# ON LONG INCREASING CHAINS MODULO FLAT IDEALS
# SH908

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Abstract. We prove that e.g. in ${}^{(\omega_3)}(\omega_3)$ there is no sequence of length $\omega_4$ increasing modulo the ideal of countable sets.

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## §0 Introduction

We hope sometime to prove: e.g.

*0.1 Conjecture.* For every $\mu > \theta$ in ${}^{(\theta^{+3})}\mu$ there is no increasing sequence of length $\mu^+$ modulo $[\theta^{+3}]^{\le \theta}$.

Let $\kappa = \text{cf}(\kappa) > \aleph_0$. If $\mu = \kappa$, then $\text{Length}({}^\kappa\mu, <_{J^{\text{bd}}_\kappa}) = \sup\{|\mathscr{F} : \mathscr{F} \subseteq {}^\kappa\kappa$ linearly ordered by $<_{J^{\text{bd}}_\kappa}\}$ can be (forced to be) large. But for $\mu >$ $\text{Length}({}^\kappa\mu,$ $<_{J^{\text{bd}}_\kappa})$ this implies pcf results (see [Sh 410], [Sh 589]).

However, e.g. for the ideal $\mathscr{I} = [\omega_3]^{\le \aleph_0}$ it is harder to get long increasing sequence, as above for "high $\mu$", this leads to pcf results so let $\bar\lambda = \langle \lambda_i : i < \omega_3\rangle \in {}^{\omega_3}\text{Reg}, (\pi\bar\lambda, <_{\mathscr{I}})$ has cofinality $> \sup\{\lambda_i; i < \omega_3\}$ which are much stronger than known consistency results. Even for $I = [\omega_1]^{\le \aleph_0}$ we do not know, for $I = [\beth_\omega]^{\le \aleph_0}$ we know ([Sh 460]), so even $[\aleph_\omega]^{\le \aleph_0}$ would be interesting good news

The following may be a first step and anyhow stand by itself.

## §1 Increasing sequences modulo countable

**1.1 Claim.** *Assume $\partial^+ < \kappa \le \theta$ and $\mathscr{J} = [\kappa]^{<\partial}$, an ideal and $\mathrm{cf}([\theta]^\partial, \subseteq) \le \theta$. Then there is no $<_{\mathscr{J}}$-increasing sequence $\langle f_\alpha : \alpha < \theta^+ \rangle$ of functions from ${}^\kappa\theta$ (or even ${}^\kappa\gamma(*)$ for some $\gamma(*) < \theta^+$).*

*Remark.* E.g. $\partial = \aleph_1, \kappa = \theta = \aleph_3$ was discussed above.

*Proof.* It is enough to treat the version with $\gamma(*)$. So toward contradiction assume $\langle f_\alpha : \alpha < \theta^+ \rangle$ is a counterexample.

Let $\mathscr{S} \subseteq [\gamma(*)]^\partial$ be cofinal of cardinality $\le \theta$ exists as $|\gamma(*)| = \theta$ and $\mathrm{cf}([\theta]^\partial, \subseteq) \le \theta$. Now for every $s \in \mathscr{S}$ and $\beta < \kappa$ let $I_\beta = I(\beta) := [\beta, \beta + \partial)$ and we define

$(*)_0$ for $\zeta < \theta^+$ let $f^{s,\beta}_\zeta \in {}^{I(\beta)}(\gamma(*) + 1)$ be defined as $f^s_\zeta \restriction I(\beta)$ where $f^s_\zeta \in {}^\kappa(\gamma(*) + 1)$ is defined by $f^s_\zeta(i) = \min(s \cup \{\gamma(*)\} \backslash f_\zeta(i))$.

Now

$(*)_1$ for $s \in \mathscr{S}$ we have
- $(a)$ $\zeta < \theta^+ \Rightarrow f^{s,\beta}_\zeta \in {}^{I(\beta)}(s \cup \{\gamma(*)\})$
- $(b)$ $\zeta < \xi < \theta^+ \Rightarrow f^{s,\beta}_\zeta \le f^{s,\beta}_\xi \mod \mathscr{J} \restriction I_\beta$.

For $s \in \mathscr{S}$, let

$(*)_2$ $B_s = \{\beta < \kappa :$ for every $\zeta < \theta^+$ there is $\xi \in (\zeta, \theta^+)$ such that $\neg(f^{s,\beta}_\zeta = f^{s,\beta}_\xi \mod \mathscr{J} \restriction I_\beta)\}$.

For $s \in \mathscr{S}$ and $\beta < \kappa$ clearly we can choose $C^s_\beta$ such that

$(*)_3$
- $(a)$ $C^s_\beta$ is a club of $\theta^+$
- $(b)$ if $\beta \in B_s$ and $\xi \in C^s_\beta$ then $\zeta < \xi \Rightarrow \neg(f^{s,\beta}_\zeta = f^{s,\beta}_\xi \mod \mathscr{J} \restriction I_\beta)$
- $(c)$ if $\beta \in \kappa \backslash B_s$ then $\theta^+ > \xi \ge \zeta \ge \mathrm{Min}(C^s_\beta) \Rightarrow (f^{s,\beta}_\zeta = f^{s,\beta}_\xi \mod \mathscr{J} \restriction I_\beta)$.

So

$(*)_4$ $C := \cap\{C^s_\beta : s \in \mathscr{S}$ and $\beta < \kappa\}$ is a club of $\theta^+$ as $|\mathscr{S}| \le \theta, \kappa \le \theta$.

