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SPECTRA OF MONADIC SECOND ORDER SENTENCES SH817

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Abstract. For a monadic sentence ψ in the finite vocabulary we show that the spectra, the set of cardinalities of models of ψ is almost periodic under reasonable conditions. The first is that every model is so called "weakly k-decomposable". The second is that we restrict ourselves to a nice class of models constructed by some recursion.

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Introduction

This continues [GuSh 536] and was announced there. For a monadic second order sentence ψ in the language with one unary functions and unary predicates, the spectra of the sentence (i.e., the set $\{||M|| : M$ a finite model of $\psi\}$ is (see [GuSh: 536]) periodic, but this fail badly when we allow, e.g. two unary functions. In the second section we characterize the family of finite structures which really behave like the unary function case, i.e., the proof works.
In section one we assume that a monadic second order sentence satisfies: every model is not indecomposable, i.e., has a non trivial decoposition in a weak sense (see Definition 1.2). We conclude that the specra is not arbitrary, mainly - there are no big gaps in it (from some point on). This is of course considerably weaker conclusion than what we know for the languages with only a unary function (under a much weaker assumption) or in §2.
This work was done when Gurevich was writing [GuSh 536], but he at first did not include an announcement in the version he circulated insisting that I rewrote it to his satisfaction. Meanwhile Fischer and Makowsky [FiMw03] started [FiMw03] to work from the earlier version of [GuSh 536] continuing it in a different direction, using counting monadic logic and dealing with tree and clique width of graphs (and of models). It seems that Definition 2.2 maybe a variant of "clique width of models"; see on this [FiMw03].
Clearly we can in §1 use operations like 2.2 instead of $M_1 \cup M_2$.

Note that in the definition of weakly k-decomposable we do not require that the "component" M_1, M_2 belongs to $\mathfrak{K}$. As it was indirectly asked and to clarify Definition 1.3, we add:
0.1 Example: The class $\mathfrak{K}$ of finite incidence (or edge) graphs is not weakly k-decomposable for every k.

Why? Let m be such that $m \geq k + \binom{k}{2}, m \geq 2k+2$ and it suffices to prove that for every $n > m$, the statement $\circledast$ of Definition 1.2(1) fail. Let G be the complete graph with set of nodes $\{b_1,\dots,b_n\}$, so the incidence graph G' has set of nodes $A = \{b_1,\dots,b_n\} \cup \{c_{i,j} : 1 \leq i < j \leq n\}$ and set of edges $\{\{b_i,c_{i,j}\} : 1 \leq i < j \leq n\} \cup \{\{b_j,c_{i,j}\} : 1 \leq i < j \leq n\}$. So toward contradiction assume $A = A_1 \cup A_2, |A_1 \cap A_2| \leq k$ and $|A_1| \geq m, |A_2| \geq m$ with no edge (of G') between $A_1 \backslash A_2$ and $A_2 \backslash A_1$. Let $u = \{i : b_i \in A_1 \cap A_2$ or for some $j, c_{i,j} \in A_1 \cap A_2$ or $c_{j,i} \in A_1 \cap A_2\}$. So $|u| \leq 2k$. Let $c_{\{i,j\}} = c_{\{j,i\}} = c_{i,j}$ when $1 \leq i < j \leq n$.

Case 1: For some i_1, i_2 we have $b_{i_1} \in A_1 \backslash A_2, b_{i_2} \in A_2 \backslash A_1$.

So for every $i \in \{1,\dots,n\} \backslash \{i_1,i_2\}$ for some $\ell \in \{1,2\}, b_i \in A_\ell$, so $\{b_i, c_{\{i,i_{3-\ell}\}}\}$ is an edge of G' hence either $b_i \in A_1 \cap A_2$ or $c_{i,i_{3-\ell}} \in A_1 \cap A_2$ hence (in both possibilities) $i \in u$ so $n-2 \leq |u|$ but $|u| \leq 2k$ and $2k+2 \leq m < n$, contradiction.

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Case 2: Not Case 1.

So for some $\ell \in \{1,2\}$ we have $\{b_1,\dots,b_n\} \subseteq A_\ell$, hence $A_{3-\ell} \backslash A_\ell \subseteq \{c_{i,j} : b_i, b_j \in A_1 \cap A_2\}$, without loss of generality $\ell = 1$ so $|A_2| \le |A_2 \cap A_1| + |A_2 \backslash A_1| \le k + \binom{k}{2} < m$, contradiction. $\square_{0.1}$

0.2 Notation. 1) Let n, m, ℓ, k, i, j, d be natural numbers.
2) τ is a vocabulary (i.e., set of predicates, individual constants and function symbols, the last are not used here).

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§1 Weakly decomposable

We can deal just with graphs just as this is traditional. The restriction to relational vocabulary is for simplifying our statements.

