THE HEIGHT OF THE AUTOMORPHISM TOWER OF A GROUP
SH810

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Abstract. For a group G with trivial center there is a natural embedding of G into its automorphism group, so we can look at the latter as an extension of the group. So an increasing continuous sequence of groups, the automorphism tower, is defined, the height is the ordinal where this becomes fixed, arriving to a complete group. We show that for many such κ there is such a group of cardinality κ which is of height $> 2^\kappa$, so proving that the upper bound essentially cannot be improved.

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§0 Introduction

For a group G with trivial center there is a natural embedding of G into its automorphism group $\mathrm{Aut}(G)$ where $g \in G$ is mapped to the inner automorphism $x \mapsto gxg^{-1}$ which is defined and is not the identity for $g \neq e_G$ as G has a trivial center, so we can view $\mathrm{Aut}(G)$ as a group extending G. Also the extension $\mathrm{Aut}(G)$ is a group with trivial center, so we can continue defining $G^{<\alpha>}$ increasing with α for every ordinal α; let τ_G be when we stop, i.e., the first α such that $G^{<\alpha+1>} = G^{<\alpha>}$ (or $\alpha = \infty$ but see below) hence $\beta > \alpha \Rightarrow G^{<\beta>} = G^{<\alpha>}$, (see Definition 0.2). How large can τ_G be?

Weilandt [Wel39] proves that for finite G, τ_G is finite. Thomas [Th85] celebrated work proves for infinite G that $\tau_G \leq (2^{|G|})^+$, in fact as noted by Felgner and Thomas $\tau_G < (2^{|G|})^+$. Thomas shows also that $\tau_\kappa \geq \kappa^+$. Later he ([Th98]) showed that if $\kappa = \kappa^{<\kappa}, 2^\kappa = \kappa^+$ (hence $\tau_\kappa \leq \kappa^{++}$ in $\mathbf{V}$) and $\lambda \geq \kappa^{++}$ and we force by $\mathbb{P}$, the forcing of adding λ Cohen subsets to κ, then in $\mathbf{V}^{\mathbb{P}}$ we still have $\tau_\kappa \leq \kappa^{++}$ though 2^κ is $\geq \lambda$ (and $\mathbf{V}, \mathbf{V}^{\mathbb{P}}$ has the same cardinals).

Just, Shelah and Thomas [JShT 654] proved that when $\kappa = \kappa^{<\kappa} < \lambda$, in some forcing extension (by a specially constructed κ-complete κ^+-c.c. forcing notion) we have $\tau_\kappa \geq \lambda$, so consistently $\tau_\kappa > 2^\kappa > \kappa^+$ for some κ. An important lemma there which we shall use (see 0.6 below) is that if G is the automorphism group of a structure of cardinality $\kappa, H \subseteq G, |H| \leq \kappa$ then $\tau'_{G,H}$, the normalizer length of H in G (see Definition 0.3(2)), is $< \tau_\kappa$. Concerning groups with center Hamkins show that $\tau_G <$ the first strongly inaccessible cardinal $> |G|$. On the subject see the forthcoming book of Thomas.

We shall show, e.g.

0.1 Theorem. *If κ is strong limit singular of uncountable cofinality <u>then</u> $\tau_\kappa > 2^\kappa$.*

It would have been nice if the lower bound for τ_κ, κ^+ would (consistently) be the correct one for all κ simultaneously, but Theorem 0.1 shows that this is not so. Note that Theorem 0.1 shows that provably in ZFC, in general the upper bound $(2^\kappa)^+$ cannot be improved. See Conclusion 3.12 for proof of the theorem, quoting results from pcf theory. We thank Simon Thomas, the referee, Itay Kaplan and Daniel Herden for many valuable complaints detecting serious problems in earlier versions.

The program, described in a simplified way, is that for each so called "κ-parameter $\mathbf{p}$" which includes a partial order $I = I_{\mathbf{p}}$, we define a group $G_{\mathbf{p}}$ and a two element subgroup $H_{\mathbf{p}}$ such that $\langle \mathrm{nor}^\alpha_{G_{\mathbf{p}}}(H_{\mathbf{p}}) : \alpha \leq \mathrm{rk}_{\mathbf{p}} \rangle$ "reflects" $\mathrm{rk}_{\mathbf{p}} = \mathrm{rk}^{<\infty}(I_{\mathbf{p}})$, the natural rank on I (see Definition 1.1), so in particular $\tau'_{G_{\mathbf{p}},H_{\mathbf{p}}} = \mathrm{rk}^{<\infty}(I_{\mathbf{p}})$. (Actually in the end we shall get only "H of cardinality $\leq \kappa$").

We use an inverse system $\mathfrak{s} = \langle J, \mathbf{p}_u, \pi_{u,v} : u \le_J v\rangle$ of κ-parameters where $\pi_{u,v}$ maps $I_{\mathbf{p}_v}$ to $I_{\mathbf{p}_u}$; however, in general the $\pi_{u,v}$'s do not preserve order (but do preserve it in some weak global sense) where J is an $\aleph_1$-directed partial order. Now for each $u \in J$, we can define the group $G_{\mathbf{p}_u}$; and we can take inverse limit in two ways.

Way 1: The inverse limit $\mathbf{p}_{\mathfrak{s}}$ (with $\pi_{u,\mathfrak{s}}$ for $u \in J$ of $\mathfrak{s}$) is a κ-parameter and so the group $G_{\mathbf{p}_{\mathfrak{s}}}$ is well defined.

Way 2: The inverse system $\langle G_{\mathbf{p}_u}, \hat{\pi}_{u,v} : u \le_J v\rangle$ of groups, where $\hat{\pi}_{u,v}$ is the (partial) homomorphism from $G_{\mathbf{p}_v}$ to $G_{\mathbf{p}_u}$ induced by $\pi_{u,v}$, has an inverse limit $G_{\mathfrak{s}}$.
Now

- (A) concerning $G_{\mathbf{p}_{\mathfrak{s}}}$, we normally have good control over $\mathrm{rk}_{\mathbf{p}_{\mathfrak{s}}}$ hence on the normalizer length of $H_{\mathbf{p}_{\mathfrak{s}}}$ inside $G_{\mathbf{p}_{\mathfrak{s}}}$
- (B) $G_{\mathfrak{s}}$ is (more exactly can be represented good enough as) inverse limit of groups of cardinality $\le \kappa$ hence is isomorphic to $\mathrm{Aut}(\mathfrak{A})$ for some structure $\mathfrak{A}$ of cardinality $\le \kappa$
- (C) in the good case $G_{\mathbf{p}_{\mathfrak{s}}} = G_{\mathfrak{s}}$ so we are done (by 0.6).

In §3 we work to get the main result.
There are obvious possible improvement of the results here, say trying to prove $\delta_\kappa \le \tau_\kappa$ (see Definition 0.5) for every κ. But more importantly, a natural conjecture, at least for me was $\tau_\kappa = \delta_\kappa$ because all the results so far on τ_κ have a parallel for δ_κ (though not inversely). In particular it seemed reasonable that for $\kappa = \aleph_0$ the lower bound was right, i.e., $\tau_\kappa = \omega_1$. See more in Kaplan-Shelah [KpSh 882].

0.2 Definition. 1) For a group G with trivial center, define the group $G^{<\alpha>}$ with trivial center for an ordinal α, increasing continuous with α such that $G^{<0>} = G$ and $G^{<\alpha+1>}$ is the group of automorphisms of $G^{<\alpha>}$ identifying $g \in G^{<\alpha>}$ with the inner automorphisms it defines. We may stipulate $G^{<-1>} = \{e_G\}$.
[We know that $G^{<\alpha>}$ is a group with trivial center increasing continuous with α and for some $\alpha < (2^{|G|+\aleph_0})^+$ we have $\beta > \alpha \Rightarrow G^{<\beta>} = G^{<\alpha>}$.]
2) The automorphism tower height of the group G is $\tau_G = \tau_G^{\mathrm{atw}} = \mathrm{Min}\{\alpha : G^{<\alpha>} = G^{<\alpha+1>}\}$; clearly $\beta \ge \alpha \ge \tau_G \Rightarrow G^{<\beta>} = G^{<\alpha>}$, atw stands for automorphism tower.
3) Let $\tau_\kappa = \tau_\kappa^{\mathrm{atw}}$ be the least ordinal τ such that $\tau_G < \tau$ for every group G of cardinality $\le \kappa$; we call it the group tower ordinal of κ.

Now we define normalizer (group theorist write $N_G(H)$, but probably for others $\mathrm{nor}_G(H)$ will be clearer, at least this is so for the author).

0.3 Definition. 1) Let H be a subgroup of G.

We define $\text{nor}^\alpha_G(H)$, a subgroup of G, by induction on the ordinal α, increasing continuous with α. We may add $\text{nor}^{-1}_G(H) = \{e_G\}$.

Case 1: $\alpha = 0$.
$\text{nor}^0_G(H) = H$.

Case 2: $\alpha = \beta + 1$.
$\text{nor}^\alpha_G(H) = \text{nor}_G(\text{nor}^\beta_G(H))$, see below.

Case 3: α a limit ordinal

$$\text{nor}^\alpha_G(H) = \cup\{\text{nor}^\beta_G(H) : \beta < \alpha\}$$

where

$$\text{nor}_G(H) = \{g \in G : g \text{ normalizes } H, \text{ i.e. } gHg^{-1} = H, \text{ equivalently } (\forall x \in H)[gxg^{-1} \in H \ \& \ g^{-1}xg \in H]\}.$$

2) Let $\tau'_{G,H} = \tau^{\text{nlg}}_{G,H}$, the normalizer length of H in G, be $\text{Min}\{\alpha : \text{nor}^\alpha_G(H) = \text{nor}^{\alpha+1}_G(H)\}$; so $\beta \geq \alpha \geq \tau'_{G,H} \Rightarrow \text{nor}^\beta_G(H) = \text{nor}^\alpha_G(H)$; nlg stands for normalizer length.
3) Let $\tau'_\kappa = \tau^{\text{nlg}}_\kappa$ be the least ordinal τ such that $\tau > \tau'_{G,H}$ whenever $G = \text{Aut}(\mathfrak{A})$ for some structure $\mathfrak{A}$ on κ and $H \subseteq G$ is a subgroup satisfying $|H| \leq \kappa$.
4) $\tau''_\kappa = \tau^{\text{nlf}}_\kappa$ is the least ordinal τ such that $\tau > \tau^{\text{nlg}}_{G,H}$ wherever $G = \text{Aut}(\mathfrak{A}), \mathfrak{A}$ a structure of cardinality $\leq \kappa, H$ a subgroup of G of cardinality $\leq \kappa$ and $\text{nor}^\infty_G(H) = \cup\{\text{nor}^\alpha_G(H) : \alpha \text{ an ordinal}\} = G$.

0.4 Definition. We say that G is a κ-automorphism group if G is the automorphism group of some structure of cardinality $\leq \kappa$.

0.5 Definition. Let $\delta_\kappa = \delta(\kappa)$ be the first ordinal α such that there is no sentence $\psi \in \mathbb{L}_{\kappa^+,\omega}$ satisfying:

- (a) $\psi \vdash$ "$<$ is a linear order"
- (b) for every $\beta < \alpha$ there is a model M of ψ such that $(|M|, <^M)$ has order type $\geq \beta$
- (c) for every model M of $\psi, (|M|, <^M)$ is a well ordering.

See on this, e.g. [Sh:c, VII,§5].

Our proof of better lower bounds rely on the following result from [JShT 654].

0.6 Lemma. $\tau'_\kappa \le \tau_\kappa$.

0.7 Question: 1) Is it consistent that for some $\kappa, \tau'_\kappa < \tau_\kappa$? Is this provable in ZFC? Is the negation consistent?
2) Similarly for the inequalities $\delta_\kappa < \tau'_\kappa$, (and $\delta_\kappa < \tau'_\kappa < \tau_\kappa$).

0.8 Observation. For every $\kappa \ge \aleph_0$ we have $\tau^{\rm atw}_\kappa \ge \tau^{\rm nlg}_\kappa \ge \tau^{\rm nlf}_\kappa$.

Proof. By 0.6 and checking the definitions of $\tau^{\rm nlg}_\kappa, \tau^{\rm nlf}_\kappa$. In fact we mostly work on proving that in 0.1, $\tau^{\rm nlf}_\kappa > 2^\kappa$.

Notation: For a group G and $A \subseteq G$ let $\langle A \rangle_G$ be the subgroup of G generated by A.

* * *

A more detailed explanation of the proof:

We would like to derive the desired group from a partial order I representing the ordinal desired as $\tau'_{G,H}$ in some way and the tower of normalizers of an appropriate subgroup of this length. It seems natural to say that if $t \in I$ represent the ordinal α then the $s <_I t$ will represent ordinals $< \alpha$ so we use the depth in I

$$\mathrm{dp}_I(t) = \cup\{\mathrm{dp}_I(s) + 1 : s <_I t\}.$$

For each $t \in I$ we would like to have a generator g_t of the group (we denote the group by K_I and g_t is really denoted by $g_{(<t>,<>)}$) exemplifying that the normalizer tower does not stop at $\alpha = \mathrm{dp}_I(t)$, say g_t will be in the $(\alpha+1)$-th normalizer but not in the α-th normalizer. But we need a witness for g_t not being in the earlier $(\beta+1)$-th normalizer, $\beta < \alpha$.

Now β is represented by some $s <_I t$, so we have witnesses $g_{(<t,s>,<0>)}, g_{(<t,s>,<1>)}$, the first in the first member of the normalizer sequence, the second in the $(\beta+1)$-th normalizer not in the β-th normalizer. So we have a long normalizer tower of the subgroup $G^{<0}_I$, the one generated by $\{g_{(\bar t,\eta)} : \eta(\ell) = 0$ for some $\ell < \ell g(\eta) = \ell g(\bar t) - 1, \bar t \in {}^{\ell g(\bar t)}I$ $<_I$-decreasing$\}$. Now §1 is dedicated to defining and investigating those groups.

However $G^{<0}_I = \langle g_{(\bar t,\eta)} : \bar t$ ends with a $<_I$-minimal member$\rangle$ which by this scheme will be the first in the normalizer tower described above is too big. So in 2.1 we use a semi-direct product $K_I = G_I *_{\mathbf h} L_I$, where L_I is an abelian group with every element of order two, generated by $\{h_{gG^{<0}_I} : g \in G_I\}$ with $(\mathbf{h}(g_1))h_{gG^{<0}_I} = h_{(g_1 g)G^{<0}_I}$

and try to show that the normalizer tower of the subgroup $H_I = \{e, h_{eG_I^{<0}}\}$ of K_I has the same height.

But we have to make K_I a κ-automorphism group. We only almost have it: (and under the present description necessarily fail) we will represent it as $\mathrm{aut}(M)/N$ for some structure M of cardinality $\le \kappa$ and normal subgroup N of it of cardinality $\le \kappa$; this suffices.

From where will M come from? We will represent I as an inverse limit of some kind of $\mathfrak{t} = \langle I_u, \pi_{u,v} : u \le_J v\rangle$ where I_u is a partial order of cardinality $\le \kappa, \pi_{u,v}$ a mapping from I_v to I_u (commuting). It seemed a priori natural to demand that $\pi_{u,v}$ is order preserving but it seemingly does not work out. It seemed a priori natural to prove that whenever $\mathfrak{t}$ is as above there is an inverse limit, etc. We find it more transparent to treat the matter axiomatically: the limit is given inside, i.e. as $\mathfrak{s}$ which is $\mathfrak{t}$ + a limit v^*; and $J^{\mathfrak{t}} = J^{\mathfrak{s}} \backslash \{v^*\}$ is directed.

Also we demand that $J^{\mathfrak{t}}$ is $\aleph_1$-directed (otherwise in the limit of the groups we have elements represented as infinite products of limits of the generators).
We shall derive the structure M from $\mathfrak{t}$ so its automorphisms come from members of K_{I_u} (for $u \in J^{\mathfrak{t}}$). Well, not exactly by formal terms for it, to enable us to project to $u' \le_{J[\mathfrak{t}]} u$; recalling that $\pi_{u,v}$ does not necessarily preserve order. To make things smooth we demand that $J^{\mathfrak{t}}$ is a linear order (say $\mathrm{cf}(\kappa)$) when, as in the main case, κ is singular strong limit of uncountable cofinality.

More specifically, if $s,t \in I$ then for every large enough $u \in J^{\mathfrak{t}}, s <_{I_{v^*}} t \Leftrightarrow \pi_{u,v^*}(s) <_{I_u} \pi_{u,v^*}(t)$; note the order of the quantifiers. Then we define a structure M derived from $\mathfrak{t}$. So the automorphism group of M is the inverse limit of groups which comes from the formal definitions of elements of K_{I_u}'s. Each depend on finitely many generators, which in different u's give different reduced forms.

Now they are defined from some $\bar{t} \in {}^k(I_u)$ using "I_{v^*} is the inverse limit..." the "important" t_u's, those which really affect, will form an inverse system (without loss of generality the length k is constant on an end segment, here we use "$J^{\mathfrak{t}}$ is $\aleph_1$-directed) so for those ℓ's the sequence $\langle t_{u,\ell} : u \in J^{\mathfrak{t}}\rangle$ has limit $t_{v^*,\ell}$ say for $\ell < k_*$.

So $\langle t_{u_*,\ell} : \ell < k_*\rangle$ has the same quantifier type in I_u whenever $u_* \le u \le v^*$ for some $u_* < v^*$. The other t's still has influence, so it is enough to find for them a pseudo limit: $t_{v^*,\ell}$ such that they will have the same affect on how the "important" $t_{u,\ell}$ are used (this is the essential limit).

All this gives an approximation to $\mathrm{aut}(M) \cong K_{I_{v^*}}$. The "almost" means that we divide by the subgroup of the automorphism of M which are id_{K_u} for every $u \in J^{\mathfrak{t}}$ large enough. This is a normal subgroup of cardinality $\le \kappa$ so we are done except constructing such systems.

§1 Constructing groups from partial orders and long normalizer sequences

Discussion: Our aim is, for a partial order I, to define a group $G = G_I$ and a subgroup $H = H_I$ such that the normalizer length of H inside G reflects the depth of the well founded part of I. Eventually we would like to use I of large depth such that $|H_I| \le \kappa$ and the normalizer length of H inside G_I is $> 2^\kappa$, even equal to the depth of I.

For clarity we first define an approximation, in particular, H appears only in §2. How do we define the group $G = G_I$ from the partial order I? For each $t \in I$ we would like to have an element associated with it (it is $g_{(<t>,<>)}$) such that it will "enter" $\mathrm{nor}^\alpha_G(H)$ exactly for $\alpha = \mathrm{rk}_I(t)+1$. We intend that among the generators of the group commuting is the normal case, and we need witnesses that $g_{(<t>,<>)} \notin \mathrm{nor}^{\beta+1}_G(H)$ wherever $\beta < \mathrm{rk}_I(t), \beta > 0$. It is natural that if $\mathrm{rk}_I(t_1) = \beta$ and $t_1 <_I t_0 := t$ then we use t_1 to represent β, as witness; more specifically, we construct the group such that conjugation by $g_{(<t>,<>)}$ interchanges $g_{(<t_0,t_1>,<0>)}$ and $g_{(<t_0,t_1>,<1>)}$ and one of them, say $g_{(<t_0,t_1>,<1>)}$ belongs to $\mathrm{nor}^{\beta+1}_G(H) \setminus \mathrm{nor}^\beta_G(H)$ whereas the other one, $g_{(<t_0,t_1>,<0>)}$, belongs to $\mathrm{nor}^1_G(H)$. Iterating we get the elements $x \in X_I$ defined below.

To "start the induction", we add to G an element g_* of order 2 getting K_I, commuting with $g \in G$ iff g is intended to be in the low level (e.g. $g_{(\bar t,\eta)}, t_n \in I$ is minimal, see notation below).

We could have in this section considered only a partial order I, and the groups G_I (and later K_I) derived from it. But as anyhow we shall use it in the context of κ-p.o.w.i.s., we do it in this frame (of course if $J^{\mathfrak s} = \{u\}$, then $\mathfrak s$ is essentially just I_u).

Note that for our main result it suffices to deal with the case $\mathrm{rk}(I) < \infty$.

1.1 Definition. Let I be a partial order (so $\ne \emptyset$).
1) $\mathrm{rk}_I : I \to \mathrm{Ord} \cup \{\infty\}$ is defined by $\mathrm{rk}_I(t) \ge \alpha$ iff $(\forall \beta < \alpha)(\exists s <_I t)[\mathrm{rk}_I(s) \ge \beta]$.
2) $\mathrm{rk}^{<\infty}_I(t)$ is defined as $\mathrm{rk}_I(t)$ if $\mathrm{rk}_I(t) < \infty$ and is defined as $\cup\{\mathrm{rk}_I(s)+1 : s$ satisfies $s <_I t$ and $\mathrm{rk}_I(s) < \infty\}$ in general.
3) Let $\mathrm{rk}(I) = \cup\{\mathrm{rk}_I(t)+1 : t \in I\}$ stipulating $\alpha < \infty = \infty + 1$.
4) $\mathrm{rk}^{<\infty}(I) = \cup\{\mathrm{rk}_I(t)+1 : t \in I$ and $\mathrm{rk}_I(t) < \infty\}$.
5) Let $I_{[\alpha]} = \{t \in I : \mathrm{rk}_I(t) = \alpha\}$.
6) I is non-trivial <u>when</u> $\{s : s \le_I t$ and $\mathrm{rk}_I(s) \ge \beta\}$ is infinite for every $t \in I$ satisfying $\mathrm{rk}^{<\infty}_I(t) > \beta$ (used in the proof of 1.9(1); if $\mathrm{rk}(I) < \infty$ then it is equivalent to demand "$\mathrm{rk}_I(s) = \beta$").
7) I is explicitly non-trivial <u>when</u> each E_I-equivalence class is infinite where $E_I = \{(t_1,t_2) : t_1 \in I, t_2 \in I$ and $(\forall s \in I)(s <_I t_1 \Leftrightarrow s <_I t_2 \wedge t_1 <_I s \Leftrightarrow t_2 <_I s)\}$.

1.2 Definition. 1) $\mathfrak{s}$ is a κ-p.o.w.i.s. (partial order weak inverse system) <u>when</u>:

- (a) $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ so $J = J^{\mathfrak{s}} = J[\mathfrak{s}], \bar{I} = \bar{I}^{\mathfrak{s}}, \bar{\pi} = \bar{\pi}^{\mathfrak{s}}$
- (b) J is a directed partial order of cardinality $\le \kappa$
- (c) $\bar{I} = \langle I_u : u \in J \rangle = \langle I^{\mathfrak{s}}_u : u \in J \rangle$
- (d) $I_u = I^{\mathfrak{s}}_u$ is a partial order of cardinality $\le \kappa$
- (e) $\bar{\pi} = \langle \pi_{u,v} : u \le_J v \rangle$
- (f) $\pi_{u,v}$ is a partial mapping from I_v into I_u (no preservation of order is required!)
- (g) if $u \le_J v \le_J w$ then $\pi_{u,w} = \pi_{u,v} \circ \pi_{v,w}$.

2) $\mathfrak{s}$ is a p.o.w.i.s. means κ-p.o.w.i.s. for some κ.
3) For $u \in J$ let $X_u = X^{\mathfrak{s}}_u$ be the set of x such that for some $n < \omega$:

- (a) $x = (\bar{t}, \eta) = (\bar{t}^x, \eta^x)$
- (b) η^x is a function from $\{0, \dots, n-1\}$ to $\{0, 1\}$
- (c) $\bar{t} = \langle t_\ell : \ell \le n \rangle = \langle t^x_\ell : \ell \le n \rangle$ where $t_\ell \in I^{\mathfrak{s}}_u$ is $<_{I^{\mathfrak{s}}_u}$-decreasing, i.e., $t_n <_{I^{\mathfrak{s}}_u} t_{n-1} <_{I^{\mathfrak{s}}_u} \dots <_{I^{\mathfrak{s}}_u} t_0$.

