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THE DEPTH OF ULTRAPRODUCTS OF BOOLEAN ALGEBRAS SH853

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Abstract. We show that if μ is a compact cardinal then the depth of ultraproducts of less than μ many Boolean Algebras is at most μ plus the ultraproduct of the depths of those Boolean Algebras.

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§0 Introduction

Monk has looked systematically at cardinal invariants of Boolean Algebras. In particular, he has looked at the relations between $\text{inv}(\prod_{i<\kappa} \mathbf{B}_i/D)$ and $\prod_{i<\kappa} \text{inv}(\mathbf{B}_i)/D$, i.e., the invariant of the ultraproducts of a sequence of Boolean Algebras vis the ultraproducts of the sequence of the invariants of those Boolean Algebras for various cardinal invariants inv of Boolean Algebras. That is: is it always true that $\text{inv}(\prod_{i<\kappa} \mathbf{B}_i/D) \le \prod_{i<\kappa}(\text{inv}(\mathbf{B}_i/D)$? is it consistently always true? Is it always true that $\prod_{i<\kappa} \text{inv}(\mathbf{B}_i)/D \le \text{inv}(\prod_{i<\kappa} \mathbf{B}_i/D)$? is it consistenly always true? See more on this in Monk [Mo96]. Roslanowski Shelah [RoSh 534] deals with specific inv and with more on kinds of cardinal invariants and their relationship with ultraproducts. Monk [Mo90a], [Mo96], in his list of open problems raises the question for the central cardinal invariants, most of them have been solved by now; see Magidor Shelah [MgSh 433], Peterson [Pe97], Shelah [Sh 345], [Sh 462], [Sh 479], [Sh 589, §4], [Sh 620], [Sh 641], [Sh 703], Shelah and Spinas [ShSi 677].

We here throw some light on problem 12 of [Mo96], pg.287 and will be continued in [Sh:F683].

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0.1 Definition. For a Boolean Algebra $\mathbf{B}$ let

- (a) $\text{Depth}(\mathbf{B}) = \sup\{\theta$: in $\mathbf{B}$ there is an increasing sequence of length $\theta\}$
- (b) $\text{Depth}^+(\mathbf{B}) = \sup\{\theta^+$: in $\mathbf{B}$ there is an increasing sequence of length $\theta\}$.

0.2 Remark. So $\text{Depth}^+(\mathbf{B}) = \lambda^+ \Rightarrow \text{Depth}(\mathbf{B}) = \lambda$ and if $\text{Depth}^+(\mathbf{B})$ is a limit cardinal then $\text{Depth}^+(\mathbf{B}) = \text{Depth}(\mathbf{B})$.

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§1 Above a compact cardinal

The following claim gives severe restrictions on any try to build a ZFC example for $\mathrm{Depth}(\prod_{\varepsilon<\kappa} \mathbf{B}_\varepsilon)/D > \prod_{\varepsilon<\kappa} \mathrm{Depth}(\mathbf{B}_\varepsilon)/D$ if $\mathbf{V}$ is near $\mathbf{L}$, see [Sh 652] for complimentary to §1.

1.1 Claim. *1) Assume*

- (a) $\kappa < \mu \le \lambda$
- (b) μ *is a compact cardinal*
- (c) D *is an ultrafilter on* κ
- (d) $\lambda = \mathrm{cf}(\lambda)$ *such that* $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$
- (e) $\mathbf{B}_i\ (i < \kappa)$ *is a Boolean Algebra with* $\mathrm{Depth}^+(\mathbf{B}_i) \le \lambda$
- (f) $\mathbf{B} = \prod_{i<\kappa} \mathbf{B}_i/D$.

Then $\mathrm{Depth}^+(\mathbf{B}) \le \lambda$.
2) Instead $(\forall \alpha < \lambda)|\alpha|^\kappa < \lambda)$ *it suffices that* $(\forall \alpha < \lambda)(|\alpha^\kappa/D| < \lambda = \mathrm{cf}(\lambda))$.
3) We can weaken clause (e) (for parts (1) and (2)) to

- (g) $\{i < \kappa : \mathbf{B}_i$ *is a Boolean Algebra with* $\mathrm{Depth}^+(\mathbf{B}_i) \le \lambda\} \in D$.