1.1 Context. 1) Let τ be a finite relational vocabulary, i.e., a finite set of predicates.
2) Let $\mathfrak{K}^*_\tau$ be the class of τ-models and recall $\|M\|$ is the number of elements of $M \in \mathfrak{K}^*_\tau, R^M$ is the interpretation of the predicate $R \in \tau$.
3) Let $\mathfrak{K}$ denote a family of τ-models closed under isomorphisms.

1.2 Definition. 1) We say that $\mathfrak{K}$ is weakly k-decomposable <u>if</u>: for every m there is n such that

$\circledast_{k,m,n}$ if $M \in \mathfrak{K}, \|M\| \geq n$ <u>then</u> we can find submodels M_1, M_2 (for graphs-induced subgraphs G_1, G_2) such that

- (a) $M_1 \cup M_2 = M$ i.e., $a \in M \Leftrightarrow a \in M_1 \vee a \in M_2$ and $R^M = R^{M_1} \cup R^{M_2}$ for any $R \in \tau$ (for graphs: G, G_1, G_2 let $G = G_1 \cup G_2$ mean that the set of nodes is the union of the set of nodes of G_1 and of G_2, and the set of edges of G is the union of the set of edges of G_1 and of G_2)
- (b) $|M_1 \cap M_2| \leq k$
- (c) $\|M_\ell\| \geq m$ for $\ell = 1, 2$.

2) For a monadic second order sentence ψ (in a vocabulary τ) we say that ψ is weakly k-decomposable <u>if</u> $\mathfrak{K}^\tau_\psi$ is (see part (3)).
3) For a vocabulary τ (as in 1.1) and sentence ψ (in this vocabulary) let $\mathfrak{K}^\tau_\psi = \{M : M$ is a finite τ-model such that $M \models \psi\}$. We may suppress τ, when clear from the context.

1.3 Claim. *Assume*

$(*)^{k^*}_\psi$ *ψ a monadic second order sentence, in the vocabulary τ such that $\mathfrak{K} = \mathfrak{K}^\tau_\psi$ is weakly k^*-decomposable*

<u>*then*</u> $\mathrm{Sp}(\psi) = \{\|M\| : M \in \mathfrak{K}\}$ *satisfies for some n^*, that*

$\circledast$ *if $n_1 < n_2$ are successive members of* $\mathrm{Sp}(\psi)$ *and $n^* < n_1$ then $n_2 < 2n_1$.*

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Proof. Let ψ have quantifier depth $\leq d^*$.
Let $m_1^* > k^*$ be large enough such that

$\boxdot_1$ if $M_1 \in \mathfrak{K}_\tau^*$ (yes $\mathfrak{K}_\tau^*$ and not $\mathfrak{K}$), $\|M_1\| > k^*$ and $a_1, \ldots, a_k \in M_1$ and $k \leq k^*$ <u>then</u> there are $M_2 \in \mathfrak{K}_\tau^*$ and $b_1, \ldots, b_k \in M_2$ such that (see [GuSh 536] or 2.6 below)
$\mathrm{Th}^{d^*}(M_1, a_1, \ldots, a_k) = \mathrm{Th}^{d^*}(M_2, b_1, \ldots, b_k)$
and $k^* < \|M_2\| < m_1^*$.

Let m_2^* be such that the statement $\circledast_{k^*, m_1^*, m_2^*}$ from Definition 1.2 holds (for $\mathfrak{K}$).

Now assume that $n_1 < n_2$ are successive members of $\mathrm{Sp}(\psi)$ and $n_2 > m_2^*$. Hence there is $M \in \mathfrak{K}$ with exactly n_2 members. So applying $\circledast_{k^*, m_1^*, m_2^*}$ to M we can find M_1, M_2 as in Definition 1.2 and let $\{a_1, \ldots, a_k\}$ list $M_1 \cap M_2$; so $k \leq k^*$ and $\|M_1\|, \|M_2\| \geq m_1^*$. Without loss of generality $\|M_1\| \leq \|M_2\|$, still $\|M_1\| \geq m_1^*$. By the choice of m_1^* there is $(M_1', b_1, \ldots, b_k)$ such that $k^* < \|M_1'\| < m_1^*$ and

$$\mathrm{Th}^{d^*}(M_1', b_1, \ldots, b_k) = \mathrm{Th}^{d^*}(M_1, a_1, \ldots, a_2).$$

Without loss of generality $\ell \in \{1, \ldots, k\} \Rightarrow b_\ell = a_\ell$ and no member (for graphs - node) of M_1' belongs to $M_2 \backslash \{a_1, \ldots, a_k\}$. Let $M' = M_1' \underset{\{a_1, \ldots, a_k\}}{+} M_2$ be defined naturally (set of elements of M' = union of set of elements of M_1' and set of elements of M_2 and $R^M = R^{M_1'} \cup R^{M_2}$ for $R \in \tau$).