3A) In fact for every partial order I we define X_I similarly, so $X^{\mathfrak{s}}_u = X_{I^{\mathfrak{s}}_u}$.
4) In part (3) for $x \in X^{\mathfrak{s}}_u$ let $n(x) = \lg(\bar{t}^x) - 1$ and so $t^x_{n(x)}$ is the last in the sequence $\bar{t}$.
5) For $x \in X^{\mathfrak{s}}_u$ and $n \le n(x)$ let $y = x \upharpoonright n \in X^{\mathfrak{s}}_u$ (with $n(y) = n$) be defined by:

$$\bar{t}^y := \bar{t}^x \upharpoonright (n+1) = \langle t^x_0, \dots, t^x_n \rangle$$

$$\eta^y := \eta^x \upharpoonright n = \eta^x \upharpoonright \{0, \dots, n-1\}.$$

6) We define $\text{rk}^2_u = \text{rk}^{2,\mathfrak{s}}_u$ to be the function from X_u to $\{-1\} \cup \text{ Ord} \cup \{\infty\}$ as follows:

- (α) if $x \in X_u$ and $\{\eta^x(\ell) : \ell < n(x)\} \subseteq \{1\}$ (e.g., $n(x) = 0$) then let $\text{rk}^2_u(x) := \text{rk}_{I_u}(t^x_{n(x)})$
- (β) if $x \in X_u$ and $\{\eta^x(\ell) : \ell < n(x)\} \nsubseteq \{1\}$ then let $\text{rk}^{2,\mathfrak{s}}_u(x) = -1$ (yes,-1).

7) We say that $\mathfrak{s}$ is nice <u>when</u> every $I^{\mathfrak{s}}_u$ is non-trivial and $\pi_{u,v}$ is a function from I_v into I_u, i.e., the domain of $\pi^{\mathfrak{s}}_{u,v}$ is I_v.
8) $X^{<\alpha}_u := \{x \in X^{\mathfrak{s}}_u : \text{rk}^2_u(x) < \alpha\}$ and $X^{\le\alpha}_u := \{x \in X^{\mathfrak{s}}_u : \ \text{rk}^2_u(x) \le \alpha\}$. Note that $X^{\le\alpha}_u = X^{<\alpha+1}_u$ when $\alpha < \infty$. Of course, we may write $X^{<\alpha,\mathfrak{s}}_u, X^{\le\alpha,\mathfrak{s}}_u$ and note that $X^{<0}_u = \{x \in X^{\mathfrak{s}}_u : 0 \in \text{ Rang}(\eta^x)\}$.

1.3 Definition. Assume $\mathfrak{s}$ is a κ-p.o.w.i.s. and $u \in J^{\mathfrak{s}}$.
1) Let $G_u = G_u^{\mathfrak{s}}$ be the group generated by $\{g_x : x \in X_u^{\mathfrak{s}}\}$ freely except the equations in $\Gamma_u = \Gamma_u^{\mathfrak{s}}$ where Γ_u consists of

- (a) $g_x^{-1} = g_x$, that is g_x has order 2, for each $x \in X_u$
- (b) $g_{y_1} g_{y_2} = g_{y_2} g_{y_1}$ when $y_1, y_2 \in X_u$ and $n(y_1) = n(y_2)$
- (c) $g_x g_{y_1} g_x^{-1} = g_{y_2}$ when $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$, see below.

1A) Let $\circledast_{x,y} = \circledast^u_{x,y} = \circledast^{u,\mathfrak{s}}_{x,y}$ mean that $\circledast_{x,y_1,y_2}$ for some y_1, y_2 such that $y \in \{y_1, y_2\}$, see below.
1B) Let $\circledast_{x,y_1,y_2} = \circledast^u_{x,y_1,y_2} = \circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ mean that:

- (a) $x, y_1, y_2 \in X_u$
- (b) $n(x) < n(y_1) = n(y_2)$
- (c) $y_1 \restriction n(x) = y_2 \restriction n(x)$
- (d) $\bar{t}^{y_1} = \bar{t}^{y_2}$
- (e) for $\ell < n(y_1)$ we have: $\eta^{y_1}(\ell) \neq \eta^{y_2}(\ell)$ <u>iff</u> $\ell = n(x) \wedge x = y_1 \restriction n(x)$.

2) Let $G_u^{<\alpha} = G_u^{<\alpha,\mathfrak{s}}$ be defined similarly to $G_u^{\mathfrak{s}}$ except that it is generated only by $\{g_x : x \in X_u^{<\alpha}\}$, freely except the equations from $\Gamma_u^{<\alpha} = \Gamma_u^{<\alpha,\mathfrak{s}}$, where $\Gamma_u^{<\alpha}$ is the set of equations from Γ_u among $\{g_x : x \in X_u^{<\alpha}\}$.

Similarly $G_u^{\leq\alpha}, \Gamma_u^{\leq\alpha}$; note that $G_u^{\leq\alpha} = G_u^{<\alpha+1}, \Gamma_u^{\leq\alpha} = \Gamma_u^{<\alpha+1}$ if $\alpha < \infty$.
3) For $X \subseteq X_u$ let $G_{u,X} = G^{\mathfrak{s}}_{u,X}$ be the group generated by $\{g_y : y \in X\}$ freely except the equations in $\Gamma_{u,X} = \Gamma^{\mathfrak{s}}_{u,X}$ which is the set of equations from Γ_u mentioning only generators among $\{g_y : y \in X\}$.

1.4 Observation. 1) The sequence $\langle X_u^{<\alpha} : \alpha \leq \text{rk}(I_u^{\mathfrak{s}})\rangle$ is $\subseteq$-increasing continuous.
2) If $x, y \in X_u$ are such that $x \neq y = x \restriction n$ then $\text{rk}^2_u(y) \geq \text{rk}^2_u(x)$ and if equality holds then $\text{rk}^2_u(x) = \infty = \text{rk}^2_u(y)$ <u>or</u> both are -1.
3) If a partial order I is explicitly non-trivial <u>then</u> I is non-trivial.

Proof. Check.

1.5 Observation. For a κ-p.o.w.i.s. $\mathfrak{s}$.
1) $\circledast^{u,\mathfrak{s}}_{x,y}$ holds <u>iff</u>:

- (α) $x, y \in X_u$ and
- (β) $n(y) \geq n(x) + 1$.

2) If $x \in X^{\mathfrak{s}}_u$ then $\{(y_1,y_2) : \circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ holds$\}$ is a permutation of order two of $\{y \in X^{\mathfrak{s}}_u : n(y) > n(x)\}$.
3) Moreover, the permutation in part (2) maps each $\{y \in X^{\mathfrak{s}}_u : n(y) = k\}$ onto itself when $k \in (n(x),\omega)$ and it maps $\Gamma_{u\{y \in X^{\mathfrak{s}}_u : n(y) > k\}}$ onto itself when $n(x) \le k < \omega$.
4) $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ iff $\circledast^{u,\mathfrak{s}}_{x,y_2,y_1}$.
5) For $x,y \in X^{\mathfrak{s}}_u$, in the group $G^{\mathfrak{s}}_u$ the elements g_x, g_y commute except when $x \ne y \wedge (x = y \restriction n(x) \vee y = x \restriction n(y))$. In this case, if $n(x) < n(y)$ there is $y' \ne y$ such that $\circledast_{x,y,y'}$ and $\eta^y(\ell) = \eta^{y'}(\ell) \Leftrightarrow \ell \ne n(x)$.

Proof. Straight (details on (2),(3) see the proof of 1.6). $\square_{1.5}$

We first sort out how elements in $G^{\mathfrak{s}}_u$ and various subgroups can be (uniquely) represented as products of the generators.

1.6 Claim. *Assume that $\mathfrak{s}$ is a κ-p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $<^*$ is any linear order of X_u such that*

$\boxdot$ *if $x \in X_u, y \in X_u$ and $n(x) > n(y)$ <u>then</u> $x <^* y$.*

1) Any member of G_u is equal to a product of the form $g_{x_1} \ldots g_{x_m} (x_\ell \in X_u)$ where $x_\ell <^* x_{\ell+1}$ for $\ell = 1,\ldots,m-1$. Moreover, this representation is unique.
2) Similarly for $G^{\le\alpha}_u, G^{<\alpha}_u$ (using $X^{\le\alpha}_u, X^{<\alpha}_u$ respectively instead X_u) hence $G^{\le\alpha}_u, G^{<\alpha}_u$ are subgroups of G_u.
3) In part (1) we can replace G_u and X_u by $G = G_{u,X}$ and X respectively when $X \subseteq X_u$ is such that $[\{x,y_1,y_2\} \subseteq X_u \wedge \circledast^{u,\mathfrak{s}}_{x,y_1,y_2} \wedge \{x,y_1\} \subseteq X \Rightarrow y_2 \in X]$. Hence $G_{u,X}$ is equal to $\langle g_x : x \in X\rangle_{G_u}$.
4) If $g = g_{y_1} \ldots g_{y_m}$ where $y_1,\ldots,y_m \in X_u$ and $g = g_{x_1} \ldots g_{x_n} \in G_u$ and $x_1 <^* \ldots <^* x_n$ <u>then</u> $n \le m$.
5) $\langle G^{<\alpha}_u : \alpha \le \mathrm{rk}(I^{\mathfrak{s}}_u), \alpha$ an ordinal$\rangle$ is an increasing continuous sequence of groups with last element $G^{<\infty}_u$.
6) $\{gG^{<0}_u : g \in G_u\}$ is a partition of G_u (to right cosets of G_u over $G^{<0}_u$).
7) If $<^1,<^2$ are two linear orders of X_u as in $\boxdot$ above and $G_u \models$ "$g_{x_1} \ldots g_{x_k} = g_{y_1} \ldots g_{y_m}$" and $x_1 <^1 \ldots <^1 x_k$ and $y_1 <^2 \ldots <^2 y_m$ (or just $x_1 <^1 \ldots <^1 x_k, n(y_1) \ge n(y_2) \ge \ldots \ge n(y_m)$ and $\langle y_\ell : \ell = 1,\ldots,m\rangle$ is with no repetitions), <u>then</u>:

- (α) *$k = m$*
- (β) *for every i we have $\{\ell : n(x_\ell) = i\} = \{\ell : n(y_\ell) = i\}$ and this set is a convex subset of $\{1,\ldots,m\}$.*

(So the only difference is permuting $g_{x_{\ell(1)}}, g_{x_{\ell(2)}}$ when $n(x_{\ell(1)}) = n(x_{\ell(2)})$.
8) If $I \subseteq I_u$ and $X = X_I$ then $G_{u,X} \cap G^{<0}_u$ is the subgroup of $G_{u,X}$ generated by $\{g_x :$

$x \in X, \text{Rang}(\eta^x) \nsubseteq \{1\}\}$, *i.e., the (naturally defined)* $G^{<0}_I, (G_I := G_{u,X_I}, G^{<0}_I := G^{<0}_{u,X_I})$.
9) If $I_\ell \subseteq I^{\mathfrak{s}}_u$ *for* $\ell = 1,2,3$ *(so* $\le_{I_\ell} = \le_I \restriction I_\ell$*) and* $I_1 \cap I_2 = I_3$ *then* $G_{I_1} \cap G_{I_2} = G_{I_3}$ *and* $G^{<0}_{I_1} \cap G^{<0}_{I_2} = G^{<0}_{I_3}$.

Proof. 1),2),3) Recall that each generator has order two. We can use standard combinatorial group theory (but in the rewriting process below we do not assume knowledge of it); the point is that in the rewriting the number of generators in the word does not increase (so no need of $<^*$ being a well ordering).
We now give a full self-contained proof reducing everything to (3). For part of (2) we consider $G = G^{<\alpha}_u, X = X^{<\alpha}_u \subseteq X_u, \Gamma = \Gamma^{<\alpha}_u$ for α an ordinal or infinity and for part (1) and the rest of part (2) consider $G = G^{\le\beta}_{\bar u}, X = X^{\le\beta}_{\bar u} \subseteq X_u, \Gamma = \Gamma^{\le\beta}_{\bar u}$ for β an ordinal or infinity (recall that G_u, X_u is the case $\beta = \infty$). Now in parts (1),(2) for the set X, the condition from part (3) holds by 1.4(2).
[Why? So assume $\circledast^u_{x,y_1,y_2}$ and e.g. $x, y_1 \in X^{<\alpha}_u$ and we should prove that $y_2 \in X^{<\alpha}_u$. If $y_1 = y_2$ this is trivial so assume $y_1 \ne y_2$, hence necessarily $y_1 \restriction n(x) = x = y_2 \restriction n(x)$ and $n(x) < n(y_1) = n(y_2)$ and $\bar t^{y_1} = \bar t^{y_2}$ and $\eta^{y_1}(\ell) = \eta^{y_2}(\ell) \Leftrightarrow \ell \ne n(x)$. If η^x is not constantly one then also η^{y_2} is not constantly one hence $y_2 \in X^{<0}_u$ so fine. If η^x is constantly one then $\alpha > \text{rk}^2_u(x) = \text{rk}_{I_u}(t^x_{n(x)}) \ge \text{rk}_{I_u}(t^{y_1}_{n(y_1)}) = \text{rk}_{I_u}(t^{y_2}_{n(y_2)}) \ge \text{rk}^2_u(y_2)$ hence $y_2 \in X^{\le\alpha}_{\bar u}$ so fine.]
So it is enough to prove part (3). Now recall that $G = G_{u,X}$ and

- $\circledast_1$ every member of G can be written as a product $g_{x_1} \dots g_{x_n}$ for some $n < \omega, x_\ell \in X$
  [Why? As the set $\{g_x : x \in X\}$ generates G and $G \models$ "$g^{-1}_x = g_x$".]
- $\circledast_2$ if in $g = g_{x_1} \dots g_{x_n}$ we have $x_\ell = x_{\ell+1}$ then we can omit both
  [Why? As $g_x g_x = e_G$ for every $x \in X$ by clause (a) of Definition 1.3(1)]
- $\circledast_3$ if $1 \le \ell < n$ and $g = g_{x_1} \dots g_{x_n}$ and we have $x_{\ell+1} <^* x_\ell$ and $[m \in \{1,\dots,n\} \backslash \{\ell,\ell+1\} \Rightarrow y_m = x_m]$ then we can find $y_\ell, y_{\ell+1} \in X$ such that $g = g_{y_1} \dots g_{y_n}$ and $y_\ell <^* y_{\ell+1}$ and, in fact, $y_{\ell+1} = x_\ell$.

[Why does $\circledast_3$ hold? By Definition 1.3(1) and Observation 1.5(5) one of the following cases occurs.

Case 1: $g_{x_\ell}, g_{x_{\ell+1}}$ commutes.
Let $y_\ell = x_{\ell+1}, y_{\ell+1} = x_\ell$.

Case 2: Not Case 1 but $\circledast^{u,\mathfrak{s}}_{x_{\ell+1},x_\ell}$, see Definition 1.3(1A).
By clause (b) of Definition 1.3(1B) we have $n(x_{\ell+1}) < n(x_\ell)$. So by $\boxdot$ of the assumption of the present claim we have $x_\ell <^* x_{\ell+1}$, contradiction.

Case 3: Not Case 1 but $\circledast^{u,\mathfrak{s}}_{x_\ell,x_{\ell+1}}$, see Definition 1.3(1A).

By 1.5(5) there is $y_\ell \in X$ such that $n(y_\ell) = n(x_{\ell+1}) > n(x_\ell), \bar t^{y_\ell} = \bar t^{x_{\ell+1}}, [i < n(x_{\ell+1}) \Rightarrow (\eta^{y_\ell}(i) = \eta^{x_{\ell+1}}(i) \Leftrightarrow i \neq n(x_\ell))]$ and $\circledast_{x_\ell,x_{\ell+1},y_\ell}$.

Let $y_{\ell+1} = x_\ell$, clearly $y_{\ell+1}, y_\ell \in X$. By Definition 1.3(1), we have $g_{x_\ell} g_{x_{\ell+1}} g^{-1}_{x_\ell} = g_{y_\ell}$ hence $g_{x_\ell} g_{x_{\ell+1}} = g_{y_\ell} g_{x_\ell} = g_{y_\ell} g_{y_{\ell+1}}$ and clearly $n(y_{\ell+1}) = n(x_\ell) < n(y_\ell)$ hence $y_\ell <^* x_\ell = y_{\ell+1}$, so we are done.

The three cases exhaust all possibilities (according to whether $n(x_\ell) = n(x_{\ell+1}), n(x_\ell) > n(x_{\ell+1})$ or $n(x_\ell) < n(x_{\ell+1})$ hence $\circledast_3$ is proved.]

$\circledast_4$ every $g \in G$ can be represented as $g_{x_1} \ldots g_{x_n}$ with $x_1 <^* x_2 <^* \ldots <^* x_n$.

[Why? Really the proofs below of $\circledast_4$ and $\circledast_5$ are incredibly detailed, but try to serve complaints about the proof being only implicit, not to mention errors in earlier versions; so a reader who "sees" those assertions (or parts) can jump ahead.

Without loss of generality g is not the unit of G. By $\circledast_1$ we can find $x_1, \ldots, x_n \in X$ such that $g = g_{x_1} \ldots g_{x_n}$ and $n \geq 1$. Choose such a representation satisfying

$\otimes$ (a) with minimal n and

(b) for this n, with minimal $m \in \{1, \ldots, n+1\}$ such that $x_m <^* \ldots <^* x_n$ and $1 \leq \ell < m \leq n \Rightarrow x_\ell \leq^* x_m$, and

(c) for this pair (n, m) if $m > 2$ then with maximal ℓ where $\ell \in \{1, \ldots, m-1\}$ satisfies x_ℓ is $<^*$-maximal among $\{x_1, \ldots, x_{m-1}\}$ that is $k \in \{1, \ldots, m-1\} \Rightarrow x_k \leq^* x_\ell$.

Easily there is such a sequence $(x_1, \ldots, x_n)$, noting that $m = n+1$ is O.K. for (b) and there is ℓ as in $\otimes(c)$.

By $\circledast_2$ and clause (a) of $\otimes$ we have $x_\ell \neq x_{\ell+1}$ when ℓ from $\otimes(c)$ is well defined (i.e., if $m > 2$).

Now $m = 2$ is impossible (as then $m = 1$ can serve), if $m = 1$ we are done, and if $m > 2$ then ℓ is well defined and $\ell = m-1$ is impossible (as then $m-1$ can serve instead m). Lastly by $\circledast_3$ applied to this ℓ, we could have improved ℓ to $\ell+1$, contradiction.]

$\circledast_5$ the representation in $\circledast_4$ is unique.

[Why does $\circledast_5$ hold? Assume toward contradiction that $g_{x'_1} \ldots g_{x'_{n_1}} = g_{y'_1} \ldots g_{y'_{n_2}}$ where $x'_1 <^* \ldots <^* x'_{n_1}$ and $y'_1 <^* \ldots <^* y'_{n_2}$ and $(x'_1, \ldots, x'_{n_1}) \neq (y'_1, \ldots, y'_{n_2})$. For $k \leq m < \omega$ let $X^{<k,m>} = \{x \in X : k \leq n(x) < m\}$ and let $G^{<k,m>} = G^{\mathfrak{s}}_{u,X^{\langle k,m\rangle}}$, i.e. be the group generated by $\{g_x : x \in X^{<k,m>}\}$ freely except the equations in $\Gamma^{<k,m>}$, i.e., the equations from $\Gamma_{u,X^{<k,m>}}$, i.e., the equations from Definition 1.3(1) mentioning only its generators, i.e. generators from $\{g_x : x \in X^{<k,m>}\}$. Now clearly if $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$, see Definition 1.3(1B) then $n(y_1) = n(y_2) \Rightarrow$

$[y_1 \in X^{<k,m>} \Leftrightarrow y_2 \in X^{<k,m>}]$ so the set $X^{<k,m>} \subseteq X$ satisfies the requirement in part (3) of 1.6 which we are proving; so what we have proved for X holds for $X^{<k,m>}$. In particular $\circledast_1 - \circledast_4$ above gives that for every $g \in G^{<k,m>}$ there are n and $x_1 <^* \ldots <^* x_n$ from $X^{<k,m>}$ such that $G^{<k,m>} \models$ "$g = g_{x_1} \cdots g_{x_n}$". Also it is enough to prove the uniqueness for $G^{<k,m>}$ (for every $k \leq m < \omega$), i.e., we can assume $x'_1, \ldots, x'_{n_1}, y'_1, \ldots, y'_{n_2} \in X$ as if the equality holds (though $(x'_1, \ldots, x'_{n_1}) \neq (y'_1, \ldots, y'_{n_2})$), finitely many equations of $\Gamma_{u,X}$ imply the undesirable equation and for some $k \leq m < \omega$ they are all from $\Gamma^{<k,m>}$ and $\{x'_1, \ldots, x'_{n_1}, y'_1, \ldots, y'_{n_2}\} \subseteq X^{<k,m>}$, hence already in $G^{\langle k,m \rangle}$ we get this undesirable equation.

Now for $k < m < \omega$ and $x \in X^{<k,k+1>}$ let $\pi^{k,m}_x$ be the following permutation of $X^{\langle k+1,m \rangle}$:

$\boxdot_0$ $\pi^{k,m}_x$ maps $y_1 \in X^{\langle k+1,m \rangle}$ to y_2 if $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$.

It is easy but we shall check that

$\boxdot_1$ For k, m, x as above,
- (i) $\pi^{k,m}_x$ is a permutation of order 2 of $X^{\langle k+1,m \rangle}$ which maps $\Gamma^{\langle k+1,m \rangle}$ onto itself
- (ii) $\pi^{k,m}_x$ induces an automorphism $\hat{\pi}^{k,m}_x$ of $G^{\langle k+1,m \rangle}$: the one mapping g_{y_1} to g_{y_2} when $\pi^{k,m}_x(y_1) = y_2$
- (iii) the automorphisms $\hat{\pi}^{k,m}_x$ of $G^{\langle k+1,m \rangle}$ for $x \in X^{<k,k+1>}$ pairwise commute
- (iv) the automorphism $\hat{\pi}^{k,m}_x$ of $G^{\langle k+1,m \rangle}$ is of order two.

Why $\boxdot_1$? By Definition 1.3(1B) we have $\circledast_{x,y,y_1} \wedge \circledast_{x,y,y_2} \Rightarrow y_1 = y_2$ hence $\pi^{k,m}_x$ is a partial function. Next if $y \in X^{<k+1,m>}$ then $n(y) \geq k+1 > k = n(x)$ hence by 1.5(1) we have $\circledast_{x,y}$, which by Definition 1.3(1A) there is $y_1 \in X$ such that $\circledast_{x,y,y_1}$, this implies $n(y_1) = n(y)$ so as $y \in X^{<k+1,m>}$ also $y_1 \in X^{<k+1,m>}$, so $[y \in X^{<k+1,m>} \Rightarrow \pi^{k,m}_x(y) = y_1 \in X^{<k+1,m>}]$. So $\pi^{k,m}_x$ is a function from $X^{<k+1,m>}$ onto itself. By 1.5(4) we have $\pi^{k,m}_x(y_1) = y_2 \Rightarrow \pi^{k,m}_x(y_2) = y_1$ hence $\pi^{k,m}_x$ is one to one (so is a permutation) and has order two, so the first phrase of (i) holds. For the second phrase it suffices to show that every equation from $\Gamma^{<k+1,m>}$ is mapped to an equation from the same set. If the equation is from Definition 1.3(1)(a), i.e. $g^{-1}_y = g_y$ it follows from "$\pi^{k,m}_x$ is a permutation of order 2 of $X^{<k+1,m>}$". If the equation is from Definition 1.3(1)(b), i.e. $g_{y_1} g_{y_2} = g_{y_2} g_{y_1}$ where $y_1, y_2 \in X^{<k+1,m>}$ and $n(y_1) = n(y_2)$ then it suffices to note $n(\pi^{k,m}_x(y_1)) = n(y_1) = n(y_2) = n(\pi^{k,m}_x(y_2))$.

