P-NDOP and **P**-decompositions of $\aleph_\epsilon$-saturated models of superstable theories

Saharon Shelah*
The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
Jerusalem, 91904, Israel

Department of Mathematics
Hill Center-Busch Campus
Rutgers, the State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-9019 USA

Michael C. Laskowski†
Department of Mathematics
University of Maryland
College Park, MD 20742 USA

September 15, 2020

Abstract

*Partially supported by the US-Israel Binational Science Foundation and by NSF grants DMS-0600940 and DMS-1101597. Publication 933.

†Partially supported by NSF grant DMS-0901336.

Given a complete, superstable theory, we distinguish a class **P** of regular types, typically closed under automorphisms of $\mathfrak{C}$ and non-orthogonality. We define the notion of **P**-NDOP, which is a weakening of NDOP. For superstable theories with **P**-NDOP, we prove the existence of **P**-decompositions and derive an analog of [Sh401]. In this context, we also find a sufficient condition on **P**-decompositions that imply non-isomorphic models. For this, we investigate natural structures on the types in $\mathbf{P} \cap S(M)$ modulo non-orthogonality.

1 Introduction

Results by the first author, most notably Chapter X of [4] and the first half of [Sh401] demonstrate that $\aleph_\epsilon$-saturated models of superstable theories with NDOP admit very desirable decompositions. In this paper, we generalize these results in three ways. First, we always assume that the theory T is superstable, but we only have NDOP for a class **P** of regular types. Second, we show that the tree structure of a decomposition of an $\aleph_\epsilon$-saturated model M can be read off from the non-orthogonality classes of regular types in $S(M)$. Third, we show that these results for $\aleph_\epsilon$-saturated models give information about weak decompositions of arbitrary models of such theories.

In more detail, throughout the paper we assume we have a fixed, complete, **superstable** theory and we work within a monster model $\mathfrak{C}$. We fix a set **P** of stationary, regular types over small subsets of $\mathfrak{C}$ that is closed under automorphisms of $\mathfrak{C}$ and the equivalence relation of nonorthogonality, and additionally assume that our theory satisfies **P**-NDOP. Typically, we fix a model M that is at least $\aleph_\epsilon$-saturated (i.e., M contains a realization of every strong type over every finite subset of M) and study **P**-decompositions inside M of many varieties. Of primary interest are prime, $(\aleph_\epsilon, \mathbf{P})$-decompositions $\mathfrak{d}$ of M over $\binom{B}{A}$ (see Definition 4.16) where $A \subseteq B$ are ϵ-finite and every regular type p non-orthogonal to $\text{stp}(B/A)$ is in **P**. We associate a subset $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}, M)$ of $S(M) \cap \mathbf{P}$ (see Definition 5.1) to such a pair. The main theorem of the paper, Theorem 5.12, asserts that this set of regular types depends only on $\binom{B}{A}$. In particular, it is independent of the decomposition $\mathfrak{d}$, and successive results show that these sets have a tree structure under inclusion.

In the final section of the paper, we show how this result, which holds only for $\aleph_\epsilon$-saturated models, gives positive information for much weaker decompositions of models M_0 without any saturation assumption.

2 Preliminaries

As mentioned above, we always work in a class of models of a complete, superstable, first-order theory T. We fix a monster model $\mathfrak{C}$, and all models and sets we discuss will be small subsets of $\mathfrak{C}$. We assume that T eliminates quantifiers, so any model M will be an elementary submodel of $\mathfrak{C}$, and we additionally assume that '$T = T^{\mathrm{eq}}$', so that every type over an algebraically closed set is stationary.

Definition 2.1 A set A is *ϵ-finite* if $\mathrm{acl}(A) = \mathrm{acl}(a)$ for some $a \in \mathfrak{C}^{\mathrm{eq}}$.

Recall that as we are working in $\mathfrak{C}^{\mathrm{eq}}$, it would be equivalent to say that $\mathrm{acl}(A) = \mathrm{acl}(\bar{a})$ for some finite tuple. It is easily seen that the union of two ϵ-finite sets is ϵ-finite. Furthermore, since T is superstable, any subset $B \subseteq A$ of an ϵ-finite set is ϵ-finite. [Why? If $B \subseteq A$ with $\mathrm{acl}(A) = \mathrm{acl}(a)$, choose a finite $\bar{b}$ from B such that $B \underset{\bar{b}}{\perp} a$. Then $\mathrm{acl}(B) = \mathrm{acl}(\bar{b})$.] Thus, the set of ϵ-finite subsets of $\mathfrak{C}$ form an ideal.

Convention 2.2 $\aleph_\epsilon$ is a cardinal strictly between $\aleph_0$ and $\aleph_1$.

Thus, if we write 'M is λ-saturated for some $\lambda \geq \aleph_\epsilon$' we mean that either M is $\aleph_\epsilon$-saturated (i.e., realizes all types over ϵ-finite subsets) or M is λ-saturated for some $\lambda \geq \aleph_1$. Recall that by e.g., IV 2.2(7) of [4], that for $\lambda \geq \aleph_1$, M is λ-saturated if and only if for every subset $A \subseteq M$ of size less than λ, M realizes every type over $\mathrm{acl}(A)$.

We record several facts from [4] that will be used throughout this paper. The first is the Second Characterization Theorem, IV 4.18, the second is X Claim 1.6(5), the third is V 1.12, and (4) follows easily from (2) and (3).

