THE ERDÖS-RADO ARROW FOR SINGULAR

SAHARON SHELAH

Abstract. We prove that if $\mathrm{cf}(\lambda) > \aleph_0$ and $2^{\mathrm{cf}(\lambda)} < \lambda$ then $\lambda \to (\lambda, \omega + 1)^2$ in ZFC

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0. INTRODUCTION

For regular uncountable κ, the Erdös-Dushnik-Miller theorem, Theorem 11.3 of [1], states that $\kappa \to (\kappa,\ \omega+1)^2$. For singular cardinals, κ, they were only able to obtain the weaker result, Theorem 11.1 of [1], that $\kappa \to (\kappa,\ \omega)^2$. It is not hard to see that if $\mathrm{cf}(\kappa) = \omega$ then $\kappa \not\to (\kappa,\ \omega{+}1)^2$. If cf $(\kappa) > \omega$ and κ is a strong limit cardinal, then it follows from the General Canonization Lemma, Lemma 28.1 in [1], that $\kappa \to (\kappa,\ \omega+1)^2$. Question 11.4 of [1] is whether this holds without the assumption that κ is a strong limit cardinal, e.g., whether, in ZFC,

$$(1)\ \ \aleph_{\omega_1} \to (\aleph_{\omega_1},\ \omega+1)^2.$$

In [5] it was proved that $\lambda \to (\lambda, \omega+1)^2$ if $2^{\mathrm{cf}(\lambda)} < \lambda$ and there is a nice filter on κ, (see [3, Ch.V]: follows from suitable failures of SCH). Also proved there are consistency results when $2^{\mathrm{cf}(\lambda)} \geq \lambda$

Here continuing [5] but not relying on it, we eliminate the extra assumption, i.e, we prove (in ZFC)

Theorem 0.1. *If $\aleph_0 < \kappa = \mathrm{cf}(\lambda)$ and $2^\kappa < \lambda$ <u>then</u> $\lambda \to (\lambda,\ \omega+1)^2$.*

Before starting the proof, let us recall the well known definition:

Definition 0.2. Let D be an $\aleph_1$-complete filter on Y, and $f \in {}^Y\mathrm{Ord}$, and $\alpha \in \mathrm{Ord} \cup \{\infty\}$.

We define when $\mathrm{rk}_D(f) = \alpha$ by induction on α (it is well known that $\mathrm{rk}_D(f) < \infty$):

(*) $\mathrm{rk}_D(f) = \alpha$ iff $\beta < \alpha \Rightarrow \mathrm{rk}_D(f) \neq \beta$, and for every $g \in {}^Y\mathrm{Ord}$ satisfying $g <_D f$, there is $\beta < \alpha$ such that $\mathrm{rk}_D(g) = \beta$.

Notice that we will use normal filters on $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, so the demand of $\aleph_1$- completeness in the definition, holds for us.
Recall also

Definition 0.3. Assume Y, D, f are as in definition 0.2.

$$J[f,D] = \{Z \subseteq Y : Y \setminus Z \in D \text{ or } \mathrm{rk}_{D+(Y\setminus Z)}(f) > \mathrm{rk}_D(f)\}$$

Lastly, we quote the next claim (the definition 0.3 and claim are from [2], and explicitly [4](5.8(2),5.9)):

Claim 0.4. *Assume $\kappa > \aleph_0$ is realized, and D is a κ-complete (a normal) filter on Y.*

<u>Then</u> $J[f,D]$ is a κ-complete (a normal) ideal on Y disjoint to D for any $f \in {}^Y\mathrm{Ord}$

1. The proof

In this section we prove Theorem 0.1 of the Introduction, which, for convenience, we now restate.

Theorem 1.1. *If* $\aleph_0 < \kappa = \mathrm{cf}(\lambda),\ 2^\kappa < \lambda$ *then* $\lambda \to (\lambda,\ \omega+1)^2$.

Proof.
Stage A We know that $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda,\ 2^\kappa < \lambda$ We will show that $\lambda \to (\lambda,\ \omega+1)^2$.

So, towards a contradiction, suppose that

$(*)_1$ $c : [\lambda]^2 \ \to\ \{\text{red, green}\}$ but has no red set of cardinality λ and no green set of order type $\omega+1$.

Choose $\bar\lambda$ such that:

$(*)_2$ $\bar\lambda = \langle \lambda_i : i < \kappa\rangle$ is increasing and continuous with limit λ, and for $i = 0$ or i a successor ordinal, λ_i is a successor cardinal. We also let $\Delta_0 = \lambda_0$ and for $i < \kappa,\ \Delta_{1+i} = [\lambda_i,\ \lambda_{i+1})$. For $\alpha < \lambda$ we will let $\mathbf{i}(\alpha) =$ the unique $i < \kappa$ such that $\alpha \in \Delta_i$.

