

**TWO CARDINALS MODELS
WITH GAP ONE REVISITED
SH824**

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ABSTRACT. We succeed to say something on the identities of (μ^+, μ) when $\mu > \theta > \text{cf}(\mu)$, μ strong limit θ -compact or even μ limit of compact cardinals.

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ANNOTATED CONTENT

§0 Introduction

[We give the basic definitions.]

§1 2-simplicity for gap one

[We prove that if $\mu = 2^{<\mu}$ then the family of identities of (μ^+, μ) is 2-simple. So this applies to μ singular strong limit but also, e.g., to triples (μ^+, μ, κ) , $\mu = 2^{<\mu} > \kappa$.]

§2 Successor of strong limit above supercompact:2-identities

[Consider a pair (μ^+, μ) with μ strong limit singular $> \theta > \text{cf}(\mu)$, θ a compact cardinal. We point out quite simple 2-identities which belong to $\text{ID}_2(\mu^+, \mu)$ but not to $\text{ID}_2(\aleph_1, \aleph_0)$.]

§0 INTRODUCTION

There has been much work on κ -compactness of pairs (λ, μ) of cardinals, i.e., when: if T is a set of first order sentences of cardinality $\leq \kappa$ and every finite subset has a (λ, μ) -model M (i.e., a model M of cardinality λ , $|P^M| = \mu$ for a fixed unary P). Then T has a (λ, μ) -model.

A particularly important case is $\lambda = \mu^+$ in which case this can be represented as a problem on the κ -compactness of the logic $\mathbb{L}(\mathbf{Q}_\lambda^{\text{card}})$, i.e., $(\mathbf{Q}_{\geq \lambda}^{\text{card}} x)\varphi$ says that there are at least λ element x satisfying φ . We deal here only with this case. See Furkhen [Fu65], Morley and Vaught [MoVa62], Keisler [Ke70], Mitchel [Mi72]; for more history see [Sh 604].

Now two cardinal theorems can be translated to partition problems so-called identities (0.2): see [Sh 8], [Sh:E17] which also give more on compactness, lately Shelah and Vaananan [ShVa 790] or [ShVa:E47].

Restricting ourselves to pairs (μ^+, μ) , the identities of (\aleph_1, \aleph_0) were sorted out in [Sh 74], so giving an alternative proof for Vaught's two cardinal theorem. But we do not know of the identities of any really different pair (μ^+, μ) , i.e., one for which $(\aleph_1, \aleph_0) \not\rightarrow (\mu^+, \mu)$. We know that (consistently) some pairs (μ^+, μ) have a different set of identities than (\aleph_1, \aleph_0) but we do not have a characterization in any of those cases. By Mitchel [Mi72] this applies to (\aleph_2, \aleph_1) in the universe gotten by forcing: suitably collapsing of a Mahlo strongly inaccessible to \aleph_2 . The other such case is when there is a compact cardinal in the interval $(\text{cf}(\mu), \mu)$ by Litman and Shelah. So it would be nice to know (taking the extreme case):

0.1 Question: Assume μ is a singular cardinal the limit of compact and even super-compact cardinals.

- 1) What are the identities of (μ^+, μ) ? Is it the biggest one?
- 2) If $\text{cf}(\mu) = \aleph_0$, is (μ^+, μ) \aleph_0 -compact (equivalently μ -compact)?

Note that though we already know that there are some identities of (μ^+, μ) which are not identities of (\aleph_1, \aleph_0) we have no explicit example. We give here a partial solution to 0.1(1) by finding families of such identities.

Another problem is consistency of failure of compactness. In [Sh 604] we have dealt with the simplest case for pairs (λ, μ) by a reasonable criterion: including no use of large cardinals. From another perspective the simplest case is the consistency of non-compactness of $\mathbb{L}(\mathbf{Q})$, \mathbf{Q} one cardinality quantifier, and the simplest one is $\mathbf{Q} = \exists^{\geq \mu^+}$. So we are again drawn to pairs (μ^+, μ) , that is gap one instead of gap 2 as in [Sh 604], so necessarily we need to use large cardinals as if, e.g., $-0^\#$ then every such pair is compact.

0.2 Definition. 1) A partial identity¹ \mathbf{s} is a pair $(a, e) = (\text{Dom}_{\mathbf{s}}, e_{\mathbf{s}})$ where a is a

¹identification in the terminology of [Sh 8]

finite set and e is an equivalence relation on a subfamily of the family of the finite subsets of a , having the property

$$b e c \Rightarrow |b| = |c|.$$

The equivalence class of b with respect to e will be denoted b/e .

1A) We say \mathbf{s} is a full identity or identity if $\text{Dom}(e) = \mathcal{P}(a)$.

1B) We say that partial identities $\mathbf{s}_1 = (a_1, e_1), \mathbf{s}_2 = (a_2, e_2)$ are isomorphic if there is an isomorphism h from \mathbf{s}_1 onto \mathbf{s}_2 which mean that h is a one-to-one function from a_1 onto a_2 such that for every $b_1, c_1 \subseteq a_1$ we have $(b_1 e_1 c_1) \equiv h(b_1) e_2 h(c_1)$ (so h maps $\text{Dom}(e_1)$ onto $\text{Dom}(e_2)$). We define similarly “ h is an embedding of \mathbf{s}_1 into \mathbf{s}_2 ” when $b_1 e_1 c_1 \Rightarrow h(b_1) e_2 h(c_1)$.

2) We say that $\lambda \rightarrow (a, e)_\mu$, if (a, e) is an identity or a partial identity and for every function $f : [\lambda]^{<\aleph_0} \rightarrow \mu$, there is a one-to-one function $h : a \rightarrow \lambda$ such that

$$b e c \Rightarrow f(h''(b)) = f(h''(c)).$$

(Instead $\text{Rang}(f) \subseteq \mu$ we may just require $|\text{Rang}(f)| \leq \mu$, this is equivalent).

3) We define

$$\text{ID}(\lambda, \mu) =: \{(n, e) : n < \omega \ \& \ (n, e) \text{ is an identity and } \lambda \rightarrow (n, e)_\mu\}$$

and for $f : [\lambda]^{<\aleph_0} \rightarrow X$ we let

$$\begin{aligned} \text{ID}(f) =: \{ & (n, e) : (n, e) \text{ is an identity such that for some one-to-one function} \\ & h \text{ from } n = \{0, \dots, n-1\} \text{ to } \lambda \text{ we have} \\ & (\forall b, c \subseteq n)(b e c \Rightarrow f(h''(b)) = f(h''(c))) \}. \end{aligned}$$

Clearly two-place functions are easier to understand; this motivates:

0.3 Definition. 1) A two-identity or 2-identity² is a pair (a, e) where a is a finite set and e is an equivalence relation on $[a]^2$. Let $\lambda \rightarrow (a, e)_\mu$ mean $\lambda \rightarrow (a, e^+)_\mu$ where $b e^+ c \leftrightarrow [(b e c) \vee (b = c \subseteq a)]$ for any $b, c \subseteq a$.

2) We define

$$\text{ID}_2(\lambda, \mu) =: \{(n, e) : (n, e) \text{ is a 2-identity and } \lambda \rightarrow (n, e)_\mu\}$$

²it is not an identity as e is an equivalence relation on too small set but it is a partial identity

we define $ID_2(f)$ when $f : [\lambda]^2 \rightarrow X$ as

$$\left\{ (n, e) : (n, e) \text{ is a two-identity such that for some } h, \right. \\ \left. \begin{array}{l} \text{a one-to-one function from } \{0, \dots, n-1\} \text{ into } \lambda \\ \text{we have } \{\ell_1, \ell_2\} e \{k_1, k_2\} \text{ implies that } \ell_1 \neq \ell_2 \in \{0, \dots, n-1\}, \\ k_1 \neq k_2 \in \{0, \dots, n-1\} \text{ and } f(\{h(\ell_1), h(\ell_2)\}) = f(\{h(k_1), h(k_2)\}) \end{array} \right\}.$$

3) Let us define

$$ID_2^{\otimes} =: \{({}^n 2, e) : ({}^n 2, e) \text{ is a two-identity and if} \\ \{\eta_1, \eta_2\} \neq \{\nu_1, \nu_2\} \text{ are } \subseteq {}^n 2, \text{ then} \\ \{\eta_1, \eta_2\} e \{\nu_1, \nu_2\} \Rightarrow \eta_1 \cap \eta_2 = \nu_1 \cap \nu_2\}.$$

4) In parts (1) and (2) we may replace 2 by $k < \omega$ (only $k < |a_s|$ is interesting) and by $(\leq k)$.