Fact 2.3 *Suppose T is superstable and $\lambda \geq \aleph_\epsilon$.*

1. *A model M is λ-prime over a set A if and only if (1) $M \supseteq A$ and is λ-saturated; (2) M is λ-atomic over A; and (3) every A-indiscernible sequence $I \subseteq M$ has length at most λ. (When $\lambda = \aleph_\epsilon$, the λ occurring in (3) should be replaced by $\aleph_0$.)*

2. *If M is λ-saturated, $A \supseteq M$, and N is λ-prime over $M \cup A$, then N is dominated by A over M.*

3. *If $M \preceq N$ are both λ-saturated, $p \in S(M)$ is regular, and there is some $c \in N \setminus M$ such that* $\mathrm{tp}(c/M) \not\perp p$, *then p is realized in N.*

4. *If $M_0 \preceq M_1 \preceq M_2$ are all λ-saturated and there is $e \in M_2 \setminus M_1$ such that* $\mathrm{tp}(e/M_1)$ *is regular and non-orthogonal to M_0, then there is $e^* \in M_2 \setminus M_1$ such that e and e^* are domination equivalent over M_1, with $e^* \underset{M_0}{\perp} M_1$.*

3 P-NDOP

Our story begins by localizing the notion of DOP around a single parallelism class of stationary, regular types.

Definition 3.1 An *independent triple* of models (M_0, M_1, M_2) satisfy $M_0 = M_1 \cap M_2$ and $\{M_1, M_2\}$ are independent over M_0. For $\lambda \geq \aleph_\epsilon$, a λ-*quadruple* is a sequence (M_0, M_1, M_2, M_3) of λ-saturated models, where (M_0, M_1, M_2) form an independent triple, and M_3 is λ-prime over $M_1 \cup M_2$. A λ-*DOP witness* for a stationary, regular type p is a λ-quadruple (M_0, M_1, M_2, M_3) for which $Cb(p) \subseteq M_3$, but $p \perp M_1$ and $p \perp M_2$. We say that p *has a DOP witness* if it has a λ-DOP witness for some $\lambda \geq \aleph_\epsilon$.

Visibly, whether a specific λ-quadruple is a λ-DOP witness for p depends only on the parallelism class of p. To understand the consequences of this notion, we recall that a set A is *self-based* on an independent triple (M_0, M_1, M_2) of models if $A \underset{A \cap M_i}{\perp} M_i$ holds for each $i < 3$. The concept of self-basedness was defined explicitly in [2] and was used implicitly in the proof of X 2.2(iii$\rightarrow$iv) of [4]. The fact that for any independent triple (M_0, M_1, M_2), any finite set A can be extended to a finite, self-based $B \subseteq AM_1M_2$ follows from Lemma 2.4 of [2]. The Claim in the proof of Theorem 1.3 of [2] establishes the following Fact.

Fact 3.2 *If A is self-based on the independent triple (M_0, M_1, M_2), $p \in S(A)$ is stationary, $p \perp M_1$, and $p \perp M_2$, then $p \vdash p|AM_1M_2$.*

Using this Fact, an easy examination of the proof of [4], X 2.2 yields:

Fact 3.3 *Let p be any stationary, regular type with a DOP witness. Then:*

1. *For every $\lambda \geq \aleph_\epsilon$, p has a λ-DOP witness;*

2. *For every λ-DOP witness (M_0, M_1, M_2, M_3) for p, there is an infinite, indiscernible set $I \subseteq M_3$ over $M_1 \cup M_2$ whose average type $Av(I, M_3)$ is parallel to p; and*

3. *For every λ-DOP witness (M_0, M_1, M_2, M_3) for p, there is a subset $A \subseteq M_3$, $|A| < \lambda$ over which p is based and stationary and a Morley sequence $\langle b_i : i < \lambda \rangle$ from M_3 in $p|AM_1M_2$.*

We isolate one Corollary from this that will be crucial for us later.

Corollary 3.4 *For any $\lambda \geq \aleph_\epsilon$, if (M_0, M_1, M_2, M_3) is a λ-DOP witness for a stationary, regular $p \in S(M_3)$, then for any realization c of p, any λ-prime model $M_3[c]$ over $M_3 \cup \{c\}$ is isomorphic to M_3 over $M_1 \cup M_2$. In particular, $M_3[c]$ is λ-prime over $M_1 \cup M_2$.*

Proof. By the uniqueness of λ-prime models, both statements will follow once we establish that $M_3 \cup \{c\}$ is the universe of a λ-construction sequence over $M_1 \cup M_2$. To see this, first fix a λ-construction sequence $\langle b_i : i < \delta \rangle$ of M_3 over $M_1 \cup M_2$. As notation, for each $i < \delta$, let $B_i = M_1 \cup M_2 \cup \{b_j : j < i\}$ and fix a subset $X_i \subseteq B_i$, $|X_i| < \lambda$ such that $\text{stp}(b_i/X_i) \vdash \text{stp}(b_i/B_i)$.

Next, choose a subset $A \subseteq M_3$, $|A| < \lambda$ over which p is based and stationary. By forming an increasing ω-chain, we can increase A slightly (still maintaining $|A| < \lambda$) so that A is self-based on (M_0, M_1, M_2) and $X_i \subseteq A$ whenever $b_i \in A$.

Let $\langle a_i : i < \gamma \rangle$ be the enumeration of A given by the ordering of the original construction. Easily, $\langle a_i : i < \gamma \rangle$ is λ-constructible over $M_1 \cup M_2$.

