X \\ 0 & \text{otherwise.} \end{cases}$$

It is a routine (but somewhat lengthy) exercise to show that there indeed is such a function h. The key observations are that X and X' are unions of E_n and $E_{n'}$-classes respectively, and that for $k = n$ or $k = n'$, for all $l_1, l_n \in V_k$,

$$\Gamma^*_{k,l_1} = \Gamma^*_{k,l_2} \quad \text{if and only if} \quad E_k(l_1, l_2).$$

Choose any $(\alpha, \beta) \in Z^*$ that satisfies $c(\hat{\zeta}_\alpha^{\pi_0(r)}, \zeta_\beta^{\pi_1(l)}) = h(\pi_0(r), \pi_1(l))$ for all $r \in J_d$ and all $l \in J_n \cup J_{n'}$. It follows from Conditions (1) and (2) of the constraints on h that

$$\{l \in m(n) : R(a_\alpha, b^l_{n,\beta}) \text{ holds}\} = P_n \cup X,$$

while

$$\{l' \in m(n') : R(a_\alpha, b^{l'}_{n',\beta}) \text{ holds}\} = P_{n'} \cup X'.$$

But X was chosen so that $P_n \cup X$ is false for n, hence

$$M_\delta \models \neg\varphi(e_\alpha, b_{n,\beta}).$$

It follows from elementarity and the fact that h is closed under r-extensions that

$$M_\delta \models \neg\varphi(d_\alpha, a_{n,\beta}) \Rightarrow M_\delta \models \neg\varphi(d_\alpha, a_{n',\beta}) \Rightarrow M_\delta \models \neg\varphi(e_\alpha, b_{n',\beta}),$$

so $P_{n'} \cup X'$ is false for n'.

But now, consider the function $h' : k \times k \to \mathcal{H}$, where $h' = h$ EXCEPT that

$$h'(\pi_0(r), 0)[\Gamma^*_C] = h'(\pi_0(r), 0)[\Psi] = 1$$

for all types Ψ extending Γ^*_C. Note that h' is also closed under r-extensions. Using the properties of the coloring c, choose $(\alpha', \beta') \in Z^*$ such that $c(\hat{\zeta}_{\alpha'}^{\pi_0(r)}, \zeta_{\beta'}^{\pi_1(l)}) = h'(\pi_0(r), \pi_1(l))$ for all $r \in J_d$ and all $l \in J_n \cup J_{n'}$. It is easily verified that

$$\{l \in m(n) : R(a_{\alpha'}, b^l_{n,\beta'}) \text{ holds}\} = P_n \cup X \cup C,$$

and

$$\{l' \in m(n') : R(a_{\alpha'}, b^{l'}_{n',\beta'}) \text{ holds}\} = P_{n'} \cup X' \cup D'.$$

But $M_\delta \models \varphi(e_{\alpha'}, b_{n,\beta'})$. So, arguing as above, it follows that

$$M_\delta \models \varphi(e_{\alpha'}, b_{n,\beta'}).$$

Thus, $P_{n'} \cup X' \cup D'$ is true for n'.

But now, simply write $D = \{C'_0, \ldots, C'_{s-1}\}$, where the C_i's are distinct $E_{n'}$-classes. Thus, there is $j < s$ such that $P_{n'} \cup X' \cup \bigcup_{i<j} C_i$ is false for n', while $P_n \cup X' \cup \bigcup_{i \le j} C'_i$ is true for n'. In particular, the class C'_j is n'-interesting and $\eta^C \lessdot \eta^{C'_j}$ since $C'_j \subseteq D$. ∎

Appendix A. Cleaning Lemmas

In the appendix we define a number of desirable properties of sequences and show that if the original sequence was indexed by a stationary subset of μ (which is regular) then there is a subsequence that is also indexed by a stationary set that has this desirable property. Many of these properties are unary, which makes the situation easy. For example, if every element of the sequence has one of fewer than μ colors, then there is a monochromatic stationary subsequence. It would certainly be desirable to extend this to pairs, i.e., if $S \subseteq \mu$ is stationary and every pair $(\alpha, \beta) \in S^2$ with $\alpha > \beta$ is given one of fewer than μ colors, then one could find a subsequence that is homogeneous in this sense. However, for an arbitrary coloring, this would require μ to be weakly compact. In fact, the existence of the coloring of pairs given by Theorem 2.6 can be viewed as a strong refutation of the existence in general of such a homogeneous set. However, if we restrict to relations that concentrate on tails (see Definition A.10) then Lemma A.11 provides us with a stationary homogeneous subset.