Proof. 1) Toward contradiction assume that $\langle a_\alpha : \alpha < \lambda \rangle$ is an increasing sequence in $\mathbf{B}$. So let $a_\alpha = \langle a_i^\alpha : i < \kappa \rangle/D$, so for $\alpha < \beta$, $A_{\alpha,\beta} =: \{i < \kappa : \mathbf{B}_i \models a_i^\alpha < a_i^\beta\} \in D$.
Let E be a μ-complete uniform ultrafilter on λ.

For each $\alpha < \lambda$ let A_α be such that the set $\{\beta : \alpha < \beta < \lambda$ and $A_{\alpha,\beta} = A_\alpha\}$ is a member of E so an unbounded subset of λ (exist as $\lambda = \mathrm{cf}(\lambda) \ge \mu > 2^\kappa$).
We choose C as follows

$$C =: \{\delta < \lambda : \delta \text{ is a limit ordinal and if } u \subseteq \delta \text{ is bounded of cardinality } \le \kappa \text{ then } \delta = \sup(S_u \cap \delta)\}$$

where

$$S_u =: \{\beta < \lambda : \beta > \sup(u) \text{ and } (\forall \alpha \in u)(A_{\alpha,\beta} = A_\alpha)\}.$$

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As $\lambda = \text{cf}(\lambda) > 2^\kappa = |D|$, for some $A_* \in D$ the set $S =: \{\alpha < \lambda : \text{cf}(\alpha) > \kappa$ and $A_\alpha = A_*\}$ is a stationary subset of λ.

As we have assumed $\lambda = \text{cf}(\lambda)$ and $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$, clearly C is a club of λ. Let $\{\delta_\varepsilon : \varepsilon < \lambda\} \subseteq C, \delta_\varepsilon$ increases continuous with ε and $\delta_{\varepsilon+1} \in S$. For each $\varepsilon < \lambda$ the family $\mathfrak{A}_\varepsilon = \{S_u \cap \delta_{\varepsilon+1} \backslash \delta_\varepsilon : u \in [\delta_{\varepsilon+1}]^{\leq \kappa}\}$ is a downward κ^+-directed family of non-empty subsets of $[\delta_\varepsilon, \delta_{\varepsilon+1})$ hence there is a κ^+-complete filter E_ε on $[\delta_\varepsilon, \delta_{\varepsilon+1})$ extending $\mathfrak{A}_\varepsilon$.

For $\varepsilon < \lambda$ and $i < \kappa$ let $W_{\varepsilon,i} =: \{\beta : \delta_\varepsilon \leq \beta < \delta_{\varepsilon+1}$ and $i \in A_{\beta,\delta_{\varepsilon+1}}\}$ and let $B_\varepsilon =: \{i < \kappa : W_{\varepsilon,i} \in E^+_\varepsilon\}$. As E_ε is κ^+-complete clearly $W_\varepsilon =: \cap\{[\delta_\varepsilon, \delta_{\varepsilon+1}) \backslash W_{\varepsilon,i} : i \in \kappa \backslash B_\varepsilon\} \in E_\varepsilon$ hence there is $\beta \in W_\varepsilon$; if $i \in A_{\beta,\delta_{\varepsilon+1}}$ then $\{\gamma : \delta_\varepsilon \leq \gamma < \delta_{\varepsilon+1}$ and $i \in A_{\gamma,\delta_{\varepsilon+1}}\} \in E^+_\varepsilon$, so $A_{\beta,\delta_{\varepsilon+1}}$ is a subset of B_ε and belongs to D hence $B_\varepsilon \in D$. So $B_\varepsilon \cap A_* \in D$ is non-empty.