Lastly, if the equation is from Definition 1.3(1)(c), i.e. has the form $g_y g_{y_1} g^{-1}_y = g_{y_2}$ where $y, y_1, y_2 \in X^{<k+1,m>}$ and $\circledast_{y,y_1,y_2}$ holds, let $y' = \pi^{k,m}_x(y), y'_1 = \pi^{k,m}_x(y_1), y'_2 =$

$\pi^{k,m}_x(y_2)$, and it suffices to show that $y', y'_1, y'_2 \in X^{<k+1,m>}$ and $\circledast_{y',y'_1,y'_2}$. First, $y', y'_1, y'_2 \in X^{<k+1,m>}$ as $\pi^{k,m}_x$ is a permutation of $X^{<k+1,m>}$.

Now, recalling $n(y) \ge k+1 > n(x)$, if $y \restriction n(x) \ne x, y_\ell \restriction n(y) = y$ then for $\ell = 1, 2$, as $\circledast_{y,y_1,y_2}$ we have $n(y_\ell) > n(y) > n(x)$ and $y_\ell \restriction n(x) = y \restriction n(x) \ne x$ hence by Definition 1.3(1B), $\circledast_{x,y,y}, \circledast_{x,y_1,y_1}, \circledast_{x,y_2,y_2}$ hence $\pi^{k,m}_x$ maps y, y_1, y_2 to y, y_1, y_2 respectively, so the desired conclusion is trivial. If $(y \restriction n(x) \ne x) \wedge (y_\ell \restriction n(y) \ne y)$ or $(y \restriction n(x) = x) \wedge (y_\ell \restriction n(y) \ne y)$ we can also get the result. So we can assume $y \restriction n(x) = x$ and $y_\ell \restriction n(y) = y$ and as above $y_\ell \restriction n(x) = x$ for $\ell = 1, 2$. So by Definition 1.3(1B) as $\circledast_{x,y,y'}$ we have $\bar t^y = \bar t^{y'}, \eta^y(i) = \eta^{y'}(i) \Leftrightarrow i < n(y) \wedge i \ne n(x)$ and as $\circledast_{x,y_\ell,y'_\ell}$ we have $\bar t^{y'_\ell} = \bar t^{y_\ell}, \eta^{y_\ell}(i) = \eta^{y'_\ell}(i) \Leftrightarrow i < n(y_\ell) \wedge i \ne n(x)$ for $\ell = 1, 2$ and as $\circledast_{y,y_1,y_2}$ we have $\bar t^y = \bar t^{y_\ell} \restriction (n(y)+1), \eta^{y_1} \restriction n(y) = \eta^{y_2} \restriction n(y) = \eta^y, \bar t^{y_1} = \bar t^{y_2}$ and $\eta^{y_1}(i) = \eta^{y_2}(i) \Leftrightarrow i < n(y_1) \wedge i \ne n(y)$.

Hence $\bar t^{y'} = \bar t^{y'_\ell} \restriction (n(y')+1), \bar t^{y'_1} = \bar t^{y'_2}, \eta^{y'_1} \restriction n(y') = \eta^{y'_2} \restriction n(y') = \eta^{y'}$, and $\eta^{y'_1}(i) = \eta^{y'_2}(i) \Leftrightarrow i < n(y'_1) \wedge i \ne n(y')$ recalling $\eta^{y'_1}(i) \ne 1 \Leftrightarrow \eta^{y'_1}(i) = 0$. So we have finished proving clause (i).

Clause (ii) of $\boxdot_1$ follows from clause (i).

As for clause (iii) note that for $x_1 \ne x_2 \in X$ such that $n(x_1) = k = n(x_2)$ and $y \in X^{<k+1,m>}$ we have $\pi^{k,m}_{x_1}(y) \ne y \Rightarrow y \restriction n(x_1) = x_1 \Rightarrow y \restriction n(x_2) = y \restriction n(x_1) = x_1 \ne x_2 \Rightarrow \pi^{k,m}_{x_2}(y) = y$, so "$\pi^{k,m}_{x_1}, \pi^{k,m}_{x_2}$ commute" follows, hence by (ii) it follows that "$\hat\pi^{k,m}_{x_1}, \hat\pi^{k,m}_{x_2}$ commute" as required.

Lastly, clause (iv) follows from "$\pi^{k,m}_x$ is a permutation <u>of order two of $X^{<k+1,m>}$</u>".

We prove this revised formulation of the uniqueness, the one on $G_{u,X^{<k,m>}}$ by induction on $m-k$.

Note that (recalling assumption $\boxdot$ of 1.6)

(∗) if $x \in X^{<k,k+1>}, y \in X^{<\ell,\ell+1>}$ and $x <^* y$ then $\ell \le k$.

If $m - k = 0$, then $G^{<k,m>}$ is the trivial group so the uniqueness is trivial.

Also the case $k = m-1$ is trivial too as in this case $G^{\langle k,m\rangle}$ is generated by $\{g_x : x \in X^{<k,m>}$, i.e. $x \in X$ and $n(x) = k\}$ freely except that they pairwise commute (i.e. clause (b) of Definition 1.3(1)) and each has order 2 (i.e. clause (a) of Definition 1.3(1)) because clause (c) there is empty in the present case.

So

⊙ $G^{<k,k+1>}$ is actually a vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $\{g_x : x \in X^{<k,k+1>}\}$, well in additive notation, so the uniqueness is clear.

So assume that $m - k \ge 2$, now we need

$\boxdot^2_{k,m}$ if $x'_1, \dots, x'_{n_1}, y'_1, \dots, y'_{n_2}$ from $X^{\langle k,m\rangle}$ are as above in $G^{<k,m>}$ then $(x'_1, \dots, x'_{n_1}) = (y'_1, \dots, y'_{n_2})$.

We can prove the induction step.

Now we define a mapping π from $\{g_x : x \in X^{<k,k+1>}\}$ to $\mathrm{Aut}(G^{<k+1,m>})$ by $x \mapsto \hat{\pi}^{k,m}_x$. Now $\odot$ above describes $G^{<k,k+1>}$ and by $\boxdot_1$ the mapping π maps $\Gamma^{<k,k+1>}$ to equations which are satisfied by $\mathrm{Aut}(G^{<k+1,m>})$, hence there is a homomorphism $\hat{\pi}$ from $G^{<k,k+1>}$ into $\mathrm{Aut}(G^{<k+1,m>})$.

Hence by 1.8 the twisted product $\hat{G} = G^{<k,k+1>} *_{\hat{\pi}} G^{<k+1,m>}$ is well defined. Let $\varkappa$ be the following mapping from $\{g_x : x \in X^{<k,m>}\}$ to $\hat{G}$: if $x \in X^{<k,k+1>}$ then $\varkappa(g_x) := (g_x, e_{G^{<k+1,m>}}) \in G^{<k,k+1>} \times G^{<k+1,m>}$ and if $x \in X^{<k+1,m>}$ then $\varkappa(g_x) := (e_{G^{<k,k+1>}}, g_x) \in G^{<k,k+1>} \times G^{<k+1,m>}$.

Now easily every equation from $\Gamma^{<k,m>}$ is mapped by $\varkappa$ to an equation satisfied in $\hat{G}$ (if it is from $\Gamma^{<k+1,m>}$ then we use the definition of $G^{<k+1,m>} = G_{u,X^{<k+1,m>}}$, if it is from $\Gamma^{<k,m>} \setminus \Gamma^{<k+1,m>}$, then we check by cases according to the clauses of Definition 1.3(1), if it is clause (a) the equation has the form $g^2_x = e, x \in X^{<k,k+1>}$ and use $G^{<k,k+1>} \models$ "$g^2_x = e$". If the equation is from clause (b) then it has the form $g_x g_y = g_y g_x$ where $x, y \in X^{<k,k+1>}$ and use "$G^{<k,k+1>}$ is abelian".

Lastly, if the equation is from clause (c) then the equation has the form $g_x g_{y_1} g^{-1}_x = g_{y_2}$ where $x \in X^{<k,k+1>}, y_1, y_2 \in X^{<k+1,m>}$ and $\circledast_{x,y_1,y_2}$ holds; then we use (e) of 1.8(2).

So as $G^{<k,m>}$ is generated by $\{g_x : x \in X^{<k,m>}\}$ freely except the equations from $\Gamma^{<k,m>}$ it follows that $\varkappa$ can be (uniquely) extended to a homomorphism from $G^{<k,m>}$ into $\hat{G}$. Let us return to the statment in $\circledast_5$. So assume $x'_1 <^* \ldots <^* x'_{n_1}$ and $y'_1 <^* \ldots <^* y'_{n_2}$ are from $X^{<k,m>}$ and $G^{<k,m>} \models$ "$g_{x'_1} \cdots g_{x'_{n_1}} = g_{y'_1} \cdots g_{y'_{n_2}}$".

If $\{x'_i, y'_j : i = 1, \ldots, n_1 \text{ and } j = 1, \ldots, n_2\} \subseteq X^{<k+1,m>}$ using $\varkappa$ and recalling 1.8(2)(d) and that G_2 there stands for $G^{<k+1,m>}$ here we get a counterexample to $\circledast_5$ for $G^{<k+1,m>}$ but $m-(k+1) < m-k$ so we are done by the induction hypothesis. So by the demand on $<^*$, we have $x'_{n_1} \in X^{<k,k+1>} \vee y'_{n_2} \in X^{<k,k+1>}$. Now let $\hat{n}_1, \hat{n}_2$ be such that $g_{x_i} \in G^{<k+1,m>} \Leftrightarrow i < \hat{n}_1$ and $g_{y_j} \in G^{<k+1,m>} \Leftrightarrow j < \hat{n}_2$.

Let $\hat{\varkappa}_1 : G^{<k,m>} \to G^{<k,k+1>}$ and $\hat{\varkappa}_2 : G^{<k,m>} \to G^{<k+1,m>}$ be such that $g \in G^{<k,m>} \Rightarrow \varkappa(g) = (\hat{\varkappa}_1(g), \hat{\varkappa}_2(g))$. Applying $\hat{\varkappa}_1$ clearly $g_{x_{\hat{n}_1}} g_{x_{\hat{n}_1+1}} \cdots g_{x_{n_1}} = g_{y_{\hat{n}_2}} g_{y_{\hat{n}_2+1}} \cdots g_{y_{n_2}}$ and $(x_{\hat{n}_1}, x_{\hat{n}_1+1}, \ldots, x_{n_1}) = (y_{\hat{n}_2}, y_{\hat{n}_2+1}, \ldots, y_{n_2})$ with $\odot$, "dividing" $G^{<k,m>} \models$ "$g_{x_1} \cdots g_{x_{\hat{n}_1-1}} = g_{y_1} \cdots g_{y_{\hat{n}_2-1}}$" and we have dealt with this above. So 1),2),3) holds.

4) Included in the proof of $\circledast_4$ inside the proof of parts (1),(2),(3).

5) For $\alpha < \beta \leq \infty$, clearly $X^{<\alpha}_u \subseteq X^{<\beta}_u$ and $\Gamma^{<\alpha}_u \subseteq \Gamma^{<\beta}_u$ hence there is a homomorphism from $G^{<\alpha}_u$ into $G^{<\beta}_u$. This homomorphism is one-to-one (because of the uniqueness clause in part (2)) hence the homomorphism is the identity. So the sequence is $\subseteq$-increasing, the continuity follows by $\mathrm{rk}^2_u(x) = \alpha < \infty \Leftrightarrow g_x \in G^{<\alpha+1}_u \setminus G^{<\alpha}_u$.

6),7),8),9) Easy. $\square_{1.6}$

1.7 Observation. Assume that $\mathbf{n}$ is a natural number > 1, G a group and J a set with:

- (a) f_t is an automorphism of G of order $\mathbf{n}$ for $t \in J$ (i.e. $f_t^{\mathbf{n}} = \mathrm{id}_G$)
- (b) $f_t, f_s \in \mathrm{Aut}(G)$ commute for any $s, t \in J$.

Then there are K and $\langle g_t : t \in J \rangle$ such that

- (α) K is a group
- (β) G is a normal subgroup of K
- (γ) K is generated by $G \cup \{g_t : t \in J\}$
- (δ) if $a \in G$ and $t \in J$ then $g_t^{-1} a g_t = f_t(a)$
- (ε) if $<_*$ is a linear order of J then every member of K has a one and only one representation as $g_{t_1}^{b_1} g_{t_2}^{b_2} \dots g_{t_n}^{b_n} x$ where $x \in G, n < \omega, t_1 <_* \dots <_* t_n$ are from J and $b_1, \dots, b_n \in \{1, \dots, \mathbf{n} - 1\}$
- (ζ) $g_t^{\mathbf{n}} = e_G$.

Proof. A case of twisted product, see below. (Compare also with the proof of 1.6(3), $\boxdot^2_{k,m}$). Set $K = \bigoplus_{t \in J} \mathbb{Z}/\mathbf{n}\mathbb{Z} g_t *_\pi G$, where $\pi(g_t) = f_t \in \mathrm{Aut}(G)$. $\square_{1.7}$

1.8 Definition/Claim. 1) Assume G_1, G_2 are groups and π is a homomorphism from G_1 into $\mathrm{Aut}(G_2)$, we define the twisted product $G = G_1 *_\pi G_2$ as follows:

- (a) the set of elements is $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$
- (b) the product operation is $(g_1, g_2) * (h_1, h_2) = (g_1 h_1, g_2^{\pi(h_1)} h_2)$ where
  - (α) $g_2^{\pi(h_1)}$ is the image of g_2 by the automorphism $\pi(h_1)$ of G_2
  - (β) $g_1 h_1$ is a G_1-product
  - (γ) $g_2^{\pi(h_1)} h_2$ is a G_2-product.

2)

- (a) such group G exists
- (b) in G every member has one and only one representation as $g_1' g_2'$ where $g_1' \in G_1 \times \{e_{G_2}\}, g_2' \in \{e_{G_1}\} \times G_2$
- (c) the mapping $g_1 \mapsto (g_1, e)$ embeds G_1 into G
- (d) the mapping $g_2 \mapsto (e, g_2)$ embeds G_2 into G
- (e) so up to renaming, for each $h_1 \in G_1$ conjugating by it (i.e. $g \mapsto h_1^{-1} g h_1$) inside G acts on G_2 as the automorphism $\pi(h_1)$ of G_2.

3) If H_1, H_2 are subgroups of G_1, G_2 respectively, and $g_1 \in H_1 \Rightarrow \pi(g_1)$ maps H_2 onto itself and $\pi' : H_1 \to \mathrm{Aut}(H_2)$ is $\pi'(x) = \pi(x) \restriction H_2$ then $\{(h_1, h_2) : h_1 \in H_1, h_2 \in H_2\}$ is a subgroup of $G_1 *_\pi G_2$ and is in fact $H_1 *_{\pi'} H_2$; we denote π' by $\pi[H_1, H_2]$.
4) If the pairs (H_1^a, H_2^a) and (H_1^b, H_2^b) are as in part (3) and $H_1^c := H_1^a \cap H_1^b, H_2^c := H_2^a \cap H_2^b$ then the pair (H_1^c, H_2^c) is as in part (3) and $(H_1^a *_{\pi[H_1^a, H_2^a]} H_2^a) \cap (H_1^b *_{[H_1^b, H_2^b]} H_2^b) = (H_1^c *_{\pi[H_1^c, H_2^c]} H_2^c)$.

Proof. Known and straight. $\square_{1.8}$

1.9 Claim. *Let $\mathfrak{s}$ be a κ-p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $I_u = I_u^{\mathfrak{s}}$ be non-trivial, see Definition 1.1(6).*
1) If $0 \le \alpha < \infty$ then the normalizer of $G_u^{<\alpha}$ in G_u is $G_u^{<\alpha+1}$.
2) If $\alpha = \mathrm{rk}^{<\infty}(I_u)$ then the normalizer of $G_u^{<\alpha}$ in G_u is $G_u^{<\infty} = G_u^{<\alpha}$.

Proof. 1) First

$(*)_1$ if $x \in X_u$ and $\mathrm{rk}_u^2(x) = \alpha$ then conjugation by g_x in G_u maps $\{g_y : y \in X_u^{<\alpha}\} = \{g_y : y \in X_u \text{ and } \mathrm{rk}_u^2(y) < \alpha\}$ onto itself.

[Why? As $g_x = g_x^{-1}$ it is enough to prove that conjugation by g_x maps the set into itself, i.e. to prove for every $y \in X_u^{<\alpha}$ that: $g_x g_y g_x^{-1} \in \{g_z : z \in X_u^{<\alpha}\}$. As $\mathrm{rk}_u^2(x) = \alpha$ and $\alpha \ge 0$ by the assumptions of the claim it follows that $\mathrm{Rang}(\eta^x) \subseteq \{1\}$.

Now for each such y, one of the following cases occurs.

Case *(i)*: g_x, g_y commutes so $g_x g_y g_x^{-1} = g_y \in \{g_z : z \in X_u^{<\alpha}\}$.
In this case the desired conclusion holds trivially.

Case *(ii)*: $n(y) \le n(x)$ and not case (i).
As case (i) does not occur, necessarily $n(y) < n(x)$ and $y = x \upharpoonleft n(y)$ by 1.5(5). Also it follows that $t_{n(x)}^x <_{I_u^{\mathfrak{s}}} t_{n(y)}^y$, so as $\mathrm{rk}_{I_u}(t_{n(x)}^x) = \mathrm{rk}_u(x) = \alpha < \infty$ (recalling $\mathrm{Rang}(\eta^x) \subseteq \{1\}$) we have $\mathrm{rk}_{I_u}(t_{n(y)}^y) > \alpha$. Now $\mathrm{Rang}(\eta^y) \subseteq \mathrm{Rang}(\eta^x) \subseteq \{1\}$, so necessarily $\mathrm{rk}_u^2(y) > \alpha$, contradiction.

Case *(iii)*: $n(y) > n(x)$ and not case (i).
As in case (ii) by 1.5(5) we have $x = y \upharpoonleft n(x)$.
Clearly $t_{n(y)}^y <_{I_u^{\mathfrak{s}}} t_{n(x)}^y = t_{n(x)}^x$ so as $\mathrm{rk}_u^2(x) \ge 0$ necessarily $\mathrm{rk}_{I_u}(t_{n(x)}^x) = \mathrm{rk}_u^2(x) = \alpha \in [0, \infty)$ hence $\mathrm{rk}_{I_u}(t_{n(y)}^y) < \mathrm{rk}_{I_u}(t_{n(x)}^x) = \alpha$ and so $\mathrm{rk}_u^2(y) \le \mathrm{rk}_{I_u}(t_{n(y)}^y) < \alpha$.

Let $y_1 = y$ and by 1.5(1),(5) and Definition 1.3(1A) there is y_2 such that $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ hence $G_u \models$ "$g_x g_y g_x^{-1} = g_{y_2}$" and $\bar t^y = \bar t^{y_1} = \bar t^{y_2}$, so $\mathrm{rk}^2_u(y_2) \le \mathrm{rk}_{I_u}(t^{y_2}_{n(y_2)}) = \mathrm{rk}_{I_u}(t^{y_1}_{n(y_1)}) < \alpha$ hence $y_2 \in X^{<\alpha}_u$ and so $g_{y_2} \in G^{<\alpha}_u$ so we are done.
So $(*)_1$ holds.]

Now by $(*)_1$ it follows that g_x normalizes $G^{<\alpha}_u$ for every member g_x of $\{g_x : \mathrm{rk}^2_u(x) = \alpha\}$, hence clearly $\mathrm{nor}_{G_u}(G^{<\alpha}_u) \supseteq (G^{<\alpha}_u) \cup \{g_x : \mathrm{rk}^2_u(x) = \alpha \text{ and } x \in X_u\}$ but the latter generates $G^{<\alpha+1}_u$ hence

$(*)_2$ $\mathrm{nor}_{G_u}(G^{<\alpha}_u) \supseteq G^{<\alpha+1}_u$.

Second assume $g \in G_u \backslash G^{<\alpha+1}_u$, let $<^*$ be a linear ordering of X_u as in $\boxdot$ of 1.6. We can find $k < \omega$ and $x_1, \dots, x_k$ from X_u such that $g = g_{x_1} g_{x_2} \dots g_{x_k}$ and so it suffices to prove by induction on k that: if $g = g_{x_1} \dots g_{x_k} \in G_u \backslash G^{<\alpha+1}_u$ then $g \notin \mathrm{nor}_{G_u}(G^{<\alpha}_u)$. By 1.6(1),(4) without loss of generality $x_1 <^* \dots <^* x_k$. As $g \notin G^{<\alpha+1}_u$ necessarily not all the x_m's are from $X^{<\alpha+1}_u$ hence for some $m, g_{x_m} \notin G^{<\alpha+1}_u$.

$(*)_3$ without loss of generality $g_{x_1}, g_{x_k} \notin G^{<\alpha+1}_u$.
[Why? So assume $g_{x_k} \in G^{<\alpha+1}_u$ hence
- (a) $g_{x_k} \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$ (as we have already proved $G^{<\alpha+1}_u \subseteq \mathrm{nor}_{G_u}(G^{<\alpha}_u)$)
- (b) $\mathrm{nor}_{G_u}(G^{<\alpha}_u)$ is a subgroup of G_u hence
- (c) $g = g_{x_1} \dots g_{x_{k-1}} g_{x_k} \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$ iff $g_{x_1} \dots g_{x_{k-1}} \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$.

By the induction hypothesis on k we are done. Similarly if $g_{x_1} \in G^{<\alpha+1}_u$ then derive $g \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$ iff $g_{x_2} \dots g_{x_k} \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$ to finish.]

As $\mathrm{rk}^2_u(x_1) \ge \alpha+1$ and I_u is non-trivial (recall Definition 1.1(6)) we can find $t^* \in I_u$ such that

$(*)_4$
- (a) $t^* <_{I_u} t^{x_1}_{n(x_1)}$
- (b) $\mathrm{rk}_{I_u}(t^*) \ge \alpha$
- (c) $t^* \notin \{t^x_\ell : x \in \{x_1, \dots, x_k\} \text{ and } \ell \in \{0, \dots, n(x)\}\}$.

Let $m(*)$ be maximal such that $1 \le m(*) \le k$ and $(\exists i)(x_{m(*)} = x_1 \restriction i)$.
Now we choose $y \in X^{\mathfrak{s}}_u$ as follows:

$(*)_5$
- (a) $\bar t^y = \bar t^{x_{m(*)}} {}^\frown \langle t^* \rangle$
- (b) $\eta^y \restriction n(x_{m(*)}) = \eta^{x_{m(*)}}$
- (c) $\eta^y(n(x_{m(*)})) = 0$.

Note that

$(*)_6$ $x_{m(*)} = y \restriction n(x_{m(*)})$ and $y \in X_u^{<0}$ and $n(y) = n(x_{m(*)}) + 1$ and

$(*)_7$ $n(x_1) \geq \ldots \geq n(x_{m(*)}) \geq n(x_{m(*)+1}) \geq \ldots \geq n(x_k)$.

[Why? Recall that the sequence $\langle x_\ell : 1 \leq \ell \leq k\rangle$ is $<^*$-increasing hence by property $\boxdot$ of $<^*$ the sequence $\langle n(x_\ell) : 1 \leq \ell \leq k\rangle$ is non-increasing.]

We now try to define $\langle y_\ell : \ell = 1, \ldots, k+1\rangle$ by induction on ℓ as follows :

$(*)_8$ $y_1 = y$ and $G_u \models$ "$g_{x_\ell}^{-1} g_{y_\ell} g_{x_\ell} = g_{y_{\ell+1}}$" if well defined.