Furthermore, it follows from Fact 3.2 that for any Morley sequence I in $p|A$ with $|I| < \lambda$, we have $p|AI \vdash p|AIM_1M_2$. Using this, we have a λ-construction sequence $\langle a_i : i < \gamma \rangle \hat{} \langle c_j : j < \lambda \rangle$ over $M_1 \cup M_2$, where $\langle c_j : j < \lambda \rangle$ is any Morley sequence in $p|A$ from M_3 (the existence of such a sequence follows from Fact 3.3(4)). It follows from the uniqueness of λ-prime models and the fact that such models are λ-constructible that there is another λ-construction sequence of M_3 over $M_1 \cup M_2$ in which $\langle a_i : i < \gamma \rangle \hat{} \langle c_j : j < \lambda \rangle$

is an initial segment. As notation, let $\langle b_k : k < \nu\rangle$ be the tail of this sequence. For each $k < \nu$, let $B^*_k = M_1 \cup M_2 \cup A \cup \{c_j : j < \lambda\} \cup \{b_\ell : \ell < k\}$ and choose $Y_k \subseteq B^*_k$, $|Y_k| < \lambda$ such that $\mathrm{stp}(b_k/Y_k) \vdash \mathrm{stp}(b_k/B^*_k)$. Without loss, we may assume $A \subset Y_k$ for each k. To complete the proof, it suffices to prove that

$$\langle a_i : i < \gamma\rangle\hat{}\langle c\rangle\hat{}\langle c_i : i < \lambda\rangle\hat{}\langle b_k : k < \nu\rangle$$

is a λ-construction sequence over $M_1 \cup M_2$.

We already know that $\langle a_i : i < \gamma\rangle$ is a λ-construction sequence over $M_1 \cup M_2$. Using the first sentence of the previous paragraph, combined with the fact that $\{c\} \cup \{c_j : j < \lambda\}$ is independent over A, we inductively obtain that $\langle a_i : i < \gamma\rangle\hat{}\langle c\rangle\hat{}\langle c_j : j < \lambda\rangle$ is also a λ-construction sequence over $M_1 \cup M_2$. Thus, it suffices to prove that $\mathrm{stp}(b_k/Y_k) \vdash \mathrm{stp}(b_k/B^*_k c)$ for each $k < \nu$. For this, since both $\mathrm{tp}(c/B^*_k)$ and $\mathrm{tp}(b_k/B^*_k)$ do not fork over Y_k, it suffices to show that $\mathrm{tp}(c/Y_k)$ is almost orthogonal to $\mathrm{stp}(b_k/Y_k)$. To see this, choose $j < \lambda$ such that $\mathrm{tp}(c_j/A)$ does not fork over Y_k. Now, $\mathrm{tp}(c/Y_k) = \mathrm{tp}(c_j/Y_k)$ and $\mathrm{tp}(c_j/Y_k)$ is almost orthogonal to $\mathrm{stp}(b_k/Y_k)$ since $\mathrm{stp}(b_k/Y_k) \vdash \mathrm{stp}(b_k/Y_k c_j)$, so we finish.

Next, we show additional closure properties of DOP witnesses.

Definition 3.5 A regular type q *lies directly above* p if there is a non-forking extension $p' \in S(M)$ of p with M $\aleph_\epsilon$-saturated, a realization c of p', and an $\aleph_\epsilon$-prime model $M[c]$ over $M \cup \{c\}$ such that $q \not\perp M[c]$, but $q \perp M$. A regular type q *lies above* p if there is a sequence $p_0, \dots, p_n$ of types such that $p_0 = p$, $p_n = q$, and p_{i+1} lies directly above p_i for each $i < n$. (We allow $n = 0$, so in particular, any regular type lies above itself.)

We say that p *supports* q if q lies above p.

The nomenclature above is apt if one considers a branch of a decomposition tree. Suppose $M_0 \preceq \dots \preceq M_n$ is a sequence of $\aleph_\epsilon$-saturated models such that for each $i < n$ there is $a_i \in M_{i+1}$ such that $\mathrm{tp}(a_i/M_i)$ is regular (and orthogonal to M_{i-1} when $i > 0$) and M_{i+1} is $\aleph_\epsilon$-prime over $M_i \cup \{a_i\}$. Then any regular $q \not\perp M_n$ lies over any regular type p non-orthogonal to $\mathrm{tp}(a_0/M_0)$. Similarly, any such p supports any such q.

Proposition 3.6 *Fix a stationary, regular type p with a DOP witness. Then:*

1. *Every type parallel to p has a DOP witness;*

2. *Every automorphic image of p has a DOP witness;*

3. *Every stationary, regular q non-orthogonal to p has a DOP witness;*

4. *Every stationary, regular q lying above p has a DOP witness.*

Proof. (1) and (2) are immediate. For (3), choose $\lambda \geq \aleph_\epsilon$ and a λ-quadruple (M_0, M_1, M_2, M_3) witnessing that p has λ-DOP. Let q be any stationary, regular type non-orthogonal to p. As q is non-orthogonal to M_3, there is $q' \in S(M_3)$ non-orthogonal to q (and hence to p) and conjugate to q. But now, $q' \perp M_1$ and $q' \perp M_2$, so (M_0, M_1, M_2, M_3) witnesses that q' has λ-DOP. Thus, q has a DOP witness by (2).

(4) It suffices to prove this for q lying directly above p. As both notions are parallelism invariant, we may assume that $p \in S(N)$, where N is $\aleph_\epsilon$-saturated. Choose c realizing p and $N[c]$ $\aleph_\epsilon$-prime over $N \cup \{c\}$ such that $q \not\perp N[c]$, but $q \perp N$. Choose $q' \in S(N[c])$ nonorthogonal to q. Fix a cardinal $\lambda > |N|$, and choose a λ-DOP witness (M_0, M_1, M_2, M_3) for p. Without loss, we may assume that $N \preceq M_3$ and that $c \underset{N}{\perp} M_3$. Let M^* be λ-prime over $N[c] \cup M_3$ and let q^* be the non-forking extension of q' to M^*. We argue that (M_0, M_1, M_2, M^*) is a λ-DOP witness for q^*.