0.4 Discussion: By [Sh 49], under the assumption $\aleph_\omega < 2^{\aleph_0}$, the families $ID_2(\aleph_\omega, \aleph_0)$ and ID_2^{\otimes} coincide (up to an isomorphism of identities). In Gilchrist and Shelah [GcSh 491] and [GcSh 583] we considered the question of the equality between these $ID_2(2^{\aleph_0}, \aleph_0)$ and ID_2^{\otimes} under the assumption $2^{\aleph_0} = \aleph_2$. We showed that consistently the answer may be “yes” and may be “no”.

Note that $(\aleph_n, \aleph_0) \not\rightarrow (\aleph_\omega, \aleph_0)$ so $ID(\aleph_2, \aleph_0) \neq ID(\aleph_\omega, \aleph_0)$, but for identities for pairs (i.e. ID_2) the question is meaningful.

We can look more at ordered identities

0.5 Definition. 1) An ord-identity or order identity is an identity \mathbf{s} such that $a_s \subseteq \text{Ord}$ or just: a is an ordered set.

2) $\lambda \rightarrow_{\text{ord}} (\mathbf{s})_\mu$ if \mathbf{s} is an ord-identity and for every $\mathbf{c} : [\lambda]^{<\aleph_0} \rightarrow \mu$ we have $\mathbf{s} \in \text{OID}(\mathbf{c})$, see below (equivalently $\text{Dom}(\mathbf{c}) = [\lambda]^{<\aleph_0}$, $|\text{Rang}(\mathbf{c})| \leq \mu$).

3) For $\mathbf{c} : [\lambda]^{<\aleph_0} \rightarrow \mu$ let $\text{OID}(\mathbf{c}) = \{(a, e) : a \text{ is a set of ordinals and there is an order preserving function } f : a \rightarrow \lambda \text{ such that } b_1 e b_2 \Rightarrow \mathbf{c}(f''(b_1)) = \mathbf{c}(f''(b_2))\}$.

4) $\text{OID}(\lambda, \mu) = \{(n, e) : (n, e) \in \text{OID}(\mathbf{c}) \text{ for every } \mathbf{c} : [\lambda]^{<\aleph_0} \rightarrow \mu\}$.

5) Similarly OID_2 , OID_k , $\text{OID}_{\leq k}$.

Of course,

0.6 Claim. 1) $ID(\lambda, \mu)$ can be computed from $OID(\lambda, \mu)$.

2) Let a be a finite set of ordinals and e an equivalence relation. If (a, e) is an identity, a a set of ordinals and $\lambda > \mu$, then $(a, e) \in ID(\lambda, \mu)$ iff for some permutation π of a we have $(a, e^\pi) \in OID(\lambda, \mu)$ where $e^\pi = \{(b, c) : (\pi''(b), \pi''(c)) \in e\}$.

3) Let A be a set of ordinals, (a, e) an ord-identity and \mathbf{c} a function with domain $[A]^{<\aleph_0}$. Then $(a, e) \in ID(\mathbf{c})$ iff for some permutation π of a , $(a, e^\pi) \in OID(\mathbf{c})$.

4) Similarly for 2-identities and k -identities and $(\leq k)$ -identities and partial identities.

0.7 Claim. If $n \in [1, \omega)$ and \mathbf{s} an ordered partial identity then there is a first order sentence $\psi_{\mathbf{s}}$ such that: $\psi_{\mathbf{s}}$ has a (μ^{+n}, μ) -model iff $\mathbf{s} \notin OID(\mu^{+n}, \mu)$.

Proof. Easy as for some first order ψ sentence if M is a (μ^{+n}, μ) -model of ψ then $<^M$ is a linear order of M (of cardinality μ^{+n}) which is μ^{+n} -like (i.e. every initial segment has cardinality). $\square_{0.7}$

We define simplicity:

0.8 Definition. 1) For $k < \aleph_0$, we say (λ, μ) has k -simple identities when $(a, e) \in ID(\lambda, \mu) \Rightarrow (a, e') \in ID(\lambda, \mu)$ whenever:

$(*)_k$ $a \subseteq \omega$, (a, e) is an identity of (λ, μ) and e' is defined by

$be'c$ iff $|b| = |c|$ and $(\forall b', c')$ [if $b' \subseteq b$ and $|b'| \leq k$ and $c' = OP''_{c,b}(b')$ then $b'ec'$];

recall $OP_{A,B}(\alpha) = \beta$ iff $\alpha \in A$ and $\beta \in B$ and $otp(\alpha \cap A) = otp(\beta \cap B)$.

2) We define “ (λ, μ) has k -simple ordered identities”, similarly.

We can ask

0.9 Question: 1) Define reasonably a pair (λ, μ) such that consistently

\otimes $ID(\lambda, \mu)$ is not recursive

\otimes' $ID(\lambda, \mu)$ is not, in a reasonable way, finitely generated.

2) Similarly for $ID_2(\lambda, \mu)$.

3) Restrict yourself to (μ^+, μ) .

§1 2-SIMPLICITY FOR GAP ONE

1.1 Claim. 1) If $\mu = \mu^{<\mu}$, then $ID_2(\mu^+, \mu)$ is 2-simple.

2) If $\mu = \mu^{<\mu}$ and $c_0 : [\mu^+]^{<\aleph_0} \rightarrow \mu$ then we can find $c^* : [\mu^+]^2 \rightarrow \mu$ such that:

- (α) if $n \in [2, \omega)$ and $\alpha_0, \dots, \alpha_{n-1} < \mu^+$ are with no repetitions and $\beta_0, \dots, \beta_{n-1} < \mu^+$ are with no repetitions and $\ell < k < n \Rightarrow c^*\{\alpha_\ell, \alpha_k\} = c^*\{\beta_\ell, \beta_k\}$ then $c_0\{\alpha_0, \dots, \alpha_{n-1}\} = c_0\{\beta_0, \dots, \beta_{n-1}\}$
- (β) if in addition $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1}$ then $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-2}$ and $\beta_{n-3} < \beta_{n-1}$.

1.2 Remark. 1) By the subsequent Garti-Shelah [GaSh 891] we cannot assume just $2^{<\mu} = \mu$. We may wonder what is the gain in 1.1 for two cardinal theorems, as if $\mu = 2^{<\mu}$ is regular then we know all relevant theory on (μ^+, μ) ? The answer is that it clarifies identities of triples (μ^+, μ, κ) , e.g.

- (a) (μ^+, μ, κ) , μ strong limit singular $> \kappa \geq \text{cf}(\mu)$
- (b) (μ^+, μ, κ) , $\mu = \mu^{\beth_\omega(\kappa)}$.

2) Replacing $\mu^+, 2$ by $\mu^{+k}, k+1 \geq 2$ is similar and easier.

Proof. 1) By part (2).

2) By subclaims 1.3 - 1.8 below the claim is easy (see details in the end).

So

⊠ for the rest of this section we assume $\mu^{<\mu} = \mu$ (though not all subclaims use it).