Nothing in this appendix is at all deep. The arguments simply rely on standard methods of manipulating clubs and stationary sets, with Fodor's lemma playing a prominent role. The notation in the appendix is consistent with the body of the paper. In particular, the μ, δ, I_δ and M_δ that appear in the Appendix are the same entities as in Section 3.

Lemma A.1. *Suppose that $S \subseteq \mu$ is stationary and f is any ordinal-valued function with domain S. Either there is a stationary subset $S' \subseteq S$ such that $f|_{S'}$ is constant or there is a stationary subset $S' \subseteq S$ such that $f|_{S'}$ is strictly increasing.*

Proof. Choose δ^* least such that there is a stationary $S' \subseteq S$ such that $f(\alpha) < \delta^*$ for all $\alpha \in S'$. Without loss, we may assume that $S' = S$, i.e., $f(\alpha) < \delta^*$ for all $\alpha \in S$. Let

$$T = \{\alpha \in S : f(\alpha) < f(\beta) \text{ for some } \beta \in S \cap \alpha\}.$$

We claim that T is not stationary. Indeed, if T were stationary, then the function $g : T \to \mu$ defined by $g(\alpha)$ is the least $\beta \in S$ such that $f(\alpha) < f(\beta)$ would be pressing down. Thus, by Fodor's lemma there would be a stationary $T' \subseteq T$ and $\beta^* \in S$ such that $g(\alpha) = \beta^*$ for all $\alpha \in T'$. But then, $\alpha \in T'$ would imply $f(\alpha) < f(\beta^*) < \delta^*$, which contradicts our choice of δ^*. Thus, T is not stationary. So by replacing S by $S \setminus T$, we may assume that $f(\alpha) \geq f(\beta)$ for all $\alpha < \beta$ from S. Let

$$U = \{\alpha \in S : f(\alpha) = f(\beta) \text{ for some } \beta \in S \cap \alpha\}.$$

There are now two cases. If U is stationary then it follows from Fodor's lemma that f is constant on some stationary subset of U. On the other hand, f is strictly increasing on $S \setminus U$, so if U is non-stationary then the second clause of the conclusion of the lemma holds. ∎

Definition A.2. For $X \subseteq \mu$, a sequence $\bar{\eta} = \langle \eta_\alpha : \alpha \in X \rangle$ of elements from $\mathrm{des}(\delta)$ forms a *Δ-system indexed by X* if

(1) $lg(\eta_\alpha) = lg(\eta_\beta)$ for all $\alpha, \beta \in X$. This common value, called the *length of $\bar{\eta}$,* is denoted $lg(\bar{\eta})$;
(2) For each $i < lg(\bar{\eta})$, $\langle \eta_\alpha(i) : \alpha \in X \rangle$ is either constant or strictly increasing;
(3) For all $i < j < lg(\bar{\eta})$, $\eta_\alpha(i) \neq \eta_\beta(j)$ for all $\alpha, \beta \in X$.

We call $i < lg(\bar{\eta})$ *constant* if the sequence $\langle \eta_\alpha(i) : \alpha \in X \rangle$ is constant.

