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THE SPECTRUM OF CHARACTERS OF ULTRAFILTERS ON ω SH846

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ABSTRACT. We show the consistency of statement: "the set of regular cardinals which are the character of some ultrafilter on ω is not convex". We also deal with the set of π-characters of ultrafilters on ω.

Key words and phrases. characters, ultrafilter, forcing, set theory.

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§0 Introduction

Some cardinal invariants of the continuum are actually the minimum of a natural set of cardinals $\leq 2^{\aleph_0}$ which can be called the spectrum of the invariant. Such a case is Sp_χ, the set of characters $\chi(D)$ of non-principal ultrafilters on ω (the minimal number of generators). On the history see [BnSh 642]; there this spectrum and others were investigated and it was asked if Sp_χ can be non-convex (formally 0.2(2) below).

The main result here is 1.1, it solves the problem (starting with a measurable). This was presented in a conference in honor of Juhasz, quite fitting as he had started the investigation of consistency on $\chi(D)$. In §2 we note what we can say on the strict π-character of ultrafilters.

The investigation is continued in [Sh:915] trying to get more "disorderly" behaviours in smaller cardinals and in particular answering negatively the original question, 0.2(2).

Recall

0.1 Definition. 1) $\mathrm{Sp}_\chi = \mathrm{Sp}(\chi)$ is the set of cardinals θ such that: $\theta = \chi(D)$ for some non-principal ultrafilter D on ω where
2) For D an ultrafilter on ω let $\theta = \chi(D)$ be the minimal cardinality θ such that D is generated by some family of θ members, i.e. $\mathrm{Min}\{|\mathscr{A}| : \mathscr{A} \subseteq D$ and $(\forall B \in D)(\exists A \in \mathscr{A})[A \subseteq^* B]\}$, it does not matter if we use "$A \subseteq B$".

Now, Brendle and Shelah [BnSh 642, Problem 5], asked the question formulated in 0.2(2) below, but it seems to me, at least now that the question is really 0.2(1)+(3).

<u>0.2 Problem</u> 1) Can $\mathrm{Sp}(\chi) \cap \mathrm{Reg}$ have gaps, i.e., can it be that $\theta < \mu < \lambda$ are regular, $\theta \in \mathrm{Sp}(\chi), \mu \notin \mathrm{Sp}(\chi), \lambda \in \mathrm{Sp}(\chi)$?
2) In particular does $\aleph_1, \aleph_3 \in \mathrm{Sp}(\chi)$ imply $\aleph_2 \in \mathrm{Sp}(\chi)$?
3) Are there any restrictions on $\mathrm{Sp}(\chi) \cap \mathrm{Reg}$?

We thank the referee for helpful comments and in particular 2.5(1).

<u>Discussion</u>: This rely on [Sh 700, §4], there is no point to repeat it but we try to give a description. Let $\aleph_0 < \kappa < \mu < \lambda$ be regular cardinals, κ measurable.

Let $S = \{\alpha < \lambda : \mathrm{cf}(\alpha) \neq \kappa\}$ or any unbounded subset of it. We define ([Sh 700, 4.3]) the class $\mathfrak{K} = \mathfrak{K}_{\lambda,S}$ of objects $\mathfrak{t}$ approximating our final forcing. Each $\mathfrak{t} \in K$ consists mainly of a finite support iteration $\langle \mathbb{P}^{\mathfrak{t}}_i, \underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_i : i < \mu\rangle$ of c.c.c. forcing of cardinality $\leq \lambda$ with limit $\mathbb{P}^*_{\mathfrak{t}} = \mathbb{P}^{\mathfrak{t}} = \mathbb{P}^{\mathfrak{t}}_\mu$, but also $Q^{\mathfrak{t}}_i$-names $\underset{\sim}{\tau}^{\mathfrak{t}}_i$ $(i < \mu)$ so formally

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$\mathbb{P}^{\mathfrak{t}}_{i+1}$-names, satisfying a strong version of the c.c.c. and for $i \in S$, also $\underset{\sim}{D}^{\mathfrak{t}}_i$, a $\mathbb{P}^t_i$-name of a non-principal ultrafilter on ω from which $\mathbb{Q}^{\mathfrak{t}}_i$ is nicely defined and $\underset{\sim}{A}^{\mathfrak{t}}_i$, a $\underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_i$-name (so $\mathbb{P}^t_{i+1}$-name) of a pseudo-intersection (and $\mathbb{Q}_i, i \in S$, nicely defined) of $\underset{\sim}{D}^t_i$ such that $i < j \in S \Rightarrow \underset{\sim}{A}^{\mathfrak{t}}_i \in D^{\mathfrak{t}}_j$. So $\{\underset{\sim}{A}_i : i \in S\}$ witness $\mathfrak{u} \le \mu$ in $\mathbf{V}^{\mathbb{P}_{\mathfrak{t}}}$; not necessarily we have to use nicely defined $\mathbb{Q}_i$, though for $i \in S$ we do.