1.3 Subclaim. There is $c_1 : [\mu^+]^2 \rightarrow \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0, \beta_1, \beta_2 < \mu^+$ are with no repetitions and $c_1\{\beta_\ell, \beta_k\} = c_1\{\alpha_\ell, \alpha_k\}$ for $\ell < k < 3$ then at least two of the following inequalities holds $\beta_0 < \beta_1, \beta_0 < \beta_2, \beta_1 < \beta_2$.

Notice, that we have only three possibilities (not four):

- (i) $\beta_0 < \beta_1 < \beta_2$
- (ii) $\beta_1 < \beta_0 < \beta_2$
- (iii) $\beta_0 < \beta_2 < \beta_1$.

Proof. Let $\eta_\alpha \in {}^\mu 2$ for $\alpha < \mu^+$ be pairwise distinct and for $\alpha \neq \beta < \mu^+$ let $\varepsilon\{\alpha, \beta\} = \text{Min}\{\varepsilon : \eta_\alpha \upharpoonright \varepsilon \neq \eta_\beta \upharpoonright \varepsilon\}$ and define the function c'_1 with domain $[\mu^+]^2$ by $c'_1\{\alpha, \beta\} = \{\eta_\alpha \upharpoonright \varepsilon\{\alpha, \beta\}, \eta_\beta \upharpoonright \varepsilon\{\alpha, \beta\}\}$, now $|\text{Rang}(c'_1)| \leq \mu$ holds because $\mu = 2^{<\mu}$. For $\alpha \neq \beta$, let $c''_1\{\alpha, \beta\}$ be 1 if $(\eta_\alpha <_{\text{lex}} \eta_\beta) \equiv (\alpha < \beta)$ and 0 otherwise (the Sierpinski colouring). Lastly, define c_1 by: $c_1\{\alpha, \beta\} = (c'_1\{\alpha, \beta\}, c''_1\{\alpha, \beta\})$, it is a function with domain $[\mu^+]^2$ and range of cardinality $\leq \mu$ and easily it is as required.

That is

- (*)₁ if $\ell g(\eta_{\alpha_0} \cap \eta_{\alpha_1}) > \ell g(\eta_{\alpha_0} \cap \eta_{\alpha_1}) = \ell g(\eta_{\alpha_1} \cap \eta_{\aleph_2})$, then clause (i) or (ii) holds
- (*)₂ if $\ell g(\eta_{\alpha_0} \cap \eta_{\alpha_2}) > \ell g(\eta_{\alpha_0} \cap \eta_{\alpha_1}) = \ell g(\eta_{\alpha_1} \cap \eta_{\aleph_0})$, then clause (i) holds
- (*)₃ if $\ell g(\eta_{\alpha_0} \cap \eta_{\alpha_1}) = \ell g(\eta_{\alpha_0} \cap \eta_{\alpha_1}) > \ell g(\eta_{\alpha_1} \cap \eta_{\aleph_2})$, then clause (i) or (iii) holds.

□_{1.3}

1.4 Subclaim. For every $c : [\mu^+]^{<\aleph_0} \rightarrow \mu$ there is $c_2 : [\mu^+]^2 \rightarrow \mu$ such that: if $n \geq 2, \alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \mu^+, \beta_0 < \beta_1 < \dots < \beta_{n-1} < \mu^+$ and $\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_k\} = c_2\{\beta_\ell, \beta_k\}$ then $c\{\alpha_0, \dots, \alpha_{n-1}\} = c\{\beta_0, \dots, \beta_{n-1}\}$.

- 1.5 Remark. 1) This is enough for $\text{OID}(\mu^+, \mu)$ being 2-simple.
 2) Note that here we use $\mu^{<\mu} = \mu$ and not just $2^{<\mu} = \mu$.

Proof. We are given $c : [\mu^+]^{<\aleph_0} \rightarrow \mu$ and for each $\alpha < \mu^+$ let f_α be a one-to-one function from α onto the ordinal $|\alpha| \leq \mu$ and we shall use those f_α 's also later. We define an equivalence relation E on $[\mu^+]^2$

- (*) for $\alpha_1 < \beta_1 < \mu^+$ and $\alpha_2 < \beta_2 < \mu^+$ we have $\{\alpha_1, \beta_1\} E \{\alpha_2, \beta_2\}$ iff
 - (a) $f_{\beta_1}(\alpha_1) = f_{\beta_2}(\alpha_2)$ and
 - (b) for any $n < \omega$ and $\gamma_0, \dots, \gamma_{n-1} < f_{\beta_1}(\alpha_1)$ we have

$$c\{\alpha_1, \beta_1, f_{\beta_1}^{-1}(\gamma_0), \dots, f_{\beta_1}^{-1}(\gamma_{n-1})\} = c\{\alpha_2, \beta_2, f_{\beta_2}^{-1}(\gamma_0), \dots, f_{\beta_2}^{-1}(\gamma_{n-1})\}$$

and similarly if we omit α_1, α_2 and/or β_1, β_2 .

So $[\mu^+]^2/E$ has cardinality $\leq {}^\mu 2 = \mu$ and let $c_2 : [\mu^+]^2 \rightarrow \mu$ be such that $c_2\{\alpha_1, \beta_1\} = c_2\{\alpha_2, \beta_2\}$ iff $\{\alpha_1, \beta_1\}/E = \{\alpha_2, \beta_2\}/E$. We now check that it is as required in 1.4. Let $n, \langle \alpha_\ell : \ell < n \rangle, \langle \beta_\ell : \ell < n \rangle$ be as in 1.4; so $\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_k\} = c_2\{\beta_\ell, \beta_k\}$, hence by (*) (a) above (for $k = n - 1$) we have $\ell < n - 1 \Rightarrow f_{\alpha_{n-1}}(\alpha_\ell) = f_{\beta_{n-1}}(\beta_\ell)$, call it γ_ℓ ; as $f_{\alpha_{n-1}}$ is one to one, clearly

$\langle \gamma_\ell : \ell < n - 2 \rangle$ is with no repetitions. Let $\ell(*) < n$ be such that $\gamma_{\ell(*)}$ is maximal and for $\ell < n - 2$ let γ'_ℓ be γ_ℓ if $\ell < \ell(*)$ and by $\gamma'_{\ell+1}$ if $\ell \in [\ell(*), n - 1)$. Now apply $(*)$ (b) with $\alpha_{\ell(*)}, \alpha_{n-1}, \beta_{\ell(*)}, \beta_{n-2}, \langle \gamma'_\ell : \ell < n - 2 \rangle$ here standing for $\alpha_1, \beta_1, \alpha_2, \beta_2, \langle \gamma_\ell : \ell < n - 2 \rangle$ there and we get the desired result. $\square_{1.4}$

1.6 Subclaim. *In 1.4, using $f_\alpha : \alpha \rightarrow \mu$ as in its proof, we have $c\{\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}\} = c\{\beta_0, \dots, \beta_{n-2}, \beta_{n-1}\}$ also when*

- (*) $n \geq 2, \alpha_0 < \alpha_1 < \dots < \alpha_{n-3} < \alpha_{n-2} < \alpha_{n-1} < \mu^+, \beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2}$ and
 - (a) $\ell < n - 2 \Rightarrow f_{\alpha_{n-1}}(\alpha_\ell) = f_{\alpha_{n-2}}(\alpha_\ell)$
 - (b) $\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_k\} = c_2\{\beta_\ell, \beta_k\}$.