By the addition theorem (for local monadic theories, see 2.7(c)) we have $M' \models \psi$, i.e., $M' \in \mathfrak{K}$ and

$\boxdot$ $\frac{1}{2}\|M\| \leq \|M_2\| < \|M'\| = \|M_1'\| + \|M_2\| - k < m_1^* + \|M_2\| - k \leq \|M_1\| + \|M_2\| - k = \|M\|$.

That is $n_2/2 < \|M'\| < n_2$ but $M' \in \mathfrak{K}$ so $\|M'\| \in \mathrm{Sp}(\mathfrak{K})$ so there is $n' \in \mathrm{Sp}(\mathfrak{K})$ such that $n_2/2 < n' < n_2$ so we are done. $\square_{1.3}$

1.4 Conclusion. If φ is a second order monadic sentence and $(*)_\varphi^{k^*}$ of 1.3 holds and α is a real > 0 <u>then</u> for some n^* we have

$$\boxtimes = \boxtimes_{\varphi, \alpha, n^*} \qquad n^* < n \in \mathrm{Sp}(\varphi) = (\exists m \in (\mathrm{Sp}(\varphi))[n < m < (1+\alpha)n].$$

Proof. Let Ξ be the family of positive reals α such that

$\circledast_1$ for every monadic second order sentence ψ (for any vocabulary τ as in 1.1) such that $(*)_\psi^{k^*}$ holds, the conclusion $\boxtimes_{\varphi, \alpha, n^*}$ of 1.4 holds for some n^* (no harm in varying k^*, too).

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Note that allowing individual constants in τ is O.K. (either allow them or code them by unary predicates); for a vocabulary τ let τ^{+k} be $\tau \cup \{P_\ell : \ell = 1, \ldots, n\}$, where the P_ℓ are distinct unary predicates not from τ.
Clearly $0 < \beta < \alpha$ & $\beta \in \Xi \Rightarrow \alpha \in \Xi$. By Claim 1.3 we have $1 \in \Xi$.

We shall now prove that

$\circledast_2$ if $\alpha \in \Xi \Rightarrow \alpha/2 \in \Xi$.

This clearly suffices. So let $\alpha, \tau, \psi, \mathfrak{K} = \mathfrak{K}^\tau_\psi$ be given. Let d be above the quantifier depth of ψ. For $k \le k^*$ let

$$\begin{aligned}\mathfrak{K}'_k = \mathfrak{K}'_{\psi,k} = \{(M', P_1, \ldots, P_k) :& \text{for some } M \in \mathfrak{K} \text{ and } M_1, M_2 \text{ as in 1.2}\\ & \text{in particular } |M_1 \cap M_2| \le k^* \text{ we have } M' = M_1\\ & \text{and } \{a_1, \ldots, a_k\} \text{ lists } M_1 \cap M_2 \text{ and } P_\ell = \{a_\ell\}\}.\end{aligned}$$

This is a class of τ^{+k} models. Let $\{\mathrm{Th}^d(M', P_1, \ldots, P_k) : (M', P_1, \ldots, P_k) \in \mathfrak{K}'_k\}$ be listed as $\mathbf{t}^k_1, \ldots, \mathbf{t}^k_m$ and for $\ell \in \{1, \ldots, m\}$ let $\mathfrak{K}'_{k,\ell} = \mathfrak{K}'_{\psi,k,\ell} = \{(M', P_1, \ldots, P_k) \in \mathfrak{K}'_k : \mathrm{Th}^d(M', P_1, \ldots, P_k) = \mathbf{t}^k_\ell\}$. So $N \in \mathfrak{K}'_{k,\ell} \Rightarrow |P^N_1| = \ldots = |P^N_k| = 1$.
It is not hard to see

$\circledast_3$ for some monadic second order sentence $\psi_{k,\ell}$ of quantifier depth d in the vocabulary $\tau^{+k}, \mathfrak{K}'_{k,\ell}$ is the class of models of $\psi_{k,\ell}$, i.e., is $\mathfrak{K}^{\tau^{+k}}_{\psi_{k,\ell}}$ for every relevant pair (k, ℓ).
[Why? By direct checking $\mathfrak{K}'_{k,\ell}$ is a class of τ^{+k}-models and there is such monadic sentences by the definition of Th^d.]