The order $\le_{\mathfrak{K}}$ is natural order, we prove the existence of the so-called canonical limit.

Now a major point of [Sh 700] is: for $\mathfrak{s} \in \mathfrak{K}$ letting $\mathscr{D}$ be a uniform κ-complete ultrafilter on κ, (or just κ_1-complete $\aleph_0 < \theta < \kappa$), we can consider $\mathfrak{t} = \mathfrak{s}^\kappa/\mathscr{D}$; by Łos theorem, more exactly by Hanf's Ph.D. Thesis, (the parallel of) Łos theorem for $\mathbb{L}_{\kappa,\kappa}$ apply, it gives that $\mathfrak{t} \in \mathfrak{K}$, well if $\lambda = \lambda^\kappa/\mathscr{D}$; and moreover $\mathfrak{s} \le_{\mathfrak{K}} \mathfrak{t}$ under the canonical embedding.

The effect is that, e.g. being "a linear order having cofinality $\theta \ne \kappa$" is preserved, even by the same witness whereas having cardinality $\theta < \lambda$ is not necessarily preserved, and sets of cardinality $\ge \kappa$ are increased. As $\mathfrak{d}$ is the cofinality (not of a linear order but) of a partial order there are complications, anyhow as $\mathfrak{d}$ is defined by cofinality whereas $\mathfrak{a}$ by cardinality of sets this helps in [Sh 700], noting that as we deal with c.c.c. forcing, names reals are represented by ω-sequences of conditions, the relevant thing are preserved. So we use a $\le_{\mathfrak{K}}$-increasing sequence $\langle \mathfrak{t}_\alpha : \alpha \le \lambda \rangle$ such that for unboundedly many $\alpha < \lambda, \mathfrak{t}_{\alpha+1}$ is essentially $(\mathfrak{t}^\alpha_\alpha)^\kappa/\mathscr{D}$.

What does "nice" $\mathbb{Q} = \mathbb{Q}(D)$, for D a non-principal ultrafilter over ω mean? We need that

- (α) $\mathbb{Q}$ satisfies a strong version of the c.c.c.
- (β) the definition commute with the ultra-power used
- (γ) if $\mathbb{P}$ is a forcing notion then we can extend D to an ultrafilter $\underset{\sim}{D}^+$ for every (or at least some) $\mathbb{P}$-name of an ultrafilter $\underset{\sim}{D}$ extending D we have $\mathbb{Q}(D) \lessdot \mathbb{P} * \mathbb{Q}(\underset{\sim}{D}^+)$ (used for the existence of canonical limit).

Such a forcing is combining Laver forcing and Mathias forcing for an ultrafilter D on ω, that is: if $p \in D$ iff p is a subtree of ω with trunk $\mathrm{tr}(p) \in p$ such that for $\eta \in p$ we have $\ell g(\eta) < \ell g(\mathrm{tr}(p)) \to (\exists! n)(\eta\hat{\ }\langle n \rangle \in p)$ and $\ell g(\eta) \ge \ell g(\mathrm{tr}(p)) \Rightarrow \{n : \eta\hat{\ }\langle n \rangle \in p\} \in D$.

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§1 Using measurables and FS iterations with non-transitive memory

We use [Sh 700] in 1.1 heavily. We use measurables (we could have used extenders to get more). The question on $\aleph_1, \aleph_2, \aleph_3$, i.e. Problem 0.2(2) remains open.