Proof. Just the same proof. $\square_{1.6}$

1.7 Subclaim. *There is $c_4 : [\mu^+]^2 \rightarrow \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0, \beta_1, \beta_2 < \mu^+$ with no repetitions and $c_4\{\beta_\ell, \beta_k\} = c_4\{\alpha_\ell, \alpha_k\}$ for $\ell < k < 3$ then $\beta_0 < \beta_1$ & $\beta_0 < \beta_2$.*

Proof. For $\alpha < \beta < \mu^+$ we let $c'\{\alpha, \beta\} = \{f_\beta(\gamma) : \gamma < \alpha \text{ \& } f_\beta(\gamma) < f_\beta(\alpha)\}$ and let $c_4\{\alpha, \beta\} = (c'\{\alpha, \beta\}, c_1\{\alpha, \beta\}, f_\beta(\alpha))$ where c_1 is from 1.3 and $\langle f_\gamma : \gamma < \mu^+ \rangle$ is from the proof of 1.4. Clearly $|\text{Rang}(c')| \leq \sum_{\zeta < \mu} 2^{|\zeta|} = \mu$ hence $|\text{Rang}(c_4)| \leq \mu^3 = \mu$.

If $\alpha_\ell, \beta_\ell (\ell < 3)$ form a counterexample, then $c_1\{\alpha_\ell, \alpha_k\} = c_1\{\beta_\ell, \beta_k\}$ for $\ell < k < 3$ hence by 1.3 we have three cases according to which (if any) one of the inequalities $\beta_\ell < \beta_k, \ell < k < 3$ fail.

Case (i): $\beta_0 < \beta_1 < \beta_2$.

Trivial: the desired conclusion holds.

Case (ii): $\beta_1 < \beta_0$ so $\beta_1 < \beta_0 < \beta_2$.

Let $\zeta_\ell = f_{\alpha_2}(\alpha_\ell)$ for $\ell = 0, 1$ hence $\zeta_0 \neq \zeta_1$ as f_{α_2} is one to one and $\zeta_\ell = f_{\beta_2}(\beta_\ell)$. Now on the one hand if $\zeta_0 < \zeta_1$ then $c'\{\alpha_1, \alpha_2\} \neq c'\{\beta_1, \beta_2\}$ (as $\zeta_0 \in c'\{\alpha_1, \alpha_2\}, \zeta_0 \notin c'\{\beta_1, \beta_2\}$), contradiction. On the other hand if $\zeta_1 < \zeta_0$ then $c'\{\alpha_0, \alpha_2\} \neq c'\{\beta_0, \beta_2\}$ (as $\zeta_1 \in c'\{\beta_0, \beta_2\}, \zeta_1 \notin c'\{\alpha_0, \alpha_2\}$), a contradiction, too.

Case (iii): $\beta_2 < \beta_1$.

By Subclaim 1.3 we have $\beta_0 < \beta_2 < \beta_1$.

This is O.K. for 1.7. $\square_{1.7}$

1.8 Subclaim. For every $c : [\mu^+]^2 \rightarrow \mu$ there is $c_5 : [\mu^+]^2 \rightarrow \mu$ such that

- (a) $c_5\{\alpha_1, \beta_1\} = c_5\{\alpha_2, \beta_2\} \Rightarrow c_2\{\alpha_1, \beta_1\} = c_2\{\alpha_2, \beta_2\}$
- (b) there are no $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0 < \beta_1 < \beta_2 < \mu^+$ such that $f_{\alpha_2}(\alpha_0) \neq f_{\alpha_1}(\alpha_0)$, $c_5\{\alpha_0, \alpha_1\} = c_5\{\beta_0, \beta_2\}$, $c_5\{\alpha_0, \alpha_2\} = c_5\{\beta_0, \beta_1\}$ and $c_5\{\alpha_1, \alpha_2\} = c_5\{\beta_1, \beta_2\}$
- (c) $c_5\{\alpha_1, \beta_1\} = c_5\{\alpha_2, \beta_2\} \Rightarrow c_4\{\alpha_1, \beta_1\} = c_4\{\alpha_2, \beta_2\}$ where c_4 is from Subclaim 1.7.

Proof. Let $\kappa = \text{cf}(\mu) \leq \mu$ and $\mu = \sum_{i < \kappa} \lambda_i$ be such that if μ is a limit cardinal then λ_i is (strictly) increasing continuous and if μ is a successor cardinal then $\mu = \lambda^+$, $\kappa = \mu$ and $\lambda_i = \lambda$ for $i < \kappa$. We can find $d : [\mu^+]^2 \rightarrow \kappa$ and \bar{g} such that

- ⊗₀ (i) for $\beta < \mu^+$, $i < \kappa$ the set $A_{\beta, i} =: \{\alpha < \beta : d\{\alpha, \beta\} \leq i\}$ has cardinality $\leq \lambda_i$
- (ii) if $\alpha < \beta < \gamma < \mu^+$ then $d\{\alpha, \gamma\} \leq \max\{d\{\alpha, \beta\}, d\{\beta, \gamma\}\}$
- (iii) \bar{g} is a sequence $\langle g_\alpha : \alpha < \mu^+ \rangle$
- (iv) $g_\alpha : \alpha \rightarrow \mu$ is one to one and $\lambda_i^+ < \mu$ & $i < \kappa$ & $\alpha < \beta \Rightarrow ((g_\beta(\alpha) < \lambda_i^+) \equiv (d\{\alpha, \beta\} \leq i))$
- (v) if $\alpha < \beta$, $d\{\alpha, \beta\} = i$ and $\lambda_i^+ = \mu$ then $g_\beta(\alpha) < d\{\alpha, \beta\}$.

[Why we can find them? By induction on $\beta < \mu^+$ by induction on $i < \mu$ for $\alpha = f_\beta^{-1}(i)$ we choose $d\{\alpha, \beta\}$ and $g_\beta(\alpha)$ as required.]

Define the functions c'_6 and c'_7 with domain $[\mu^+]^2$ as follows: if $\alpha < \beta$ then $c'_6\{\alpha, \beta\} = \{(t, \zeta_1, \zeta_2) : \zeta_1, \zeta_2 \leq g_\beta(\alpha), t < 2 \text{ and } t = 0 \Rightarrow g_\beta^{-1}(\zeta_1) < g_\beta^{-1}(\zeta_2) \text{ and } t = 1 \Rightarrow g_\beta^{-1}(\zeta_1) > g_\beta^{-1}(\zeta_2)\}$ and $c'_7\{\alpha, \beta\} = \{(t, \zeta, \xi) : \zeta \in \lambda_{d\{\alpha, \beta\}}^+ \cap \text{Rang}(g_\alpha) \text{ and } \xi \in \lambda_{d\{\alpha, \beta\}}^+ \cap \text{Rang}(g_\beta) \text{ and } [\lambda_{d\{\alpha, \beta\}}^+ = \mu \Rightarrow \zeta < d\{\alpha, \beta\} \text{ \& \ } \xi < d\{\alpha, \beta\}] \text{ and } g_\alpha^{-1}(\zeta) < g_\beta^{-1}(\xi) \text{ \& \ } t = 0 \text{ or } g_\alpha^{-1}(\zeta) = g_\beta^{-1}(\xi) \text{ \& \ } t = 1 \text{ or } g_\alpha^{-1}(\zeta) > g_\beta^{-1}(\xi) \text{ \& \ } t = 2\}$.

Now for $\alpha < \beta < \mu^+$ we define $c'_5\{\alpha, \beta\} \in \Pi\{\lambda_j^+ : j \leq d\{\alpha, \beta\}\}$, we do this by induction on β and for a fixed β by induction on $i = d\{\alpha, \beta\}$ and for a fixed β and i by induction on α .

Arriving to $\alpha < \beta$, for each $j \leq d\{\alpha, \beta\}$, let $(c'_5\{\alpha, \beta\})(j)$ be the first ordinal $\xi < \lambda_j^+$ such that:

- ⊗₁ if $\gamma < \beta$ & $d\{\gamma, \beta\} \leq j$ & $(d\{\gamma, \beta\} = d\{\alpha, \beta\} \Rightarrow \gamma < \alpha)$ then $(c'_5\{\alpha, \gamma\})(j) < \xi$.