$\circledast_4$ $\mathfrak{K}^{\tau^{+k}}_{\psi_{k,\ell}}$ is weakly k^*-decomposable.
[Why? Clearly $\psi_{k,\ell}$ is a monadic sentence in the vocabulary τ^{+k}.
By the choice of $\mathbf{t}^k_\ell$ there are $M, M_1, M_2, a_1, \ldots, a_k$ as in the definition of $\mathfrak{K}'_k$.
Now expand M_1 to a τ^{+k}-model M^*_1 by $P^{M^*_1}_i = \{a_i\}$ and so $\mathbf{t}^k_\ell = \mathrm{Th}^d(M^*_1)$.
We have to prove "$\mathfrak{K}'_{k,\ell}$ weakly decomposable", i.e., Definition 1.3. So let a number m be given. Let $m' = m + \|M_2\|$ and let n be as guaranteed for m' by $(*)^{k^*}_\psi$ for $\mathfrak{K}^\tau_\psi$. We shall show that n is as required for $\mathfrak{K}^{\tau^{+k}}_{\psi_{k,\ell}}$.
Let $(M'_1, P'_1, \ldots, P'_k) \in \mathfrak{K}'_{k,\ell}$, $\|M'_1\| \ge n$ so without loss of generality (i.e. by renaming) $P'_i = \{a_i\}$ (for $i = 1, \ldots, k$) and $M'_1 \backslash \{a_1, \ldots, a_k\} \cap M_2 = \emptyset$. We can define N such that $N, M'_1, M_2, a_1, \ldots, a_k$ are as in 1.3 so by the addition theorem (see §2) $\mathrm{Th}^d(N, P'_1, \ldots, P'_k) = \mathrm{Th}^d(M, P'_1, \ldots, P'_k)$ so $N \models \psi$. As $\|N\| \ge \|M'_1\| \ge n$ by the choice of n we can find find $N_1, N_2, k' \le k^*$ and $\bar b = \langle b_\ell : \ell = 1, \ldots, k'\rangle$ such that the tuple $(N, N_1, N_2, \bar b)$ is as in 1.3, i.e.,

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$N = N_1 \underset{\{b_1,\dots,b_{k'}\}}{+} N_2$ and $\|N_1\|, \|N_2\| \geq m'$. Let $N'_\ell = N_\ell \restriction (|M'_1|)$ and $\bar{c} = \langle c_\ell : \ell \neq 1, \dots, k'' \rangle$ enumerate $\{b_\ell : \ell = 1, \dots, k'\} \cap M'_1$.

Clearly $\|N'_\ell\| \geq \|N_\ell\| - \|M_2\| \geq m' - \|M_2\| = m$ by the choice of m' above. So $(M'_1, N'_1, N'_2, \bar{c})$ is as required in the conclusion of 1.3 for $\psi_{k,\ell}$.]

$\circledast_5$ $(*)^{k^*}_{\psi_{k,\ell}}$ holds
[why? By $\circledast_3 + \circledast_4$.]

Hence by the induction hypothesis for some m^*

$\circledast_6$ the conclusion of $\boxtimes_{\psi_{k,\ell},\alpha,m^*}$ of 1.4 holds (for any relevant k, ℓ)

Let n^* be as in 1.2(1) for $k^*, m^*, \mathfrak{K} = \mathfrak{K}^\tau_\psi$.
So it is enough to prove that $\boxtimes_{\psi,\alpha,n^*}$ holds. Now for any $M \in \mathfrak{K}$ with $\geq n^*$ elements there are $M_1, M_2, k, a_1, \dots, a_k$ as in Definition 1.2(1) such that $m^* \leq \|M_1\| \leq \|M_2\|$. So for some $\ell, (M_1, P_1, \dots, P_k) \in \mathfrak{K}'_{k,\ell}, P_i = \{a_i\}$ for $i = 1, \dots, k$, so as we are assuming $\circledast_6$ clearly

$\circledast_7$ we can choose $(M'_1, P'_1, \dots, P'_k) \in \mathfrak{K}'_{k,\ell}$ such that

$$\frac{\|M_1\|}{1+\alpha} < \|M'_1\| < \|M_1\|.$$

Without loss of generality (by renaming)

$$P'_i = P_i, M'_1 \cap M_2 = \{a_1, \dots, a_k\}.$$

We can define $M' \in \mathfrak{K}^\tau_\psi$ as in the proof of 1.3 so with universe $|M'_1| \cup |M_2|$.
Hence

$$\circledast_8 \qquad \begin{aligned} \|M\| > \|M'\| &= \|M_2\| + \|M'_1\| - k > \|M_2\| + \frac{\|M_1\|}{1+\alpha} - k \\ &= \frac{1}{(1+\alpha)}(\|M_2\| + \alpha\|M_2\| + \|M_1\| - k - \alpha k) \\ &= \frac{1}{1+\alpha}(\|M\| + \alpha\|M_2\| - \alpha k) \\ &\geq \frac{1}{1+\alpha}(\|M\| + \alpha\|M\|/2) - \frac{\alpha k}{1+\alpha} = \frac{1+\alpha/2}{1+\alpha}\|M\| - \frac{\alpha k}{1+\alpha}. \end{aligned}$$

Let

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$$\beta =: \frac{1+\alpha}{1+\alpha/2} - 1 = \frac{\alpha/2}{1+\alpha/2} < \alpha/2.$$

So by $\circledast_8$ we have

$$\begin{aligned}\|M\| > \frac{1+\alpha/2}{1+\alpha}\|M\| - \frac{\alpha k}{1+\alpha} = \frac{1}{(1+\beta)}\|M\| - \frac{\alpha k}{1+\alpha} &= \frac{1}{1+\alpha/2}\|M\| \\ &+ (\frac{1}{1+\beta} - \frac{1}{1+\alpha/2})\|M\| - \frac{\alpha k}{1+k} \\ &= \frac{1}{1+\alpha/2}\|M\| + (\frac{\alpha/2-\beta}{(1+\beta)(1+\alpha/2)}\|M\| - \frac{\alpha k}{1+\alpha}).\end{aligned}$$