1.1 Theorem. *There is a* c.c.c. *forcing notion* $\mathbb{P}$ *of cardinality* λ *such that in* $\mathbf{V}^{\mathbb{P}}$ *we have* $\mathfrak{a} = \lambda, \mathfrak{b} = \mathfrak{d} = \mu, \mathfrak{u} = \mu, \{\mu, \lambda\} \subseteq \mathrm{Sp}_\chi$ *but* $\kappa_2 \notin \mathrm{Sp}(\chi)$ *if*

$\circledast$ κ_1, κ_2 *are measurable and* $\kappa_1 < \mu = \mathrm{cf}(\mu) < \kappa_2 < \lambda = \lambda^\mu = \lambda^{\kappa_2} = \mathrm{cf}(\lambda)$.

Proof. Let $\mathscr{D}_\ell$ be a normal ultrafilter on κ_ℓ for $\ell = 1, 2$. Repeat [Sh 700, §4] with (κ_1, μ, λ) here standing for (κ, μ, λ) there, getting $\mathfrak{t}_\alpha \in \mathfrak{K}$ for $\alpha \le \lambda$ which is $\le_{\mathfrak{K}}$-increasing and letting $\mathbb{P}^\alpha_i = \mathbb{P}^{\mathfrak{t}_\alpha}_i$ we have $\bar{\mathbb{Q}}^\alpha = \langle \mathbb{P}^\alpha_\varepsilon : \varepsilon < \mu \rangle$ is a $\lessdot$-increasing continuous sequence of c.c.c. forcing notions, $\mathbb{P}^\alpha_\mu = \mathbb{P}^\alpha = \mathbb{P}_{\mathfrak{t}_\alpha} := \mathrm{Lim}(\bar{\mathbb{Q}}^\alpha) = \cup\{\mathbb{P}^\alpha_\varepsilon : \varepsilon < \mu\}$; in fact $\langle \mathbb{P}^\alpha_\varepsilon, \underset{\sim}{\mathbb{Q}}^\alpha_\varepsilon : \varepsilon < \mu \rangle$ is an FS iterations, etc., but add the demand that for unboundedly many $\alpha < \lambda$

$\boxtimes^1_\alpha$ $\mathbb{P}^{\alpha+1}$ is isomorphic to the ultrapower $(\mathbb{P}^\alpha)^{\kappa_2}/\mathscr{D}_2$, by an isomorphism extending the canonical embedding.

More explicitly we choose $\mathfrak{t}_\alpha$ by induction on $\alpha \le \lambda$ such that

$\circledast_1$ (a) $\mathfrak{t}_\alpha \in \mathfrak{K}$, see Definition [Sh 700, 4.3] so the forcing notion $\mathbb{P}^{\mathfrak{t}_\alpha}_i$ for $i \le \mu$ is well defined and is $\lessdot$-increasing with i

(b) $\langle \mathfrak{t}_\beta : \beta \le \alpha \rangle$ is $\le_{\mathfrak{K}}$-increasing continuous which means that:

(α) $\gamma \le \beta \le \alpha \Rightarrow \mathfrak{t}_\gamma \le_{\mathfrak{K}} \mathfrak{t}_\beta$, see Definition [Sh 700, 4.6](1) so $\mathbb{P}^{\mathfrak{t}_\gamma}_i \lessdot \mathbb{P}^{\mathfrak{t}_\beta}_i$ for $i \le \mu$

(β) if α is a limit ordinal then $\mathfrak{t}_\alpha$ is a canonical $\le_{\mathfrak{K}}$-u.b. of $\langle \mathfrak{t}_\beta : \beta < \alpha \rangle$, see Definition [Sh 700, 4.6](2)

(c) if $\alpha = \beta + 1$ and $\mathrm{cf}(\beta) \ne \kappa_2$ <u>then</u> $\mathfrak{t}_\alpha$ is essentially $\mathfrak{t}^{\kappa_1}_\beta/\mathscr{D}_1$ (i.e. we have to identify $\mathbb{P}^{\mathfrak{t}_\beta}_\varepsilon$ with its image under the canonical embedding of it into $(\mathbb{P}^{\mathfrak{t}_\beta}_\varepsilon)^{\kappa_1}/\mathscr{D}_1$, in particular this holds for $\varepsilon = \mu$, see Subclaim [Sh 700, 4.9])

(d) if $\alpha = \beta + 1$ and $\mathrm{cf}(\beta) = \kappa_2$ <u>then</u> $\mathfrak{t}_\alpha$ is essentially $\mathfrak{t}^{\kappa_2}_\beta/\mathscr{D}_2$.