Clearly possible. The colouring we use is c_5 where for $\alpha < \beta < \mu^+$ we let $c_5\{\alpha, \beta\} = (d\{\alpha, \beta\}, g_\beta(\alpha), f_\beta(\alpha), c_2\{\alpha, \beta\}, c'_5\{\alpha, \beta\}, c'_6\{\alpha, \beta\}, c'_7\{\alpha, \beta\}, c_4\{\alpha, \beta\})$, recalling c_4 is from Subclaim 1.7 and c_2 is from Subclaim 1.4. Obviously, $|\text{Rang}(c_5)| \leq \mu$ and clauses (a) + (c) of Subclaim 1.8 holds. So assume $\alpha_0 < \alpha_1 < \alpha_2, \beta_0 < \beta_1 < \beta_2$ form a counterexample to clause (b) of Subclaim 1.8 and we shall eventually derive a contradiction.

Clearly

$$\begin{aligned} \textcircled{*}_2 \text{ (i)} \quad & d\{\alpha_0, \alpha_2\} = d\{\beta_0, \beta_1\}, d\{\alpha_0, \alpha_1\} = d\{\beta_0, \beta_2\}, d\{\alpha_1, \alpha_2\} = d\{\beta_1, \beta_2\} \\ \text{(ii)} \quad & \text{similarly for } c_4, c'_5, c'_6, c'_7. \end{aligned}$$

By clause $\textcircled{*}_0(ii)$ above we have $d\{\alpha_0, \alpha_2\} \leq \max\{d\{\alpha_0, \alpha_1\}, d\{\alpha_1, \alpha_2\}\}$, and applying clause $\textcircled{*}_0(ii)$ to $\beta_0 < \beta_1 < \beta_2$ and using $\textcircled{*}_2$ we have $d\{\alpha_0, \alpha_1\} = d(\beta_0, \beta_2) \leq \max\{d(\beta_0, \beta_1), d(\beta_1, \beta_2)\} = \max\{d\{\alpha_0, \alpha_2\}, d\{\alpha_1, \alpha_2\}\}$. Hence $d\{\alpha_0, \alpha_1\} = d\{\alpha_0, \alpha_2\} > d\{\alpha_1, \alpha_2\}$ or $\bigwedge_{\ell=1}^2 [d\{\alpha_0, \alpha_\ell\} \leq d\{\alpha_1, \alpha_2\}]$; we deal with those two cases separately.

Case 1: $\varepsilon = d\{\alpha_0, \alpha_1\} = d\{\alpha_0, \alpha_2\} > d\{\alpha_1, \alpha_2\}$.

So (see the definition of c'_5 , with $\alpha_0, \alpha_2, \alpha_1, \varepsilon$ here standing for α, β, γ, j there recalling that $\alpha_0 < \alpha_1 < \alpha_2$) we have $\lambda_\varepsilon^+ > (c'_5\{\alpha_0, \alpha_2\})(\varepsilon) > (c'_5\{\alpha_0, \alpha_1\})(\varepsilon)$. Similarly, $\lambda_\varepsilon^+ > (c'_5\{\beta_0, \beta_2\})(\varepsilon) > (c'_5\{\beta_0, \beta_1\})(\varepsilon)$. This contradicts $c'_5\{\alpha_0, \alpha_\ell\} = c'_5\{\beta_0, \beta_{3-\ell}\}$ for $\ell = 1, 2$.

Case 2: $d\{\alpha_0, \alpha_\ell\} \leq d\{\alpha_1, \alpha_2\}$ for $\ell = 1, 2$.

Let $\varepsilon = d\{\alpha_1, \alpha_2\}$. Let $\zeta_\ell = g_{\alpha_\ell}(\alpha_0)$ for $\ell = 1, 2$ so $\zeta_\ell = g_{\beta_{3-\ell}}(\beta_0)$ for $\ell = 1, 2$. By the assumption toward contradiction, i.e., by a demand in clause (b) of 1.8 we have $\zeta_1 \neq \zeta_2$. Clearly $\zeta_\ell < \lambda_{d\{\alpha_0, \alpha_\ell\}}^+ \leq \lambda_{d\{\alpha_1, \alpha_2\}}^+ = \lambda_\varepsilon^+$ and $\lambda_\varepsilon^+ = \mu \Rightarrow \zeta_\ell < d\{\alpha_0, \alpha_\ell\} \leq d\{\alpha_1, \alpha_2\} = \varepsilon$.

As $c'_7\{\alpha_1, \alpha_2\} = c'_7\{\beta_1, \beta_2\}$ and $g_{\alpha_1}^{-1}(\zeta_1) = \alpha_0 = g_{\alpha_2}^{-1}(\zeta_2)$ clearly $g_{\beta_1}^{-1}(\zeta_1) = g_{\beta_2}^{-1}(\zeta_2)$ and they are well defined.

For $\ell = 1, 2$ as $c_5\{\alpha_0, \alpha_\ell\} = c_5\{\beta_0, \beta_{3-\ell}\}$ by the choice of ζ_ℓ (that is $\zeta_\ell = g_{\alpha_\ell}(\alpha_0)$) we have $g_{\beta_\ell}(\beta_0) = \zeta_{3-\ell}$ so $g_{\beta_\ell}^{-1}(\zeta_{3-\ell}) = \beta_0$ for $\ell = 1, 2$ hence $g_{\beta_1}^{-1}(\zeta_2) = g_{\beta_2}^{-1}(\zeta_1)$. As $c_5\{\alpha_1, \alpha_2\} = c_5\{\beta_1, \beta_2\}$ we have $c'_6\{\alpha_1, \alpha_2\} = c'_6\{\beta_1, \beta_2\}$ but $\zeta_1, \zeta_2 \leq g_{\alpha_2}(\alpha_1)$ hence

$$\textcircled{*}_3 (g_{\alpha_\ell}^{-1}(\zeta_1) < g_{\alpha_\ell}^{-1}(\zeta_2)) \equiv (g_{\beta_\ell}^{-1}(\zeta_1) < g_{\beta_\ell}^{-1}(\zeta_2)) \text{ for } \ell = 1, 2.$$

As $\zeta_1 \neq \zeta_2$ we have $g_{\alpha_1}^{-1}(\zeta_1) \neq g_{\alpha_1}^{-1}(\zeta_2)$.

By symmetry without loss of generality $\zeta_1 > \zeta_2$. We can form an equivalence chain, starting with $g_{\beta_1}^{-1}(\zeta_1) < g_{\beta_1}^{-1}(\zeta_2)$ and arriving to $g_{\beta_1}^{-1}(\zeta_2) < g_{\beta_1}^{-1}(\zeta_1)$, a clear contradiction. Well, $g_{\beta_1}^{-1}(\zeta_1) < g_{\beta_1}^{-1}(\zeta_2)$ iff $g_{\beta_2}^{-1}(\zeta_2) < g_{\beta_2}^{-1}(\zeta_1)$ (by the equalities

above) $\text{iff } g_{\alpha_2}^{-1}(\zeta_2) < g_{\alpha_2}^{-1}(\zeta_1)$ (by $\textcircled{*}_3$) $\text{iff } g_{\beta_1}^{-1}(\zeta_2) < g_{\beta_1}^{-1}(\zeta_1)$ (by $c'_6\{\alpha_0, \alpha_2\} = c'_6\{\beta_0, \beta_1\}$ and use the parameter t in the triple (t, ζ_1, ζ_2)).

So we have proved Subclaim 1.8. $\square_{1.8}$

We can now sum up, i.e.:

Proof of 1.1(2) from Subclaims 1.3-1.8. We are given $c_0 : [\mu^+]^{<\aleph_0} \rightarrow \mu$. First we apply Subclaim 1.4 for $c = c_0$ and get $c_2 : [\mu^+]^2 \rightarrow \mu$ as there and then let c_4 be as in 1.7.