We can clearly assume, in addition, that

$(*)_3$ $\lambda_0 > 2^\kappa$, for $i < \kappa,\ \lambda_{i+1} \geq \lambda_i^{++}$, and that each Δ_i is homogeneously red for c.

The last is justified by the Erdös-Dushnik-Miller theorem for λ_{i+1}, i.e., as $\lambda_{i+1} \to (\lambda_{i+1}, \omega+1)^2$ because λ_{i+1} is regular.

Stage B: For $0 < i < \kappa$, we define Seq_i to be $\{\langle \alpha_0, ..., \alpha_{n-1}\rangle : \mathbf{i}(\alpha_0) < ... < \mathbf{i}(\alpha_{n-1}) < i\}$. For $\zeta \in \Delta_i$ and $\langle \alpha_0, ..., \alpha_{n-1}\rangle = \bar\alpha \in \mathrm{Seq}_i$, we say $\bar\alpha \in \mathcal{T}^\zeta$ iff $\{\alpha_0, ..., \alpha_{n-1},\ \zeta\}$ is homogeneously green for c. Note that an infinite $\triangleleft$-increasing branch in $\mathcal{T}^\zeta$ violates the non-existence of a green set of order type $\omega+1$, so,

$(*)_4$ $\mathcal{T}^\zeta$ is well-founded, that is we cannot find $\eta_0 \triangleleft \eta_1 \triangleleft \ldots \triangleleft \eta_n \triangleleft \ldots$.

Therefore the following definition of a rank function, rk^ζ, on Seq_i can be carried out.

If $\eta \in \mathrm{Seq}_i \setminus T^\zeta$ then $\mathrm{rk}^\zeta(\eta) = -1$. We define $\mathrm{rk}^\zeta : \mathrm{Seq}_i \to \mathrm{Ord} \cup \{-1\}$ as follows by induction on the ordinal ξ, we have $\mathrm{rk}^\zeta(\bar\alpha) = \xi$ iff for all $\epsilon < \xi, \mathrm{rk}^\zeta(\bar\alpha)$ was not defined as ϵ but there is β such that $\mathrm{rk}^\zeta(\bar\alpha{}^\frown\langle\beta\rangle) \geq \epsilon$. Of course, if ξ is a successor ordinal, it is enough to check for $\epsilon = \xi - 1$, and for limit ordinals, δ, if for all $\xi < \delta,\ \mathrm{rk}^\zeta(\bar\alpha) \geq \xi$, then $\mathrm{rk}^\zeta(\bar\alpha) \geq \delta$. In fact, it is clear that the range of rk^ζ is a proper initial segment of μ_i^+, where $\mu_i := \mathrm{card}(\bigcup\{\Delta_\epsilon : \epsilon < i\})$, and so, in particular, the range of rk^ζ has cardinality at most λ_i. Note that $\lambda_{i+1} \geq \lambda_i^{++} > \mu_i^+$.

Now we can choose B_i, an end-segment of Δ_i such that for all $\bar\alpha \in \mathrm{Seq}_i$ and all $0 \leq \gamma < \mu_i^+$, if there is $\zeta \in B_i$ such that $\mathrm{rk}^\zeta(\bar\alpha) = \gamma$, then there are λ_{i+1} such ζ-s. Recall that Δ_i and therefore also B_i are of order type λ_{i+1}, which is a successor cardinal $> \mu_i^+ > |\mathrm{Seq}_i|$ hence such B_i exists. Everything is now in place for the main definition.

Stage C: $(\bar{\alpha},\ Z,\ D,\ f) \in K$ iff

(1) D is a normal filter on κ,
(2) $f : \kappa \to \mathrm{Ord}$,
(3) $Z \in D$
(4) for some $0 < i < \kappa$ we have $\bar{\alpha} \in \mathrm{Seq}_i$ and Z is disjoint to $i+1$ and for every $j \in Z$ (hence $j > i$) there is $\zeta \in B_j$ such that $\mathrm{rk}^{\zeta}(\bar{\alpha}) = f(j)$ (so, in particular, $\bar{\alpha} \in \mathcal{T}^{\zeta}$).

Stage D: Note that $K \neq \emptyset$, since if we choose $\zeta_j \in B_j$, for $j < \kappa$, take $Z = \kappa \setminus \{0\}$, $\bar{\alpha} =$ the empty sequence, choose D to be any normal filter on κ and define f by $f(j) = \mathrm{rk}^{\zeta_j}(\bar{\alpha})$, then $(\bar{\alpha},\ Z,\ D,\ f) \in K$.