So for each ε for some $i_{\delta_{\varepsilon+1}} \in A_*$ we have

$$\{\beta : \delta_\varepsilon \leq \beta < \delta_{\varepsilon+1} \text{ and } i_{\delta_{\varepsilon+1}} \in A_{\beta,\delta_{\varepsilon+1}}\} \in E^+_\varepsilon.$$

We can find $i_* \in A_*$ such that

$$Y = \{\varepsilon < \lambda : \varepsilon \text{ is an even ordinal and } i_{\delta_{\varepsilon+1}} = i_*\}$$

has cardinality λ, and let $Z = \{\delta_{\varepsilon+1} : \varepsilon \in Y\}$ so $Z \in [\lambda]^\lambda$. Now

- $(*)_0$ $\varepsilon \in Y \Rightarrow A_{\delta_{\varepsilon+1}} = A_*$
  [why? as $\delta_{\varepsilon+1} \in S$]
- $(*)_1$ $i_* \in A_* \in D$
  [trivial; note if $\forall \alpha < \lambda, |\alpha|^{2^\kappa} < \lambda$ we can have E_ε is $(2^\kappa)^+$-complete filter so we have $B_{\delta_{\varepsilon+1}}$ instead of i_{δ_ε} so we can weaken "D ultrafilter" to: $D \subseteq \mathscr{P}(\kappa)$ upward closed and the intersection of any two non-empty]
- $(*)_2$ if $\alpha < \beta$ are from Z then $i_* \in A_{\alpha,\beta}$
  [why? let $\alpha = \delta_{\varepsilon+1}, \beta = \delta_{\zeta+1}$ so $\varepsilon < \zeta$; let

  $$\mathscr{U}_1 := \{\gamma : \delta_\zeta < \gamma < \delta_{\zeta+1}, A_{\alpha,\gamma} = A_\alpha (= A_{\delta_{\varepsilon+1}})\}$$

  so

  $$\mathscr{U}_1 = S_{\{\delta_{\varepsilon+1}\}} \cap [\delta_\zeta, \delta_{\zeta+1}) \in \mathfrak{A}_\zeta \subseteq E_\zeta$$

  and let

  $$\mathscr{U}_2 := \{\gamma : \delta_\zeta \leq \gamma < \delta_{\zeta+1}, i_* \in A_{\gamma,\delta_{\zeta+1}}\} \in E^+_\zeta.$$

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[Why? As this is how $i_{\delta_{\zeta+1}}$ is defined.]

So for any $\alpha < \beta$ from Z as $\mathscr{U}_1 \in E_\zeta$ and $\mathscr{U}_2 \in E^+_\zeta$ clearly there is $\gamma \in \mathscr{U}_1 \cap \mathscr{U}_2$ hence ($\alpha = \delta_{\varepsilon+1} < \delta_\zeta \le \gamma < \delta_{\zeta+1} = \beta$ and) for $i = i_*$ we have $\mathbf{B}_i \models a_i^{\delta_{\varepsilon+1}} < a_i^\gamma$ (because $\gamma \in \mathscr{U}_1$) and $\mathbf{B}_i \models a_i^\gamma < a_i^{\delta_{\zeta+1}}$ (because $\gamma \in \mathscr{U}_2$) so together $\mathbf{B}_i \models a_i^{\delta_{\varepsilon+1}} < a_i^{\delta_{\zeta+1}}$ but $\alpha = \delta_{\varepsilon+1}, \beta = \delta_{\zeta+1}$ so we have gotten $\mathbf{B}_i \models a_i^\alpha < a_i^\beta$ so we are done.

2) We change the choice of the club C. By the assumption, for each $\alpha < \lambda$ let $\langle f^\alpha_\gamma/D : \gamma < \gamma_\alpha\rangle$ be a list of the members of α^κ/D without repetitions, so $\gamma_\alpha < \lambda$. Let

$$
\begin{aligned}
C = \{\delta : & (i) \quad \delta < \lambda \text{ is a limit ordinal} \\
& (ii) \quad \text{if } \alpha < \delta \text{ then } \gamma_\alpha < \delta \\
& (iii) \quad \text{if } \alpha < \delta \text{ and } \gamma < \gamma_\alpha \text{ and} \\
& \qquad \bar{A} = \langle A_i : i < \kappa\rangle \in {}^\kappa D \text{ and there is } \xi \in [\delta, \lambda) \text{ such that} \\
& \qquad i < \kappa \Rightarrow A_{f^\alpha_\gamma(i),\xi} = A_i \text{ then there is} \\
& \qquad \xi \in (\alpha, \delta) \text{ such that } i < \kappa \Rightarrow A_{f^\alpha_\gamma(i),\xi} = A_i\}.
\end{aligned}
$$

Clearly C is a club of λ. The only additional point in the proof is

- $(*)$ if $\delta_1 < \delta_2$ are from C and $A_{\delta_2} = A_*$ <u>then</u> there is $i_* \in A_*$ such that: for every $\alpha \in S \cap \delta_1$ there is $\beta \in [\delta_1, \delta_2)$ satisfying $A_{\alpha,\beta} = A_* \wedge i_* \in A_{\beta,\delta_2}$.