Second, we apply Subclaim 1.8 for $c = c_2$ and (recalling 1.8, clause (c) relate to c_4) get c_5 as there. Let us check that c_5 is as required on c^* in 1.1(2). So assume $(*)_0 + (*)_1$ below and (as the case $n = 2$ is trivial) assume $n \geq 3$ where

- $(*)_0$ $\{\alpha_0, \dots, \alpha_{n-1}\} \in [\mu^+]^n$ and $\{\beta_0, \dots, \beta_{n-1}\} \in [\mu^+]^n$ and
- $(*)_1$ $\ell < k < n \Rightarrow c_5\{\alpha_\ell, \alpha_k\} = c_5\{\beta_\ell, \beta_k\}$.

Without loss of generality (by renaming)

- $(*)_2$ $\alpha_0 < \dots < \alpha_{n-1}$.

and it is enough to prove that $c_0\{\alpha_0, \dots, \alpha_{n-1}\} = c_0\{\beta_0, \dots, \beta_{n-1}\}$. By clause (a) of Subclaim 1.8 we have

- $(*)_3$ $\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_k\} = c_2\{\beta_\ell, \beta_k\}$.

By clause (c) of Subclaim 1.8 we have

- $(*)_4$ $\ell < k < n \Rightarrow c_4\{\alpha_\ell, \alpha_k\} = c_4\{\beta_\ell, \beta_k\}$.

Hence by Subclaim 1.7 we have

- $(*)_5$ if $\ell < k < n$ and $\ell < n - 2$ then $\beta_\ell < \beta_k$.

[Why? Apply Subclaim 1.7 to $\alpha_\ell, \alpha_{\ell+1}, \alpha_k; \beta_\ell, \beta_{\ell+1}, \beta_k$ if $\ell + 1 < k$, and apply 1.7 to $\alpha_\ell, \alpha_{\ell+1}, \alpha_{\ell+2}; \beta_\ell, \beta_{\ell+1}, \beta_{\ell+2}$ if $\ell + 1 = k$.]

So

- $(*)_6(i)$ $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-2} < \beta_{n-1}$ or
- (ii) $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2}$.

So clause (β) of 1.1 holds.

If (i) of $(*)_6$ holds, then the choice of c_2 , i.e., by Subclaim 1.4 and $(*)_3$ above we get $c_0\{\alpha_0, \dots, \alpha_{n-1}\} = c_0\{\beta_0, \dots, \beta_{n-1}\}$ so we are done. Otherwise we have (ii) of $(*)_6$ so by clause (b) of Subclaim 1.8 we have

- $(*)_7$ if $\ell < n - 2$ then $f_{\alpha_{n-1}}(\alpha_\ell) = f_{\alpha_{n-2}}(\alpha_\ell)$.

[Why? Apply clause (b) of Subclaim 1.8 to $\alpha_\ell, \alpha_{n-2}, \alpha_{n-1}; \beta_\ell, \beta_{n-1}, \beta_{n-2}$.]
So by Subclaim 1.6 we get $c_0\{\alpha_0, \dots, \alpha_{n-1}\} = c_0\{\beta_0, \dots, \beta_{n-1}\}$ finishing. $\square_{1.1}$

1.9 Claim. *Defining $ID(\lambda, \mu)$, we can restrict ourselves to $c : [\lambda]^{<\aleph_0} \rightarrow \mu$ such that $c \upharpoonright [\lambda]^1$ is constant if $cf(\lambda) > \mu$.*

1.10 Claim. *Assume $\mu = 2^{<\mu}$ and $n \in [1, \omega)$. The identities of $ID(\mu^{+n}, \mu)$ are $(n+1)$ -simple (and also $OID(\mu^+, \mu)$).*

Proof. As in 1.1, only easier in the additional cases. $\square_{2.1}$

§2 SUCCESSOR OF STRONG LIMIT ABOVE SUPERCOMPACT: 2-IDENTITIES

So we know that if μ is strong limit singular and there is a compact cardinal in $(\text{cf}(\mu), \mu)$ then $\text{ID}_2(\mu^+, \mu) \neq \text{ID}_2(\aleph_1, \aleph_0)$. It seems desirable to find explicitly such 2-identities.

The proof of the following does much more.

2.1 Claim. *Assume*

- (a) $\mathfrak{s}_k = (k + \binom{k}{2}, e_{\mathfrak{s}_k})$ where the non-singleton $e_{\mathfrak{s}_k}$ -equivalence classes are the sets:
 $\{\{\ell_0, \ell_2\} : \ell_0 < k \text{ and for some } \ell_1 \in \{\ell_0 + 1, \dots, k - 1\} \text{ we have}$
 $\ell_2 = k + \binom{\ell_1}{2} + \ell_0\}$ and $\{\{\ell_1, \ell_2\} : \ell_1 < k \text{ and for some } \ell_0 < \ell_1 \text{ we have}$
 $\ell_2 = k + \binom{\ell_1}{2} + \ell_0\}$.

We stipulate $\binom{1}{2} = 0$ here.

- (b) μ is strong limit, θ a compact cardinal and $\text{cf}(\mu) < \theta < \mu$.

Then

- 1) $\mathfrak{s}_k \in \text{ID}_2(\mu^+, \mu)$, moreover $\mathfrak{s}_k \in \text{OID}_2(\mu^+, \mu)$.
 2) $\mathfrak{s}_k \notin \text{ID}_2(\aleph_1, \aleph_0)$ for $k \geq 3$ so for $k = 3$ we have $\mathfrak{s}_k = (6, e_{\mathfrak{s}_k})$ and the non-singleton equivalence classes, $\mathfrak{a}_0 = \{\{0, 3\}, \{0, 4\}, \{1, 5\}\}$ and $\mathfrak{a}_1 = \{\{1, 3\}, \{2, 4\}, \{2, 5\}\}$.

Proof. Part (1) follows from subclaim 2.2(3) below and part (2) follows from 2.3 below. □_{2.1}

2.2 Claim. *Assume*

- (a) μ is strong limit,
 (b) θ is compact and $\text{cf}(\mu) < \theta < \mu$
 (c) $\kappa = \text{cf}(\mu)$, $\langle \lambda_i : i < \kappa \rangle$ is increasing with limit μ
 (d) $c : [\mu^+]^2 \rightarrow \mu$
 (e) $d\{\alpha, \beta\} = \text{Min}\{i : c\{\alpha, \beta\} < \lambda_i\}$.

1) We can find $i(*)$, A , f such that

- (*) (i) $i(*) < \kappa$, $A \in [\mu^+]^{\mu^+}$ and $i(*) < \kappa$
 (ii) for every set $B \subseteq A$ of cardinality $< \theta$ there are μ^+ ordinals $\gamma \in A$ satisfying $(\forall \alpha \in B)[d\{\alpha, \gamma\} = i(*)]$.

- 2) In part (1) we also have: if $A_1 \subseteq A, |A_1| \geq \beth_n(\lambda)^+$ and $\lambda_{i(*)} \leq \lambda < \mu$, then there are $\langle \gamma_\ell : \ell < n \rangle \in {}^n(\lambda_{i(*)})$ and $B \in [A_1]^\lambda$ such that for every $\alpha_0 < \dots < \alpha_{n-1}$ from B for arbitrarily large $\beta < \lambda$ we have $\ell < n \Rightarrow c\{\alpha_\ell, \beta\} = \gamma_\ell$.
- 3) $\mathbf{s}_k \in \text{ID}_2(c)$ where \mathbf{s}_k is from clause (a) of 2.1.

Proof. 1) Let D be a uniform θ -complete ultrafilter on μ^+ .