So we conclude: if conclusion 1.4 holds for $\alpha > 0$ it holds for $\alpha/2$ provided that $(\alpha/2-\beta)\|M\| > \frac{\alpha k}{1+\alpha}(1+\beta)(1+\alpha/2)$, of course which holds if $\|M\|$ is large enough. So we can prove by induction on i that $\boxtimes$ holds for $\alpha \geq \frac{1}{2^i}$. $\square_{1.4}$

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§2 WHAT THE METHOD OF [GuSh 536] GIVES

2.1 Discussion: The result above is interesting but leave us unsatisfied. For trees we get essentially sharp results. Here the spectra is not characterized. We know that it is quite restricted but, e.g. is it almost periodic?

The problem is that we do not see here a parallel to the operations generating the class.

We may consider such classes:

2.2 Definition. Let τ and k^* be fixed, τ a finitary vocabulary with predicates only (coding function and individual constants by them if necessary) and let $\mathfrak{K}_{k^*} = \mathfrak{K}^{c\ell}_{\tau,k^*}$ be the minimal family of $(M, a_1, \ldots, a_k), M$ a finite τ-model, $k \le k^*, a_\ell \in M$ such that

- (a) $\mathfrak{K}_{k^*} = \mathfrak{K}_{\tau,k^*}$ includes all the $(M, c_1, \ldots, c_k), k \le k^*, c_\ell \in M$ with M a τ-structure with $\le k^*$ elements
- (b) if $(M_\ell, a^\ell_1, \ldots, a^\ell_{k_\ell}) \in \mathfrak{K}_{k^*}$ for $\ell = 1, 2$ and $x \in M_1 \wedge x \in M_2 \Rightarrow x \in \{a^1_1, \ldots, a^1_{k_1}\} \cap \{a^2_1, \ldots, a^2_{k_2}\}$ then $(M, b_1, \ldots, b_k) \in \mathfrak{K}_{k^*}$ when:
  - $\circledast$ (i) x an element of $M \Rightarrow x$ an element (= node) of M_1 or of M_2
  - (ii) x an element of $M_\ell, x \notin \{a^\ell_1, \ldots, a^\ell_{k_\ell}\} \Rightarrow x$ an element of M
  - (iii) $\{b_1, \ldots, b_k\} \subseteq \{a^1_1, \ldots, a^1_{k_1}\} \cup \{a^2_1 \ldots a^2_{k_2}\}$
  - (iv) if $R \in \tau$ is m-place predicate, and $y_1, \ldots, y_m \in M, z_1, \ldots, z_m \in M$ then $\langle y_1, \ldots, y_m\rangle \in R^M \equiv \langle z_1, \ldots, z_m\rangle \in R^M$ when:
    - $(\boxdot)$ $(z_i = z_j) \equiv (y_i = y_j), (z_i = a^1_\ell) \equiv (y_i = a^1_\ell)$ $(z_i = a^2_\ell) \equiv (y_i = a^2_\ell), (z_i \in M_\ell) \equiv (y_i \in M_\ell)$ and letting $w_\ell = \{i : y_i \in M_\ell\}$ the quantifier free type of $\langle y_i : i \in w_\ell\rangle$ in M_ℓ is equal to the quantifier free type of $\langle z_i : i \in w_\ell\rangle$ in M_ℓ for $\ell = 1, 2$.

2.3 Claim. *We can prove for $\mathfrak{K}_{\tau,k^*}$ what we have proved for trees in [GuSh 536]; including almost periodically of the spectrum for monadic sentences (see 2.7(f)).*

Proof. This is clause (f) of Claim 2.7 proved below (as in [GuSh 536]).

2.4 Question: Is the class K_{k^*} known? Interesting? (see §0)

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Of course, e.g., the result on the spectrum is inherited by reducts. After second thoughts I decide to add details (but naturally in the style of [Sh 42] rather than [GuSh 536] as far as there is a difference).

2.5 Notation: τ is a vocabulary which has only predicates and possibly individual constants. If $R \in \tau$ is an m-place predicate we write $\text{arity}(R) = m$; let $\text{arity}(\tau) = \max(\{1\} \cup \{\text{arity}(R) : R \in \tau\})$.

Let $\tau_m = \tau + \{P_0, \dots, P_{m-1}\}$ be the vocabulary τ when we add the (new and pairwise distinct) predicates $P_0, \dots, P_{m-1}$ which below will be unary, similarly $\tau + \{c_0, \dots, c_{k-1}\}$ for c_ℓ individual constants and $\tau_{m,k} = \tau_m + \{\tau_0, \dots, c_{k-1}\}$.