[Why $(*)$ holds? If not, then for every $i \in A_*$ there is $\alpha_i \in S \cap \delta_1$ satisfying $\beta \in [\delta_1, \delta_2) \wedge A_{\alpha_i,\beta} = A_* \Rightarrow i \notin A_{\beta,\delta_2}$. Let $f \in {}^\kappa\alpha$ be defined by: $f(i) = \alpha_i$, if $i \in A_*, f(i) = 0$ otherwise, so for some $\gamma < \gamma_{\delta_1}$ we have $f = f^{\delta_1}_\gamma$ mod D hence $A =: \{i \in A_* : f(i) = f^{\delta_1}_\gamma(i)\} \in D$. As $\kappa < \mu$ and D is μ-complete there is $\xi_1 \in (\delta_2, \lambda)$ such that $i < \kappa \Rightarrow A_{f^{\delta_1}_\gamma(i),\xi_1} = A_{f^{\delta_1}_\gamma(i)}$ hence by the choice of C there is $\xi_2 \in (\delta_1, \delta_2)$ such that $i < \kappa \Rightarrow A_{f^{\delta_1}_\gamma(i),\xi_2} = A_{f^{\delta_1}_\gamma(i),\xi_1} = A_{f^{\delta_1}_\gamma(i)}$. But $i \in A \Rightarrow f^{\delta_1}_\gamma(i) = f(i) = \alpha_i \in S \Rightarrow A_{\alpha_i,\xi_2} = A_{f^{\delta_1}_\gamma(i),\xi_2} = A_{f^{\delta_1}_\gamma(i)} = A_*$ so $i \in A \Rightarrow A_{\alpha_i,\xi_2} = A_*$. Now $A_{\xi_2,\delta_2} \in D$ hence there is $i_* \in A_* \cap A_{\xi_1,\delta_2}$ and for it we get contradiction.]

Of course, the set of such i_*'s belongs to D.

3) Obvious. $\square_{1.1}$

<u>1.2 Conclusion</u>: Let μ be a compact cardinal. If $\kappa < \mu$ and D is an ultrafilter on $\kappa, \mathbf{B}_i$ is a Boolean Algebra for $i < \kappa$ <u>then</u>

- $(*)$ (a) if D is a regular ultrafilter then $\text{Depth}(\prod\limits_{i<\kappa} \mathbf{B}_i/D) \le \mu + \prod\limits_{i<\kappa} \text{Depth}(\mathbf{B}_i)/D$
  - (b) this holds if $\kappa = \aleph_0$.

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Proof. If this fails, let $\lambda = (\mu + \prod\limits_{i<\kappa} \text{Depth}(\mathbf{B}_i)/D)^+$, so λ is a regular cardinal $> \mu$ and $(\forall \alpha < \lambda)[|\alpha^\kappa/D| < \lambda]$ - see below and $\lambda \le \text{Depth}(\prod\limits_{i<\kappa} \mathbf{B}_i/D)$, so by 1.1 we get a contradiction. $\square_{1.2}$

1.3 Remark. 1) Actually we prove that if μ is a compact cardinal, $\kappa < \mu \le \lambda = \text{cf}(\lambda)$ and $\mathbf{c} : [\lambda]^2 \to \kappa$ then we can find an increasing sequence $\langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ of ordinals $< \lambda$ and $i, j < \kappa$ such that for every $\varepsilon < \zeta < \lambda$ for some γ satisfying $\alpha_\varepsilon < \gamma < \alpha_\zeta$ we have $\mathbf{c}\{\alpha_\varepsilon, \gamma\} = i, \mathbf{c}\{\gamma, \alpha_\zeta\} = j$ (the result follows using $\mathbf{c} : [\lambda]^2 \to D$).
2) We use i_* rather than some $B \in D$ in order to help clarify what we need.
3) Note that if D is a normal ultrafilter on $\kappa > \aleph_0$ and $\langle \lambda_i : i < \kappa \rangle$ is increasing continuous with limit $\lambda, i < \kappa \Rightarrow \prod\limits_{j \le i} \lambda_j < \lambda_{i+1}$ then $\lambda = \prod\limits_{i<\kappa} \lambda_i/D$ but $\lambda^\kappa/D > \lambda$. This is essentially the only reason for the undesirable extra assumption "D is regular" in 1.2.