Define $f : \mu^+ \rightarrow \kappa$ by $f(\alpha) = i \Leftrightarrow \{\gamma < \mu^+ : d\{\alpha, \gamma\} = i\} \in D$, note that the function f is well defined as D is a θ -complete ultrafilter on μ^+ and $\theta > \kappa \supseteq \text{Rang}(d)$. So for some $i(*)$, the set $A =: \{\alpha < \mu^+ : f(\alpha) = i(*)\}$ belongs to D and check that $(*)$ holds, that is (i) + (ii) hold.

2) Define $c^* : [A]^n \rightarrow {}^n(\lambda_{i(*)})$ such that

- ⊗ if $\alpha_0 < \dots < \alpha_{n-1}$ are from A then for μ^+ ordinals $\beta < \mu^+$ we have $\langle c\{\alpha_\ell, \beta\} : \ell < n \rangle = c^*\{\alpha_0, \dots, \alpha_{n-1}\}$.

So $\text{Rang}(c^*)$ has cardinality $\leq (\lambda_{i(*)})^n = \lambda_{i(*)}$ hence by the Erdős-Rado theorem there is $B \subseteq A_1$ infinite (even of any pregiven cardinality $< \lambda$) such that $c^* \upharpoonright [B]^n$ is constant.

3) Straightforward: in part (2) use $n = 2, A_1 = A$ and get B and $\langle \gamma_0, \gamma_1 \rangle \in {}^2(\lambda_{i(*)})$ as there and choose $\alpha_0 < \dots < \alpha_{k-1}$ from B . Next choose α_ℓ for $\ell = 0, 1, \dots, \binom{k}{2} - 1$, choosing β_ℓ by induction on ℓ . If $\ell = \binom{\ell_1}{2} + \ell_0$ and $\ell_0 < \ell_1 < k$ choose $\beta_\ell \in A$ satisfying $\beta_\ell > \alpha_{k-1}$ and $\beta_\ell > \beta_m$ for $m < \ell$ such that $c\{\alpha_{\ell_0}, \beta_\ell\} = \gamma_0, c\{\alpha_{\ell_1}, \beta_\ell\} = \gamma_1$.

Now let $\alpha_{k+\ell} = \beta_\ell$ for $\ell < \binom{k}{2}$, and clearly $\langle \alpha_\ell : \ell < k + \binom{k}{2} \rangle$ realize the identity \mathbf{s}_k . □_{2.2}

2.3 Subclaim. 1) If $\mathbf{s} \in \text{ID}_2(\aleph_1, \aleph_0)$, then we can find a function $h : [\text{Dom}_{\mathbf{s}}]^2 \rightarrow \omega$ respecting $e_{\mathbf{s}}$ (i.e. $\{\ell_1, \ell_2\} e_{\mathbf{s}} \{\ell_3, \ell_4\} \Rightarrow h\{\ell_1, \ell_2\} = h\{\ell_3, \ell_4\}$) and there is a linear order $<$ of $\text{Dom}_{\mathbf{s}}$ satisfying:

- ⊗ for any equivalence class \mathbf{a} of $e_{\mathbf{s}}$ there are a_0, a_1 such that
- (i) a_0, a_1 are disjoint finite subsets of $\text{Dom}_{\mathbf{s}}$
 - (ii) if $\{\ell_0, \ell_1\} \in \mathbf{a}$ and $\ell_0 < \ell_1$ then $\ell_0 \in a_0$ & $\ell_1 \in a_1$
 - (iii) if $\ell_0 \neq \ell_1$ are from $a_0 \cup a_1$ and $\{\ell_0, \ell_1\} \notin \mathbf{a}$ and $\{\ell^0, \ell^1\} \in \mathbf{a}$ then $h(\{\ell_0, \ell_1\}) > h(\{\ell^0, \ell^1\})$.

2) We can add in ⊗

- (iv) if $\mathbf{a}_0, \mathbf{a}_1$ are distinct $e_{\mathbf{s}}$ -equivalence classes then for some $m \in \{0, 1\}$ we have $[\cup \mathbf{a}_m]^2 \setminus \mathbf{a}_m$ is disjoint to \mathbf{a}_{1-m}

(v) *in* \otimes above a_0, a_1 can be defined as $\{\ell_0 : \{\ell_0, \ell_1\} \in \mathbf{a}, \ell_0 < \ell_1\}, \{\ell_1 : \{\ell_0, \ell_1\} \in \mathbf{a}, \ell_0 < \ell_1\}$ respectively.

3) If $k \geq 3$, \mathbf{s}_k from 2.1 clause (a) then \mathbf{s}_k does not belong to $\text{ID}_2(\aleph_1, \aleph_0)$.

Proof. 1) Remember that by 0.6 we can deal with $\text{OID}(\aleph_1, \aleph_0)$. By [Sh 74] we know what is $\text{OID}(\aleph_1, \aleph_0)$, i.e., the family of identities in $\text{OID}(\aleph_1, \aleph_0)$ is generated by two operations; one is called duplication and the other called restriction (see below) from the trivial identity (i.e. $|\text{dom}_{\mathbf{s}}| = 1$) and we prove \otimes by induction on n , the number of times we need to apply the operations.

Recall that (a, e) is gotten by duplication if we can find sets a_0, a_1, a_2 and a function g such that

- $\otimes_1(a)$ $a_0 < a_1 < a_2$ (i.e. $\ell_0 \in a_0, \ell_1 \in a_1, \ell_2 \in a_2 \Rightarrow \ell_0 < \ell_1 < \ell_2$)
- (b) $a = a_0 \cup a_1 \cup a_2$
- (c) g a one-to-one order preserving function from $a_0 \cup a_1$ onto $a_0 \cup a_1$ (so $g \upharpoonright a_0 = \text{id}_{a_0}$; let $g_1 = g, g_2 = g^{-1}$)
- (d) for $\ell_0 \neq \ell_1 \in (a_0 \cup a_1)$ we have $\{\ell_0, \ell_1\}e\{g(\ell_0), g(\ell_1)\}$
- (e) if $\ell_1 \in a_1, \ell_2 \in a_2$ then $\{\ell_1, \ell_2\}/e$ is a singleton
- (f) $\mathbf{s}_\ell = (a_0 \cup a_\ell, e \upharpoonright [a_0 \cup a_\ell]^2)$ is from a lower level (up to isomorphism), for $\ell \in \{1, 2\}$.

Recall that (a, e) is gotten by restriction from (a', e') if $a \subseteq a', e = e' \upharpoonright [a]^2$.

Now we prove the existence of h as required by induction on the level. If $|\text{Dom}_{\mathbf{s}}| = 1$ this is trivial. If \mathbf{s} is gotten by restriction it is trivial too, (as if $\mathbf{s} = (a, e), \mathbf{s}' = (a', e'), a' \subseteq a, e' = e \upharpoonright a'$ and $h : [a]^2 \rightarrow \omega$ is as guaranteed then we let $h'(\{\ell_0, \ell_1\}) = h(\{\ell_0, \ell_1\})$ for $\ell_0 < \ell_1$ from a'' . Easily h' is as required). So assume $\mathbf{s} = (a, e)$ is gotten by duplication, so let a_0, a_1, a_2, g_1, g_2 be as in \otimes_1 and let h_1 be as required for $\mathbf{s}_1 = (a_0 \cup a_1, e \upharpoonright [a_0 \cup a_1]^2)$ and similarly define h_2 by $h_2\{\alpha, \beta\} = h_1\{g_2(\alpha), g_2(\beta)\}$. Let $n^* = \sup \text{Rang}(h_1)$ and define $h : [a_0 \cup a_1 \cup a_2]^2 \rightarrow \omega$ by $h \supseteq h_1, h \supseteq h_2$ and if $k \in a_1, \ell \in a_2$ then we let $h\{k, \ell\} = n^* + 1$. Now check.

2) Again by induction, for clause (iv), by symmetry, without loss of generality $h(\mathbf{a}_0) < h(\mathbf{a}_1)$ and now $m = 1$ satisfies the requirement by applying \otimes_1 to the equivalence class $\mathbf{a} = \mathbf{a}_1$.