Lemma A.3. *If $S \subseteq \mu$ is stationary, then for any sequence $\langle \eta_\alpha : \alpha \in S \rangle$ from* $\mathrm{des}(\delta)$ *there is a stationary $S' \subseteq S$ such that $\langle \eta_\alpha : \alpha \in S' \rangle$ is a Δ-system indexed by S'.*

Proof. The first clause of Definition A.2 follows easily from the fact that the countable union of non-stationary sets is non-stationary and the second clause follows by iterating Lemma A.1 finitely often. To obtain the third clause, assume that the original sequence satisfies the first two clauses and fix $i < j < lg(\bar{\eta})$. By the definition of $\mathrm{des}(\delta)$, $\eta_\alpha(i) > \eta_\alpha(j)$ for all $\alpha \in S$. If both i and j are constant there is nothing to do. If i is constant and j is strictly increasing then necessarily $\eta_\alpha(i) > \eta_\beta(j)$ for all $\alpha, \beta \in S$ and if j is constant then again $\eta_\alpha(i) > \eta_\beta(j)$ for all $\alpha, \beta \in S$. So assume that both sequences $\langle \eta_\alpha(i) : \alpha \in S \rangle$ and $\langle \eta_\alpha(j) : \alpha \in S \rangle$ are strictly increasing. It suffices to show that the set

$$T = \{\alpha \in S : \eta_\alpha(j) \in \{\eta_\beta(i) : \beta \in S \cap \alpha\}\}$$

is non-stationary. However, if T were stationary then for each $\alpha \in T$, choose $\beta \in S$ least such that $\eta_\beta(i) = \eta_\alpha(j)$. Since $\eta_\beta(j) \lessdot \eta_\beta(i) = \eta_\alpha(j)$ and since $\langle \eta_\alpha(j) : \alpha \in S \rangle$ is strictly increasing, $\alpha > \beta$. Thus, Fodor's lemma would give us $\alpha \neq \alpha'$ such that $\eta_\alpha(j) = \eta_{\alpha'}(j)$, which contradicts the fact that $\langle \eta_\alpha(j) : \alpha \in S \rangle$ is strictly increasing. ∎

Definition A.4. A sequence $\langle s_\alpha : \alpha \in X \rangle$ of elements from I_δ is *tidy* if the following conditions hold:

(1) The sequence $\langle \zeta^{s_\alpha} : \alpha \in X \rangle$ is either constant or is strictly increasing with $\zeta^{s_\alpha} \geq \alpha$ for all $\alpha \in X$;
(2) The sequence $\langle \eta^{s_\alpha} : \alpha \in X \rangle$ is a Δ-system indexed by X;
(3) The sequence $\langle p^{s_\alpha} : \alpha \in X \rangle$ is constant; and

(4) The uniform functions g^{s_α} and g^{s_β} have the same shape for all $\alpha, \beta \in X$.

Lemma A.5. *If $S \subseteq \mu$ is stationary and $\langle s_\alpha : \alpha \in S\rangle$ is any sequence of elements from I_δ, then there is a stationary $S' \subseteq S$ such that the subsequence $\langle s_\alpha : \alpha \in S'\rangle$ is tidy.*

Proof. The first condition can be obtained by applying Lemma A.1 to the sequence $\langle \zeta^{s_\alpha} : \alpha \in S\rangle$ to get a subsequence indexed by a stationary subset $S_1 \subseteq S$ that is either constant or strictly increasing. If the subsequence is strictly increasing, then it follows easily from Fodor's lemma that $\{\alpha \in S_1 : \zeta^{s_\alpha} < \alpha\}$ is non-stationary so by trimming S_1 further we may assume it is empty. The second condition follows immediately from Lemma A.3 and the final two conditions can be obtained by noting that the union of countably many non-stationary subsets of μ is non-stationary. ∎

Definition A.6. A sequence $\langle b_\alpha : \alpha \in X\rangle$ of elements from M_δ is *clean* if there is a term $\tau(x_0, \ldots, x_{m-1})$ with m free variables and sequences $\langle s^l_\alpha : \alpha \in X\rangle$ from the skeleton I_δ for each $l < m$ such that

$$b_\alpha = \tau(s^0_\alpha, \ldots, s^{m-1}_\alpha) \text{ for each } \alpha \in X$$

and satisfy the following conditions (as notation we let $(\zeta^l_\alpha, \eta^l_\alpha, g^l_\alpha, p^l_\alpha)$ denote the four components of s^l_α):