So

$(*)_9$ $y_\ell = y$ for $\ell = 1, \ldots, m(*)$ and so is well defined.
[Why? We prove it by induction on ℓ. For $\ell = 1$ this is given. So assume that this holds for ℓ and we shall prove it for $\ell + 1$ when $\ell + 1 \leq m(*)$. Now $\neg(\bar t^y = \bar t^{x_\ell} \restriction (n(y)+1))$, i.e. $\bar t^y$ is not an initial segment of $\bar t^{x_\ell}$ by the choice of t^* (and y) and hence $y \neq x_\ell \restriction n(y)$ hence $\neg(y = x_\ell \restriction n(y) \wedge n(y) < n(x_\ell))$ and we also have $\neg(x_\ell = y \restriction n(x_\ell) \wedge n(x_\ell) < n(y))$ as otherwise $x_\ell = x_{m(*)} \restriction n(x_\ell)$ but $n(x_\ell) \geq n(x_{m(*)})$ as $x_\ell <^* x_{m(*)}$ hence $x_\ell = x_{m(*)}$, but $\ell \neq m(*)$ hence $x_\ell \neq x_{m(*)}$, contradiction. Together by 1.5(5) the elements g_y, g_{x_ℓ} commute so as by the induction hypothesis $y_\ell = y$ it follows that $y_{\ell+1} = y$ so we are done.]

Now:

$(*)_{10}$ $y_{m(*)+1}$ is well defined and satisfies $(*)_5(a),(b)$ and also $(*)_5(c)$ when we replace 0 by 1.
[Why? By the definition of G_u in 1.3(1),(1B).]

$(*)_{11}$ $y_{m(*)+1} \notin X_u^{<\alpha}$.
[Why? By $(*)_3, x_1 \notin X_u^{<\alpha+1}$ hence η^{x_1} is constantly one; but $x_{m(*)} = x_1 \restriction n(x_{m(*)})$ hence $\eta^{x_{m(*)}}$ is constantly one. Now $\eta^{y_{m(*)+1}} = \eta^{x_{m(*)}} {}^\frown \langle 1\rangle$ by $(*)_{10}$ hence $\eta^{y_{m(*)+1}}$ is constantly one. So $\mathrm{rk}_u^2(y_{m(*)+1}) = \mathrm{rk}_{I_u}(t^{y_{m(*)+1}}_{n(y_{m(*)+1})}) = \mathrm{rk}_{I_u}(t^*) \geq \alpha$ recalling $(*)_4$, so we are done.]

$(*)_{12}$ if $\ell \in \{m(*)+1, \ldots, k+1\}$ then $y_\ell = y_{m(*)+1}$ and y_ℓ is well defined.
[Why? We prove this by induction on ℓ. For $\ell = m(*)+1$ this is trivial by $(*)_{10}$. For $\ell+1 \in \{m(*)+2, \ldots, k+1\}$, it is enough to prove that $y_{m(*)+1}, x_\ell$ commute. Now $\neg(\bar t^{y_{m(*)+1}} = \bar t^{x_\ell} \restriction (n(y)+1))$ because $n(y_{m(*)+1}) = n(y) = n(x_{m(*)}) + 1 \geq n(x_\ell) + 1 > n(x_\ell)$ hence $\neg\big(y_{m(*)+1} = x_\ell \restriction n(y_{m(*)+1}) \wedge n(y_{m(*)+1}) < n(x_\ell)\big)$; also $\neg\big(x_\ell = y_{m(*)+1} \restriction n(x_\ell) \wedge n(x_\ell) < n(y_{m(*)+1})\big)$ as otherwise this contradicts the choice of $m(*)$. So by 1.5(5) they commute indeed.]

$(*)_{13}$ $g^{-1} g_y g = g_{y_{k+1}}$.
[Why? We can prove by induction on $\ell = 1, \dots, k+1$ that $(g_{x_1} \dots g_{x_{\ell-1}})^{-1} g_y (g_{x_1} \dots g_{x_{\ell-1}}) = g_{y_\ell}$, by the definition of the y_ℓ's, i.e., by $(*)_8$ and they are well defined by $(*)_9 + (*)_{10} + (*)_{12}$.]

$(*)_{14}$ $g^{-1} g_y g = g_{m(*)+1}$.
[Why? By $(*)_{12}$ and $(*)_{13}$.]

$(*)_{15}$ $g^{-1} g_y g \notin G_u^{<\alpha}$.
[Why? By $(*)_{14} + (*)_{11}$.]

So by $(*)_6$ we have $g_y \in G_u^{<0} \subseteq G_u^{<\alpha}$ and by $(*)_{15}$ we have $g^{-1} g_y g \notin G_u^{<\alpha}$ hence g does not normalize $G_u^{<\alpha}$, so we have carried the induction on k. As g was any member of $G_u \backslash G_u^{<\alpha+1}$ we get $\mathrm{nor}_{G_u}(G_u^{<\alpha}) \subseteq G_u^{<\alpha+1}$.

Together with $(*)_2$ we are done.

2) Follows. $\square_{1.9}$

§2 Correcting the group

The $G^{\mathfrak{s}}_u$'s from §1 have long towers of normalizers but the "base", $G^{<0,\mathfrak{s}}_u$ is in general of large cardinality. Hence we replace below $G^{\mathfrak{s}}_u$ by $K^{\mathfrak{s}}_u$ and $G^{<0,\mathfrak{s}}_u$ by $H^{\mathfrak{s}}_u$.

2.1 Definition. Let $\mathfrak{s}$ be a κ-p.o.w.i.s.
1) For $u \in J^{\mathfrak{s}}$:

- (a) recall 1.6(6): $\mathscr{A}_u = \mathscr{A}^{\mathfrak{s}}_u := \{gG^{<0}_u : g \in G_u\}$ is a partition of G (to right cosets of $G^{<0}_u$ inside G_u);
- (b) for every $f \in G_u$ a permutation ∂_f of $\mathscr{A}_u$ is defined by $\partial_f(g_1G^{<0}_u) = (fg_1)G^{<0}_u$, we may write it also as $f(g_1G^{<0}_u)$
- (c) let $L_u = L^{\mathfrak{s}}_u$ be the group generated by $\{h_{\mathbf{a}} : \mathbf{a} \in \mathscr{A}_u\}$ freely except $h_{\mathbf{a}}h_{\mathbf{b}} = h_{\mathbf{b}}h_{\mathbf{a}}$ and $h^{-1}_{\mathbf{a}} = h_{\mathbf{a}}$ for $\mathbf{a},\mathbf{b} \in \mathscr{A}_u$; for $g \in G_u$ let $h_g = h_{gG^{<0}_u}$
- (d) let $\mathbf{h}_u = \mathbf{h}^{\mathfrak{s}}_u$ be the homomorphism from G_u into the automorphism group of L_u such that $f \in G_u \wedge \mathbf{a} \in \mathscr{A}_u \Rightarrow (\mathbf{h}_u(f))(h_{\mathbf{a}}) = h_{f\mathbf{a}}$
- (e) let $K_u = K^{\mathfrak{s}}_u$ be $G_u *_{\mathbf{h}_u} L_u$, the twisted product of G_u, L_u with respect to the homomorphism $\mathbf{h}_u$, see 1.8, and we identify G_u with $G_u \times \{e_{L_u}\}$ and L_u with $\{e_{G_u}\} \times L_u$
- (f) let $H_u = \{(e_{G_u}, h_{e_{G_u}G^{<0}_u}), (e_{G_u}, e_{L_u})\}$ a subgroup of K_u and let $h_* := h_{e_{G_u}} = h_{e_{G_u}G^{<0}_u} \in L_u$, i.e. the pair (e_{G_u}, h_*) is the unique member of H_u which is not the unit.

2) For $\alpha \le \infty$ let $K^{<\alpha}_u = K^{<\alpha,\mathfrak{s}}_u$ be the subgroup $\{(g,h) : g \in G^{<\alpha}_u \text{ and } h \in L_u\}$ of K_u. Similarly $K^{\le\alpha}_u = K^{\le\alpha,\mathfrak{s}}_u$.
3) For $u \in J^{\mathfrak{s}}$ let

- (a) $D_u = D^{\mathfrak{s}}_u = \{(v,g) : v \le_{J[\mathfrak{s}]} u \text{ and } g \in K^{\mathfrak{s}}_v\}$
- (b) $Z^0_u = Z^{0,\mathfrak{s}}_u := \{(\bar t,\eta) : \bar t = \langle t_\ell : \ell \le n\rangle, n < \omega, t_\ell \in I_u$ for each $\ell \le n$ and $\eta \in {}^n 2\}$ and let $z = (\bar t^z, \eta^z) = (\langle t^z_\ell : \ell \le n\rangle, \eta^z)$ and $n(z) = n$ for $z \in Z^0_u$; this is compatible with Definition 1.2(3); note that here $\bar t$ is not necessarily decreasing
- (c) $Z^1_u = Z^{1,\mathfrak{s}}_u := \{\langle x_\ell : \ell < k\rangle : k < \omega$, each x_ℓ is from $Z^0_u\}$ and let $z = (\langle x^z_\ell : \ell < k(z)\rangle)$ if $z \in Z^1_u$
- (d) $Z_u := Z^0_u \cup Z^1_u$
- (e) for $z \in Z_u$ we define his(z), a finite subset of I_u by
  - (α) if $z = (\langle t_\ell : \ell \le n\rangle, \eta) \in Z^0_u$ then his$(z) = \{t_\ell : \ell \le n\}$
  - (β) if $z \in Z^1_u$ say $z = \langle(\langle t^k_\ell : \ell \le \ell_k\rangle, \eta^k) : k < k^*\rangle \in Z^1_u$ then his$(z) = \{t^k_\ell : k < k^* \text{ and } \ell \le \ell_k\}$

(*f*) for $z \in Z_u$ let $n(z) = \Sigma\{\ell_k : k < k^*\}$ if $z = \langle((\langle t^k_\ell : \ell \leq \ell_k\rangle, \eta^k) : k < k^*\rangle \in Z^1_u$ and $n(z)$ is already defined if $z \in Z^0_u$ in clause (b).

2.2 Observation. In Definition 2.1:
1) For $u \in J^{\mathfrak{s}}, K_u$ is well defined and G_u, L_u are subgroups of K_u (after the identification).
2) For $I \subseteq I^{\mathfrak{s}}_u$ let $L^{\mathfrak{s}}_{u,I}$ be the subgroup of $L^{\mathfrak{s}}_u$ generated by $\{h_{gG^{<0}_u} : g \in G^{\mathfrak{s}}_{u,X_I}\}$. If $I_1, I_2 \subseteq I^{\mathfrak{s}}_u$ then $L^{\mathfrak{s}}_{u,I_1} \cap L^{\mathfrak{s}}_{u,I_2} = L^{\mathfrak{s}}_{u,I_1 \cap I_2}$. (Saharon says: The latter should be wrong!)
3) For $I \subseteq I^{\mathfrak{s}}_u$ let $K^{\mathfrak{s}}_{u,I}$ be the subgroup of $K^{\mathfrak{s}}_u$ generated by $G^{\mathfrak{s}}_{u,X_I} \cup L^{\mathfrak{s}}_{u,I}$. Then

(*a*) $K^{\mathfrak{s}}_{u,I}$ normalizes $L^{\mathfrak{s}}_{u,I}$ inside $K^{\mathfrak{s}}_u$

(*b*) $K^{\mathfrak{s}}_{u,I}$ is $G^{\mathfrak{s}}_{u,X_I} *_\pi L^{\mathfrak{s}}_{u,I}$ for the natural π, i.e. $\pi = \mathbf{h}^{\mathfrak{s}}_u \restriction G^{\mathfrak{s}}_{u,X_I}$.

Also

(*c*) if $I_1, I_2 \subseteq I^{\mathfrak{s}}_u$ then $K^{\mathfrak{s}}_{u,I_1} \cap K^{\mathfrak{s}}_{u,I_2} = K^{\mathfrak{s}}_{u,I_1 \cap I_2}$.

Proof. Easy (recall 1.6(8),(9), 1.8(2),(3)).

We want to point out that in the proof of clause (2) the following theorem is needed:

If $I_\ell \subseteq I^{\mathfrak{s}}_u$ for $\ell = 1, 2, g_\ell \in G_{u,X_{I_\ell}}$ with $h_{g_1} = h_{g_2}, I_3 = I_1 \cap I_2$ then there exists some $g_3 \in G_{u,X_{I_3}}$ with $h_{g_1} = h_{g_2} = h_{g_3}$.

Its proof is similar to 1.6(3) and is left to the reader.
SAHARON FILL! (Daniel)

Please observe:

2.2.2) "If $I_1, I_2 \subseteq I^{\mathfrak{s}}_u$ then $L^{\mathfrak{s}}_{n,I_1} \cap L^{\mathfrak{s}}_{n,I_2} = L^{\mathfrak{s}}_{n,I_1 \cap I_2}$" being wrong implies that also the following is wrong:
2.2.3)(c)
2.6.3)(c)(β)
2.7.2) and (2.7.3) – (otherwise add/give proof!)
$\boxdot_9$ on p.38, proof 3.4 (uses 2.7.3)!
3.4 GAME OVER!

Saharon, please break the above chain of conclusions!!

2.3 Definition. 1) If I is a partial order <u>then</u> ${}^k I$ is the set of $\bar t = \langle t_\ell : \ell < k\rangle$ where $t_\ell \in I$.
2) If $\bar t \in {}^k I$ then $\mathrm{tp}_{\mathrm{qf}}(\bar t, \emptyset, I) = \{(\iota, \ell_1, \ell_2) : \iota = 0$ and $I \models$ "$t_{\ell_1} < t_{\ell_2}$" or $\iota = 1$ and $t_{\ell_1} = t_{\ell_2}$ or $\iota = 2$ and $I \models$ "$t_{\ell_1} > t_{\ell_2}$" and $\iota = 3$ if none of the previous cases$\}$.
2A) Let $\mathscr{S}^k = \{\mathrm{tp}_{\mathrm{qf}}(\bar t, \emptyset, I) : \bar t \in {}^k I$ and I is a partial order$\}$.
3) We say $\bar t \in {}^k I$ realizes $p \in \mathscr{S}^k$ when $p = \mathrm{tp}_{\mathrm{qf}}(\bar t, \emptyset, I)$.
4) If $k_1 < k_2$ and $p_2 \in \mathscr{S}^{k_2}$ then $p_1 := p_2 \restriction k_1$ is the unique $p_1 \in \mathscr{S}^{k_1}$ such that if $p_2 = \mathrm{tp}_{\mathrm{qf}}(\bar t, \emptyset, I)$ then $p_1 = \mathrm{tp}_{\mathrm{qf}}(\bar t \restriction k_1, \emptyset, I)$.

Remark. Below each member of $\Lambda^0_k, \Lambda^1_k, \Lambda^2_k$ will be a description of an element of $G^{\mathfrak s}_u, \mathscr{A}^{\mathfrak s}_u, K^{\mathfrak s}_u$ respectively from a k-tuple of members of $I^{\mathfrak s}_u$. Of course, a member of $Z^{\mathfrak s}_u$ is a description of a generator of $K^{\mathfrak s}_u$.

2.4 Definition. 1) For $k < \omega$ let $\Lambda^0_k = \cup\{\Lambda^0_{k,p} : p \in \mathscr{S}^k\}$ where for $p \in \mathscr{S}^k$ we let $\Lambda^0_{k,p}$ be the set of sequences of the form $\langle(\bar\ell_j, \eta_j) : j < j(*)\rangle$ such that:

- (a) for each j for some $n = n(\bar\ell_j, \eta_j)$ we have $\bar\ell_j = \langle \ell_{j,i} : i \le n(\bar\ell_j, \eta_j)\rangle$ is a sequence of numbers $< k$ of length $n+1$ such that $p = \mathrm{tp}_{\mathrm{qf}}(\bar t, \emptyset, I) \Rightarrow \langle t_{\ell_{j,i}} : i \le n(\bar\ell_j, \eta_j)\rangle$ is $<_I$-decreasing
- (b) for each $j, \eta_j \in {}^n 2$ where $n = n(\bar\ell_j, \eta_j)$.

2) For any p.o.w.i.s. $\mathfrak s, u \in J^{\mathfrak s}, \bar t \in {}^k(I_u)$ and $\rho = \langle(\bar\ell_j, \eta_j) : j < j(*)\rangle \in \Lambda^0_k$, let $g^u_{\bar t,\rho} = g^{u,\mathfrak s}_{\bar t,\rho} = (\ldots g_{(\bar t^j, \eta_j)} \ldots)_{j<j(*)}$, the product taken in $G_u \subseteq K_u$ (so if $j(*) = 0$ it is $e_{G_u} = e_{K_u}$) where

- (a) $\bar t^j = \mathrm{seq}_{\rho,j}(\bar t) := \langle t_{\ell_{j,i}} : i \le n(\bar\ell_j, \eta_j)\rangle$
- (b) if $\bar t^j$ is decreasing (in I_u) then $g_{(\bar t^j, \eta_j)} \in G_u \subseteq K_u$ is already well defined, if not then $g_{(\bar t^j, \eta_j)} := e_{K_u}$.

2A) For a p.o.w.i.s. $\mathfrak s, u \in J^{\mathfrak s}, \bar t \in {}^k(I^{\mathfrak s}_u)$ and $\rho = \langle(\bar\ell_j, \eta_j) : j < j(*)\rangle \in \Lambda^0_k$ let $z^u_{\bar t,\rho} = z^{u,\mathfrak s}_{\bar t,\rho}$ be the following member of $Z^{1,\mathfrak s}_u$: it is $\langle x_{\bar t,\rho,j} : j < j(*)\rangle$ where $x_{\bar t,\rho,j} = x_{\bar t,(\bar\ell_j,\eta_j)} = (\langle t_{\ell_{j,i}} : i \le n(\bar\ell_j, \eta_j)\rangle, \eta_j)$. For $p \in \mathscr{S}^k$ and $\rho = \langle(\bar\ell_j, \eta_j) : j < j(*)\rangle \in \Lambda^0_{k,p}$ let $\mathrm{supp}(\rho) = \cup\{\mathrm{Rang}(\bar\ell_j) : j < j(*)\}$ and if $\bar t \in {}^k(I^{\mathfrak s}_u)$ let $\sup(\bar t, \rho) = \{t_\ell : \ell \in \mathrm{supp}(\rho)\}$.
2B) We say $\rho \in \Lambda^0_{k,p}$ is p-reduced when: $p \in \mathscr{S}^k$ and for every p.o.w.i.s. $\mathfrak s, u \in J^{\mathfrak s}$ and $\bar t \in {}^k(I^{\mathfrak s}_u)$ realizing p (in $I^{\mathfrak s}_u$), for no $\rho' \in \Lambda^0_{k,p}$ do we have $\mathrm{supp}(\rho') \subset \mathrm{supp}(\rho)$ and $g^{u,\mathfrak s}_{\bar t,\rho'} = g^{u,\mathfrak s}_{\bar t,\rho}$.
2C) We say that $\rho \in \Lambda^0_{k,p}$ is explicitly p-reduced when the sequence is with no repetitions and $\langle n(\bar\ell_j, \eta_j) : j < j(*)\rangle$ is non-increasing (the length can be zero).

3) For $k < \omega$ let $\Lambda^1_k = \cup\{\Lambda^1_{k,p} : p \in \mathscr{S}^k\}$ where for $p \in \mathscr{S}^k$ we let $\Lambda^1_{k,p}$ be the set of $\rho = \langle(\bar\ell_j,\eta_j) : j < j(*)\rangle \in \Lambda^0_{k,p}$ such that: for every $\mathfrak{s}$ and $u \in J^{\mathfrak{s}}$ if $\bar t \in {}^k(I^{\mathfrak{s}}_u)$ realizes p <u>then</u> there is no $\rho' \in \Lambda^0_{k,p}$ with $\mathrm{supp}(\rho') \subset \mathrm{supp}(\rho)$ and satisfying $g^{u,\mathfrak{s}}_{\bar t,\rho} G^{<0}_u = g^{u,\mathfrak{s}}_{\bar t,\rho'} G^{<0}_u$.
4) For $k < \omega$ and $p \in \mathscr{S}^k$ let $\Lambda^2_{k,p}$ be the set of finite sequences ϱ of length ≥ 1 such that $\varrho(0) \in \Lambda^0_{k,p}$ and $0 < i \Rightarrow \varrho(i) \in \Lambda^1_{k,p}$. Let $\Lambda^2_k = \cup\{\Lambda^2_{k,p} : p \in \mathscr{S}^k\}$.
5) For any $\mathfrak{s}$, if $u \in J^{\mathfrak{s}}, \bar t \in {}^k(I_u)$ and $\varrho = \langle\rho_i : i < i(*)\rangle \in \Lambda^2_k$ then $g_{\bar t,\varrho} \in K_u$ (recalling $i(*) \geq 1$) is $g_{\bar t,\rho_0} h_{g_{\bar t,\rho_1}} h_{g_{\bar t,\rho_2}} \ldots h_{g_{\bar t,\rho_{i(*)-1}}}$ (product in K_u) where $g_{\bar t,\rho_\ell}$ is from clause (2), recalling that $h_g = h_{gG^{<0}_u}$ is from clause (c) of Definition 2.1(1).
5A) For any p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}, \bar t \in {}^k(I^{\mathfrak{s}}_u)$ and $\varrho = \langle\rho_i : i < i(*)\rangle \in \Lambda^2_k$, let $z^u_{\bar t,\varrho} = z^{u,\mathfrak{s}}_{\bar t,\varrho}$ be $\langle z^u_{\bar t,\rho_i} : i < i(*)\rangle$.
5B) For $p \in \mathscr{S}^k$ and $\varrho \in \Lambda^2_{k,p}$ let $\mathrm{supp}(\varrho) = \cup\{\mathrm{supp}(\varrho(i)) : i < i(*)\}$.
5C) We say $\varrho \in \Lambda^2_{k,p}$ is p-reduced when for every p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}$ and $\bar t \in {}^k(I^{\mathfrak{s}}_u)$ realizing p, for no $\varrho' \in \Lambda^2_{k,p}$ do we have (in $K^{\mathfrak{s}}_u$) $g^{u,\mathfrak{s}}_{\bar t,\varrho'} = g^{u,\mathfrak{s}}_{\bar t,\varrho}$ and $\mathrm{supp}(\varrho') \subset \mathrm{supp}(\varrho)$.

2.5 Definition. 1) For $\rho_1,\rho_2 \in \Lambda^0_{k,p}$ we say $\rho_1 \mathscr{E}^0_{k,p} \rho_2$ or ρ_1,ρ_2 are 0-p-equivalent when: for every p.o.w.i.s. $\mathfrak{s}$ and $u \in J^{\mathfrak{s}}$ and $\bar t \in {}^k(I^{\mathfrak{s}}_u)$ realizing p the elements $g^{u,\mathfrak{s}}_{\bar t,\rho_1}, g^{u,\mathfrak{s}}_{\bar t,\rho_2}$ of $G^{\mathfrak{s}}_u$ are equal.
2) For $\rho_1,\rho_2 \in \Lambda^1_{k,p}$ we say $\rho_1 \mathscr{E}^1_{k,p} \rho_2$ or ρ_1,ρ_2 are 1-p-equivalent when: for every p.o.w.i.s. $\mathfrak{s}$ and $u \in J^{\mathfrak{s}}$ and $\bar t \in {}^k(I_u)$ realizing p we have $g^{u,\mathfrak{s}}_{\bar t,\rho_1} G^{<0}_u = g^{u,\mathfrak{s}}_{\bar t,\rho_2} G^{<0}_u$.
3) For $\varrho_1,\varrho_2 \in \Lambda^2_{k,p}$ we say that $\varrho_1 \mathscr{E}^2_{k,p} \varrho_2$ or ϱ_1,ϱ_2 are 2-p-equivalent, <u>when</u>: for every p.o.w.i.s. $\mathfrak{s}$ and $u \in J^{\mathfrak{s}}$ and $\bar t \in {}^k(I_u)$ realizing p the element $g^{u,\mathfrak{s}}_{\bar t,\varrho_1}$ and $g^{u,\mathfrak{s}}_{\bar t,\varrho_2}$ of $K^{\mathfrak{s}}_u$ are equal.