Note

1.4 Claim. *1) In 1.1 instead "$\mu \in (\kappa, \lambda]$ is a compact cardinal" it suffices to demand: $\circledast_{\kappa^+, 2^\kappa, \lambda}$ where*

$\circledast_{\sigma,\theta,\lambda}$ *if $\mathbf{c} : [\lambda]^2 \to \theta$ then we can find a stationary $S \subseteq \lambda$ and $\gamma < \theta$ such that for every $u \in [S]^{<\sigma}$ the set $S_u = \{\beta < \lambda : (\forall \alpha \in u)[\mathbf{c}\{\alpha, \beta\} = \gamma]\}$ is unbounded in λ.*

2) If μ is supercompact $\sigma < \theta = \text{cf}(\theta) < \mu < \lambda = \text{cf}(\lambda)$ and $\mathbb{Q} =$ adding μ Cohen subsets of θ then in $\mathbf{V}, \circledast_{\sigma,\mu,\lambda}$ holds (even $\circledast_{\sigma,\mu_1,\lambda}$ if $\mu_1^{<\sigma} < \lambda$ in $\mathbf{V}$).

In 1.4 we cannot get such results for $\kappa > \mu$ because for μ supercompact Laver indestructible and regular $\lambda > \kappa > \mu$ we can force $\{\delta < \lambda\text{:cf}(\delta) > \mu\}$ to have a square preserving the supercompactness.

1.5 Claim. *Assume $\lambda = \text{cf}(\lambda) > \kappa^+$ and $\kappa = \text{cf}(\kappa)$, and there is a square on $S = \{\delta < \lambda\text{: cf}(\delta) \ge \kappa\}$ (see 1.6 below). Then*

- (a) *there is a sequence $\langle \mathbf{B}_i : i < \kappa \rangle$ of Boolean Algebras such that*
  - (α) $\text{Depth}^+(\mathbf{B}_i) \le \lambda$
  - (β) *for any uniform ultrafilter D on κ,* $\text{Depth}^+(\prod\limits_{i<\kappa} \mathbf{B}_i/D) > \lambda$

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(b) *the proof of [Sh 652, 5.1] can be carried.*

Where

1.6 Definition. For $\lambda = \text{cf}(\lambda) > \aleph_0, S \subseteq \lambda = \sup(S)$ we say that S has a square when we can find S^+ and $\langle C_\alpha : \alpha \in S^+ \rangle$ such that

(a) $S \backslash S^+$ is not a stationary subset of λ
(b) C_α is a closed subset of α
(c) $\beta \in C_\alpha \Rightarrow \beta \in S \cap C_\beta = C_\alpha \cap \beta$
(d) we stipulate $C_\alpha = \{\emptyset\}$ for $\alpha \notin S^+$.

Proof of 1.5. As in [Sh 652, 5.1] using $\bar{C} = \langle C_\alpha : \alpha \in S^+ \rangle$ from 1.6 instead $\langle \text{acc}(C_\alpha) : \alpha < \lambda^+ \rangle$. The only change being that in the proof of [Sh 652, Fact 5.3] in case 3 we have just $\text{cf}(\alpha) \le \kappa$ and let $\langle \beta_\zeta : \zeta < \text{cf}(\alpha)$ be increasing continuous with limit α. If $\text{cf}(\alpha) < \kappa$ we can find $\varepsilon(*) < \kappa$ such that $\zeta_1 < \zeta_2 < \kappa \Rightarrow \beta_{\zeta_1} \in A_{\beta_{\zeta_2}, \varepsilon(*)}$ and let $A_{\alpha,\varepsilon} = \emptyset$ if $\varepsilon < \varepsilon(*)$ and $A_{\alpha,\varepsilon} = \cup\{A_{\beta_\zeta,\varepsilon} : \zeta < \text{cf}(\kappa)\}$ if $\varepsilon \in [\varepsilon(*), \kappa)$. $\square_{1.6}$

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