(1) For each $l < m$ the sequence $\langle s^l_\alpha : \alpha \in X\rangle$ is tidy;
(2) For each $\alpha \in X$ the sequence $\langle s^l_\alpha : l < m\rangle$ is strictly $<_{I_\delta}$-increasing;
(3) For all $l, l' < m$ and all $\alpha, \beta \in X$, $\zeta^l_\alpha < \zeta^{l'}_\alpha \Leftrightarrow \zeta^l_\beta < \zeta^{l'}_\beta$ and $\zeta^l_\alpha > \zeta^{l'}_\alpha \Leftrightarrow \zeta^l_\beta > \zeta^{l'}_\beta$;
(4) For all $l, l' < m$ and all $\alpha, \beta \in X$,
   - $\eta^l_\alpha = \eta^{l'}_\alpha$ if and only if $\eta^l_\beta = \eta^{l'}_\beta$;
   - $\eta^l_\alpha = \eta^{l'}_{\zeta^l_\alpha}$ if and only if $\eta^l_\beta = \eta^{l'}_{\zeta^l_\beta}$;
   - $\eta^l_\alpha \lessdot \eta^{l'}_{\zeta^l_\alpha}$ if and only if $\eta^l_\beta \lessdot \eta^{l'}_{\zeta^l_\beta}$;
   - $\eta^{l'}_{\zeta^l_\alpha} \lessdot \eta^l_\alpha$ if and only if $\eta^{l'}_{\zeta^l_\beta} \lessdot \eta^l_\beta$;
(5) For $\alpha > \beta$, $\alpha > \zeta^l_\beta$ for all $l < m$;
(6) For all $l, l' < m$ such that $\zeta^l_\alpha > \zeta^{l'}_\alpha$ for some $\alpha \in X$, $\langle c(\zeta^l_\alpha, \zeta^{l'}_\alpha) : \alpha \in X\rangle$ is constant;
(7) For all $l < m$ and all ordinals β^*, if $\zeta^l_\beta = \beta^*$ for all $\beta \in X$ then $\langle c(\zeta^l_\alpha, \beta^*) : \alpha \in X\rangle$ is constant.

It is readily checked that if $\langle b_\alpha : \alpha \in X\rangle$ is clean and $b_\alpha = \tau(s^l_\alpha : l < m)$ for all $\alpha \in X$ then $R(s^l_\alpha, s^{l'}_\alpha) \leftrightarrow R(s^l_\beta, s^{l'}_\beta)$ for all $l, l' < m$ and all $\alpha, \beta \in X$.

Lemma A.7. *If $S \subseteq \mu$ is stationary and $\langle b_\alpha : \alpha \in S\rangle$ is any sequence of elements from M_δ, then there is a stationary $S' \subseteq S$ such that the subsequence $\langle b_\alpha : \alpha \in S'\rangle$ is clean.*

Proof. Since M_δ is an Ehrenfeucht-Mostowski model built from the skeleton I_δ, for each $\alpha \in S$ there is a term τ_α with $m(\alpha)$ free variables and elements $s^0_\alpha, \dots, s^{m(\alpha)-1}_\alpha$ from I_δ such that $b_\alpha = \tau_\alpha(s^l_\alpha : l < m(\alpha))$. Since $|L'| < \mu$, we can shrink S to a smaller stationary set on which our choice of τ (and hence m) is constant. By applying Lemma A.5 to $\langle s^l_\alpha : \alpha \in S\rangle$ for each $l < m$, we obtain Condition (1). As well, Conditions (2)–(4) and (6)–(7) are obtainable since the union of fewer than μ non-stationary subsets of μ is non-stationary. To obtain Condition (5), it suffices to note that the set

$$C = \{\alpha \in \mu : \alpha > \zeta^l_\beta \text{ for all } \beta \in S \cap \alpha \text{ and all } l < m\}$$

is club in μ (hence $S \cap C$ is stationary). ∎

Next we want to relate pairs of clean sequences from M_δ.