2.6 Claim. *1) In Definition 2.4 parts (2B),(3),(5C) saying "for every p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}$ and $\bar t \in {}^k(I_u)$ realizing p" is equivalent to saying "for some ...".*
2) In Definition 2.5, $\mathscr{E}^\iota_{k,p}$ is an equivalence relation on $\Lambda^\iota_{k,p}$ for $\iota = 0,1,2$. For $\iota = 0,2$ every $\mathscr{E}^\iota_{k,p}$-equivalence class contains a p-reduced member and for $\iota = 0$ even an explicitly p-reduced one. Explicitly p-reduced implies p-reduced.
3) For every p.o.w.i.s. $\mathfrak{s}$, <u>if</u> $u \in J^{\mathfrak{s}}$ and $\bar t \in {}^k(I^{\mathfrak{s}}_u)$ realizes $p \in \mathscr{S}^k$ then

- (a) *for $\rho_1,\rho_2 \in \Lambda^0_{k,p}$ we have*
  - (α) $g^{u,\mathfrak{s}}_{\bar t,\rho_1} = g^{u,\mathfrak{s}}_{\bar t,\rho_2}$ *iff* $\rho_1 \mathscr{E}^0_{k,p} \rho_2$
  - (β) *if $\bar t$ is with no repetition and ρ_1,ρ_2 are explicitly p-reduced, <u>then</u> they are $\rho_1 \mathscr{E}^0_{k,p} \rho_2$ <u>iff</u> letting $\rho_i = \langle(\bar\ell^i_j,\eta^i_j) : j < j_i\rangle$ for $i = 1,2$ we have*
    - (i) $j_1 = j_2$
    - (ii) *for some permutation π of $\{0,\ldots,j_1-1\}$ we have*

$(\bar\ell^2_j, \eta^2_j) = (\bar\ell^1_{\pi(j)}, \eta^1_{\pi(j)})$ *(so ρ_2 is a permutation of ρ_1, compare 1.6(7))*

(b) *for $\rho_1, \rho_2 \in \Lambda^1_{k,p}$ we have*
 (α) $g^{u,\mathfrak{s}}_{\bar t,\rho_1} G^{<0}_u = g^{u,\mathfrak{s}}_{\bar t,\rho_2} G^{<0}_u$ *iff* $\rho_1 \mathscr{E}^1_{k,p} \rho_2$

(c) *for $\varrho_1, \varrho_2 \in \Lambda^2_{k,p}$ we have*
 (α) $g^{u,\mathfrak{s}}_{\bar t,\varrho_1} = g^{u,\mathfrak{s}}_{\bar t,\varrho_2}$ *iff* $\varrho_1 \mathscr{E}^2_{k,p} \varrho_2$
 (β) *if $\bar t$ is with no repetition, $\varrho_1 \mathscr{E}^2_{k,p} \varrho_2$ and ϱ_1, ϱ_2 are p-reduced then* $\mathrm{supp}(\varrho_1) = \mathrm{supp}(\varrho_2)$.

Proof. Straight, (recalling Claim 1.6(3),(7), Observation 2.2(2) and note that (3) elaborates (1)). $\square_{2.6}$

2.7 Claim. *Assume $k < \omega, p \in \mathscr{S}^k, \mathfrak{s}$ is a p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $\bar t_1, \bar t_2 \in {}^k I$ satisfy $p = \mathrm{tp}_{\mathrm{qf}}(\bar t_\ell, \emptyset, I^{\mathfrak{s}}_u)$ for $\ell = 1,2$.*
1) If $\rho \in \Lambda^0_{k,p}$ and ρ is p-reduced and $g_{\bar t_1,\rho} = g_{\bar t_2,\rho} \in G^{\mathfrak{s}}_u$, then $\bar t_2 \restriction \mathrm{supp}(\rho)$ is a permutation of $\bar t_1 \restriction \mathrm{supp}(\rho)$.
2) If $\rho \in \Lambda^1_{k,p}$ and $g^{u,\mathfrak{s}}_{\bar t_1,\rho} G^{<0}_u = g^{u,\mathfrak{s}}_{\bar t_2,\rho} G^{<0}_u$ then $\bar t_1 \restriction \mathrm{supp}(\rho)$ is a permutation of $\bar t_2 \restriction \mathrm{supp}(\rho)$.
3) If $\varrho \in \Lambda^2_{k,p}$ is p-reduced and $g^{u,\mathfrak{s}}_{\bar t_1,\varrho} = g^{u,\mathfrak{s}}_{\bar t_2,\varrho}$ then similarly $\bar t_1 \restriction \mathrm{supp}(\varrho)$ is a permutation of $\bar t_2 \restriction \mathrm{supp}(\varrho)$ and both are with no repetition.
4) For every $\varrho_1 \in \Lambda^2_{k,p}$ there is a p-reduced ϱ_2 such that for every p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $\bar t \in {}^k(I^{\mathfrak{s}}_u)$ realizing p we have $g^{u,\mathfrak{s}}_{\bar t,\varrho_1} = g^{u,\mathfrak{s}}_{\bar t,\varrho_2}$. (Similarly for $\Lambda^0_{k,p}, \Lambda^1_{k,p}$).

Proof. Straight.

2.8 Definition. Let $\mathfrak{s}$ be a κ-p.o.w.i.s.
1) For $u \le_{J[\mathfrak{s}]} v$ let $\hat\pi^0_{u,v}$ be the following partial mapping from $Z^{0,\mathfrak{s}}_v$ to $Z^{0,\mathfrak{s}}_u$, recalling Definition 2.1(3)(b):
 $x \in \mathrm{Dom}(\hat\pi^0_{u,v})$ iff $x \in Z^{0,\mathfrak{s}}_v$ and $\pi_{u,v}(t^x_\ell)$ is well defined for $\ell \le n(x)$ and then $\hat\pi_{u,v}(x) = (\langle \pi_{u,v}(t^x_\ell) : \ell \le n(x)\rangle, \eta^x)$.
2) For $u \le_{J[\mathfrak{s}]} v$ let $\hat\pi^1_{u,v} = \hat\pi^{1,\mathfrak{s}}_{u,v}$ be the following partial mapping from Z^1_v to Z^1_u: if $z \in Z^1_v$ so $z = \langle (\bar t^k, \eta^k) : k < k^*\rangle$ and $\bar t^k = \langle t^k_\ell : \ell \le \ell_k\rangle, t^k_\ell \in I_v$ for $k < k^*, \ell \le \ell_k$ then $\hat\pi^1_{u,v}(z) = \langle (\langle \pi_{u,v}(t^k_\ell) : \ell \le \ell_k\rangle, \eta^k) : k < k^*\rangle$ when each $\pi_{u,v}(t^k_\ell)$ is well defined.

3) For $u \le_{J[\mathfrak{s}]} v$ let $\hat{\pi}_{u,v}$ be $\hat{\pi}^0_{u,v} \cup \hat{\pi}^1_{u,v}$.
4) For $u \in J^{\mathfrak{s}}$ and $z \in Z_u$ let $\partial_{u,z}$ be the following permutation of $D_u = D^{\mathfrak{s}}_u$ where D_u is from Definition 2.1(3)(a).

For each $(v,g) \in D_u$ we define $\partial_{u,z}((v,g))$ as follows:

Case 1: $z \in \mathrm{Dom}(\hat{\pi}^0_{v,u}) \subseteq Z^0_u$ and $\hat{\pi}_{v,u}(z) \in X^{\mathfrak{s}}_v$, i.e., $\langle \pi_{v,u}(t^z_\ell) : \ell \le n(z)\rangle$ is $<_{I_u}$-decreasing.

Then let $\partial_{u,z}((v,g)) = (v, g_{\hat{\pi}_{v,u}(z)} g)$ noting $g_{\hat{\pi}_{v,u}(z)} \in G_v \subseteq K_v$.

Case 2: $z \in \mathrm{Dom}(\hat{\pi}^1_{v,u}) \subseteq Z^1_u$ so $z = \langle x_\ell : \ell < k\rangle$ and $x_\ell \in \mathrm{Dom}(\hat{\pi}^0_{v,u})$ for $\ell < k$ and let $x'_\ell := \hat{\pi}^0_{v,u}(x_\ell) \in X^{\mathfrak{s}}_v$ for $\ell < k$.

Then let $\partial_{u,z}((v,g)) = (v,g')$ where $g' \in K_v$ is defined by $h_{g_{x'_0} \cdots g_{x'_{k-1}}} g$, as product in K_v noting $g_{x'_\ell} \in G_v \subseteq K_v$ for $\ell < k$.

Case 3: Neither Case 1 nor Case 2.

Then let $\partial_{u,z}((v,g)) = (v,g)$.

2.9 Observation. In Definitions 2.1, 2.8:
1) If $u \le_{J[\mathfrak{s}]} v$ then $\hat{\pi}_{u,v}$ is a partial mapping from Z_v to Z_u.
2) In part (1), $\hat{\pi}_{u,v}$ maps Z^0_v, Z^1_v to Z^0_u, Z^1_u respectively, that is it maps $Z^\ell_v \cap \mathrm{Dom}(\hat{\pi}_{u,v})$ into Z^ℓ_u for $\ell = 0,1$.
3) If $u \le_{J[\mathfrak{s}]} v$ and $\mathfrak{s}$ is nice or just $\mathrm{Dom}(\pi_{u,v}) = I_v$ then $\mathrm{Dom}(\hat{\pi}_{u,v}) = Z_v$.

Proof. 1),2),3) Check. $\square_{2.9}$

2.10 Claim. *1)* $\mathrm{nor}_{K_u}(H_u)$ *is* $K^{<0}_u$ *where* H_u *is from Definition 2.1(1)(f).*
2) $\mathrm{nor}^{1+\alpha}_{K_u}(H_u)$ *is* $K^{<\alpha}_u$ *for* $\alpha \ge 0$ *if* I_u *is non-trivial.*

Proof. 1) As H_u has two elements e_{K_u} and (e_{G_u}, h_*) clearly an element of K_u normalizes H_u iff it commutes with $h_* \in L_u \subseteq K_u$. Now when does $(g,h) \in G_u *_{\mathbf{h}_u} L_u$ commute with $(e_{G_u}, h_{e_{G_u} G^{<0}_u})$. Note that

$$(g,h)(e_{G_u}, h_{e_{G_u} G^{<0}_u}) = (g, h + h_{e_{G_u} G^{<0}_u})$$

$$(e_{G_u}, h_{e_{G_u} G^{<0}_u})(g,h) = (g, (\mathbf{h}_u(g))(h_{e_{G_u} G^{<0}_u}) + h).$$

As L_u is commutative, "h_* and (g,h) commute in K_u" iff in L_u

$$(\mathbf{h}_u(g))(h_{e_{G_u}G_u^{<0}}) = h_{e_{G_u}G_u^{<0}}.$$

By the definition of $\mathbf{h}_u \in \text{Hom}(G_u, \text{Aut}(L_u))$ in 2.1(1)(d),(e) this means

$$(ge_{G_u})G_u^{<0} = e_{G_u}G_u^{<0}.$$

i.e.

$$g \in G_u^{<0}.$$

We can sum that: $(g,h) \in G_u *_{\mathbf{h}_u} L_u$ belongs to $\text{nor}_{K_u}(H_u)$ iff (g,h) commutes with h_* iff $g \in G_u^{<0}$ iff $(g,h) \in K_u^{<0}$, as required.
2) Let $\mathbf{f}_u : K_u \to G_u$ be defined by $\mathbf{f}_u((g,h)) = g$. Clearly

$(*)_1$ $\mathbf{f}_u$ is a homomorphism from K_u onto G_u and for every ordinal $\alpha \geq 0$, it maps $K_u^{<\alpha}$ onto $G_u^{<\alpha}$ so $\mathbf{f}_u(K_u^{<\alpha}) = G_u^{<\alpha}$ and moreover $\mathbf{f}_u^{-1}(G_u^{<\alpha}) = K_u^{<\alpha}$ (see the definition of $K_u^{<\alpha}$ in 2.1(2)).

Also

$(*)_2$ $\text{Ker}(\mathbf{f}_u) = \{e_{G_u}\} \times L_u \subseteq K_u^{<0}$.

Now we prove by induction on the ordinal $\alpha \geq 0$ that $\text{nor}_{K_u}^{1+\alpha}(H_u) = K_u^{<\alpha}$. For $\alpha = 0$ this holds by part (1). For α limit this holds as both $\langle \text{nor}_{K_u}^{\beta}(H_u) : \beta \leq \alpha \rangle$ and $\langle K_u^{<\beta} : \beta \leq \alpha \rangle$ are increasing continuous.

Lastly, for $\alpha = \beta + 1 > 0$ we have for any $f \in K_u$

$$\begin{aligned} f \in \text{nor}_{K_u}^{1+\alpha}(H_u) &\Leftrightarrow f \in \text{nor}_{K_u}(\text{nor}_{K_u}^{1+\beta}(H_u)) \\ &\Leftrightarrow f \in \text{nor}_{K_u}(\mathbf{f}_u^{-1}(G_u^{<\beta})) \\ &\Leftrightarrow f(\mathbf{f}_u^{-1}(G_u^{<\beta}))f^{-1} = \mathbf{f}_u^{-1}(G_u^{<\beta}) \\ &\Leftrightarrow \mathbf{f}_u(f)G_u^{<\beta}\mathbf{f}_u(f)^{-1} = G_u^{<\beta} \\ &\Leftrightarrow \mathbf{f}_u(f) \in \text{nor}_{G_u}(G_u^{<\beta}) \\ &\Leftrightarrow \mathbf{f}_u(f) \in G_u^{<\alpha} \Leftrightarrow f \in K_u^{<\alpha}. \end{aligned}$$

[Why? The first $\Leftrightarrow$ by the definition of $\text{nor}_{K_u}^{1+\alpha}(-)$, the second $\Leftrightarrow$ by the induction hypothesis, the third $\Leftrightarrow$ by the definition of $\text{nor}_{K_u}(-)$, the fourth $\Leftrightarrow$ by $(*)_1$, the fifth $\Leftrightarrow$ by the definition of $\text{nor}_{G_u}(-)$, the sixth $\Leftrightarrow$ by 1.9(1), the seventh $\Leftrightarrow$ by $(*)_1$.] $\square_{2.10}$

2.11 Observation. Let $\mathfrak{s}$ be a p.o.w.i.s.
1) For $u \in J^{\mathfrak{s}}$ and $x \in Z^{\mathfrak{s}}_u$ we have: $\partial_{u,x}$ is a well defined function and is a permutation of $D^{\mathfrak{s}}_u$.
2) If $u \le_{J[\mathfrak{s}]} v$ then $D^{\mathfrak{s}}_u \subseteq D^{\mathfrak{s}}_v$.
3) If $u \le_{J[\mathfrak{s}]} v$ and $y \in Z^{\mathfrak{s}}_v$ and $x = \hat{\pi}_{u,v}(y)$ then $\partial_{u,x} = \partial_{v,y} \restriction D_u$.

Proof. Straight.

2.12 Definition. Let $\mathfrak{s}$ be a κ-p.o.w.i.s.
1) Let $\mathbf{S}^k = \{\mathbf{q} : \mathbf{q}$ is a function with domain $\mathscr{S}^k$ and for $p \in \mathscr{S}^k, \mathbf{q}(p) \in \Lambda^2_{k,p}\}$, on $\Lambda^2_{k,p}$, see Definition 2.4(4) above.
2) We say that $\mathbf{q} \in \mathbf{S}^k$ is disjoint when $\langle \text{supp}(\mathbf{q}(p)) : p \in \mathscr{S}^k \rangle$ is a sequence of pairwise disjoint sets. We say that $\mathbf{q}$ is reduced when $\mathbf{q}(p)$ is p-reduced for every $p \in \mathscr{S}^k$.
3) Let $Z^2_u = Z^{2,\mathfrak{s}}_u$ be $\cup\{Z^{2,k}_u : k < \omega\}$, where $Z^{2,k}_u = Z^{2,k,\mathfrak{s}}_u$ is the set of pairs $(\bar{t}, \mathbf{q})$ where $\bar{t} \in {}^k(I^{\mathfrak{s}}_u)$ and $\mathbf{q} \in \mathbf{S}^k$.
4) For $z = (\bar{t}, \mathbf{q}) \in Z^2_u$ let $\partial_{u,z} = \partial^{\mathfrak{s}}_{u,z}$ be the following permutation of D_u: if $v \le_{J[\mathfrak{s}]} u$ and $(v, g) \in \{v\} \times K_v$ then $\partial^{\mathfrak{s}}_{u,z}((v, g)) = (v, g'g)$ where $g' = g^{v,\mathfrak{s}}_{\pi_{v,u}(\bar{t}),\mathbf{q}(p)}$ where $p = \text{tp}_{\text{qf}}(\pi_{v,u}(\bar{t}), \emptyset, I^{\mathfrak{s}}_v)$, and, of course, $\pi_{v,u}(\langle t_\ell : \ell < k \rangle) = \langle \pi_{v,u}(t_\ell) : \ell < k \rangle$. If $\pi_{v,u}(\bar{t})$ is not well-defined set $g' = 1$ trivially again.
5) For $(\bar{t}, \mathbf{q}) \in Z^2_u$ let $g_{\bar{t},\mathbf{q}} = g^u_{\bar{t},\mathbf{q}} = g^{u,\mathfrak{s}}_{\bar{t},\mathbf{q}} = g_{\bar{t},\mathbf{q}(p)}$ where $p = \text{tp}_{\text{qf}}(\bar{t}, \emptyset, I_u)$. Let $g^v_{\bar{t},\mathbf{q}} = g^{v,\mathfrak{s}}_{\bar{t},\mathbf{q}} = g^v_{\pi_{v,u}(\bar{t}),\mathbf{q}}$ when $v \le_{J[\mathfrak{s}]} u$ and $\pi_{v,u}(\bar{t})$ is well-defined.

2.13 Remark. We can add $\{\partial^{\mathfrak{s}}_{u,z} : z \in Z^{2,\mathfrak{s}}_u\}$ to the generators of $F^{\mathfrak{s}}_u$ defined in 2.15 below.

2.14 Observation. In Definition 2.12(4), $\partial^{\mathfrak{s}}_{u,z}$ is a well defined permutation of $D^{\mathfrak{s}}_u$.

Proof. Easy.

2.15 Definition. Let $\mathfrak{s}$ be a p.o.w.i.s.
1) Let $F_u = F^{\mathfrak{s}}_u$ be the subgroup of the group of permutations of $D^{\mathfrak{s}}_u$ generated by $\{\partial_{u,z} : z \in Z^{\mathfrak{s}}_u\}$.
2) For a p.o.w.i.s. $\mathfrak{s}$ let $M_{\mathfrak{s}}$ be the following model:

set of elements: $\{(u,g) : u \in J^{\mathfrak{s}}$ and $g \in K^{\mathfrak{s}}_u\} \cup \{(1,u,f) : u \in J^{\mathfrak{s}}$ and $f \in F^{\mathfrak{s}}_u\}$.

relations: $P^{M_{\mathfrak{s}}}_{1,u}$, a unary relation, is $\{(u,g) : g \in K_u\}$ for $u \in J^{\mathfrak{s}}$,

$P^{M_{\mathfrak{s}}}_{2,u}$, a unary relation is $\{(1,u,f) : f \in F_u\}$ for $u \in J^{\mathfrak{s}}$

$R^{M_{\mathfrak{s}}}_{u,v,h}$, a binary relation, is $\{((v,g),(1,u,f)) : f \in F_u, g \in K_v$ and $f((v,h)) = (v,g)\}$ for $u \in J^{\mathfrak{s}}$ and $v \le_{J[\mathfrak{s}]} u$ and $h \in K_v$.

2.16 Observation. If $\mathfrak{s}$ is a κ-p.o.w.i.s. and $v \le_{J[\mathfrak{s}]} u$ and $f \in F_u$ then f maps $\{v\} \times K_v = P^{M_{\mathfrak{s}}}_{1,v}$ onto itself.

Remark. If $\pi \in F^{\mathfrak{s}}_u$ and $v \le_{J[\mathfrak{s}]} u$ then $\pi \restriction (\{v\} \times K_v)$ comes directly from $K^{\mathfrak{s}}_v$, but the relation between the $\langle \pi \restriction (\{v\} \times K_v) : v \le_{J[\mathfrak{s}]} u\rangle$ are less clear.

2.17 Claim. *Let $\mathfrak{s}$ be a p.o.w.i.s.*
1) $\varkappa$ is an automorphism of $M_{\mathfrak{s}}$ iff:

$\circledast$ (a) *$\varkappa$ is a function with domain $M_{\mathfrak{s}}$*

(b) *for every $u \in J^{\mathfrak{s}}$ we have:*

(α) $\varkappa \restriction D_u \in F^{\mathfrak{s}}_u$

(β) *letting $f_u = \varkappa \restriction D_u$ we have $(1,u,f) \in P^{M_{\mathfrak{s}}}_{2,u} \Rightarrow \varkappa((1,u,f)) = (1,u,f_u f)$ where $f_u f$ is the product in F_u.*

2) If $f_u \in F_u$ for $u \in J^{\mathfrak{s}}$ and $f_u \subseteq f_v$ for $u \le_{J[\mathfrak{s}]} v$ then there is one and only one automorphism $\varkappa$ of $M_{\mathfrak{s}}$ such that $u \in J^{\mathfrak{s}} \Rightarrow f_u \subseteq \varkappa$.

Proof. First assume that $\bar f = \langle f_u : u \in J^{\mathfrak{s}}\rangle$ is as in part (2). We define $\varkappa_{\bar f}$, a function with domain $M_{\mathfrak{s}}$ by:

$\circledast_1$ (a) if $a = (u,g) \in P^{M_{\mathfrak{s}}}_{1,u}$ and $u \in J^{\mathfrak{s}}$ then $\varkappa_{\bar f}(a) = f_u(a)$

(b) if $a = (1,u,f) \in P^{M_{\mathfrak{s}}}_{2,u}$ then $\varkappa_{\bar f}(a) = (1,u,f_u f)$.

So

$\circledast_2$ (a) $\varkappa_{\bar f}$ is a well defined function

(b) $\varkappa_{\bar f}$ is one to one

(c) $\varkappa_{\bar f}$ is onto $M_{\mathfrak s}$

(d) $\varkappa_{\bar f}$ maps $P^{M_{\mathfrak s}}_{1,u}$ onto $P^{M_{\mathfrak s}}_{1,u}$ and $P^{M_{\mathfrak s}}_{2,u}$ onto $P^{M_{\mathfrak s}}_{2,u}$ for $u \in J^{\mathfrak s}$

(e) also $\bar f' = \langle f^{-1}_u : u \in J^{\mathfrak s}\rangle$ satisfies the condition of part (2) and $\varkappa_{\bar f'}$ is the inverse of $\varkappa_{\bar f}$

(f) $\varkappa_{\bar f}$ maps $R^{M_{\mathfrak s}}_{u,v,h}$ onto itself.

[Why? The only non-trivial one is clause (f) and in it by clause (e) it is enough to prove that $\varkappa_{\bar f}$ maps $R^{M_{\mathfrak s}}_{u,v,h}$ into $R^{M_{\mathfrak s}}_{u,v,h}$. So assume $v \le_{J[\mathfrak s]} u, h \in K_v$ and $((v,g),(1,u,f)) \in R^{M_{\mathfrak s}}_{u,v,h}$ hence $f \in F_u, g \in K_v$ and $f((v,h)) = (v,g)$. So $\varkappa_{\bar f}((v,g)) = f_v((v,g))$ and $\varkappa_{\bar f}(1,u,f) = (1,u,f_u f)$ and we would like to show that $(f_v((v,g)),(1,u,f_u f)) \in R^{M_{\mathfrak s}}_{u,v,h}$.
This means that $(f_u f)((v,h)) = f_v((v,g))$. We know that $f((v,h)) = (v,g)$ hence $(f_u f)((v,h)) = f_u(f((v,h))) = f_u((v,g))$ so we have to show that $f_u((v,g)) = f_v((v,g))$. But $v \le_{J[\mathfrak s]} u$ hence (by the assumption on $\bar f$) we have $f_v \subseteq f_u$ hence $f_u((v,g)) = f_v((v,g))$ so we are done.]