To see this, first note that $N[c]$ is $\aleph_\epsilon$-constructible over $N \cup \{c\}$, N is $\aleph_\epsilon$-saturated, and $c \underset{N}{\perp} M_3$, so $N[c]$ is $\aleph_\epsilon$-constructible (hence λ-constructible) over $M_3 \cup \{c\}$. Since M^* is λ-constructible over $N[c] \cup M_3$, it follows that M^* is λ-constructible over $M_3 \cup \{c\}$, hence is λ-prime over $M_3 \cup \{c\}$. Thus, by Corollary 3.4, M^* is λ-prime over $M_1 \cup M_2$. That is, (M_0, M_1, M_2, M^*) is a λ-quadruple.

As well, $q' \in S(N[c])$ is orthogonal to N and $N[c] \underset{N}{\perp} M_3$, so $q' \perp M_3$. As $M_1 \cup M_2 \subseteq M_3$, it follows immediately that $q^* \perp M_1$ and $q^* \perp M_2$.

Throughout the remainder of this paper, we consider sets $\mathbf{P}$ of stationary, regular types over small subsets of the monster model $\mathfrak{C}$. We typically require $\mathbf{P}$ to be closed under automorphisms of $\mathfrak{C}$ and nonorthogonality.

Definition 3.7 Let $\mathbf{T}^{\mathrm{reg}}$ denote the set of all stationary, regular types over small subsets of $\mathfrak{C}$ and fix a subset $\mathbf{P} \subseteq \mathbf{T}^{\mathrm{reg}}$ that is closed under automorphisms of $\mathfrak{C}$ and nonorthogonality.

As notation,

- A stationary type q is orthogonal to $\mathbf{P}$, written $q \perp \mathbf{P}$, if q is orthogonal to every $p \in \mathbf{P}$. $\mathbf{P}^\perp = \{q \in \mathbf{T}^{\text{reg}} : q \perp \mathbf{P}\}$;
- $\mathbf{P}^{\mathbf{active}}$ is the closure of $\mathbf{P}$ in $\mathbf{T}^{\text{reg}}$ under automorphisms, nonorthogonality, and supporting (i.e., if $p \in \mathbf{T}^{\text{reg}}$ supports some $q \in \mathbf{P}$, then $p \in \mathbf{P}^{\mathbf{active}}$;
- $\mathbf{P}^{\mathbf{dull}} = \mathbf{T}^{\text{reg}} \setminus \mathbf{P}^{\mathbf{active}}$.

Definition 3.8 Let $\mathbf{P} \subseteq \mathbf{T}^{\text{reg}}$ be any set of regular types. A theory T has $\mathbf{P}$-*NDOP* if no $p \in \mathbf{P}$ has a DOP witness.

The following Corollary is merely a restatement of Proposition 3.6.

Corollary 3.9 *For any* $\mathbf{P} \subseteq \mathbf{T}^{\text{reg}}$*,* T *has* $\mathbf{P}$*-NDOP if and only if* T *has* $\mathbf{P}^{\mathbf{active}}$*-NDOP.*

Definition 3.10 Given a class $\mathbf{P}$ of regular types, we define the $\mathbf{P}$-*depth* of a stationary, regular type p, $\text{dp}_{\mathbf{P}}(p) \in \mathbf{ON} \cup \{-1\}$, by (1) $\text{dp}_{\mathbf{P}}(p) = -1$ if and only if $p \in \mathbf{P}^{\mathbf{dull}}$; and (2) $\text{dp}_{\mathbf{P}}(p) \geq \alpha$ if and only if $p \in \mathbf{P}^{\mathbf{active}}$ and for every $\beta \in \alpha$ there is a triple (M, N, a), where M is $\aleph_\epsilon$-saturated, N is $\aleph_\epsilon$-prime over $M \cup \{a\}$, p is parallel to $\text{tp}(a/M)$, and there is $q \in S(N)$ orthogonal to M with $\text{dp}_{\mathbf{P}}(q) \geq \beta$.

As in Chapter X of [4], in the preceding definition it would be equivalent to replace '$\aleph_\epsilon$-saturation' by 'λ-saturation' for any uncountable cardinal λ. The proof of the following Lemma is identical to the proof of Lemma X 7.2 of [4].

Lemma 3.11 *If* T *has* $\mathbf{P} - NDOP$*, then any regular* p *with* $\text{dp}_{\mathbf{P}}(p) > 0$ *is trivial, i.e., the set* $p(\mathfrak{C})$ *has a trivial pre-geometry with respect to the dependence relation of forking.*

We close this section with two technical Lemmas that will be used later. Note that a type q (not necessarily regular) is orthogonal to $\mathbf{P}^{\mathbf{dull}}$ if and only if every regular type non-orthogonal to q is an element of $\mathbf{P}^{\mathbf{active}}$.