Now choose

$(*)_5$ $\alpha_\varepsilon \in C$ for $\varepsilon < \partial^+$ increasing with $\varepsilon$

hence

$(*)_6$ $u_{\varepsilon,\zeta} := \{i < \kappa : f_{\alpha_\varepsilon}(i) \ge f_{\alpha_\zeta}(i)\} \in \mathscr{J}$ for $\varepsilon < \zeta < \partial^+$.

As we are assuming $\partial^+ < \kappa$ we can find $\beta(*)$ such that

$(*)_7$ (a) $\beta(*) < \kappa$
(b) $I_{\beta(*)} = [\beta(*), \beta(*) + \partial]$ is disjoint to $\cup\{u_{\varepsilon,\zeta} : \varepsilon < \zeta < \partial^+\}$ hence
(c) the sequence $\langle f_{\alpha_\varepsilon}(i) : \varepsilon < \partial^+\rangle$ is increasing for each $i \in I_{\beta(*)}$.

As $|I_{\beta(*)}| = \partial$ and $\mathscr{S} \subseteq [\gamma(*)]^{\le\partial}$ is cofinal (for the partial order $\subseteq$), we can find $s$ such that

$(*)_8$ $s \in \mathscr{S}$ and $s \supseteq \{f_{\alpha_0}(i), f_{\alpha_1}(i) : i \in I_{\beta(*)}\}$

hence by $(*)_6 + (*)_7 + (*)_8$ we have

$(*)_9$ $f^s_{\alpha_0}(i) = f_{\alpha_0}(i) < f_{\alpha_1}(i) = f^s_{\alpha_1}(i)$ for every $i \in I_{\beta(*)}$.

As $\alpha_0 < \alpha_1$ are from $C$ and $I_{\beta(*)} \in \mathscr{J}^+$, recalling $(*)_2 + (*)_3 + (*)_4$, clearly

$(*)_{10}$ $\beta(*) \in B_s$,

recalling that $\alpha_\varepsilon \in C \subseteq C^s_{\beta(*)}$ for $\varepsilon < \partial^+$ and $\alpha_\varepsilon$ is increasing with $\varepsilon$, clearly (by $(*)_{10}$)

$(*)_{11}$ for every $\varepsilon < \partial^+$ there is $i_\varepsilon \in I_{\beta(*)}$ such that
($\alpha$) $f^s_{\alpha_\varepsilon}(i_\varepsilon) < f^s_{\alpha_{\varepsilon+1}}(i_\varepsilon)$
hence there is $j_\varepsilon \in s$ such that
($\beta$) $f^s_{\alpha_\varepsilon}(i_\varepsilon) \le j_\varepsilon < f^s_{\alpha_{\varepsilon+1}}(i_\varepsilon)$ hence
($\gamma$) $f_{\alpha_\varepsilon}(i_\varepsilon) \le j_\varepsilon < f_{\alpha_{\varepsilon+1}}(i_\varepsilon)$.

But $|I_{\beta(*)}| + |s| = \partial < \partial^+$ hence for some pair $(j_*, i_*) \in s \times I_{\beta(*)}$ we have

$(*)_{12}$ $\partial^+ = \sup(\mathscr{U})$ where $\mathscr{U} := \{\varepsilon < \theta^+ : j_\varepsilon = j_* \text{ and } i_\varepsilon = i_*\}$.

Hence we can find $\varepsilon_1 < \varepsilon_2$ from $\mathscr{U}$, but $\langle f_{\alpha_\varepsilon}(i_*) : \varepsilon < \theta^+\rangle$ is increasing by $(*)_7(c)$, i.e. the choice of $\beta(*)$, so $f_{\alpha_{\varepsilon_1}}(i_*) < f_{\alpha_{\varepsilon_1+1}}(i_*) \le f_{\alpha_{\varepsilon_2}}(i_*) < f_{\alpha_{\varepsilon_2+1}}(i_*)$ but by $(*)_{11}(\gamma)$+$(*)_{12}$ the ordinal $j_*$ belongs to $[f_{\alpha_{\varepsilon_1}}(i_*), f_{\alpha_{\varepsilon_1+1}}(i_*))$ and to $[f_{\alpha_{\varepsilon_2}}(i_*), f_{\alpha_{\varepsilon_2+1}}(i_*))$, which are disjoint intervals, contradiction. $\square_{1.1}$

Similarly

**1.2 Claim.** *There is no $<_{\mathscr{J}}$-increasing sequence of functions from $\kappa$ to $\gamma(*)$ of length $\lambda$ when*

$\circledast$ (a) $\mathscr{J}$ *is an ideal on* $\kappa$

(b) $I_\beta \in [\kappa]^\partial$ *for* $\beta < \kappa$

(c) $I_\beta \notin \mathscr{J}$ *for* $\beta < \kappa$

(d) $\mathrm{cf}([\theta]^\partial, \subseteq) < \lambda$ *where* $\theta = |\gamma(*)| + \kappa$

(e) *if* $u_\varepsilon \in \mathscr{J}$ *for* $\varepsilon < \partial^+$ *then* $I_\beta$ *is disjoint to* $\bigcup\limits_{\varepsilon<\partial^+} u_\varepsilon$ *for some* $\beta < \kappa$.

*Proof.* Without loss of generality $\lambda$ is the successor of $\mathrm{cf}([\theta]^\partial, \subseteq)$ hence is regular. The proof is similar to the proof of 1.1. $\square_{1.2}$

We may wonder could we have used same one "+", i.e.

1.3 Question: Is it consitent that ${}^\theta\theta$ contains $<_{\mathscr{I}}$-increasing sequence of length $\theta^+$ when $\theta = \kappa^+, I = [\theta]^{<\kappa}$?

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