2.6 Definition. 1) Let τ be a finite vocabulary τ consisting of predicates only; $P_0, P_1, \dots$ be unary predicates $\notin \tau$; (for notational simplicity). For a τ-model M and sequence $\bar{\mathscr{U}}^m = \langle \mathscr{U}_\ell : \ell < m \rangle$ of subsets of M we define $\text{Th}^n(M, \bar{\mathscr{U}}^m)$ by induction on n:

- (a) $n = 0$ it is the set of sentences $\psi = (\exists x_0, \dots, x_{r-1}) \wedge \Phi$ where $r \le \text{arity}(\tau) + 1, \Phi$ a set of basic formulas of $\tau + \{P_0, \dots, P_{m-1}\}$ such that $(M, \bar{\mathscr{U}}^m) \models \psi$
- (b) $\text{Th}^{n+1}(M, \bar{\mathscr{U}}^m) = \{\text{Th}^n(M, \bar{\mathscr{U}}^m {}^\frown \langle \mathscr{U}_m \rangle) : \mathscr{U}_m \subseteq M\}$.

2) $\text{TH}^{n,m}(\tau)$ is the set of formally possible $\text{Th}^n(M, \langle \mathscr{U}_\ell : \ell < m \rangle)$; see below; if $m = 0$ we may omit m.
3) For a class $\mathfrak{K}$ of models let $\text{TH}^{n,m}(\mathfrak{K})$ be $\{\text{Th}^n(M, \mathscr{U}_0, \dots, \mathscr{U}_{m-1}) : M \in \mathfrak{K}, \mathscr{U}_0, \dots, \subseteq M\}$; if $m = 0$ we may omit it.
4) Let $\mathfrak{K}_{\tau,k^*,k}$ be the set of models $(M, c_1, \dots, c_k) \in \mathfrak{K}_{\tau,k^*}$, M a τ-structure.

2.7 Claim. *Let τ have predicates only, define $\tau_k = \tau + \{c_0, \dots, c_{k-1}\}$ and let $k^* \ge \text{arity}(\tau)$ and n be given.*

We can compute the following (from τ, k^*, n^*, m^*)

- (a) *for $n \le n^*, m \le m^* - n, k \le k^*$, $\text{TH}^n(\tau_{m,k})$ which, the set of formally possible $\text{Th}^n(M, \langle \mathscr{U}_0, \dots, \mathscr{U}_{m-1} \rangle, a_0, \dots, a_k), M$ a τ-model*
- (b) *the set $\mathscr{S}_{\tau_m,k^*,k_1,k_2,k}$ of schemes defined implicitly in Definition 2.2*
- (c) *we can compute the functions*
  $F =: F^n_{\tau_m,k_1,k_2,k} : \text{TH}^n(\tau_{m,k_1}) \times \text{TH}^n(\tau_{m,k_2}) \times \mathscr{S}_{\tau_m,k^*,k_1,k_2,k} \to \text{TH}^n(\tau_{m,k})$ *such that (where M_1, M_2, M are τ_m-models) if $(M_\ell, a^\ell_0, \dots, a^\ell_{k_\ell - 1})$ for $\ell = 1, 2$ and $(M, a_0, \dots, a_{k-1})$ are as in Definition 2.2 for the scheme* $\mathbf{s}$ *and $t_\ell = \text{Th}^n(M_\ell, a^\ell_0, \dots, a^\ell_{k_\ell - 1})$ for $\ell = 1, 2$ and $t = \text{Th}^n(M, a_0, \dots, a_{k-1})$ then $t = F^n(t_1, t_2, \mathbf{s})$, in particular the representative models does not matter*
- (d) *we can compute $T^n_{\tau_m,k^*,k} = \{\text{Th}^n(M, c_0, \dots, c_{k-1}) : M$ is a τ_m-model with $\le k^*$ elements$\}$*

(e) *the sequence* $\langle \mathrm{TH}^n(\mathfrak{K}_{\tau,k^*,k}) : k \leq k^* \rangle$ *can be computed (could use* τ_m*)*

(f) *for each* $t \in \mathrm{TH}^n(\mathfrak{K}_{\tau,k^*,k})$*, the set*

$$\mathrm{Sp}_t = \{\|M\| : (M, a_0, \dots, a_{k-1}) \in \mathfrak{K}_{\tau,k^*,k} \text{ and } \mathrm{Th}^n(M, a_0, \dots, a_{k-1}) = t\}$$

is eventually periodic; i.e. for some $m_1, m_2 \in M$ *we have: if* $\ell_1, \ell_2 > m_1$ *are equal modulo* m_2 *then* $\ell_1 \in \mathrm{Sp}_t \equiv \ell_2 \in \mathrm{Sp}_t$*;*

Proof. Straight by now, e.g.,

Clause (a): For $n = 0$ trivial and for $n + 1$ we can let

$$\mathrm{TH}^{n+1}(\tau_{m,k}) = \mathscr{P}(\mathrm{TH}^n(\tau_{m+1,k})).$$

Clause (c): This is the addition theorem proved by induction on n.