Lemma 3.12 *($\mathbf{P}$-NDOP, $\lambda \geq \aleph_\epsilon$) Suppose that M is λ-prime over an independent triple (M_0, M_1, M_2) of λ-saturated models, a is ϵ-finite satisfying* $\mathrm{tp}(a/M) \perp \mathbf{P}^{\mathbf{dull}}$ *and* $\mathrm{tp}(a/M) \perp M_2$. *Let $M[a]$ be any λ-prime model over $M \cup \{a\}$. For any subset $N \subseteq M[a]$ that is maximal such that $N \underset{M_1}{\perp} M$ we have:*

1. *$N \preceq M[a]$, N is λ-saturated, and $M[a]$ is λ-prime over $N \cup M$; and*
2. *For any $a^* \subseteq N$ such that $N \underset{M_1 a^*}{\perp} a$, N is λ-prime over $M_1 \cup \{a^*\}$.*

Proof. To see that $N \preceq M[a]$ and N is λ-saturated, choose $N^+ \preceq M[a]$ to be λ-prime over N. As M_1 is λ-saturated, it follows from Fact 2.3(2) that N^+ is dominated by N over M_1, hence $N^+ \underset{M_1}{\perp} M$, so $N^+ = N$ by the maximality of N.

Next, choose $M^* \preceq M[a]$ to be maximal such that M^* is λ-saturated and λ-atomic over $N \cup M$. (Since T is superstable, the union of a continuous chain of λ-saturated models is λ-saturated, so M^* exists.) Since a is ϵ-finite, any subset $I \subseteq M[a]$ that is indiscernible over M has size at most λ (when $\lambda = \aleph_\epsilon$, I must be countable). It follows at once that every subset $I \subseteq M^*$ that is indiscernible over $N \cup M$ has size at most λ, so by Fact 2.3(1) M^* is λ-prime over $N \cup M$. We complete the proof of (1) by showing that $M^* = M[a]$.

Suppose not. Choose $c \in M[a] \setminus M^*$ such that $q = \mathrm{tp}(c/M^*)$ is regular. The argument splits into cases. First, if $q \perp N$ and $q \perp M$, then (M_1, N, M, M^*) is a DOP witness for q, so by Corollary 3.4, any λ-prime model over $M^* \cup \{c\}$ is λ-prime over $N \cup M$, which contradicts the maximality of M^*. Second, if $q \not\perp N$, then choose a regular $r \in S(M^*)$ that does not fork over N but $q \not\perp r$. Choose $d \in M[a] \setminus M^*$ realizing r. Then, by symmetry and transitivity of non-forking, $Nd \underset{M_1}{\perp} M$, which contradicts the maximality of N. Finally, suppose that $q \not\perp M$. As before, there is a regular $p \in S(M^*)$ that does not fork over M but $q \not\perp p$, and an element $e \in M[a] \setminus M^*$ realizing p. As p is regular, based on M, and non-orthogonal to $\mathrm{tp}(a/M)$, $p \in \mathbf{P}^{\mathbf{active}}$ and $p \perp M_2$. So, by $\mathbf{P}$-NDOP it must be that $p \not\perp M_1$. But then, $p \not\perp N$, so arguing as above we contradict the maximality of N. This proves (1).

For (2), choose any such a^*. We show that N is λ-prime over $M_1 \cup \{a^*\}$ via Fact 2.3(1). We already know that N is λ-saturated. To see that N is λ-atomic over $M_1 \cup \{a^*\}$, choose any finite set c from N. As $N \subseteq M[a]$,

$\operatorname{tp}(c/Ma)$ is λ-isolated. But $c \underset{M_1 a^*}{\perp} Ma$, so $\operatorname{tp}(c/M_1 a^*)$ is λ-isolated as well (see e.g., [4] IV 4.1). Finally, if $I \subseteq N$ is indiscernible over $M_1 \cup \{a^*\}$, then I is indiscernible over M_1. But $N \underset{M_1}{\perp} M$, so I is indiscernible over $N \cup M$. As $M[a]$ is λ-prime over $N \cup M$, it follows that I has size at most λ, completing the proof of (2).

Lemma 3.13 *($\mathbf{P}$-NDOP, $\lambda \geq \aleph_\epsilon$) Suppose that $M_1 \preceq M$ are both λ-saturated, a is ϵ-finite, $\operatorname{tp}(a/M) \perp \mathbf{P}^{\mathbf{dull}}$, and either $\operatorname{tp}(a/M)$ does not fork over M_1, or $\operatorname{tp}(a/M)$ is regular and non-orthogonal to M_1. Let $M[a]$ be any λ-prime model over $M \cup \{a\}$. For any subset $N \subseteq M[a]$ that is maximal such that $N \underset{M_1}{\perp} M$ we have:*

1. *$N \preceq M[a]$, N is λ-saturated, and $M[a]$ is λ-prime over $N \cup M$; and*
2. *For any $a^* \subseteq N$ such that $N \underset{M_1 a^*}{\perp} a$, N is λ-prime over $M_1 \cup \{a^*\}$.*

Proof. The proof is similar to the proof of Lemma 3.12, only easier. The hypotheses on $\operatorname{tp}(a/M)$ ensure that for any $e \in M[a] \setminus M$, as e is dominated by a over M, it follows that $\operatorname{tp}(e/M) \not\perp M_1$.

To see (1), take $N^+ \preceq M[a]$ to be λ-prime over N. As before, the maximality of N implies that $N^+ = N$, so $N \preceq M[a]$ and N is λ-saturated. As well, choose $M^* \preceq M[a]$ that is maximal such that M^* is λ-saturated and λ-atomic over $N \cup M$. As before, indiscernible subsets of M^* over $N \cup M$ have size at most λ, so M^* is λ-prime over $N \cup M$.