So we need

$\circledast_2$ Subclaim [Sh 700, 4.9] applies also to the ultrapower $\mathfrak{t}^{\kappa_2}_\beta/D$. [Why? The same proof applies as $\mu^{\kappa_2}/\mathscr{D}_2 = \mu$, i.e., the canonical embedding of μ into $\mu^{\kappa_2}/\mathscr{D}_2$ is one-to-one and onto (and $\lambda^{\kappa_1}/\mathscr{D}_1 = \lambda^{\kappa_2}/\mathscr{D}_2 = \lambda$, of course).]

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Let $\mathbb{P}^\alpha_\varepsilon = \mathbb{P}^{\mathfrak{t}_\alpha}_\varepsilon$ for $\varepsilon \le \mu$ so $\mathbb{P}^\alpha = \cup\{\mathbb{P}^\alpha_\varepsilon : \varepsilon < \mu\}$ and $\mathbb{P} = \mathbb{P}^\lambda$. It is proved in [Sh 700, 4.10] that in $\mathbf{V}^{\mathbb{P}}$, by the construction, $\mu \in \text{Sp}(\chi), \mathfrak{a} \le \lambda$ and $\mathfrak{u} = \mu, 2^{\aleph_0} = \lambda$. By [Sh 700, 4.11] we have $\mathfrak{a} \ge \lambda$ hence $\mathfrak{a} = \lambda$, and always $2^{\aleph_0} \in \text{Sp}(\chi)$ hence $\lambda = 2^{\aleph_0} \in \text{Sp}(\chi)$. So what is left to be proved is $\kappa_2 \notin \text{Sp}(\chi)$. Assume toward contradiction that $p^* \Vdash$ "$\underset{\sim}{D}$ is a non-principal ultrafilter on ω and $\chi(\underset{\sim}{D}) = \kappa_2$ and let it be exemplified by $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \kappa_2\rangle$".

Without loss of generality $p^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_\varepsilon \in \underset{\sim}{D}$ does not belong to the filter on ω generated by $\{\underset{\sim}{A}_\zeta : \zeta < \varepsilon\} \cup \{\omega\backslash n : n < \omega\}$, for each $\varepsilon < \kappa_2$ and trivially also $\omega\backslash\underset{\sim}{A}_\varepsilon$ does not belong to this filter".

As λ is regular $> \kappa_2$ and the forcing notion $\mathbb{P}^\lambda$ satisfies the c.c.c., clearly for some $\alpha < \lambda$ we have $p^* \in \mathbb{P}^\alpha$ and $\varepsilon < \kappa_2 \Rightarrow \underset{\sim}{A}_\varepsilon$ is equivalent to a $\mathbb{P}^\alpha$-name.

So for every $\beta \in [\alpha, \lambda)$ we have

$\boxtimes^2_\beta$ $p^* \Vdash_{\mathbb{P}^\beta}$ "for each $i < \kappa_2$ the set $\underset{\sim}{A}_i \in [\omega]^{\aleph_0}$ is not in the filter on ω which $\{\underset{\sim}{A}_j : j < i\} \cup \{\omega\backslash n : n < \omega\}$ generates, and also the complement of $\underset{\sim}{A}_i$ is not in this filter (as $\underset{\sim}{D}$ exemplifies this)".

But for some such β, the statement $\boxtimes^1_\beta$ holds, i.e. $\circledast_1(d)$ apply, so in $\mathbb{P}^{\beta+1}$ which essentially is a $(\mathbb{P}^\beta)^{\kappa_2}/\mathscr{D}_2$ we get a contradiction. That is, let $\mathbf{j}_\beta$ be an isomorphism from $\mathbb{P}^{\beta+1}$ onto $(\mathbb{P}^\beta)^{\kappa_2}/\mathscr{D}_2$ which extends the canonical embedding of $\mathbb{P}^\beta$ into $(\mathbb{P}^\beta)^{\kappa_2}/\mathscr{D}_2$. Now $\mathbf{j}_\beta$ induces a map $\hat{\mathbf{j}}_\beta$ from the set of $\mathbb{P}^{\beta+1}$-names of subsets of ω into the set of $(\mathbb{P}^\beta)^{\kappa_2}/\mathscr{D}_2$-names of subsets of ω, and let $\underset{\sim}{A}^* = \hat{\mathbf{j}}_\beta^{-1}(\langle \underset{\sim}{A}_i : i < \kappa_2\rangle/\mathscr{D}_2)$ so $p^* \Vdash_{\mathbb{P}^{\beta+1}}$ "$\underset{\sim}{A}^* \in [\omega]^{\aleph_0}$ and the sets $\underset{\sim}{A}^*, \omega\backslash\underset{\sim}{A}^*$ do not include any finite intersection of some members of $\{\underset{\sim}{A}_\varepsilon : \varepsilon < \kappa_2\} \cup \{\omega\backslash n : n < \omega\}$". So $p^* \Vdash_{\mathbb{P}^{\beta+1}}$ "$\{\underset{\sim}{A}_\varepsilon : \varepsilon < \kappa_2\}$ does not generate an ultrafilter on ω" but $\mathbb{P}^{\beta+1} \lessdot \mathbb{P}$, contradiction. $\square_{1.1}$