So we have shown that

$\circledast_3$ if $\bar f = \langle f_u : u \in J^{\mathfrak s}\rangle$ is as in part (2) then $\varkappa_{\bar f}$ is an automorphism of $M_{\mathfrak s}$.

Next

$\circledast_4$ if $\varkappa \in \mathrm{Aut}(M_{\mathfrak s})$ and $\varkappa \restriction D_u$ is the identity for each $u \in J^{\mathfrak s}$ then $\varkappa = \mathrm{id}_{M_{\mathfrak s}}$.

[Why? By the $P^{M_{\mathfrak s}}_{2,u}$'s, $R^{M_{\mathfrak s}}_{u,v,h}$'s and $F^{\mathfrak s}_u$ being a group of permutations of D_u.]

$\circledast_5$ the mapping $\varkappa \mapsto \langle \varkappa \restriction D_u : u \in J^{\mathfrak s}\rangle$ is a homomorphism from $\mathrm{Aut}(M_{\mathfrak s})$ into $\{\bar f : \bar f$ as above$\}$ with coordinatewise product, with kernel $\{\varkappa \in \mathrm{Aut}(M_{\mathfrak s}) : \varkappa \restriction D_u = \mathrm{id}_{D_u}$ for every $u \in J^{\mathfrak s}\}$.

[Why? Easy. Observe that $\varkappa \restriction D_u \in F_u$ for every $u \in J^{\mathfrak s}$.]

$\circledast_6$ the mapping above is onto.

[Why? Easy by $\circledast_3$.

Given $\varkappa \in \mathrm{Aut}(M_{\mathfrak s})$, let $f_u = \varkappa \restriction D_u$. Clearly $f_u \in F_u$ and $u \le_{J[\mathfrak s]} v \Rightarrow f_u \subseteq f_v$ so $\bar f = \langle f_u : u \in J^{\mathfrak s}\rangle$ is as above so by $\circledast_3$ we know $\varkappa_{\bar f}$ is an automorphism of $M_{\mathfrak s}$

and $\varkappa_{\bar f}\varkappa^{-1}$ is an automorphism of $M_{\mathfrak s}$ which is the identity on each D_u hence by $\circledast_4$ is $\mathrm{id}_{M_{\mathfrak s}}$. So $\varkappa = \varkappa_{\bar f}$, is as required.]

$\circledast_7$ the mapping above is one to one.

[Why? Easy by $\circledast_4$.]

Together both parts should be clear. $\square_{2.17}$

2.18 Definition. 1) We say that $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{S}^k$ are $\mathscr{S}$-equivalent where $\mathscr{S} \subseteq \mathscr{S}^k$ when $p \in \mathscr{S} \Rightarrow \mathbf{q}_1(p)\mathscr{E}^2_{k,p}\mathbf{q}_2(p)$.
2) Omitting $\mathscr{S}$ means $\mathscr{S} = \mathscr{S}^k$.

2.19 Claim. *Let $\mathfrak s$ be a nice κ-p.o.w.i.s. (or just $Dom(\pi_{u,v}) = I_v$ for all $u \le_{J[\mathfrak s]}$).*
1) If $u \in J^{\mathfrak s}$ and $f \in F^{\mathfrak s}_u$ <u>then</u> for some k and $\bar t = \langle \bar t_\ell : \ell < k\rangle \in {}^k(I^{\mathfrak s}_u)$ and $\mathbf{q} \in \mathbf{S}^k$ we have:

$(*)$ *$f = \partial_{u,(\bar t,\mathbf{q})}$ (so if $v \le_{J[\mathfrak s]} u$ then $f \restriction (\{v\} \times K^{\mathfrak s}_v)$ is moving by multiplication by $g^v_{\pi_{v,u}(\bar t),\mathbf{q}}$, e.g. $g \in K_v \Rightarrow f((v,g)) = (v, g^v_{\pi_{v,u}(\bar t),\mathbf{q}}g)$.*

2) $\{\partial_{u,(\bar t,\mathbf{q})} : (\bar t,\mathbf{q}) \in Z^2_u\}$ is a group of permutations of $D^{\mathfrak s}_u$ which includes $F^{\mathfrak s}_u$.
3) For every $\mathbf{q} \in \mathbf{S}^k$ there is a reduced $\mathbf{q}' \in \mathbf{S}^k$ which is equivalent to it (see Definition 2.12(2)).

Proof. 2),3) Straight.
1) We use freely Definition 2.12. Recall that $F^{\mathfrak s}_u$ is the group of permutations of $D^{\mathfrak s}_u$ generated by $\{\partial_{u,z} : z \in Z^{\mathfrak s}_u\}$. Hence it is enough to prove that $f \in F^{\mathfrak s}_u$ satisfies the conclusion of the claim in the following cases.

<u>Case 0</u>: f is the identity.

It is enough to let $k = 0$ so $\bar t = \emptyset, \mathscr{S}^k$ is a singleton $\{\emptyset\}$ and $\mathbf{q}(\emptyset)$ is the empty sequence $<<>>\in \Lambda^2_k$ of length 1, i.e. we use in Definition 2.12(3) the case $k = 0$ and in Definition 2.4(1) the case $j(*) = 0$.

<u>Case 1</u>: $f = \partial_{u,z}$ where $z \in Z^0_u$.

So $z = (\bar t^z, \eta^z)$. We set $k = n(z)+1, \bar t = \bar t^z \in {}^k(I^{\mathfrak s}_u)$ and define $\mathbf{q}$ as follows:

(a) if $p \in \mathscr{S}^k$ describes a decreasing sequence then

$$\mathbf{q}(p) = \langle(\langle 0,1,2,\dots,k-1\rangle, \eta^z)\rangle \in \Lambda^2_k$$

as sequence of length 1

(b) if not, then $\mathbf{q}(p) =<<>>$ as in Case 0.

Case 2: $f = \partial_{u,z}$ where $z \in Z^1_u$.

Also clear.

Case 3: $f = f_1 f_2$ (product in $F^{\mathfrak{s}}_u$) where $f_1, f_2 \in F^{\mathfrak{s}}_u$ satisfy the conclusion of the claim.

Just combine the definitions. Here we make use of $\mathfrak{s}$ being a nice κ-p.o.w.i.s. and 2.9(3) to avoid those cases where it is impossible to choose $\bar{t} \in$ Dom $\pi_{v,u}$, meaning that $f = \partial_{u,(\bar{t},\mathbf{q})}$ always acts trivially on $\{v\} \times K^{\mathfrak{s}}_v$ while f_1, f_2 may not be trivial themselves.

Case 4: $f = f^{-1}$ where $f \in F^{\mathfrak{s}}_u$ satisfies the conclusion of the claim.

Easy, too. $\square_{2.19}$

2.20 Remark. If $q \in \mathscr{S}^k$ and $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{S}^k$ and $v \leq_{J[\mathfrak{s}]} u, \bar{t} \in {}^k(I_u)$ and $q = \text{tp}_{\text{qf}}(\pi^{\mathfrak{s}}_{v,u}(\bar{t}), \emptyset, I_v)$ and $\mathbf{q}_1(q), \mathbf{q}_2(q)$ are not $\mathscr{E}^2_{k,q}$-equivalent, then $g^v_{\bar{t},\mathbf{q}_1} \neq g^v_{\bar{t},\mathbf{q}_2}$.

Proof. This is by Claim 2.6(3C).

§3 The main result

We can prove that every κ-p.o.w.i.s has a limit, but for our application it is more transparent to consider κ-p.o.w.i.s $\mathfrak{s}$ which is the κ-p.o.w.i.s. $\mathfrak{t}$ + its limit.

3.1 Definition. We say that $\mathfrak{s}$ is the limit of $\mathfrak{t}$ as witnessed by v_* when (both are p.o.w.i.s. and)

- (a) $J^{\mathfrak{t}} \subseteq J^{\mathfrak{s}}$ and $J^{\mathfrak{s}} = J^{\mathfrak{t}} \cup \{v_*\}, v_* \notin J^{\mathfrak{t}}$ and $u \in J^{\mathfrak{s}} \Rightarrow u \leq_{J[\mathfrak{s}]} v_*$
- (b) $I^{\mathfrak{s}}_u = I^{\mathfrak{t}}_u$ and $\pi^{\mathfrak{s}}_{u,v} = \pi^{\mathfrak{t}}_{u,v}$ when $u \leq_{J[\mathfrak{s}]} v <_{J[\mathfrak{s}]} v_*$
- (c) if $t \in I^{\mathfrak{s}}_{v_*}$ then for some $u = u_t \in J^{\mathfrak{t}}$ we have $t \in \text{Dom}(\pi^{\mathfrak{s}}_{u_t,v_*})$
- (d) if $s, t \in I^{\mathfrak{s}}_{v_*}$ <u>then</u> for some $u = u_{s,t} \in J^{\mathfrak{t}}$ for every v satisfying $u \leq_{J[\mathfrak{s}]} v \leq_{J[\mathfrak{s}]} v_*$ we have $I^{\mathfrak{s}}_{v_*} \models$ "$s \leq t$" $\Leftrightarrow \pi^{\mathfrak{s}}_{v,v_*}(s) \leq_{I^{\mathfrak{s}}_v} \pi^{\mathfrak{s}}_{v,v_*}(t)$
- (e) if $\langle t_u : u \in J^{\mathfrak{t}}_{\geq w}\rangle$ is a sequence satisfying $w \in J^{\mathfrak{t}}, J^{\mathfrak{t}}_{\geq w} = \{u : w \leq u \in J^{\mathfrak{t}}\}; t_u \in I^{\mathfrak{s}}_u$ and $w \leq u_1 \leq u_2 \in J^{\mathfrak{t}} \Rightarrow \pi_{u_1,u_2}(t_{u_2}) = t_{u_1}$, <u>then</u> there is a unique $t \in I^{\mathfrak{s}}_{v_*}$ such that $u \in J^{\mathfrak{t}}_{\geq w} \Rightarrow \pi_{u,v_*}(t) = t_u$.

3.2 Definition. We say that $\mathfrak{s}$ is an existential limit of $\mathfrak{t}$ <u>when</u>: clauses (a)-(e) of Definition 3.1 hold and

- (f) assume that
  - (α) $u_* \in J^{\mathfrak{t}}$
  - (β) $k_1, k_2 < \omega$ and $k = k_1 + k_2$
  - (γ) $\mathscr{E}$ is an equivalence relation on $\mathscr{S}^k$
  - (δ) $\bar{e} = \langle e_u : u \in J^{\mathfrak{t}}_{\geq u_*}\rangle$, where e_u is an $\mathscr{E}$-equivalence class
  - (ε) $\bar{t} \in {}^{k_1}(I^{\mathfrak{s}}_{v_*})$
  - (ζ) for every $v \in J^{\mathfrak{t}}_{\geq u_*}$ there is $\bar{s}_v \in {}^{k_2}(I^{\mathfrak{t}}_v)$ such that: if $u_* \leq_{J[\mathfrak{t}]} u \leq_{J[\mathfrak{t}]} v$ then e_u is the $\mathscr{E}$-equivalence class of $\text{tp}_{\text{qf}}(\bar{t}^u\char`^\bar{s}^{u,v}, \emptyset, I^{\mathfrak{t}}_u)$ where $\bar{t}^u = \pi^{\mathfrak{s}}_{u,v_*}(\bar{t})$ and $\bar{s}^{u,v} = \pi^{\mathfrak{t}}_{u,v}(\bar{s}_v)$.

<u>Then</u> there are $u_* \leq u^* \in J^{\mathfrak{t}}, \bar{s} \in {}^{k_2}(I^{\mathfrak{s}}_{v_*})$ such that for every $u \in J^{\mathfrak{t}}_{\geq u^*}$, $\text{tp}_{\text{qf}}(\pi^{\mathfrak{s}}_{u,v_*}(\bar{t}\char`^\bar{s}), \emptyset, I^{\mathfrak{t}}_u)$ belongs to e_u (and is constantly p^* for some $p^* \in \mathscr{S}^k$).

3.3 Remark. We may say "$\mathfrak{s}$ is semi-limit of $\mathfrak{t}$" when in clause (d) we replace $\Leftrightarrow$ by $\Rightarrow$. We may consider using this weaker version and/or omit linearity in our main theorem, but the present version suffices.

3.4 Main Claim. *$K^{\mathfrak{s}}_{v_*}$ is an almost κ-automorphism group (see below) <u>when</u>:*

$\boxtimes$ (a) *$\mathfrak{s}, \mathfrak{t}$ are both p.o.w.i.s.*

(b) *$\mathfrak{s}$ is an existential limit of $\mathfrak{t}$ as witnessed by v_**

(c) *$J^{\mathfrak{t}}$ is $\aleph_1$-directed, linear (i.e., for every $u, v \in J^{\mathfrak{t}}$ we have $u \le_{J[\mathfrak{t}]} v$ or $v \le_{J[\mathfrak{t}]} u$) and unbounded*

(d) *$\mathfrak{t}$ is a κ-p.o.w.i.s. (so $\kappa \ge |J^{\mathfrak{t}}|$ and $\kappa \ge |I^{\mathfrak{t}}_u|$ for $u \in J^{\mathfrak{t}}$)*

(e) *$\mathfrak{t}$ is nice (see Definition 1.2(7)).*

3.5 Definition. G is an almost κ-automorphism group when: there is a κ-automorphism group G^+ and a normal subgroup G^- of G^+ of cardinality $\le \kappa$ such that G is isomorphic to G^+/G^-, i.e., there is a homomorphism from G^+ onto G with kernel G^-.

Before proving 3.4 we explain: why will being almost κ-automorphism group help us in proving our intended result?

Recalling Definition 0.3 and Observation 0.8:

3.6 Claim. *For any ordinal α, if there is an almost κ-automorphism group G with a subgroup H of cardinality $\le \kappa$ such that $\tau'_{G,H} = \alpha$ [such that $\mathrm{nor}^{\alpha}_G(H) = G \wedge (\forall \beta < \alpha)(\mathrm{nor}^{\beta}_G(H) \ne G)$] <u>then</u> there is a κ-automorphism group G' with a subgroup H' of cardinality $\le \kappa$ such that $\tau'_{G',H'} = \alpha$ [such that $\mathrm{nor}^{\alpha}_{G'}(H') = G' \wedge (\forall \beta < \alpha)(\mathrm{nor}^{\beta}_{G'}(H') \ne G')$].*

Proof. Easy.

Let G^+, G^- be as in Definition 3.5 and h be a homomorphism from G^+ onto G with kernel G^- and let $H^+ = \{x \in G^+ : h(x) \in H\}$.

So it is easy to check each of the following statements:

$\circledast$ (a) H^+ is a subgroup of G^+

(b) $|H^+| \le |H| \times |G^-| \le \kappa\kappa = \kappa$

(c) G^+ is a κ-automorphism group

(d) $\mathrm{nor}^{\beta}_{G^+}(H^+) = \{x \in G^+ : h(x) \in \mathrm{nor}^{\beta}_G(H)\}$ for every $\beta \le \infty$

(e) $\tau'_{G,H} = \tau'_{G^+,H^+}$

(f) $\mathrm{nor}^{\beta}_G(H) = G$ then $\mathrm{nor}^{\beta}_{G^+}(H^+) = G^+$ for every $\beta \le \infty$.

Together (G^+, H^+) exemplifies the desired conclusion. $\square_{3.6}$

Proof of 3.4. Let G^+ be the automorphism group of $M_{\mathfrak{t}}$ and let G^- be the following subgroup of G^+

$$\{\varkappa \in G^+ : \text{for some } u \in J^{\mathfrak{t}} \text{ we have} \\ u \leq_J v \wedge g \in K_v \Rightarrow \varkappa((v,g)) = (v,g)\}.$$

Easily

$\circledast_1$ G^- is a subgroup of G^+
[Why? As $J^{\mathfrak{t}}$ is linear.]

$\circledast_2$ for every $\varkappa \in G^+$ we can find $\bar{f}^\varkappa = \langle f^\varkappa_u : u \in J^{\mathfrak{t}}\rangle$ such that
- (a) $f^\varkappa_u \in F^{\mathfrak{t}}_u$
- (b) $\varkappa \restriction D^{\mathfrak{t}}_u = f^\varkappa_u$
- (c) $\varkappa \restriction P^{M_{\mathfrak{t}}}_{2,u}$ is $(1,u,f) \mapsto (1,u,f^\varkappa_u,f)$.
  [Why? By Claim 2.17.]

$\circledast_3$ G^- (and also $M_{\mathfrak{t}}$) has cardinality $\leq \kappa$.
[Why? As $|J^{\mathfrak{t}}| \leq \kappa$, it suffices to prove that for each $u \in J^{\mathfrak{t}}$, the subgroup $G^-_u := \{\varkappa \in G^+ : \varkappa \restriction P^{M_{\mathfrak{t}}}_{1,v}$ is the identity when $u \leq_{J[\mathfrak{t}]} v\}$ has cardinality $\leq \kappa$, but this has not more elements as $F^{\mathfrak{t}}_u$ because $\varkappa \mapsto \varkappa \restriction D^{\mathfrak{t}}_u$ is an injective function from G^-_u into $F^{\mathfrak{t}}_u$ and $J^{\mathfrak{t}}$ is linear. As $|F^{\mathfrak{t}}_u| \leq \aleph_0 + |Z^{\mathfrak{t}}_u| = \aleph_0 + |I^{\mathfrak{t}}_u| \leq \kappa$ we are done.]

$\circledast_4$ G^- is a normal subgroup of G^+.
[Why? By its definition, more elaborately
- (a) each G^-_u is a normal subgroup of G^+.
  [Why? As all members of $\mathrm{Aut}(M_{\mathfrak{t}})$ map each $\{v\} \times K_v$ onto itself.]
- (b) $u \leq_{J[\mathfrak{t}]} v \Rightarrow G^-_u \subseteq G^-_v$.
  [Why? Check the definitions.]
- (c) $G^- = \cup\{G^-_u : u \in J^{\mathfrak{t}}\}$.
  [Why? Trivially.]

Together we are done proving $\circledast_4$.]

$\circledast_5$ For $x \in Z^{\mathfrak{s}}_{v_*}$ let $\varkappa_x$ be the following automorphism of $M_{\mathfrak{t}}$, it is defined as in $\circledast_2$ by $\langle f^x_u : u \in J^{\mathfrak{t}}\rangle$ where $f^x_u = \partial^{\mathfrak{s}}_{v_*,x} \restriction D^{\mathfrak{t}}_u$ is from Definition 2.8(4).

$\circledast_6$ For every $x \in Z^{\mathfrak{s}}_{v_*}$, $\varkappa_x$ is a well defined automorphism of $M_{\mathfrak{t}}$.
[Why? Look at the definitions and 2.17.]

The main point is

$\circledast_7$ G^+ is generated by $\{\varkappa_x : x \in Z^{\mathfrak{s}}_{v_*}\} \cup G^-$.

Why? Clearly the set is a set of elements of G^+. So assume $\varkappa \in G^+$ and let $\bar f^\varkappa = \langle f^\varkappa_u : u \in J^{\mathfrak{t}}\rangle$ be as in $\circledast_2$, they are fixed for awhile.

By 2.19 for each $u \in J^{\mathfrak{t}}$ there are $k = k^u$ and $\bar t = \bar t^u \in {}^{k^u}(I^{\mathfrak{t}}_u)$ and $\mathbf{q} = \mathbf{q}^u \in \mathbf{S}^{k^u}$ such that (the "disjoint" as we can replace $\bar t$ by $\bar t\char`^\bar t$ or even $\bar t\char`^\bar t\char`^\ldots\char`^\bar t$ with $|\mathscr{S}^{k^u}|$ copies, the "reduced" by 2.19(3)):

$\boxdot_1$ $f^\varkappa_u = \partial_{u,(\bar t^u,\mathbf{q}^u)}$, i.e., if $v \le_{J[\mathfrak{t}]} u$ then $(\varkappa \equiv) f^\varkappa_u \restriction (\{v\} \times K^{\mathfrak{t}}_v)$ is a multiplication from the left (of the $K^{\mathfrak{t}}_v$-coordinate) by $g^v_{\pi^{\mathfrak{t}}_{v,u}(\bar t^u),\mathbf{q}^u}$, and $\mathbf{q}^u$ is reduced and disjoint, see Definition 2.12(2),(5).

The choices are not necessarily unique, in particular

$\boxdot_2$ if $u^1 \le_{J[\mathfrak{t}]} u^2$ then $(k^{u^2}, \pi_{u^1,u^2}(\bar t^{u^2}), \mathbf{q}^{u^2})$ can serve as $(k^{u^1}, \bar t^{u^1}, \mathbf{q}^{u^1})$.

Also

$\boxdot_3$ the set of possible $(k^u, \mathbf{q}^u)$ is countable.

As $J^{\mathfrak{t}}$ is $\aleph_1$-directed and linear

$\boxdot_4$ for some pair $(k^*, \mathbf{q}^*)$ the set $\{u \in J^{\mathfrak{t}} : k^u = k^* \text{ and } \mathbf{q}^u = \mathbf{q}^*\}$ is cofinal in $J^{\mathfrak{t}}$.

Together, without loss of generality for some $k^*, \mathbf{q}$

$\boxdot_5$ $k^u = k^*$ and $\mathbf{q}^u = \mathbf{q}$ for every $u \in J^{\mathfrak{t}}$.

Let E be an ultrafilter on $J^{\mathfrak{t}}$ such that $u \in J^{\mathfrak{t}} \Rightarrow \{v : u \le_{J[\mathfrak{t}]} v\} \in E$. Such an E exists as $J^{\mathfrak{t}}$ is linear. For each $u \in J^{\mathfrak{t}}$ there are $A_u, p_u, w(u)$ such that

$\boxdot_6$ (a) $A_u \in E$ and
(b) $p_u \in \mathscr{S}^{k^*}$
(c) if $v \in A_u$ then $u \le_{J[\mathfrak{t}]} v$ and $p_u = \operatorname{tp}_{\mathrm{qf}}(\pi^{\mathfrak{t}}_{u,v}(\bar t^v), \emptyset, I_u)$
(d) $w(u) \in A_u$.

For $p \in \mathscr{S}^{k^*}$ let

$\boxdot_7$ (a) $Y_p = \{u \in J^{\mathfrak{t}} : p_u = p\}$
(b) $\bar{s}^{u,v} = \pi^{\mathfrak{t}}_{u,v}(\bar{t}^v) \restriction \text{supp}(\mathbf{q}(p_u))$ for $u \in J^{\mathfrak{t}}, v \in A_u$
(c) $\bar{s}^u = \bar{s}^{u,w(u)}$.

So

$\boxdot_8$ $\langle Y_p : p \in \mathscr{S}^{k^*}\rangle$ is a partition of $J^{\mathfrak{t}}$.

Fix $p \in \mathscr{S}^{k^*}$ for awhile so for each $u \in Y_p$ and $v \in A_u$ by $\boxdot_1$, $\varkappa \restriction (\{u\} \times K^{\mathfrak{t}}_u)$ is multiplication from the left by $g^u_{\pi^{\mathfrak{t}}_{u,v}(\bar{t}^v),\mathbf{q}}$ (it was $\mathbf{q}^v$ but we have already agreed that $\mathbf{q}^v = \mathbf{q}$). But $p = \text{tp}_{\text{qf}}(\pi^{\mathfrak{t}}_{u,v}(\bar{t}^v), \emptyset, I_u)$ as $u \in Y_p, v \in A_u$ and so by Definition 2.12(5) we know that $g^u_{\pi^{\mathfrak{t}}_{u,v}(\bar{t}^v),\mathbf{q}}$ is $g^u_{\pi^{\mathfrak{t}}_{u,v}(\bar{t}^v),\mathbf{q}(p)}$.