3) It is enough to deal with \mathbf{s}_3 . That is, let $\mathbf{a}_0, \mathbf{a}_1$ be from ?(2) and apply clause \rightarrow scite{2.1} undefined

(iv) but $\cup \mathbf{a}_0 = \{0, 1, 3, 4, 5\}, \cup \mathbf{a}_1 = \{1, 2, 3, 4, 5\}$ and

(*)₁ $\{1, 3\} \subseteq \cup \mathbf{a}_0, \{1, 3\} \notin \mathbf{a}_0$ and $\{1, 3\} \in \mathbf{a}_1$

(*)₂ $\{1, 5\} \subseteq \cup \mathbf{a}_1, \{1, 5\} \notin \mathbf{a}_1$ and $\{1, 5\} \in \mathbf{a}_0$

□_{2.3}

The following is like 2.1 with μ just limit (not necessarily a strong limit cardinal) so

2.4 Claim. *Assume*

- (a) $\mathbf{s}'_n \in \text{OID}_2$ is $(2n + n^2, e_{\mathbf{s}'_n})$ where the non-singleton $e_{\mathbf{s}'_n}$ -equivalence classes are
 $\{\{\ell_0, 2n + n\ell_0 + \ell_1\} : \ell_0, \ell_1 < n\}$ and
 $\{\{n + \ell_1, 2n + n\ell_0 + \ell_1\} : \ell_0, \ell_1 < n\}$
- (b) μ is a limit cardinal, $\mu > \theta > \text{cf}(\mu)$ and θ is a compact cardinal
- (c) $\mathbf{s}''_n \in \text{OID}_2$ is $(2^n + 2^{2n}, e_{\mathbf{s}''_n})$ where the non-singleton $e_{\mathbf{s}''_n}$ -equivalence classes are: for $m < n, \eta \in {}^m 2, i = 0, 1$ let $\mathbf{a}_\eta^i = \{\{\ell_i, 2^n + \binom{2^n}{\ell_0} + \ell_1\} : \ell_0, \ell_1 < 2^n$ and for some $\nu_0, \nu_1 \in {}^n 2$ we have $\eta \hat{\ } \langle 0 \rangle \trianglelefteq \nu_0, \eta \hat{\ } \langle 1 \rangle \trianglelefteq \nu_1$ and $\ell_0 = \Sigma\{\nu_0(j)2^j : j < n\}$ and $\ell_1 = \Sigma\{\nu_1(j)2^j : j < n\}$.

Then

- 1) $\mathbf{s}'_n \in \text{ID}_2(\mu^+, \mu)$, moreover $\mathbf{s}'_n \in \text{OID}_2(\mu^+, \mu)$; similarly for \mathbf{s}''_n .
- 2) $\mathbf{s}'_n \notin \text{ID}_2(\aleph_1, \aleph_0)$ for $n \geq 2$; similarly for \mathbf{s}''_n .

Proof. 1) Like the proof of 2.2 using [Sh 49] (or just [Sh 604, 5.13]) instead of the Erdős-Rado theorem.

2) Otherwise there is $(a, e) \in \text{ID}_2(\aleph_1, \aleph_0)$ and an embedding h of \mathbf{s}'_n into (a, e) and by 0.6 without loss of generality $(a, e) \in \text{OID}_2(\aleph_1, \aleph_0)$. Now

- (*)₁ if $\ell_0 < n, \ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$ then $h(\ell_0) < h(\ell)$.
 [Why? Choose $\ell'_1 < n, \ell'_1 \neq \ell_1$ and $\ell' = 2n + n\ell_0 + \ell'_1$, so $\ell \neq \ell'$ and $\{\ell_0, \ell\} e_{\mathbf{s}'_n} \{\ell_0, \ell'\}$ hence the pairs $\{h(\ell_0), h(\ell)\}, \{h(\ell_0), h(\ell')\}$ are e -equivalent and $h(\ell) \neq h(\ell')$. But on (a, e) we know that if $\{m_0, m_1, m_2\}$ has three members and $\{m_0, m_1\} e \{m_0, m_2\}$ then $m_2 < m_1 < m_0$ and $m_2 < m_0 < m_1$ are impossible (see 2.5(2) below) so we are done.]
- (*)₂ if $\ell_0 < n, \ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$ then $h(\ell_1) < h(\ell)$.
 [Why? Like (*)₁.]

Now we apply 2.3(1) + (2) above so $\mathbf{s}'_n \notin \text{ID}_2(\aleph_1, \aleph_0)$. The conclusion about \mathbf{s}''_n follows. □_{2.4}

2.5 Observation. 1) If $k \geq 2, \mathbf{s} = (n, e) \in \text{OID}_2(\mu^+, \mu)$ then we can find $\mathbf{s}' = (n', e')$ in fact $n' = 2n - 1$ such that:

- (i) $e' \upharpoonright [n]^2 = e$
 - (ii) $\mathbf{s}' \in \text{ID}(\mu^+, \mu)$
 - (iii) for every $c : [\mu^+]^{<\aleph_0} \rightarrow \mu$ there is $c' : [\mu^+]^{<\aleph_0} \rightarrow \mu$ refining c (i.e. $c'(u_1) = c'(u_2) \Rightarrow c(u_1) = c(u_2)$) such that: if $h : \{0, \dots, 2n-2\} \rightarrow \mu^+$ is one to one and satisfies $u_1 e' u_2 \Rightarrow c'(h''(u_1)) = c'(h''(u_2))$ then $h \upharpoonright \{0, \dots, n-1\}$ is increasing.
- 2) There is $c : [\mu^+]^2 \rightarrow \mu$ such that:
 if α, β, γ are distinct and $c\{\alpha, \beta\} = c\{\alpha, \gamma\}$ then $\alpha < \beta$ & $\alpha < \gamma$.
- 3) We can replace in (1), (μ^+, μ) by (λ, μ) if there is $\mathbf{s} = (n, e) \in \text{ID}(\lambda, \mu)$ such that for some $c : [\lambda]^{<\aleph_0} \rightarrow \mu$ such that
- ⊗ if $h : n \rightarrow \lambda$ induces $e_{\mathbf{s}}$ then $h(0) < h(1)$.

Proof. 1) Define $e' : u_1 e' u_2$ iff $u_1 e u_2 \vee u_1 = u_2 \vee \bigvee_{\ell < n-1} (u_1 = \{\ell, n + \ell + 1\} \ \& \ u_2 = \{\ell, \ell + 1\}) \vee \bigvee_{\ell < n-1} (u_2 = \{\ell, n + \ell + 1\} \ \& \ u_1 = \{\ell, \ell + 1\})$. Now use (2).

2) Let $f_\alpha : \alpha \rightarrow \mu$ be one to one for $\alpha < \mu^+$ and let $<^*$ a dense linear order on μ^+ with $\{\alpha : \alpha < \mu\}$ a dense subset. Now choose $c_1 : [\mu^+]^2 \rightarrow \mu$ such that $\alpha <^* \beta \Rightarrow \alpha <^* c_1\{\alpha, \beta\} <^* \beta$ and define $c_0 : [\mu^+]^2 \rightarrow \{0, 1\}$ by $c_0\{\alpha, \beta\} = 1 \Leftrightarrow (\alpha < \beta \equiv \alpha <^* \beta)$.

Lastly, let $c : [\mu^+]^2 \rightarrow \mu$ be $\alpha < \beta \Rightarrow c\{\alpha, \beta\} = \text{pr}(2f_\beta(\alpha) + c_0\{\alpha, \beta\}, c_1\{\alpha, \beta\})$ for some pairing function pr.

3) Similar to part (1) only $|\text{Dom}_{\mathbf{s}'}|$ is larger. □_{2.5}

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