1.2 Remark. 1) As the referree pointed out we can in 1.1, if we waive "$\mathfrak{u} < \mathfrak{a}$" we can forget κ_1 (and $\mathscr{D}_1$) so not taking ultra-powers by $\mathscr{D}_1$, so $\mu = \aleph_0$ is allowed, but we have to start with $\mathfrak{t}_0$ such that $\mathbb{P}^{\mathfrak{t}_0}_0$ is adding κ_2-Cohen.
2) Moreover, in this case we can demand that $\underset{\sim}{\mathbb{Q}}^{\mathfrak{t}}_\alpha = \mathbb{Q}(\underset{\sim}{D}^{\mathfrak{t}}_\alpha)$ and so we do not need the $\underset{\sim}{\tau}^{\mathfrak{t}}_\alpha$. Still this way was taken in [Sh:915, §1]. But this gain in simplicity has a price in lack of flexibility in choosing the $\mathfrak{t}$. We use this mildly in §2; mildly as only for $\mathbb{P}_1$. See more in [Sh:915, §2,§3].

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§2 Remarks on π-bases

2.1 Definition. 1) $\mathscr{A}$ is a π-base <u>if</u>:

- (a) $\mathscr{A} \subseteq [\omega]^{\aleph_0}$
- (b) for some ultrafilter D on $\omega, \mathscr{A}$ is a π-base of D, see below, note that D is necessarily non-principal

1A) We say $\mathscr{A}$ is a π-base of D if $(\forall B \in D)(\exists A \in \mathscr{A})(A \subseteq^* B)$.
1B) $\pi\chi(D) =$ Min$\{|\mathscr{A}| : \mathscr{A}$ is a π-base of $D\}$.
2) $\mathscr{A}$ is a strict π-base <u>if</u>:

- (a) $\mathscr{A}$ is a π-base of some D
- (b) no subset of $\mathscr{A}$ of cardinality $< |\mathscr{A}|$ is a π-base.

3) D has a strict π-base <u>when</u> D has a π-base $\mathscr{A}$ which is a strict π-base.
4) $\mathrm{Sp}^*_{\pi\chi} = \{|\mathscr{A}|$: there is a non-principal ultrafilter D on ω such that $\mathscr{A}$ is a strict π-base of $D\}$.

2.2 Definition. For $\mathscr{A} \subseteq [\omega]^{\aleph_0}$ let $\mathrm{Id}_{\mathscr{A}} = \{B \subseteq \omega$: for some $n < \omega$ and partition $\langle B_\ell : \ell < n\rangle$ of B for no $A \in \mathscr{A}$ and $\ell < n$ do we have $A \subseteq^* B_\ell\}$.

2.3 Observation. For $\mathscr{A} \subseteq [\omega]^{\aleph_0}$ we have:

- (a) $\mathrm{Id}_{\mathscr{A}}$ is an ideal on $\mathscr{P}(\omega)$ including the finite sets, though may be equal to $\mathscr{P}(\omega)$
- (b) if $B \subseteq \omega$ then: $B \in [\omega]^{\aleph_0} \backslash \mathrm{Id}_{\mathscr{A}}$ iff there is a (non-principal) ultrafilter D on ω to which B belongs and $\mathscr{A}$ is a π-base of D
- (c) $\mathscr{A}$ is a π-base iff $\omega \notin \mathrm{Id}_{\mathscr{A}}$.

Proof.

<u>Clause (a)</u>: Obvious.