Now clearly by 0.2, among the quadruples $(\bar{\alpha},\ Z,\ D,\ f) \in K$, there is one with $\mathrm{rk}_D(f)$ minimal. So, fix one such quadruple, and denote it by $(\bar{\alpha}^*,\ Z^*,\ D^*,\ f^*)$. Let D^*_1 be the filter on κ dual to $J[f^*, D^*]$, so by claim 0.4 it is a normal filter on κ extending D^*.

For $j \in Z^*$, set $C_j = \{\zeta \in B_j : \mathrm{rk}^{\zeta}(\bar{\alpha}^*) = f^*(j)\}$. Thus by the choice of B_j we know that $\mathrm{card}(C_j) = \lambda_{j+1}$, and for every $\zeta \in C_j$ the set $(\mathrm{Rang}(\bar{\alpha}^*) \cup \{\zeta\})$ is homogeneously green under the colouring c. Now: suppose $j \in Z^*$. For every $\Upsilon \in Z^* \setminus (j+1)$ and $\zeta \in C_j$, let $C^+_{\Upsilon}(\zeta) = \{\xi \in C_{\Upsilon} : c(\{\zeta, \xi\}) = \mathrm{green}\}$. Also, let $Z^+(\zeta) = \{\Upsilon \in Z^* \setminus (j+1) : \mathrm{card}(C^+_{\Upsilon}(\zeta)) = \lambda_{\Upsilon+1}\}$.

Stage E: For $j \in Z^*$ and $\zeta \in C_j$, let $Y(\zeta) = Z^* \setminus Z^+(\zeta)$. Since $\lambda_0 > 2^{\kappa}$ and $\lambda_{j+1} > \lambda_0$ is regular, for each $j \in Z^*$ there are $Y = Y_j \subseteq \kappa$ and $C'_j \subseteq C_j$ with $\mathrm{card}(C'_j) = \lambda_{j+1}$ such that $\zeta \in C'_j \Rightarrow Y(\zeta) = Y_j$.

Let $\hat{Z} = \{j \in Z^* : Y_j \in D^*_1\}$. Now the proof split to two cases.

Case 1: $\hat{Z} \neq \emptyset \mod D^*_1$

Define $Y^* = \{j \in \hat{Z}$: for every $i \in \hat{Z} \cap j$, we have $j \in Y_i\}$. Notice that Y^* is the intersection of $\hat{Z}$ with the diagonal intersection of κ sets from D^*_1 (since $i \in \hat{Z} \Rightarrow Y_i \in D^*_1$), hence (by the normality of D^*_1) $Y^* \neq \emptyset \mod D^*_1$. But then, as we will see soon, by shrinking the C'_j for $j \in Y^*$, we can get a homogeneous red set of cardinality λ, which is contrary to the assumption toward contradiction.

We define $\hat{C}_j$ for $j \in Y^*$ by induction on j such that $\hat{C}_j$ is a subset of C'_j of cardinality λ_{j+1}. Now, for $j \in Y^*$, let $\hat{C}_j$ be the set of $\xi \in C'_j$ such that for every $i \in Y^* \cap j$ and every $\zeta \in \hat{C}_i$ we have $\xi \notin C^+_j(\zeta)$. So, in fact, $\hat{C}_j$ has cardinality λ_{j+1} as it is the result of removing $< \lambda_{j+1}$ elements from C'_j where $|C'_j| = \lambda_{j+1}$ by its choice. Indeed, the number of such pairs (i, ζ) is $\leq \lambda_j$ and: for $i \in Y^* \cap j$ and $\zeta \in \hat{C}_i$:

(a) $j \in Y_i$ [Why? by the definition of Y^* as $j \in Y^*$]
(b) $\zeta \in C'_i$ [Why? as $\zeta \in \hat{C}_i$ and $\hat{C}_i \subseteq C'_i$ by the induction hypothesis]
(c) $Y(\zeta) = Y_i$ [Why? as by (b) we have $\zeta \in C'_i$ and the choice of C'_i]
(d) $j \in Y(\zeta)$ [Why? by (a)+(c)]
(e) $j \notin Z^+(\zeta)$ [Why? by (d) and the choice of $Y(\zeta)$ as $Z^* \setminus Z^+(\zeta)$]

(f) $C^+_j(\zeta)$ has cardinality $< \lambda_{j+1}$ [Why? by (e) and the choice of $Z^+(\zeta)$, as $j \in \hat{Z} \subseteq Z^*$]

So $\hat{C}_j$ is a well defined subset of C'_j of cardinality λ_{j+1} for every $j \in Y^*$. But then, clearly the union of the $\hat{C}_j$ for $j \in Y^*$, call it $\hat{C}$ satisfies:

(α) it has cardinality λ [as $j \in Y^* \Rightarrow |\hat{C}_j| = \lambda_{j+1}$ and $\sup(Y^*) = \kappa$ as $Y^* \neq \emptyset \mod D^*_1$]

(β) $c{\restriction}[\hat{C}_j]^2$ is constantly red [as we are assuming $(*)_3$]

(γ) if $i < j$ are from Y^* and $\zeta \in \hat{C}_i, \xi \in \hat{C}_j$ then $c\{\zeta, \xi\} = \text{red}$ [as $\xi \notin C^+_j(\zeta)$]

So $\hat{C}$ has cardinality λ and is homogeneously red. This concludes the proof in the case $\hat{Z} \neq \emptyset \mod D^*_1$

Case 2: $\hat{Z} = \emptyset \mod D^*_1$.

In that case there are $i \in Z^*, \beta \in C_i$ such that $Z^+(\beta) \neq \emptyset \mod D^*_1$

[Why? well, $Z^* \in D^* \subseteq D^*_1$ and $\hat{Z} = \emptyset \mod D^*_1$, hence $Z^* \setminus \hat{Z} \neq \emptyset$. Choose $i \in Z^* \setminus \hat{Z}$. By the definition of $\hat{Z}$, $Y_i \notin D^*_1$. So, if $\beta \in C'_i$ then $Y(\beta) = Y_i \notin D^*_1$ and choose $\beta \in C'_i$, so $Y(\beta) \notin D^*_1$ hence by the definition of $Y(\beta)$ we have $Z^* \setminus Z^+(\beta) = Y(\beta) \notin D^*_1$. Since $Z^* \in D^*_1$, we conclude that $Z^+(\beta) \neq \emptyset \mod D^*_1$].

Let $\bar{\alpha}' = \bar{\alpha}^* {}^\frown \langle \beta \rangle, Z' = Z^+(\beta), D' = D^* + Z'$, it is a normal filter by the previous sentence as $D^* \subseteq D^*_1$ and lastly we define $f' \in {}^\kappa\text{Ord}$ by:

(a) if $j \in Z'$ then $f'(j) = \text{Min}\{\text{rk}^\gamma(\bar{\alpha}') : \gamma \in C^+_j(\beta) \subseteq B_j\}$

(b) otherwise $f'(j) = 0$

Clearly

(α) $(\bar{\alpha}', Z', D', f') \in K$, and

(β) $f' <_{D'} f^*$

[Why? as $Z' \in D'$ and if $j \in Z'$ then for some $\gamma \in C^+_j(\beta)$ we have $f'(j) = \text{rk}^\gamma(\bar{\alpha}') = \text{rk}^\gamma(\bar{\alpha}^* {}^\frown \langle \beta \rangle)$ which by the definition of rk^γ is $< \text{rk}^\gamma(\bar{\alpha}^*) = f^*(j)$, recalling (4) from stage C.]

hence

(γ) $\text{rk}_{D'}(f') < \text{rk}_{D'}(f^*)$

[Why? see Definition 0.2].

But $\text{rk}_{D'}(f^*) = \text{rk}_{D^*}(f^*)$ as $Z' = Z^+(\beta) \neq \emptyset \mod D^*_1$ by the definition of D^*_1 as extending the filter dual to $J[f^*, D^*]$, see Definition 0.3. Hence $\text{rk}_{D'}(f') < \text{rk}_{D^*}(f^*)$, so we get a contradiction to the choice of $(\bar{\alpha}^*, Z^*, D^*, f^*)$.
Clearly at least one of the two cases holds, so we are done. $\square$

References

[1] Paul Erdős, Andras Hajnal, A. Maté, and Richard Rado. *Combinatorial set theory: Partition Relations for Cardinals*, volume 106 of *Studies in Logic and the Foundation of Math.* North Holland Publ. Co, Amsterdam, 1984.

[2] Saharon Shelah. A note on cardinal exponentiation. *The Journal of Symbolic Logic*, 45:56–66, 1980. [**Sh:71**]

[3] Saharon Shelah. *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, 1994. [**Sh:g**]

[4] Saharon Shelah. Applications of PCF theory. *Journal of Symbolic Logic*, 65:1624–1674, 2000. [**Sh:589**]

[5] Saharon Shelah and Lee Stanley. Filters, Cohen Sets and Consistent Extensions of the Erdős-Dushnik-Miller Theorem. *Journal of Symbolic Logic*, 65:259–271, 2000. math.LO/9709228. [**ShSt:419**]

Institute of Mathematics The Hebrew University of Jerusalem Jerusalem 91904, Israel and Department of Mathematics Rutgers University New Brunswick, NJ 08854, USA

Email address: shelah@math.huji.ac.il

URL: http://www.math.rutgers.edu/~shelah