Clause (e): We start with the sequence $\langle T^n_{\tau,k^*,k} : k \leq k^* \rangle$ and close it under the operations $F^n_{\tau,k^*,k_1,k_2}(-,-,\mathbf{s}), \mathbf{s} \in \mathscr{S}_{\tau,k^*,k_1,k_2,k}$. That is we define $\langle T^i_k : k \leq k^* \rangle$ by induction on $i \leq \sum_{k<k^*} |\mathrm{TH}^n(\tau_k)|$ as follows:

(α) $T^0_k = T^n_{\tau,k^*,k}$ from clause (d)

(β) $T^{i+1}_k = T^i_k \cup \{F^n_{\tau,k^*,k_1,k_2}(t_1, t_2, \mathbf{s}), t_1 \in T^i_{k_1}, t_2 \in T^i_{k_2}, \mathbf{s} \in \mathscr{S}^n_{\tau,k^*,k_1,k_2,k}\}$.

By cardinality considerations we know that for the last i we have $\mathrm{TH}^n(K_{\tau,k^*,k}) = T^i_k$ (for $k \leq k^*$) as

(γ) for each $k, \emptyset \subseteq T^0_k \subseteq T^1_k \subseteq \dots \subseteq T^i_k \subseteq \dots$ and the number of i for which $T^{k_i}_k \neq T^{i+1}_k$ is $\leq |\mathrm{TH}^n(\tau_k)|$ and

(δ) if $T^i_k = T^{i+1}_k$ for every $k \leq k^*$ then $j \geq i \wedge k \leq k^* \Rightarrow T^j_k = T^i_k$.

Clause (f): By the proof above and observation 2.8 below. $\square_{2.7}$

2.8 Observation. Assume that

(a) $m \in \mathbb{N}, \mathbf{W}$ is a finite set of quadruples $\{\langle \ell_1, \ell_2, \ell_3, j\rangle : \ell_1, \ell_1, \ell_3 < m$ and $j \in \mathbb{N}\}$

(b) N^i_ℓ $(\ell < m, i \in \mathbb{N})$ is a finite set of natural numbers

(c) $N^{i+1}_\ell = N^i_\ell \cup \{n_1 + n_2 - j$: for some $(\ell_1, \ell_2, \ell_3, j) \in \mathbf{W}$ we have $n_1 \in N^i_{\ell_1}, n_2 \in N^i_{\ell_2}, \ell_3 = \ell\}$.

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Then each set $N_\ell =: \cup\{N^i_\ell : i$ a natural number$\}$ is almost periodic.

Proof. Let $i < m$, clearly $n \in N_i$ iff we can find a witness $(\mathscr{T}, \bar\ell, \bar n, \bar w)$ which means

⊛ (a) $\mathscr{T}$ is a finite set of sequences of zeroes and ones closed under initial segments

(b) $\bar\ell = \langle \ell_\eta : \eta \in \mathscr{T}\rangle$ such that $\ell_{<>} = i$

(c) $\bar n = \langle n_\eta : \eta \in \mathscr{T}\rangle$ such that $n_{<>} = n$

(d) $\bar w = \langle w_\eta : \eta \in \mathscr{T}$ not maximal$\rangle, w_\eta \in \mathbf{W}$

(e) if $\eta \in \mathscr{T}$ is $\triangleleft$-maximal then $n_\eta \in N^0_{\ell_\eta}$

(f) if $\eta \in \mathscr{T}$ is not $\triangleleft$-maximal then $\eta_0 = \eta\hat{\ }\langle 0\rangle \in \mathscr{T}$ and $\eta_1 = \eta\hat{\ }\langle 1\rangle \in \mathscr{T}$ and we have

$$w_\eta = (\ell_{\eta\hat{\ }\langle 0\rangle}, \ell_{\eta\hat{\ }\langle 1\rangle}, \ell_\eta, n_{\eta\hat{\ }\langle 0\rangle} + n_{\eta\hat{\ }\langle 1\rangle} - n_\eta)$$

(g) (follows): $n_\eta \in N_{\ell_\eta}$ for $\eta \in \mathscr{T}$.

Now let $n^*_1 =: 2^m \times n^*_0$ where $n^*_0 =: \max(\bigcup\limits_\ell N^0_\ell)$ and let $n^* =: n^*_1!$. Assume

$(*)_0$ $n^* < n \in N_\ell$ and there is no n' such that $n^* < n' \in N_\ell, n' < n$ and $n = n'$ mod n^*.