<u>Clause (b)</u>:

<u>The "if" direction</u>: Let D be a non-principal ultrafilter on ω such that $B \in D$ and $\mathscr{A}$ is a π-base of D. Now for any $n < \omega$ and partition $\langle B_\ell : \ell < n\rangle$ of B as $B \in D$ and D is an ultrafilter clearly there is $\ell < n$ such that $B_\ell \in D$ hence by Definition 2.1(1A) there is $A \in \mathscr{A}$ such that $A \subseteq^* B_\ell$. By the definition of $\mathrm{Id}_{\mathscr{A}}$ it follows that $B \notin \mathrm{Id}_{\mathscr{A}}$ but $[\omega]^{<\aleph_0} \subseteq \mathrm{Id}_{\mathscr{A}}$ so we are done.

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The "only if" direction: So we are assuming $B \notin \text{Id}_{\mathscr{A}}$ so as $\text{Id}_{\mathscr{A}}$ is an ideal of $\mathscr{P}(\omega)$ there is an ultrafilter D on ω disjoint to $\text{Id}_{\mathscr{A}}$ such that $B \in D$. So if $B' \in D$ then $B' \subseteq \omega \wedge B' \notin \text{Id}_{\mathscr{A}}$ hence by the definition of $\text{Id}_{\mathscr{A}}$ it follows that $(\exists A \in \mathscr{A})(A \subseteq^* B')$. By Definition 2.1(1A) this means that $\mathscr{A}$ is a π-base of D.

Clause (c): Follows from clause (b). $\square_{2.4}$

2.4 Observation. 1) If D is an ultrafilter on ω then D has a π-base of cardinality $\pi\chi(D)$.
2) $\mathscr{A}$ is a π-base iff for every $n \in [1, \omega)$ and partition $\langle B_\ell : \ell < n\rangle$ of ω to finitely many sets, for some $A \in \mathscr{A}$ and $\ell < n$ we have $A \subseteq^* B_\ell$.
3) $\text{Min}\{\pi\chi(D) : D$ a non-principal ultrafilter on $\omega\} = \text{Min}\{|\mathscr{A}| : \mathscr{A}$ is a π-base$\} = \text{Min}\{|\mathscr{A}| : \mathscr{A}$ is a strict π-base$\}$.

Proof. 1) By the definition.
2) For the "only if" direction, assume $\mathscr{A}$ is a π-base of D then $\text{Id}_{\mathscr{A}} \subseteq \mathscr{P}(\omega)\backslash D$ (see the proof of 2.2) so $\omega \notin \text{Id}_{\mathscr{A}}$ and we are done.

For the "if" direction, use 2.2.

3) Easy. $\square_{2.4}$

2.5 Theorem. *In $\mathbf{V}^{\mathbb{P}}$ as in 1.1, we have $\{\mu, \lambda\} \subseteq \text{Sp}^*_{\pi\chi}$ and $\kappa_2 \notin \text{Sp}^*_{\pi\chi}$.*

Proof. Similar to the proof of 1.1 but with some additions. Defining $\mathfrak{K}$ in [Sh 700, 4.1] we allow $\mathbb{Q}_0 = \mathbb{Q}_0^{\mathfrak{t}} = \mathbb{P}_1^{\mathfrak{t}}$ to be any c.c.c. forcing notion of cardinality $\leq \lambda$ (this makes no change). The main change is in the proof of $\Vdash_{\mathbb{P}}$ "$\lambda \in \text{Sp}_\chi$". The main addition is that choosing $\mathfrak{t}_\alpha$ by induciton on α we also define $\mathscr{A}_\alpha$ such that

$\circledast'_1$ $(a),(b)$ as in $\circledast_1$ in the proof of **?**
→ scite{1.1} undefined