The verification that $M^* = M[a]$ is also similar. If not, choose $c \in M[a] \setminus M^*$ such that $q = \operatorname{tp}(c/M^*)$ is regular. If $q \perp N$ and $q \perp M$, then (M_1, N, M, M^*) is a DOP witness for q, which again contradicts the maximality of M^* by Corollary 3.4. If $q \not\perp N$, then arguing as before there is a regular $r \in S(M^*)$ that does not fork over N, $q \not\perp r$, and a realization d of r, which contradicts the maximality of N. Finally, if $q \not\perp M$, then there is a regular $p \in S(M^*)$ that does not fork over M but $q \not\perp p$ and a realization e of p in $M[a]$. Our conditions on $\operatorname{tp}(a/M)$ imply that $\operatorname{tp}(e/M) \not\perp M_1$, hence $\operatorname{tp}(e/M) \not\perp N$ and we argue as above, completing the verification of (1). The verification of (2) is identical to its verification in the proof of Lemma 3.12.

4 P-decompositions

Throughout this section, assume that T is superstable, and that $\mathbf{P}$ is a class of regular types, closed under automorphisms of $\mathfrak{C}$ and non-orthogonality. We define a number of species of $\mathbf{P}$-decompositions, along with a number of ways in which one $\mathbf{P}$-decomposition can extend another.

Definition 4.1 Fix a model M. A *weak* $\mathbf{P}$*-decomposition inside* M is a sequence $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ indexed by a tree $(I, \trianglelefteq)$ satisfying:

1. $\{N_\eta : \eta \in I\}$ is an independent tree of elementary submodels of M;
2. $\eta \trianglelefteq \nu$ implies $N_\eta \preceq N_\nu$;
3. Each $a_\eta \in N_\eta$ (but $a_{\langle\rangle}$ is meaningless);
4. For all $\nu \in Succ_I(\eta)$, N_ν is dominated by a_ν over N_η;
5. If $\eta \neq \langle\rangle$, then $\mathrm{tp}(a_\nu/N_\eta) \perp N_{\eta^-}$ for each $\nu \in Succ_I(\eta)$;
6. For each $\eta \in I$, $\{a_\nu : \nu \in Succ_I(\eta)\}$ is independent over N_η and $\mathrm{tp}(a_\nu/N_\eta) \perp \mathbf{P}^\perp$ for each $\nu \in Succ_I(\eta)$.

Note that in the Definition above, we do not require that $\mathrm{tp}(a_\nu/N_\eta)$ be regular. However, the content of (6) is that any regular type $q \not\perp \mathrm{tp}(a_\nu/N_\eta)$ is necessarily in $\mathbf{P}$.

Lemma 4.2 *Suppose* $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ *is a weak* $\mathbf{P}$*-decomposition inside* M*. Then:*

1. *If* $I_1, I_2 \subseteq I$ *are both downward closed and* $I_0 = I_1 \cap I_2$*, then*
$$\left(\bigcup_{\eta\in I_1} N_\eta\right) \underset{(\bigcup_{\eta\in I_0} N_\eta)}{\perp} \left(\bigcup_{\eta\in I_2} N_\eta\right)$$
2. *If* $\eta \in I$, $\nu = \eta\hat{\ }\langle\alpha\rangle$*, where* α *is least such that* $\eta\hat{\ }\langle\alpha\rangle \notin I$*, the element* $a_\nu \in M$ *satisfies* $\mathrm{tp}(a_\nu/N_\eta) \perp \mathbf{P}^\perp$*, if* $\eta \neq \langle\rangle$ *then* $\mathrm{tp}(a_\nu/N_\eta) \perp N_{\eta^-}$*, and* $a_\nu \underset{N_\eta}{\perp} \{a_\gamma : \gamma \in Succ_I(\eta)\}$*, and* $N_\nu \preceq M$ *is dominated by* a_ν *over* N_η*, then* $\mathfrak{d}^* = \mathfrak{d}\hat{\ }\langle N_\nu, a_\nu\rangle$ *is a weak* $\mathbf{P}$*-decomposition inside* M*.*

There are two ways of defining when a weak **P**-decomposition inside a model M is 'maximal'. Fortunately, at least when both M and each of the submodels N_η are $\aleph_\epsilon$-saturated, Lemma 4.4 shows that the two notions are equivalent.

Definition 4.3 Suppose that $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ of M is a weak **P**-decomposition inside M. As notation, for each $\eta \in I$, let

$$C_\eta(M) = \{a \in M \setminus N_\eta : \mathrm{tp}(a/N_\eta) \perp \mathbf{P}^\perp \text{ and } \perp N_{\eta^-} \text{ (when } \eta \neq \langle\rangle)\}$$

- $\mathfrak{d}$ *is a weak* **P**-*decomposition* **of M** if, for every $\eta \in I$, $\{a_\nu : \nu \in Succ_I(\eta)\}$ is a maximal N_η-independent subset of $C_\eta(M)$.
- $\mathfrak{d}$ **P**-*exhausts* M if, for every $\eta \in I$ for every regular $p \in S(N_\eta) \cap \mathbf{P}$ orthogonal to N_{η^-} (when $\eta \neq \langle\rangle$) and for every $e \in p(\mathfrak{C})$, if $e \underset{N_\eta}{\perp} \{a_\nu : \nu \in Succ_I(\eta)\}$ then $e \underset{N_\eta}{\perp} M$.

Lemma 4.4 *Suppose that* $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ *is a weak* **P**-*decomposition inside an* $\aleph_\epsilon$-*saturated model* M *such that every* N_η *is* $\aleph_\epsilon$-*saturated as well. Then* $\mathfrak{d}$ *is weak* **P**-*decomposition of* M *if and only if* $\mathfrak{d}$ **P**-*exhausts* M.

Proof. For both directions, recall that if $h \in M \setminus N_\eta$, then there is a finite, N_η-independent set $\{b_i : i < n\} \subseteq M$ domination equivalent to h over N_η with $\mathrm{tp}(b_i/N_\eta)$ is regular for each $i < n$.