Definition A.8. The (ordered) pair of sequences $\langle a_\alpha : \alpha \in X\rangle$ and $\langle b_\alpha : \alpha \in X\rangle$ of elements from M_δ is a *clean pair* if both sequences are clean and the following two conditions hold (suppose that each $a_\alpha = \tau_a(s^l_\alpha : l < m(a))$ and each $b_\alpha = \tau_b(t^{l'}_\alpha : l' < m(b)))$:

(1) If $\zeta^{s_\alpha} = \alpha^*$ for all $\alpha \in X$, then $\beta > \alpha^*$ for all $\beta \in X$;
(2) If $\zeta^{t_\beta} = \beta^*$ for all $\beta \in X$ then $c(\zeta^{s_\alpha}, \beta^*) = c(\zeta^{s_{\alpha'}}, \beta^*)$ for all $\alpha, \alpha' \in X$.

Lemma A.9. *Suppose that $S \subseteq \mu$ is stationary and that $\langle a_\alpha : \alpha \in S\rangle$ and $\langle b_\alpha : \alpha \in S\rangle$ are arbitrary sequences from M_δ indexed by S. Then there is a stationary $S' \subseteq S$ such that the subsequences $\langle a_\alpha : \alpha \in S'\rangle$ and $\langle b_\alpha : \alpha \in S'\rangle$ form a clean pair.*

Proof. It follows from Lemma A.7 that we may assume that each of the sequences is clean. Now Condition (1) can be obtained simply be removing a bounded initial segment from S and Condition (2) is obtained by noting that there are only countably many choices for the value of $c(\zeta^{s_\alpha}, \beta^*)$ for each of the (finitely many) β^*'s that are relevant. ∎

Definition A.10. Suppose that $X \subseteq \mu$. A relation $D \subseteq X^2$ *concentrates on tails* if, for all $\alpha \in X$ there is $\beta(\alpha) < \alpha$ such that

$$D(\alpha, \beta) \leftrightarrow D(\alpha, \beta')$$

for all $\beta, \beta' \in X$ that satisfy $\beta(\alpha) \leq \beta, \beta' < \alpha$.

Lemma A.11. *Suppose that $S \subseteq \mu$ is stationary and a relation $D \subseteq S^2$ concentrates on tails. Then there is a stationary subset $S' \subseteq S$ such that $D(\alpha, \beta) \leftrightarrow D(\alpha', \beta')$ for all $\alpha > \beta$, $\alpha' > \beta'$ from S'.*

Proof. Fix a function $\alpha \mapsto \beta(\alpha)$ with domain S that witnesses D concentrating on tails. As this function is pressing down, it follows from Fodor's lemma that there is a β^* and a stationary $S_1 \subseteq S \setminus \beta^*$ such that $D(\alpha, \beta) \leftrightarrow D(\alpha, \beta')$ for all $\alpha \in S_1$ and all $\beta, \beta' \in S' \cap \alpha$.

Let $T = \{\alpha \in S_1 : D(\alpha, \beta)$ holds for all α, β in S_1, $\alpha > \beta\}$. Either T or $S_1 \setminus T$ is stationary and hence is an appropriate choice for S'. ∎

We finish this section with a type of 'interpolation theorem' for strictly increasing ordinal-valued functions.