- (c) as in $\circledast_1(c)$ but only if $\alpha \neq 2 \bmod \omega$ (and $\alpha = \beta + 1$)
- (d) $\underset{\sim}{A}_\alpha$ is a $\mathbb{P}_1^{\mathfrak{t}_\alpha}$-name of an infinite subset of ω
- (e) if $\alpha \neq 2 \bmod \omega$ then $\Vdash_{\mathbb{P}^{\mathfrak{t}_\alpha}} \underset{\sim}{A}_\alpha = \omega$ (or do not define $\underset{\sim}{A}_\alpha$)
- (f) if $\alpha < \beta$ are $= 2 \bmod \omega$ then $\Vdash_{\mathbb{P}_\mu^{\mathfrak{t}_\beta}}$ "$\underset{\sim}{A}_\beta \subseteq^* \underset{\sim}{A}_\alpha$
- (g) if $\beta = \alpha + 1$ and $\beta = 2 \bmod \omega$ and $\underset{\sim}{B}$ is a $\mathbb{P}_\mu^{\mathfrak{t}_\alpha}$-name of an infinite subset of ω then $\Vdash_{\mathbb{P}_\mu^{\mathfrak{t}_\beta}}$ "$\underset{\sim}{B} \not\subseteq^* A_\alpha$.

This addition requires that we also prove

$\circledast_3$ if $\mathfrak{s} \in \mathfrak{K}$ and $\underset{\sim}{D}$ is a $\mathbb{P}_1^{\mathfrak{s}}$-name of a filter on ω including all co-finite subsets of ω (such that $\emptyset \notin D$) <u>then</u> for some $(\mathfrak{t}, \underset{\sim}{A})$ we have

- (a) $\mathfrak{s} \le_{\mathfrak{K}} \mathfrak{t}$
- (b) $\Vdash_{\mathbb{P}_1^{\mathfrak{t}}}$ "$\underset{\sim}{A}$ is an infinite subset of ω
- (c) if $\underset{\sim}{B}$ is a $\mathbb{P}^{\mathfrak{s}}$-name of an infinite subset of ω then $\Vdash_{\mathbb{P}^{\mathfrak{t}}}$ "$\underset{\sim}{B} \not\subseteq^* \underset{\sim}{A}$".

[Why $\circledast_3$ holds? Without loss of generality $\Vdash_{\mathbb{P}_1^{\mathfrak{s}}}$ "$\underset{\sim}{D}$ is an ultrafilter on ω". We can find a pair $(\mathbb{P}', \underset{\sim}{A}')$

- (α) $\mathbb{P}'$ is a c.c.c. forcing notion
- (β) $\mathbb{P}_1^{\mathfrak{s}} \lessdot \mathbb{P}'$ moreover $\mathbb{P}' = \mathbb{P}_1^{\mathfrak{s}} * \mathbb{Q}(\underset{\sim}{D})$
- (γ) $|\mathbb{P}'| \le \lambda$
- (δ) $\Vdash_{\mathbb{P}'}$ "$\underset{\sim}{A}$ is an almost intersection of $\underset{\sim}{D}$ (i.e. $\underset{\sim}{A} \in [\omega]^{\aleph_0}$ and $(\forall B \in \underset{\sim}{D})(A \subseteq^* B)$)
- (ε) $\underset{\sim}{\eta}' \in {}^\omega\omega$ is the generic of $\mathbb{Q}[\underset{\sim}{D}]$ and $\underset{\sim}{A}' =$ Rang(η) so both are $\mathbb{P}'$-names.

Now we define $\mathfrak{t}' : \mathfrak{t} \le_{\mathfrak{K}} \mathfrak{t}'$ and $\mathbb{P}_1^{\mathfrak{t}'} = \mathbb{P}'$, we do it by defining $\mathbb{Q}_i^{\mathfrak{t}'}$ by inducition on i as in the proof of [Sh 700, 4.8] and we choose $\underset{\sim}{\tau}_i^{\mathfrak{t}'}$ naturally. Let $\langle \underset{\sim}{n}_\rho : \rho \in {}^{\omega>}2\rangle$ be a $\mathbb{P}_0^{\mathfrak{t}'}$-name listing the members of $\underset{\sim}{A}$.

Now we choose $\mathfrak{t}$ such that $\mathfrak{t}' \le_{\mathfrak{K}} \mathfrak{t}$ and for some $\mathbb{P}_0^{\mathfrak{t}}$-name $\underset{\sim}{\rho}$ of a member of ${}^\omega 2$ we have $\Vdash_{\mathbb{P}_{\mathfrak{t}}}$ "$\underset{\sim}{\rho} \ne \underset{\sim}{\nu}$" for any $\mathbb{P}_{\mathfrak{t}'}$-name (clearly exists, e.g. when $(\mathfrak{t}, \mathfrak{t}')$ is like $(\mathfrak{t}', \mathfrak{s})$ above, e.g. do as above with $\mathbb{P}'$, adding λ^+ such raels, and reflect). Now $\underset{\sim}{A} := \{\underset{\sim}{n}_{\underset{\sim}{\rho}\restriction k} : k < \omega\}$ is forced to be an infinite subset of $\underset{\sim}{A}'$, and if it includes a member of $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}_{\mathfrak{s}}]}$ or even $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}_{\mathfrak{t}}]}$ we get that $\underset{\sim}{\rho}$ is from $({}^\omega 2)^{\mathbf{V}[\mathbb{P}_{\mathfrak{t}'}]}$, contradiction.]