For the easy direction, suppose that $\mathfrak{d}$ is not a weak **P**-decomposition of M. Choose $\eta \in I$ such that $A = \{a_\nu : \nu \in Succ_I(\eta)\}$ is not maximal in $C_\eta(M)$. Choose $h \in C_\eta(M)$ such that $h \underset{N_\eta}{\perp} A$, and from above, choose $\{b_i : i < n\} \subseteq M$ domination equivalent to h over N_η. Then, for any $i < n$, the element b_i and the type $\mathrm{tp}(b_i/N_\eta)$ witness that $\mathfrak{d}$ does not **P**-exhaust M.

Conversely, suppose that $\mathfrak{d}$ is a weak **P**-decomposition of M. Fix any $\eta \in I$, any regular type $p \in S(N_\eta) \cap \mathbf{P}$ that is $\perp N_{\eta^-}$ when $\eta \neq \langle\rangle$, and any $e \in p(\mathfrak{C})$ with $e \underset{N_\eta}{\not\perp} M$. We will show that $e \underset{N_\eta}{\not\perp} A$, where $A = \{a_\nu : \nu \in Succ_I(\eta)\}$.

To see this, using the note above choose $n < \omega$ minimal such that there are $h \in M \setminus N_\eta$ and $B = \{b_i : i < n\} \subseteq M$ such that $e \underset{N_\eta}{\not\perp} h$, and h and B are domination equivalent over N_η with $\mathrm{tp}(b_i/N_\eta)$ regular for each i. It follows from the minimality of n that $\text{tp}(b_i/N_\eta)$ is non-orthogonal to p, hence each $b_i \in C_\eta(M)$. As A is maximal in $C_\eta(M)$, we have that $b_i \not\perp_{N_\eta} A$ for each i, hence $\text{tp}(b_i/N_\eta A)$ is hereditarily orthogonal to p (i.e., $\text{tp}(b_i/N_\eta A)$, as well as every forking extension of it is orthogonal to p). Thus, $\text{tp}(B/N_\eta A)$ is hereditarily orthogonal to p. This implies $e \not\perp_{N_\eta} A$. [Why? If not, then $\text{tp}(e/N_\eta A)$ would be parallel to p, so by orthogonality we would have $e \perp_{N_\eta A} B$. This would imply that e and B (and hence e and h) are independent over N_η, which is a contradiction.]

For our next series of results, we insist that the model M be sufficiently saturated, and we additionally require that each submodel occurring in a decomposition be sufficiently saturated as well. In most applications, $\aleph_\epsilon$-saturation would suffice, but it costs little to work in the more general context of $(\overline{\lambda}, \mathbf{P})$-saturated models, which we now introduce.

Fix, for the remainder of this section, a pair $\overline{\lambda} = (\lambda, \mu)$ of cardinals satisfying $\lambda, \mu \geq \aleph_\epsilon$. Throughout the whole of this paper, if $\lambda = \mu = \aleph_\epsilon$, we write $(\aleph_\epsilon, \mathbf{P})$ in place of $(\aleph_\epsilon, \aleph_\epsilon, \mathbf{P})$.

Definition 4.5 We say that a model M is $(\overline{\lambda}, \mathbf{P})$-*saturated* if it is $\aleph_\epsilon$-saturated, and for each finite $A \subseteq M$, $\dim(p, M) \geq \lambda$ for each $p \in \mathbf{P} \cap S(A)$, and $\dim(q, M) \geq \mu$ for all stationary, regular $q \in \mathbf{P}^\perp \cap S(A)$. (If either λ or μ is $\aleph_\epsilon$, the associated dimension is at least $\aleph_0$.)

We say that a $(\overline{\lambda}, \mathbf{P})$-saturated model N is $(\overline{\lambda}, \mathbf{P})$-*prime over a set* X if $N \supseteq X$ and N embeds elementarily over X into any $(\overline{\lambda}, \mathbf{P})$-saturated model containing X.

Note that our assumptions on $\overline{\lambda}$ guarantee that any $(\overline{\lambda}, \mathbf{P})$-saturated model is $\aleph_\epsilon$-saturated, but we include this clause for emphasis. Also, if $\lambda = \mu$, then the $(\overline{\lambda}, \mathbf{P})$-saturated models are precisely the λ-saturated models. The standard facts about the existence $(\overline{\lambda}, \mathbf{P})$-prime models extend easily to this context. To see this, call a type $\text{tp}(a/B)$ $(\overline{\lambda}, \mathbf{P})$-*isolated* if any of the three conditions hold: (1) $\text{tp}(a/B)$ is $\aleph_\epsilon$-isolated ($= \mathbf{F}^a_{\aleph_0}$-isolated) or (2) $\text{tp}(a/B) \in \mathbf{P}$ and is λ-isolated; or (3) $\text{tp}(a/B) \in \mathbf{P}^\perp$ and is μ-isolated. Next, call a set B $(\overline{\lambda}, \mathbf{P})$-*primitive over* A if $B = A \cup \{b_i : i < \alpha\}$, where $\text{tp}(b_i/A \cup \{b_j : j < i\})$ is $(\overline{\lambda}, \mathbf{P})$-isolated for every i, and call a model M

$(\overline{\lambda},\mathbf{P})$-*primary over* A if M is $(\overline{\lambda},\mathbf{P})$-saturated and its universe is $(\overline{\lambda},\mathbf{P})$-primitive over A. This notion of isolation satisfies the same axioms as for $\mathbf{F}^a_\lambda$-isolation in Chapter 4 of [4] and thus we obtain the same consequences. In particular:

- If $A\subseteq M^*$ with M^* $(\overline{\lambda},\mathbf{P})$-saturated, then there is a $M\preceq M^*$ that is $(\overline{\lambda},\mathbf{P})$-primary over A;
- M $(\overline{\lambda},\mathbf{P})$-primary over A implies M is $(\overline{\lambda},\mathbf{P})$-prime over A; and
- If M is $(\overline{\lambda},\mathbf{P})$-saturated, $M\subseteq A$, and N is $(\overline{\lambda},\mathbf{P})$-prime over A, then N is dominated by A over M.