Lemma A.12. *Suppose that $S \subseteq \mu$ is stationary and γ is any ordinal. For every strictly increasing $f : S \to \gamma$ there is a club $C \subseteq \mu$ and a strictly increasing (total) function $f^* : \mu \to \gamma$ such that $f^*|_{S \cap C} = f|_{S \cap C}$.*

Proof. First, let $B = \{\alpha \in S : f(\alpha) < f(\beta) + \alpha$ for some $\beta \in S \cap \alpha\}$. If B were stationary, then it would follow from Fodor's lemma that there would be a stationary $B' \subseteq B$ and a $\beta^* \in S$ such that $\alpha \in B'$ implies

$$f(\beta^*) < f(\alpha) < f(\beta^*) + \alpha.$$

But then, for each $\alpha \in B'$ one could choose $\gamma(\alpha) < \alpha$ such that $f(\alpha) = f(\beta^*) + \gamma(\alpha)$. Another application of Fodor's lemma would show that this contradicts the fact that f is strictly increasing. Thus, we can find a club $C_1 \subseteq \mu$ such that $f(\alpha) \geq f(\beta) + \alpha$ for every pair $\alpha > \beta$ from $S \cap C_1$. Now define a total function $g : \mu \to \gamma$ by:

$$g(\alpha) = \begin{cases} \sup\{f(\beta) + \alpha : \beta \in S \cap \alpha\} & \text{if } S \cap C_1 \cap \alpha \neq \emptyset \\ \alpha & \text{if } S \cap C_1 \cap \alpha = \emptyset \end{cases}$$

It is easy to verify that $C_2 = \{\alpha \in \mu : g(\alpha) > g(\alpha')$ for all $\alpha' < \alpha\}$ is a club subset of μ. Let $S' = S \cap C_1 \cap C_2$ and let D be the closure of S' Define a function $h : D \to \gamma$ by:

$$h(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in S' \\ g(\alpha) & \text{if } \alpha \in D \setminus S' \end{cases}$$

It is easily checked that the function h is strictly increasing on D. So, let $j : \mu \to D$ be the enumeration map (i.e., $j(\alpha)$ is the α^{th} element of D) and let $f^* : \mu \to \gamma$ be defined by $f^*(\alpha) = h(j(\alpha))$. The function f^* is strictly increasing as both h and j are. As well, the set $C_3 = \{\alpha \in \mu : j(\alpha) = \alpha\}$ is club in μ and for $\alpha \in S \cap C_1 \cap C_2 \cap C_3$,

$$f^*(\alpha) = h(j(\alpha)) = h(\alpha) = f(\alpha)$$

so f^* is as desired. ∎

Lemma A.13. *Let $S \subseteq \mu$ be stationary and let $g : S \to \text{des}^{<\omega}(\delta)$ be any function. There is a stationary $S' \subseteq S$ and a uniform function $g^* : \mu \to \text{des}^{<\omega}(\delta)$ such that $g^*|_{S'} = g|_{S'}$.*

Proof. First, by shrinking S if needed, we may assume that there is a number m so that $lg(g(\alpha)) = m$ for all $\alpha \in S$. Similarly, for each $i < m$ we may assume that there is a number $n(i)$ such that $lg(g(\alpha))(i) = n(i)$ for all $\alpha \in S$. Let $\text{des}_{n(i)}(\delta)$ denote the subset of $\text{des}(\delta)$ consisting of decreasing sequences of length $n(i)$. Note that $(\text{des}_{n(i)}(\delta), <_{lex})$ is well ordered and hence order-isomorphic to an ordinal. So, by applying Lemma A.1 once for each $i < m$ we may assume that each of the sequences $\langle g(\alpha)(i) : \alpha \in S\rangle$ is either $<_{lex}$-strictly increasing or constant. For each constant $i < m$, let g_i denote its common value. By successively applying Lemma A.12 for each non-constant $i < m$ we obtain a stationary subset $S' \subseteq S$ and strictly increasing total functions $f_i : \mu \to \text{des}^{<\omega}(\delta)$ such that $f_i(\alpha) = g(\alpha)(i)$ for all $\alpha \in S'$. So define $g^* : \mu \to \text{des}^{<\omega}(\delta)$ by:

$$g^*(\alpha)(i) = \begin{cases} f_i(\alpha) & \text{if } i \text{ is non-constant} \\ g_i & \text{if } i \text{ is constant} \end{cases}$$

Clearly, g^* is uniform and $g^*|_{S'} = g|_{S'}$. ∎

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