$(*)_1$ $\mu \in$ Sp$^*_{\pi\chi}$, in $\mathbf{V}^{\mathbb{P}}$, of course.

[Why? As there is a $\subseteq^*$-decreasing sequence $\langle B_\alpha : \alpha < \mu\rangle$ of sets which generates a (non-principle ultrafilter). We can use B_α as the generic of $\mathbb{Q}_\alpha^{\mathfrak{t}_\lambda} = \mathbb{P}_{\alpha+1}^{\mathfrak{t}_\lambda}/\mathbb{P}_\alpha^{\mathfrak{t}_\lambda}$.]

$(*)_2$ $\kappa_2 \notin$ Sp$^*_{\pi\chi}$.

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[Why? Toward contradiction assume $p^* \in \mathbb{P}$ and $p^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{D}$ is a non-principal ultrafilter on ω and $\{\underset{\sim}{\mathscr{U}}_\varepsilon : \varepsilon < \kappa_2\}$ is a sequence of infinite subsets of ω which is a strict π-base of $\underset{\sim}{D}$"; so $p^* \Vdash_{\mathbb{P}}$ "$\{\underset{\sim}{\mathscr{U}}_\varepsilon : \varepsilon < \zeta\}$ is not a π-base of any ultrafilter on ω" for every $\zeta < \kappa_2$, hence for some $\langle \underset{\sim}{B}_{\zeta,\ell} : \ell < \underset{\sim}{n}_\zeta \rangle$ we have $p^* \Vdash$ "$\underset{\sim}{n}_\ell < \omega$ and $\langle \underset{\sim}{B}_{\zeta,\ell} : \ell < \underset{\sim}{n}_\ell \rangle$ is a partition of ω and $\varepsilon < \zeta \wedge \ell < \underset{\sim}{n}_\zeta \Rightarrow \mathscr{U}_\varepsilon \nsubseteq^* \underset{\sim}{B}_{\zeta,\ell}$". We now as in the proof of 1.1, choose suitable $\beta < \lambda$ and consider $\langle \underset{\sim}{B}^*_\ell : \ell < \underset{\sim}{n} \rangle = \hat{\mathbf{j}}_\beta^{-1}(\langle \underset{\sim}{B}_{\zeta,\ell} : \ell < \underset{\sim}{n}_\zeta \rangle : \zeta < \kappa_2 \rangle / \mathscr{D}_2)$ so $p^* \Vdash_{\mathbb{P}^{\beta+1}}$ "$\langle \underset{\sim}{B}^*_\ell : \ell < \underset{\sim}{n} \rangle$ is a partition of ω to finitely many sets and $\varepsilon < \kappa_2 \wedge \ell < \underset{\sim}{n} \Rightarrow \underset{\sim}{\mathscr{U}}_\varepsilon \nsubseteq^* \underset{\sim}{B}^*_\ell$". But this contradicts $p^* \Vdash_{\mathbb{P}}$ "$\{\underset{\sim}{\mathscr{U}}_\varepsilon : \varepsilon < \kappa_2\}$ is a π-base.]

$(*)_3$ $\lambda \in \mathrm{Sp}^*_\pi$.

[Why? Clearly it is forced (i.e. $\Vdash_{\mathbb{P}_\lambda}$) that $\langle \underset{\sim}{A}_{\omega\alpha+2} : \alpha < \lambda \rangle$ is a $\subseteq^*$-decreasing sequence of infinite subsets of ω, hence there is an ultrafilter of D on ω including it. Now $\underset{\sim}{A}_{\omega\alpha+2}$ witness that $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}_{\mathfrak{t}_{\omega\alpha+2}}]}$ is not a π-base of $\underset{\sim}{D}$ (recalling clause (g) of $\circledast'_1$). As λ is regular we are done.] $\square_{2.5}$

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