Choose $(\mathscr{T}, \bar\ell, \bar n, \bar w)$ as above such that $(\ell_{<>}, n_{<>}) = (\ell, n)$ and $|\mathscr{T}|$ minimal. Let

$$\mathscr{U} = \{\nu \in \mathscr{T} : \text{if } \nu \triangleleft \rho \in \mathscr{T} \text{ then } n_\nu < n_\rho\}$$

and let

$$t(\nu) = \text{ Max}\{|\{m : \nu \trianglelefteq \rho \restriction m \text{ and } \rho \restriction m \in \mathscr{U}\}| : \nu \triangleleft \rho \in \mathscr{T}\}.$$

Now

$(*)_1$ $\nu_0 \trianglelefteq \nu_1 \in \mathscr{T} \Rightarrow t(\nu_0) \geq t(\nu_1)$ and if $\nu \in \mathscr{U}$ is not $\triangleleft$-maximal then $t(\nu) = \text{Min}\{t(\nu^\frown\langle j\rangle) + 1 : \nu^\frown\langle j\rangle \in \mathscr{T}\}$.
[Why? Look at the definitions.]

$(*)_2$ if $\nu \in \mathscr{T}$ then $n_\nu \leq 2^{t(\nu)} \times n^*_0$
[Why? We prove this by induction on $t(\nu)$ and then on $|\{\rho : \nu \trianglelefteq \rho \in \mathscr{T}\}|$. If ν is $\triangleleft$-maximal in $\mathscr{T}$ necessarily $n_\nu \in N^0_{\ell_\nu}$ hence $n_\nu \leq n^*_0$ so the conclusion of $(*)_2$ in this case is trivial. If $\nu \in \mathscr{U}, \nu$ is not $\triangleleft$-maximal in

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$\mathscr{T}$ then $n_\nu \le 2 \times \text{ Max}\{n_{\nu^\frown <j>} : \nu^\frown <j> \in \mathscr{T}, j = 0,1\}$ and clearly $t(\nu) = \text{ Max}\{t(\nu^\frown <j>) + 1 : j = 0,1\}$, so we can check easily. Lastly, if $\nu \in \mathscr{T} \backslash \mathscr{U}$ we can find ν_1 such that $n_\nu \le n_{\nu_1}, \nu \triangleleft \nu_1 \in \mathscr{T}$ so $t(\nu) \ge t(\nu_1)$ by $(*)_1$ and so $n_\nu \le n_{\nu_1} \le 2^{t(\nu_1)} \times n^*_0 \le 2^{t(\nu)} \times n^*_0$ as required.]

$(*)_3$ there are $\nu_0 \triangleleft \nu_1$ from $\mathscr{U}$ such that $\ell_{\nu_0} = \ell_{\nu_1}$.
[Why? As $n_{<>} = n > 2^m \times n^*_0$, by $(*)_2$ we know that $t(<>) > m$, hence we can find $\nu_0 \triangleleft \nu_1 \triangleleft \ldots \triangleleft \nu_m$ all from $\mathscr{U}$. As $\ell_{\nu_0}, \ldots, \ell_{\nu_m} < m$ clearly for some $j(0) < j(1) \le m$ we have $\ell_{\nu_{j(0)}} = \ell_{\nu_{j(1)}}$.]

$(*)_4$ if $\nu_0 \in \mathscr{U}$ is $\triangleleft$-maximal such that for some $\nu_1, \nu_0 \triangleleft \nu_1 \in \mathscr{U}$ & $\ell_{\nu_0} = \ell_{\nu_1}$ then $t(\nu_0) \le m$.
[Why? Look at the definition of $t(\nu_1)$, i.e., repeat the proof of $(*)_3$.]

$(*)_5$ for some $\nu_0 \triangleleft \nu_1$ from $\mathscr{U} \subseteq \mathscr{T}, \ell_{\nu_0} = \ell_{\nu_1}$ and $n_{\nu_0} < n_{\nu_1}$ and $n_{\nu_0} \le n^*_1$
[Why? By $(*)_3$ we can find $\nu_0 \triangleleft \nu_1$ from $\mathscr{U}$ with $\ell_{\nu_0} = \ell_{\nu_1}$ so without loss of generality ν_1 is $\triangleleft$-maximal, hence by $(*)_4$ we know $t(\nu_0) \le m$, so by $(*)_2$, $n_{\nu_0} \le n^*_1$.]

Now we can take a copy of $A = \{\rho \in \mathscr{T} : \nu_0 \trianglelefteq \rho$ but $\neg \nu_1 \triangleleft \rho\}$ and insert it just before ν_1 any number of times, hence $n + i(n_{\eta_0} - n_{\eta_1}) \in N_\ell$ for any i. As $n_{\nu_0} - n_{\nu_1} \le n_{\eta_1} < n^*_1$ we are done. (We can compute a bound when this starts. That is omitting A we get $n - (n_{\eta_0} - n_{\eta_1}) \in N_\ell$ hence by the assumption on n in $(*)_0, n =: n - (n_{\eta_0} - n_{\eta_1}) \le n^*$ so $n \le n^* + (n_{\eta_0} - n_{\eta_1}) \le n^* + n^*_1$.) $\square_{2.8}$

Remark. Of course, also in §1 we can use sums as in 2.2.

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