Now $\mathbf{q}(p) \in \Lambda^2_{k^*}$ so $\mathbf{q}(p) = \langle \rho^p_0, \rho^p_1, \dots, \rho^p_{i(p)-1}\rangle$ and recall

$$g^u_{\pi^{\mathfrak{t}}_{u,v}(\bar{t}^v),\mathbf{q}(p)} \text{ is } g_{\bar{t},\rho^p_0} h_{g_{\bar{t},\rho^p_1} G^{<0}_u} \dots \text{ with } \bar{t} = \pi^{\mathfrak{t}}_{u,v}(\bar{t}^v);$$

so it depends only on $\pi^{\mathfrak{t}}_{u,v}(\bar{t}^v) \restriction \text{supp}(\mathbf{q}(p))$ only.

Now consider any two members v_1, v_2 of A_u (so they are above u) comparing the two expressions for $\varkappa \restriction (\{u\} \times K^{\mathfrak{t}}_u)$ one coming from v_1 the second from v_2 we conclude that $g^u_{\pi^{\mathfrak{t}}_{u,v_1}(\bar{t}^{v_1}),\mathbf{q}(p)} = g^u_{\pi^{\mathfrak{t}}_{u,v_2}(\bar{t}^{v_2}),\mathbf{q}(p)}$. As $\mathbf{q}$ is reduced also $\mathbf{q}(p)$ is p-reduced hence by 2.7(3) we conclude that

$\boxdot_9$ if ($p \in \mathscr{S}^{k^*}, u \in Y_p \subseteq J^{\mathfrak{t}}$ and) $v_1, v_2 \in A_u$ then $\pi^{\mathfrak{t}}_{u,v_1}(\bar{t}^{v_1}) \restriction \text{supp}(\mathbf{q}(p))$ is a permutation of $\pi^{\mathfrak{t}}_{u,v_2}(\bar{t}^{v_2}) \restriction \text{supp}(\mathbf{q}(p))$
this means

$\boxdot_{10}$ if $u \in J^{\mathfrak{t}}$ and $v_1, v_2 \in A_u$ then $\bar{s}^{u,v_1}$ is a permutation of $\bar{s}^{u,v_2}$.

Hence for each $u \in J^{\mathfrak{t}}$

$\boxdot_{11}$ if $v \in A_u$ then $\bar{s}^{u,v}$ is a permutation of $\bar{s}^u = \bar{s}^{u,w(u)}$.

As there are only finitely many permutations of $\bar{s}^u$, there are $\omega(u), A'_u$ such that

$\boxdot_{12}$ for $u \in J^{\mathfrak{t}}$:
(a) $A'_u \in E$
(b) $A'_u \subseteq A_u$
(c) $\bar{s}^u = \bar{s}^{u,v}$ for every $v \in A'_u$.

Now

$\boxdot_{13}$ if $p \in \mathscr{S}^{k^*}$ and $u_1 \le_{J[\mathfrak{t}]} u_2$ are from Y_p then $\pi^{\mathfrak{t}}_{u_1,u_2}(\bar{s}^{u_2}) = \bar{s}^{u_1}$.

[Why? As E is an ultrafilter on $J^{\mathfrak{t}}$ and $A'_{u_1}, A'_{u_2} \in E$ we can find $v \in A'_{u_1} \cap A'_{u_2}$. So for $\ell = 1,2$ we have $\bar{s}^{u_\ell} = \pi^{\mathfrak{t}}_{u_\ell,v}(\bar{t}^v) \restriction \operatorname{supp}(\mathbf{q}(p)) = \pi^{\mathfrak{t}}_{u_\ell,v}(\bar{t}^v \restriction \operatorname{supp}(\mathbf{q}(p)))$.

As $\pi^{\mathfrak{t}}_{u_1,v} = \pi^{\mathfrak{t}}_{u_1,u_2} \circ \pi^{\mathfrak{t}}_{u_2,v}$ we conclude $\bar{s}^{u_1} = \pi^{\mathfrak{t}}_{u_1,u_2}(\bar{s}^{u_2})$ is as required.]

Let $\mathscr{S}' = \{p \in \mathscr{S}^{k^*} : Y_p$ is an unbound subset of $J^{\mathfrak{t}}\}$, so for some $u_* \in J^{\mathfrak{t}}$ we have

$\boxdot_{14}$ $J^{\mathfrak{t}}_{\ge u_*} \subseteq \cup\{Y_p : p \in \mathscr{S}'\}$.

Also without loss of generality

$\boxdot_{15}$ $k^* = k^*_1 + k^*_2$ and $\{0,\dots,k^*_1 - 1\} = \cup\{\operatorname{supp}(\mathbf{q}(p)) : p \in \mathscr{S}'\}$

$\boxdot_{16}$ for $p \in \mathscr{S}'$ and $\ell \in \operatorname{supp}(\mathbf{q}(p))$, so $s^u_\ell = (\bar{s}^u)_\ell$ is well defined for $u \in Y_p$, there is a unique $t_\ell \in I^{\mathfrak{s}}_{v_*}$ such that:

$$u \in Y_p \Rightarrow \pi^{\mathfrak{s}}_{u,v_*}(t_\ell) = s^u_\ell.$$

[Why? By clause (e) of Definition 3.1, $\boxdot_{13}$ and the linearity of $J^{\mathfrak{t}}$.]

Next we can find $\bar{t}$ such that

$\boxdot_{17}$ (a) $\bar{t} = \langle t_\ell : \ell < k^*_1\rangle$

(b) if $p \in \mathscr{S}'$ and $\ell \in \operatorname{supp}(\mathbf{q}(p))$ then $t_\ell \in I^{\mathfrak{s}}_{v_*}$ is as in $\boxdot_{16}$.

[Why? For $\ell \in \cup\{\operatorname{supp}(\mathbf{q}(p)) : p \in \mathscr{S}'\}$ use $\boxdot_{16}$. As $\mathbf{q}$ is disjoint (see Definition 2.12(2)) there is no case of "double definition".]

By clause (d) of Definition 3.1, possibly increasing u_*,

$\boxdot_{18}$ $p^* = \operatorname{tp}_{\rm qf}(\pi^{\mathfrak{s}}_{u,v_*}(\bar{t}), \emptyset, I_u)$ for every $u \in J^{\mathfrak{t}}_{\ge u_*}$.

$\boxdot_{19}$ let $\mathscr{E}$ be the following equivalence relation on $\mathscr{S}^{k^*}$, $p_1 \mathscr{E} p_2 \Leftrightarrow \mathbf{q}(p_1)\mathscr{E}^1_{k^*_1, p\restriction k^*_1}\mathbf{q}(p_2)$; note they are actually from $\mathscr{S}^{k^*_1}$ and so "$\mathscr{E}^1_{k^*_1,p\restriction k^*_1}$-equivalent" is meaningful, see Definition 2.3(4)

$\boxdot_{20}$ let $\bar{e} = \langle e_u : u \in J^{\mathfrak{t}}_{\ge u_*}\rangle$ be defined by $e_u = p_u/E$

$\boxdot_{21}$ $E, \bar{t}, \bar{e}, \langle \pi^{\mathfrak{t}}_{u,w(u)}(\bar{t}^{w(u)}) : u \in J^{\mathfrak{t}}_{\ge u_*}\rangle$ satisfies the demands $(f)(\alpha) - (\zeta)$ from Definition 3.2.

[Why? Check.]

Recall $p^* = \operatorname{tp}(\bar{t}, \emptyset, I^{\mathfrak{s}}_{v_*})$ here so let $\bar{s} \in {}^{(k^*_2)}(I^{\mathfrak{s}}_{v_*})$ be as guaranteed to exist by Definition 3.2. Let $\bar{t}^{v^*} := \bar{t}\hat{\ }\bar{s}$. So possibly increasing $u_* \in J^{\mathfrak{t}}$ for some p^* we have

$\boxdot_{22}$ if $u \in J^{\mathfrak{t}}_{\ge u_*}$ then $p^* = \operatorname{tp}(\pi^{\mathfrak{s}}_{u,v_*}(\bar{t}\hat{\ }\bar{s}), \emptyset, I^{\mathfrak{s}}_u) = \operatorname{tp}(\bar{t}\hat{\ }\bar{s}, \emptyset, I^{\mathfrak{s}}_{v_*})$.

Let

$\boxdot_{23}$ (a) $\varrho^* = \mathbf{q}(p^*)$ so $\varrho^* \in \Lambda^2_{k^*_1,p^*}$ and let $\varrho^* = \langle \rho_\ell : \ell < \ell(*)\rangle$
(b) $\bar t_u = \pi^{\mathfrak{s}}_{u,v_*}(\bar t)$ for $u \in J^{\mathfrak{t}}$
(c) let $z_u = z^{u,s}_{\bar t_u,\varrho} \in Z^{1,\mathfrak{s}}_u$ (see Definition 2.4(5A))
(d) let $f_u = \partial^{\mathfrak{s}}_{u,z_u} \in F^{\mathfrak{s}}_u$; (this is not the same as $f^{\varkappa}_u$!).

Now

$\boxdot_{24}$ for $u_1 \le_{J[\mathfrak{t}]} u_2$ we have $f_{u_1} \subseteq f_{u_2}$.

[Why? Check.]

$\boxdot_{25}$ $\varkappa_{\bar f}$ is a finite product of members of $\{\varkappa_x : x \in Z^{\mathfrak{s}}_{v_*}\}$.

[Why? Recall $\varkappa_x$ for $x \in Z^{\mathfrak{s}}_{v^*}$ is from $\circledast_5$. Now use $\boxdot_{23}$.]
Lastly

$\boxdot_{26}$ $(\varkappa^{-1}_{\bar f})\varkappa \in G^+ =$ Aut$(M_{\mathfrak{t}})$ is the identity on $P^{M_{\mathfrak{t}}}_u$ whenever $u \in J^{\mathfrak{t}}_{\ge u_*}$.

[Why? By $\boxdot_{24}$ and our choices.]

$\boxdot_{25}$ $(\varkappa_{\bar f}) \in (G^-_{u_*} \subseteq) G^-$.

[Why? By $\boxdot_{25}$ and the definition of $(G_{u^*}$ and) G^-.]

$\boxdot_{28}$ $\varkappa$ is the product (in G^+) of $\varkappa_{\bar f} \in G^-$ and $(\varkappa^{-1}_f)\varkappa \in \langle\{\varkappa_x : x \in Z^{\mathfrak{s}}_{v_*}\}\rangle$.

[Why? $\boxdot_{25} + \boxdot_{27}$ this is clear.]
As $\varkappa$ was any a member of G^+ we are done proving $\circledast_7$.

$\circledast_8$ there is a homomorphism $\mathbf{h}$ from $K^{\mathfrak{s}}_{v_*}$ onto G^+/G^- which maps g_x to $\varkappa_x G^-$ for $x \in Z^{\mathfrak{s}}_{v_*}$.

[Why? By $\circledast_7$ there is at most one such homomorphism and if it exists it is onto.
So it is enough to show that for any group term, σ if $K^{\mathfrak{s}}_{v_*}$ satisfies $K_{v_*} \models$ "$\sigma(g_{x_1},\dots,g_{x_{k-1}}) = e$" then $\sigma(\varkappa_{x_0},\dots,\varkappa_{x_{k-1}}) \in G^-$. Let $\langle t_\ell : \ell < \ell^*\rangle$ list $\cup\{\text{his}(x_\ell) : \ell < k\} \subseteq I^{\mathfrak{s}}_{v_*}$ and let $u_* \in J^{\mathfrak{t}}$ be such that: if $u_* \le_{J[\mathfrak{t}]} u$ and $\ell(1),\ell(2) < \ell^*$ we have $I^{\mathfrak{s}}_{v_*} \models t_{\ell(1)} <_I t_{\ell(2)}$ iff $I^{\mathfrak{t}}_u \models \pi_{u,v_*}(t_{\ell(1)}) < \pi_{u,v^*}(t_{\ell(2)})$ and similarly for equality, see clause (d) of Definition 3.1.
Let $t_{u,\ell} = \pi_{u,v_*}(t_\ell), x_{u,\ell} = \hat\pi_{u,v_*}(x_\ell)$. By the definition of G^- it is enough to show that: if $u_* \le_{J[\mathfrak{t}]} u$ then $K_u \models$ "$\sigma(g_{x_{u,0}},\dots,g_{x_{u,k_1}}) = e_{K_u}$". By the analysis in 1.6 and §2 (i.e., twisted product) this should be clear.]

$\circledast_9$ $\varkappa^*$ is one to one.

[Why? By part of the analysis as for $\circledast_7$.]

By $\circledast_8 + \circledast_9$ we are done.

The problem is in verifying clause (ζ) of (f) of Definition 3.2. Now if $u \in J^{\mathfrak{t}}_{\geq u_*}$ we can find $w_p[u] \in \mathfrak{t} \geq v$ for each $p \in \mathscr{S}'$ such that

$\odot$ (α) $v \leq_{J[\mathfrak{t}]} w_p[u] \in Y_p$

(β) $\bar{t}^{w_p[u]} \restriction \operatorname{supp}(\mathbf{q}(p)) = \pi^{\mathfrak{s}}_{w_p[u]}(\bar{t} \restriction \operatorname{supp}(\mathbf{q}(p))$.

Let $w[p] \in \cap\{A'_{w_p[u]} : p \in \mathscr{S}'\}$ be a $\leq_{J[\mathfrak{t}]}$-common upper bound of $\{w_p[u] : p \in S'\} \cup \{u\}$.

Lastly, let $\bar{s}_u = (\pi^{\mathfrak{t}}_{u,w[u]}(\bar{t}^{w[u]})) \restriction [k^*_1, k^*)$. $\square_{3.4}$

Main Claim 3.4, p.40

<u>Once more on $\boxdot_{21}$</u>:

I do not see why the definition of $\mathscr{E}$ and $\bar{s}^{u,v}$ given on pg.40A has property 3.2(ζ). Even worse: I momentarily have some doubts that this works.

<u>Try on a counter-example</u>:

Let $p_j \in \mathscr{S}', j \in \{1,2\}$ with $p_1 \neq p_2$. Thus, in particular, $\sup(\mathbf{q}(p_1)) \cap \operatorname{supp}(\mathbf{q}(p_2)) = \emptyset$. let $i(j) \in \operatorname{supp}(\mathbf{q}(p_j))$ be chosen.

There seems to be no argument preventing the following to happen: for every $p \in \mathscr{S}'$ and every $\bar{t}'$ realizing p the elements $\bar{t}'(i(1))$ and $\bar{t}'(i(2))$ are comparable, i.e. (see Definition 2.3)

$$\forall p \in \mathscr{S}' : \{(0, i(1), i(2)), (2, i(1), i(2))\} \cap p \neq \emptyset,$$

while for the constructed limit $\bar{t}$ in $\boxdot_{17}$ holds

$$(3, i(1), i(2)) \in p^*$$

(see $\boxdot_{18}$), i.e. $\bar{t}(i(1))$ and $\bar{t}(i(2))$ are incomparable.

The consequence for 3.2(ζ) is

$$\begin{aligned} \operatorname{tp}_{\rm qf}(\bar{t}^u {}^\frown \bar{s}^{u,v}, \emptyset, I^{\mathfrak{t}}_u) &= \operatorname{tp}_{\rm qf}(\pi^{\mathfrak{s}}_{u,v_*}(\bar{t}) {}^\frown \pi^{\mathfrak{t}}_{u,v}(\bar{s}_v, \emptyset, I^{\mathfrak{t}}_u) \\ &\Rightarrow p^* =_{\boxdot_{18}} \operatorname{tp}_{\rm qf}(\pi^{\mathfrak{s}}_{u,v^*}(\bar{t}), \emptyset, I^{\mathfrak{t}}_u) \\ &= \operatorname{tp}_{\rm qf}(\bar{t}^u {}^\frown \bar{s}^{u,v}, \emptyset, I^{\mathfrak{t}}_u) \\ &\Rightarrow_{(1)<(2)} \operatorname{tp}_{\rm qf}(\bar{t}^u {}^\frown \bar{s}^{u,v}, \emptyset, I^{\mathfrak{t}}_u) \notin \mathscr{S}' \end{aligned}$$

while $p_u \in_{\boxdot_{14}} \mathscr{S}'$.

In particular $\text{tp}_{\text{qf}}(\bar t^u \char`\^ \bar s^{u,v}, \emptyset, I^{\text{t}}_u) \notin_{40A} e_u =_{\boxdot_{20}} p_u/\mathscr{E} \subseteq \mathscr{S}'$ (Contradiction!)

[For me the main obstacle here seems to be $Y_{p_1} \cap Y_{p_2} =_{\boxdot_8} \emptyset$.]

Saharon please: make me see and give the missing argument!

Otherwise FIX! (Maybe 3.1 and 3.2 need additional properties?)

3.7 Theorem. *Assume*

(a) $\aleph_0 < \text{cf}(\theta) = \theta \le \kappa$

(b) $\mathscr{F}_\alpha \subseteq {}^\alpha\kappa$ *for* $\alpha < \theta$ *has cardinality* $\le \kappa$ *(also* $\mathscr{F}_\alpha \subseteq {}^\alpha\beta$ *for some* $\beta < \kappa^+$ *is O.K.)*

(c) $\mathscr{F} = \{f \in {}^\theta\kappa : f \restriction \alpha \in \mathscr{F}_\alpha$ *for every* $\alpha < \theta\}$

(d) $\gamma = \text{rk}(\mathscr{F}, <_{J^{\text{bd}}_\theta})$, *necessarily* $< \infty$ *so* $< (\kappa^\theta)^+$

(e) *if* $f_1, f_2 \in \mathscr{F}$, *then* $f_1 <_{J^{\text{bd}}_\theta} f_2$ *or* $f_2 <_{J^{\text{bd}}_\theta} f_1$ *or* $f_2 =_{J^{\text{bd}}_\theta} f_1$ *(follows from (f))*

(f) *for stationarily many* $\delta < \theta$ *we have: if* $f_1, f_2 \in \mathscr{F}_\delta$, *then for some* $\alpha < \delta$ *we have* $\beta \in (\alpha, \delta) \Rightarrow (f_1(\beta) < f_2(\beta) \Leftrightarrow f_1(\alpha) < f_2(\alpha))$.

Then $\tau^{\text{atw}}_\kappa \ge \tau^{\text{nlg}}_\kappa \ge \tau^{\text{nlf}}_\kappa > \gamma$ *(on* τ^{nlf}_κ *see Definition 0.3(4)).*

3.8 Theorem. *We can in Theorem 3.7 weaken clause (f) to*

(f)′ (α) $S \subseteq \theta$ *is a stationary set consisting of limit ordinals*

(β) D *is a normal filter on* θ

(γ) $S \in D$

(δ) $\bar J = \langle J_\delta : \delta \in S\rangle$

(ε) J_δ *is an ideal on* δ *extending* J^{bd}_δ *for* $\delta \in S$

(ζ) *if* $S' \subseteq S$ *is stationary,* $S' \in D^+$ *and* $w_\delta \in J_\delta$ *for* $\delta \in S'$, *then* $\cup\{\delta \backslash w_\delta : \delta \in S'\}$ *contains an end segment of* θ

(η) *if* $\delta \in S$ *and* $f_1, f_2 \in \mathscr{F}$, *then* $f_1 \restriction \delta <_{J_\delta} f_2 \restriction \delta$ *or* $f_2 \restriction \delta <_{J_\delta} f_1 \restriction \delta$ *or* $f_1 \restriction \delta =_{J_\delta} f_2 \restriction \delta$

Remark. 1) We can justify (f)′ by pcf theory quotation, see below.

2) We should prove that the p.o.w.i.s. being existential holds.

Note that in proving 3.7, 3.8 the main point is the "existential limit". This proof has affinity to the first step in the elimination of quantifiers in the theory of $(\omega, <)$.

For this it is better if $I_\theta = (\mathscr{F}, <_{J^{\rm bd}_\theta})$ has many cases of existence. Toward this we "padded it" in $(*)_0$ of the proof - take care of successors $(f \in \mathscr{F} \Rightarrow f+1 \in \mathscr{F})$, have zero $(0_\theta \in \mathscr{F})$ without losing the properties we have.
2) The demand of 3.7 may seem very strong, but by pcf theory it is natural.

3.9 Observation. 1) Theorem 3.8 implies Theorem 3.7.
2) If $(a)-(d)$ of 3.7 holds, <u>then</u> $(f) \Rightarrow (f)'$.
3) If $(a)-(d)$ of 3.7 holds, <u>then</u> $(f)' \Rightarrow (e)$.

Proof. 1) By 2).
2) Let

$$\begin{aligned} S := \{\delta < \theta : &\delta \text{ is a limit ordinal and if } f_1, f_2 \in \mathscr{F}_\delta, \\ &\text{then for some } \alpha < \delta \text{ we have } \beta \in (\alpha, \delta) \Rightarrow \\ &(f_1(\beta) < f_2(\beta) \Leftrightarrow f_1(\alpha) < f_2(\alpha))\}. \end{aligned}$$

By (f) we know that S is a stationary subset of θ. Let $\mathscr{D}_\theta$ be the club filter on θ and $D := \mathscr{D}_\theta + S$, it is a normal filter on θ and $S \in D$. So sub-clauses $(\alpha), (\beta), (\gamma)$ of $(f)'$ hold.

Let $J_\delta = J^{\rm bd}_\delta$ for $\delta \in S$ so $\bar J = \langle J_\delta : \delta \in S\rangle$ satisfies sub-clauses $(\delta), (\varepsilon)$ of $(f)'$. To prove (ζ) assume $S' \subseteq S$ stationary, $S' \in D^+$ and $w_\delta \in J_\delta$ for $\delta \in S'$. Then $\sup(w_\delta) < \delta$ and S' is a stationary subset of θ hence by Fodor's lemma for some $\beta(*) < \theta$ the set $S'' = \{\delta \in S' : \sup(w_\delta) = \beta(*)\}$ is a stationary subset of θ and so $[\beta(*), \theta)$ is an end segment of θ and is equal to $\cup\{[\beta(*), \delta) : \delta \in S''\}$ which is included in $\cup\{\delta \backslash w_\delta : \delta \in S'\}$, as required in (ζ) from $(f)'$, so sub-clause (ζ) really holds.

To prove sub-clause (η) of clause $(f)'$ note that what it says is what is said in (f).
3) Should be clear. Given $f_1, f_2 \in \mathscr{F}$; by sub-clause (η) of $(f)'$ for each $\delta \in S$ there are $w_\delta \in J_\delta$ and $\ell_\delta < 3$ such that $(\ell_\delta = 0 \wedge \alpha \in \delta \backslash w_\delta) \Rightarrow f_1(\alpha) < f_2(\alpha)$ and $(\ell_\delta = 1 \wedge \alpha \in \delta \backslash w_\delta) \Rightarrow f_1(\alpha) = f_2(\alpha)$ and $(\ell_\delta = 2 \wedge \alpha \in \delta \backslash w_\delta) \Rightarrow f_1(\alpha) > f_2(\alpha)$. So for some $\ell < 3$ the set $S' := \{\delta \in S : \ell_\delta = \ell\}$ is stationary ($S' \in D^+$ without loss of generality), hence $\cup\{\delta \backslash w_\delta : \delta \in S'\}$ includes an end segment of θ and we are easily done. $\square_{3.9}$

Proof of 3.8. Without loss of generality

$(*)_0$ (a) $(\forall f \in \mathscr{F})(\exists^\infty g \in \mathscr{F})\big(f \restriction [1, \theta) = g \restriction [1, \theta)\big)$;

moreover for $f \in \mathscr{F}$ we have
$\omega = \{g(0) : g \in \mathscr{F}$ and $g \restriction [1,\theta) = f \restriction [1,\theta)\}$

(b) $\alpha < \beta < \theta \Rightarrow \mathscr{F}_\alpha = \{f \restriction \alpha : f \in \mathscr{F}_\beta\}$; moreover $\alpha < \theta \Rightarrow \mathscr{F}_\alpha = \{f \restriction \alpha : f \in \mathscr{F}\}$

(c) if $f \in \mathscr{F}$, <u>then</u> $f + 1 \in \mathscr{F}$

(d) the $f \in {}^\theta\{0\}$, the constantly zero function, belongs to $\mathscr{F}$.

[Why? Let $\mathscr{F}' = \{f \in {}^\theta\kappa$: for some $n < \omega$ we have $(\forall 0 < \alpha < \theta)(f(\alpha) = u) \wedge f(0) < \omega$ or for some $f' \in \mathscr{F}$ and $n < \omega$ we have $(\forall 0 < \alpha < \theta)(f(\alpha) = \omega(1+f'(\alpha))+n) \wedge f(0) < \omega\}$ and for $\alpha < \theta$, replace $\mathscr{F}_\alpha$ by $\mathscr{F}'_\alpha = \{f \restriction \alpha : f \in \mathscr{F}'\}$. Now check that $(a) - (e), (f)'$ of the assumption still holds.]

We define $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ as follows:

$(*)_1$ (a) $J = (\theta + 1, <)$

(b)(α) let $I_\theta = (\mathscr{F}, <_{J^{\rm bd}_\theta})$ and

(β) $I_\alpha = (\mathscr{F}_{1+\alpha+1}, <_{\alpha+1})$ for $\alpha < \theta$ where

$$f_1 <_{\alpha+1} f_2 \Leftrightarrow f_1(1+\alpha) < f_2(1+\alpha)$$

(c) for $\alpha \le \beta < \theta + 1$ let $\pi_{\alpha,\beta} : I_\beta \to I_\alpha$ be

$$\pi_{\alpha,\beta}(f) = f \restriction (1 + \alpha + 1).$$

Note that

$(*)_2$ I_α is explicitly non-trivial for all $\alpha \in J$ (see Definition 1.1(7)).

[Why? By $(*)_0(a)$ and the choice of $<_{I_\alpha}$ in $(*)_1(b)$.]

$(*)_3$ $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ is a p.o.w.i.s. even nice.

$(*)_4$ $\mathfrak{s}$ is a limit of $\mathfrak{t} := \mathfrak{s} \restriction \theta = ((\theta, <), \bar{I} \restriction \theta, \bar{\pi} \restriction \theta)$.
[Why? Note that clause (d) of Definition 3.1 holds by clause (e) of Theorem 3.7. Easy to check the other clauses.]

$(*)_5$ $\mathfrak{t}$ is a nice κ-p.o.w.i.s.
[Why? This follows from clause (a),(b) of Theorem 3.7.]

Now $K^{\mathfrak{s}}_\theta$ is an almost κ-automorphism group by Claim 3.4, the "existential limit" holds by $(*)_6$ below (note: J is linear). Now ${\rm rk}^{<\infty}(I^{\mathfrak{s}}_\theta) = {\rm rk}(I^{\mathfrak{s}}_\theta) = \gamma$ and $H^{\mathfrak{s}}_\theta$ is a subgroup of $K^{\mathfrak{s}}_\theta$ of cardinality $2 \le \kappa$.

Combining Claim 1.9 and Claim 2.10 we have

$$\tau^{\rm nlg}_{K^{\mathfrak{s}}_\theta, H^{\mathfrak{s}}_\theta} = {\rm rk}^{<\infty}(I^{\mathfrak{s}}_\theta) = \gamma$$

with ${\rm nor}^\infty_{K^{\mathfrak{s}}_\theta}(H^{\mathfrak{s}}_\theta) = K^{\mathfrak{s}}_\theta$ and thus $\tau^{\rm atw}_\kappa \geq \tau^{\rm nlg}_\kappa \geq \tau^{\rm nlf}_\kappa > \tau^{\rm nlg}_{K^{\mathfrak{s}}_\theta, H^{\mathfrak{s}}_\theta} = \gamma$ by 0.8 and Claim 3.6.

We still have to check

$(*)_6$ "$\mathfrak{s}$ is an existential limit of $\mathfrak{t}$", see Definition 3.2.

That is we have to prove clause (f) of 3.2, so we should prove its conclusion, assuming its assumption which means in our case

$\circledast_1$ (a) $k = k_1 + k_2, \mathscr{E}$ is an equivalence relation on $\mathscr{S}^k$
 (b) $\bar f \in {}^{k_1}\mathscr{F}$ and $\alpha(*) < \theta$
 (c) $\bar e = \langle e_\alpha : \alpha \in [\alpha(*), \theta)\rangle$ is such that $e_\alpha \in \mathscr{S}^k/\mathscr{E}$
 (d) $\langle \bar g^\alpha : \alpha \in [\alpha(*), \theta)\rangle$ is such that $\bar g^\alpha \in {}^{k_2}(\mathscr{F}_{1+\alpha+1})$
 (e) if $\alpha(*) \leq \alpha \leq \beta < \theta$ then:

e_α is the $\mathscr{E}$-equivalence class of ${\rm tp}_{\rm qf}(\langle f_\ell \restriction (1+\alpha+1) : \ell < k_1\rangle \hat{} \langle g^\beta_\ell \restriction (1+\alpha+1) : \ell < k_2\rangle, \emptyset, I_\alpha)$.

Without loss of generality [recalling clause (e) of Theorem 3.7 and $(*)_0(c)$]

$\circledast_2$ (f) $\langle f_\ell : \ell < k_1\rangle$ is $\leq_{J^{\rm bd}_\theta}$-increasing
 (g) f_0 is constantly zero
 (h) for each $\ell < k_1 - 1$ we have: $f_{\ell+1} = f_\ell \mod J^{\rm bd}_\theta$ or $f_{\ell+1} = f_\ell + 1 \mod J^{\rm bd}_\theta$ or $f_\ell + \omega \leq f_{\ell+1} \mod J^{\rm bd}_\theta$
 (i) $\langle f_\ell : \ell < k_1\rangle$ is without repetition
 (j) $\langle f_\ell(0) : \ell < k_1\rangle$ is without repetition.

Possibly increasing $\alpha(*) < \theta$ without loss of generality

$\circledast_3$ if $\alpha \in [\alpha(*), \theta)$ and $\ell_1, \ell_2 < k_1$ then $f_{\ell_1}(\alpha) \leq f_{\ell_2}(\alpha) \Leftrightarrow f_{\ell_1}(\alpha(*)) \leq f_{\ell_2}(\alpha(*))$.

Hence by clause (f) of $\circledast_2$

$\circledast_4$ $\langle f_\ell(\alpha(*)) : \ell < k_1\rangle$ is non-decreasing.

For notational simplicity

$\circledast_5$ (a) replace $\bar g^\delta(\delta \in [\alpha(*), \theta))$ by $\langle g^\delta_\ell : \ell < k\rangle := \langle f_\ell \restriction (1+\delta+1) : \ell < k_1\rangle \hat{} \bar g^\delta$
 (b) for $\ell_1, \ell_2 < k$ let $g^\delta_{\ell_1} = g^\delta_{\ell_2} \Leftrightarrow g^\delta_{\ell_1}(0) = g^\delta_{\ell_2}(0)$ without loss of generality.

Next for some p^*

$\circledast_6$ $p^* \in \mathscr{S}^k$ and for some stationary $S' \subseteq S$ from D^+, for every $\delta \in S'$ for the J_δ-majority of $\alpha < \delta$, say $\alpha \in \delta \backslash w_\delta, w_\delta \in J_\delta$, we have $p^* = \text{tp}_{\text{qf}}(\langle g^\delta_\ell \restriction (1+\alpha+1) : \ell < k_1\rangle, \emptyset, I_\alpha)$. Without loss of generality $S' \subseteq (\alpha(*), \theta)$ and $(0, \alpha(*)) \subseteq \omega_\delta$.

[Why? By sub-clause (η) of clause $(f)'$, as $J^{\text{bd}}_\delta \subseteq J_\delta$ is an ideal (applied to $(g^\delta_{\ell_1}, g^\delta_{\ell_2})$ for every $\ell_1, \ell_2 < k$) for each $\delta \in S$ ($S \subseteq (\alpha(*), \theta)$ without loss of generality) we can choose $w_\delta \in J_\delta$ and $q_\delta \in \mathscr{S}^k$ such that for every $\alpha \in \delta \backslash w_\delta$ we have $\text{tp}_{\text{qf}}(\langle g^\delta_\ell \restriction (1+\alpha+1) : \ell < k\rangle, \emptyset, I_\alpha)$ is equal to q_δ. For each $p \in \mathscr{S}^k$ let $S_p = \{\delta \in S : q_\delta = p\}$. So $S = \cup\{S_p : p \in \mathscr{S}^k\}$, hence for some p we have S_p stationary ($S_p \in D^+$ without loss of generality). So let $S' = S_p, p^* = p$.]

So considering the way $\bar{g}^\delta$ was defined by $\circledast_5$

$\circledast_7$ there are $\mathscr{E}^*_1, \mathscr{E}^*_2, <_*$ such that
- (a) $\mathscr{E}^*_1$ is an equivalence relation on $k = \{0, \ldots, k-1\}$
- (b) $\mathscr{E}^*_2$ is an equivalence relation on k refining $\mathscr{E}^*_1$
- (c) $<_*$ linearly orders $k/\mathscr{E}^*_1$
- (d) if $\delta \in S', \alpha \in \delta \backslash w_\delta$ so $p^* = \text{tp}_{\text{qf}}(\langle g^\delta_\ell \restriction (1+\alpha+1) : \ell < k\rangle, \emptyset, I_\alpha)$ then:
  - (α) $\ell_1 \mathscr{E}^*_1 \ell_2$ iff $g^\delta_{\ell_1}(1+\alpha) = g^\delta_{\ell_2}(1+\alpha)$
  - (β) $\ell_1 \mathscr{E}^*_2 \ell_2$ iff $g^\delta_{\ell_1} \restriction (1+\alpha+1) = g^\delta_{\ell_2} \restriction (1+\alpha+1)$
  - (γ) $(\ell_1/\mathscr{E}^*_1) <_* (\ell_2/\mathscr{E}^*_1)$ iff $g^\delta_{\ell_1}(1+\alpha) < g^\delta_{\ell_2}(1+\alpha)$.

Let $\langle u_0, \ldots, u_{m-1}\rangle$ list the $\mathscr{E}^*_1$-equivalence classes in $<_*$-increasing order. Necessary $0 \in u_0$.

Using $(f')(\zeta)$ on $\circledast_6$ let be $\alpha^* \in S'$ with $[\alpha^*, \theta) \subseteq \cup\{\delta \backslash \omega_\delta : \delta \in S'\}$. Thus in particular $p^* \in e_\alpha$ for all $\alpha \in [\alpha^*, \theta)$ by $\circledast_1(e)$. We now define $g_\ell \in {}^\theta\kappa$ for $\ell < k$ as follows: necessarily for a unique $i = i(\ell), \ell \in u_i$ and let $i_1 = i_1(\ell) \le i$ be maximal such that $u_{i_1} \cap \{0, \ldots, k_1 - 1\} \ne \emptyset, j_2 = j_2(\ell) = \min(\{u_{i_1} \cap \{0, \ldots, k_1 - 1\})$. It is well defined as necessary $0 \in u_0$ because f_0 is constantly zero. Now we let

$\boxdot_0$ $g_\ell = (g^{\alpha_*}_\ell \restriction \{0\}) \cup ((f_{j_2} + (i - i_1)) \restriction [1, \theta))$.

Now

$\boxdot_1$ if $\ell < k_1$ then $g_\ell = f_\ell$
[Why? Check the definition $g^{\alpha_*}_\ell(0) = f_\ell(0)$ as $g^{\alpha_*}_\ell = f_\ell$.]

$\boxdot_2$ $g_\ell \in \mathscr{F}$ for $\ell < k$
[Why? As $f_{j_2} \in \mathscr{F}$ and clauses (a)+(c) of $(*)_0$.]

$\boxdot_3$ if $\ell_1 \mathscr{E}_2^* \ell_2$ then $g_{\ell_1} = g_{\ell_2}$
[Why? First, as $\ell_1 \mathscr{E}_2^* \ell_2$ we have $g_{\ell_1}(0) = g_{\ell_1}^{\alpha_*}(0) = g_{\ell_2}^{\alpha_*}(0) = g_{\ell_2}(0)$. Second, clearly $i(\ell_1) = i(\ell_2), i_1(\ell_1) = i_1(\ell_2)$ and $j_2(\ell_1) = j_2(\ell_2)$ hence for $\alpha \in [1, \theta)$ we have

$$\begin{aligned} g_{\ell_1}(\alpha) = & f_{j_2(\ell_1)}(\alpha) + (i(\ell_1) - i_1(\ell_1)) = \\ & f_{j_2(\ell_2)}(\alpha) + (i(\ell_2) - i_1(\ell_2)) = g_{\ell_2}(\alpha). \end{aligned}$$

So we are done.]

$\boxdot_4$ if $\ell_1, \ell_2 < k$ but $\neg(\ell_1 \mathscr{E}_2^* \ell_2)$ then $g_{\ell_1} \neq g_{\ell_2}$
[Why? As $\neg(\ell_1 \mathscr{E}_2^* \ell_2)$ by $\circledast_5(b)$ we have $g_{\ell_1}^{\alpha_*}(0) \neq g_{\ell_2}^{\alpha_*}(0)$, hence $g_{\ell_1}(0) = g_{\ell_1}^{\alpha_*}(0) \neq g_{\ell_2}^{\alpha_*}(0) = g_{\ell_2}(0)$ hence $g_{\ell_1} \neq g_{\ell_2}$.

$\boxdot_5$ if $\ell_1, \ell_2 < k, \ell_1 \mathscr{E}_1^* \ell_2$ then $\neg(g_{\ell_1} <_{J_\theta^{\rm bd}} g_{\ell_2})$
[Why? As $g_{\ell_1} \restriction [1, \theta) = g_{\ell_2} \restriction [1, \theta)$, so $g_{\ell_1} = g_{\ell_2} \mod J_\theta^{\rm bd}$, so $\neg(g_{\ell_1} <_{J_\theta^{\rm bd}} g_{\ell_2})$.]

$\boxdot_6$ if $\ell_1, \ell_2 < k$ and $(\ell_1/\mathscr{E}_1^*) <_* (\ell_2/\mathscr{E}_1^*)$ then $g_{\ell_1} <_{J_\theta^{\rm bd}} g_{\ell_2}$.
[Why? Obviously $i(\ell_1) < i(\ell_2), i_1(\ell_1) \leq i_1(\ell_2)$ and $j_2(\ell_1) \leq j_2(\ell_2)$ by $\circledast_4$. But by $\circledast_2(h)$ we have $f_{j_2(\ell_1)} + (i_1(\ell_2) - i_1(\ell_1)) \leq_{J_\theta^{\rm bd}} f_{j_2(\ell_2)}$ thus $f_{j_2(\ell_1)} + (i(\ell_1) - i_1(\ell_1)) <_{J_\theta^{\rm bd}} f_{j_2(\ell_1)} + (i(\ell_2) - i_1(\ell_1)) \leq_{J_\theta^{\rm bd}} f_{j_2(\ell_2)} + (i(\ell_2) - i_1(\ell_2))$ and $g_{\ell_1} <_{J_\theta^{\rm bd}} g_{\ell_2}$.]

Together $p^* = \text{tp}_{\rm qf}(\langle g_\ell : \ell < k\rangle, \emptyset, I_\theta) \in e_\alpha$ for all $\alpha \in [\alpha^*, \theta)$ proving the conclusion of Definition 3.2, the definition of existential limit, i.e. $(*)_6$. $\square_{3.7}$ $\square_{3.8}$

Theorem 3.8, p.48

Question concerning $\boxdot_1$:

$\boxdot_1$ seems to be wrong!

Why: Let $\ell_1, \ell_2 < k_1$ with $f_{\ell_1} =_{J_\theta^{\rm bd}} f_{\ell_2}$ (but $f_{\ell_1} \neq f_{\ell_2}$!)
Then $\circledast_7(d)(\alpha)$ implies

$$i(\ell_1) = i(\ell_2), i_1(\ell_1) = i_1(\ell_2) \text{ and } j_2(\ell_1) = j_2(\ell_2).$$

Thus $g_{\ell_1} \restriction [1, \theta) = g_{\ell_2} \restriction [1, \theta)$ follows by $\boxdot_0$ and $\boxdot_1$ would imply

$$f_{\ell_1} \restriction [1, \theta) = f_{\ell_2} \restriction [1, \theta)!$$

That does not hold in general.
Thus only

$\boxdot_1'$ if $\ell < k_1$ then $g_\ell =_{J_\theta^{\rm bd}} f_\ell$.

A possible solution: Theorem 3.8 remains true if weakening the conclusion of Definition 3.2 to:

> Then there are $\bar{t}' \in {}^{k_1}(I^{\mathfrak{s}}_{v_*})$ and $\bar{s} \in {}^{k_2}(I^{\mathfrak{s}}_{v_*}$ such that for every $u \in J^{\mathfrak{t}}_{\geq u_*}$ large enough $\text{tp}_{\text{qf}}(\pi^{\mathfrak{s}}_{u,v_*}(\bar{t}' \hat{}\, \bar{s}), \emptyset, I^{\mathfrak{t}}_u) = p_* \in e_u$ (for some constant $p_* \in \mathscr{S}^k$).

Saharon, please check: is that enough to prove 3.4? Otherwise improve $(*)_0$ of 3.8,p.44.

pg.43 in $(*)_0$, change (c) to:
if $f \in \mathscr{F}$ and $\alpha < \theta$ then $f' = f + 1_{[\alpha,\theta)} \in \mathscr{F}$, i.e.

$$f'(\beta) = \begin{cases} f(\beta) & \text{if } \beta < \alpha \\ f(\beta) + 1 & \text{if } \beta \in [\alpha, \theta) \end{cases}$$

pg.46: Let $m_\ell < k_1$ be maximal $m < k_1$ such that $(m/\mathscr{E}_0) \leq_* (\ell/\mathscr{E}_0)$ exists as $g_0 = f_0 = 0_\theta$ by $\circledast_2(g)$; let $n_\ell = \{\iota/\mathscr{E}_1 : \iota < k_2, (m/\mathscr{E}_0) \leq_* (\iota/\mathscr{E}_0) < (\ell/\mathscr{E}_0)\}$. So $n_\ell = 0$ if $\ell < k_1$ or just $u_\ell \cap \{0, \ldots, k_1\} \neq 0$

$\boxdot_1$ we define $g_\ell \in {}^\theta\kappa$ as follows:

- (a) $g_\ell \restriction \delta^* = g^{\delta^*}_\ell \restriction \delta^*$
- (b) $g_\ell \restriction [\delta^*, \theta) = g_{m_\ell} + n_\ell$, i.e. $g_{m_\ell} + 1_{[\delta^*,\theta)}$.

The rest should be clear (but we give details?).

Main Claim 3.8,pg.48:
Dear Saharon!
In 46A you gave a revised proposal for $\boxdot_0$. It is conform to replacing $\boxdot_0$ by

$\boxdot'_0$ $g_\ell = (g^{\alpha_*}_\ell \restriction \alpha_*) \cup ((f_{j_2} + (i - i_1)) \restriction [\alpha_*, \theta))$.

This most certainly solves $\boxdot_1$, but now $\boxdot_2$ is violated. This can/must be fixed by enhancing $(*)_0$, pg.44 once more:

- $(c)_1$ if $f \in \mathscr{F}$, then $f + 1 \in \mathscr{F}$
- $(c)_2$ if $f_1, f_2 \in \mathscr{F}, \alpha \in \theta$, then $(f_1 \restriction \alpha) \cup (f_2 \restriction [\alpha, \theta)) \in \mathscr{F}$.

This again seems to force replacement of (b),(c) in Theorem 3.7 + 3.8 as follows:

(b)′ $\mathscr{F}_\alpha \subseteq \bigcup\limits_{\beta \le \alpha} {}^{[\beta,\alpha)}\kappa$ for $\alpha < \theta$ has cardinality $\le \kappa$

(c)′ $\mathscr{F} = \{f \in {}^\theta\kappa | \exists \beta \in \theta$ with $f \restriction [\beta,\alpha) \in \mathscr{F}_\alpha$ for all $\beta \le \alpha < \theta\}$.

Question:
1) Does the pcf-argument with these changes still hold?
2) Does this (hopefully) fix all gaps around 3.7 and 3.8?

Saharon:
I can do 2), but 1) needs YOU!!!

We quote

3.10 Claim. *Assume* $\mathrm{cf}(\kappa) = \theta > \aleph_0, \alpha < \kappa \Rightarrow (\alpha)^\theta < \kappa$ *and* $\lambda = \kappa^\theta$. *Then we can find* $\langle \mathscr{F}_i : i < \theta\rangle, S, D, J_\delta$ *satisfying the conditions from 3.8 with* $\gamma = \lambda$ *(and more).*

Proof. By 3.11 and [Sh:g]. $\square_{3.10}$

3.11 Claim. *Assume*

⊛ (a) $\bar\lambda = \langle \lambda_i : i < \theta\rangle$ *is an increasing sequence of regular cardinals with limit* κ

(b) $\lambda = \mathrm{tcf}(\prod\limits_{i<\theta} \lambda_i, <_{J^{\mathrm{bd}}_\theta})$

(c) $\max \mathrm{pcf}\{\lambda_i : i < j\} < \kappa$ *for every* $j < \theta$.

1) Then there are D, S^*, u *such that*

(α) $u \in [\theta]^\theta, S^* \subseteq \theta$ *is stationary*

(β) *there are no* $u_\varepsilon \in [u]^\theta$ *for* $\varepsilon < \theta$ *such that for some club* E *of* $\theta, \delta \in E \cap S^*$ *for at least one* $\varepsilon < \delta$ *we have* $\max \mathrm{pcf}\{\lambda_i : i \in \delta \cap u_\varepsilon\} < \max \mathrm{pcf}\{\lambda_i : i \in \delta \cap u\}$ *hence*

(γ) D *is a normal filter on* θ *where:* D *is* $\{S \subseteq \theta$*: for every sequence* $\langle u_\varepsilon : \varepsilon < \theta\rangle$ *of subsets of* u *each of cardinality* θ *and for every club* E *of* θ*, if* $\delta \in E \cap S \cap S^*$ *then for every* $\varepsilon < \delta$ *we have* $\max \mathrm{pcf}\{\lambda_i : i \in \delta \cap u_\varepsilon\} = \max \mathrm{pcf}\{\lambda_i : i \in \delta \cap u\}\}$

(δ) *for* $\delta \in S^*$ *let* $J_\delta = \{u' \subseteq \delta :$ $\max \mathrm{pcf}\{\lambda_i : i \in \delta \backslash u' < \max \mathrm{pcf}\{\lambda_i : i < \delta\}\}$.

2) We can choose $\mathscr{F}_i \subseteq \prod\limits_{j<i} \lambda_j$ *for* $i < \theta$ *such that all the conditions in 3.8 hold.*

Proof. By [Sh:g, II,3.5], see on this [Sh:E12, §18].

3.12 Conclusion. If κ is strong limit singular of uncountable cofinality <u>then</u> $\tau^{\rm atw}_\kappa \geq \tau^{\rm nlg}_\kappa \geq \tau^{\rm nlf}_\kappa > 2^\kappa$.

Proof. By 3.8 and Claim 3.10. $\square_{3.12}$

3.13 Remark. 1) If $\kappa = \kappa^{\aleph_0}$ do we have $\tau^{\rm atw}_\kappa \geq \tau^{\rm nlg}_\kappa \geq \tau^{\rm nlf}_\kappa > \kappa^+$? But if $\kappa = \kappa^{<\kappa} > \aleph_0$ then quite easily yes.
2) In 3.12 we can weaken "κ is strong limit". E.g. if κ has uncountable cofinality and $\alpha < \kappa \Rightarrow |\alpha|^{{\rm cf}(\kappa)} < \kappa$, then $\tau^{\rm nlf}_\kappa > \kappa^{{\rm cf}(\kappa)}$; see more in [Sh:E12, §18].
3) We elsewhere will weaken the assumption in 3.7, 3.8 but deduce only that $\tau^{\rm nlg}_\kappa$ is large.

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