Definition 4.6 Suppose that M is $(\overline{\lambda},\mathbf{P})$-saturated. A *weak* $(\overline{\lambda},\mathbf{P})$*-decomposition inside M (of M)* is a weak $\mathbf{P}$-decomposition inside M (of M) for which each of the submodels N_η is an $(\overline{\lambda},\mathbf{P})$-saturated elementary substructure of M.

A salient feature of weak $(\overline{\lambda},\mathbf{P})$-decompositions is that each of the submodels is itself $\aleph_\epsilon$-saturated. The proof of the following Lemma is virtually identical to arguments in Section X.3 of [4].

Lemma 4.7 (P-NDOP) *Suppose that $\langle N_\eta,a_\eta:\eta\in I\rangle$ is a weak $(\overline{\lambda},\mathbf{P})$-decomposition of a $(\overline{\lambda},\mathbf{P})$-saturated model M. Let $\overline{M}\preceq M$ be any $\aleph_\epsilon$-prime submodel of M over $\bigcup_{\eta\in I}N_\eta$. Then if $p\in\mathbf{P}$ is non-orthogonal to $\overline{M}$, then there is a unique, $\triangleleft$-minimal $\eta\in I$ such that $p\not\perp N_\eta$.*

Proof. We first show that $p\not\perp N_\eta$ for some $\eta\in I$. As $\overline{M}$ is $\aleph_\epsilon$-saturated, there is $q\in S(\overline{M})$ that is regular and non-orthogonal to p. As any such q is in $\mathbf{P}$, we may assume that $p\in S(\overline{M})$ to begin with. Choose a finite $B\subseteq\overline{M}$ over which p is based and stationary. As B is $\aleph_\epsilon$-isolated over $\bigcup_{\eta\in I}N_\eta$, there is a finite subtree $I_0\subseteq I$ such that B is $\aleph_\epsilon$-isolated over $\bigcup_{\eta\in I_0}N_\eta$. Choose any $M_0\preceq\overline{M}$ such that $B\subseteq M_0$ and M_0 is $\aleph_\epsilon$-prime over $\bigcup_{\eta\in I_0}N_\eta$. As there is some type $p'\in S(M_0)$ parallel to p, $\mathbf{P}$-NDOP implies that $p\not\perp N_\eta$ for some $\eta\in I_0$.

Finally, using Lemma 4.2(1) it follows that there is a unique $\triangleleft$-minimal $\eta\in I$ with $p\not\perp N_\eta$.

The following definition makes sense in our context, as $(\overline{\lambda},\mathbf{P})$-decompositions have no control over types orthogonal to $\mathbf{P}$.

Definition 4.8 An $\aleph_\epsilon$-saturated model N is $\mathbf{P}$*-minimal over* X if $N \supseteq X$, but for any $\aleph_\epsilon$-saturated $N_0 \preceq N$ containing X, $\mathrm{tp}(e/N_0) \perp \mathbf{P}$ for every $e \in N \setminus N_0$.

Corollary 4.9 (P-NDOP) *Suppose that* $\langle N_\eta, a_\eta : \eta \in I\rangle$ *is a weak* $(\overline{\lambda}, \mathbf{P})$*-decomposition of a* $(\overline{\lambda}, \mathbf{P})$*-saturated model* M *and let* $\overline{M} \preceq M$ *be any* $\aleph_\epsilon$*-prime model over* $\bigcup_{\eta\in I} N_\eta$*. Then:*

1. *Every* $c \in M \setminus \overline{M}$ *satisfies* $\mathrm{tp}(c/\overline{M}) \perp \mathbf{P}$*; and*
2. $\overline{M}$ *is* $\mathbf{P}$*-minimal over* $\bigcup_{\eta\in I} N_\eta$.

Proof. (1) Assume by way of contradiction that there is $c \in M$ such that $\mathrm{tp}(c/\overline{M}) \not\perp \mathbf{P}$. As $\mathbf{P}$ is closed under non-orthogonality and automorphisms of $\mathfrak{C}$, there is $p \in \mathbf{P} \cap S(\overline{M})$ non-orthogonal to $\mathrm{tp}(c/\overline{M})$. Then, by Fact 2.3(3), there is $e \in M$ realizing p. So, by Lemma 4.7, $p \not\perp N_\eta$ for some $\eta \in I$. Thus, by Fact 2.3(4) there is $e^* \in M$ domination equivalent to e over $\overline{M}$ with $e^* \underset{N_\eta}{\perp} \overline{M}$. As $\{a_\nu : \nu \in Succ_I(\eta)\} \subseteq \overline{M}$, this contradicts the fact that $\langle N_\eta, a_\eta : \eta \in I\rangle$ is a weak $(\overline{\lambda}, \mathbf{P})$-decomposition of M.

(2) Choose any $M_1 \preceq \overline{M}$ that is $\aleph_\epsilon$-prime over $\bigcup_{\eta\in I} N_\eta$. Then (1) applies to M_1. That is, there is no $c \in M \setminus M_1$ such that $\mathrm{tp}(c/M_1) \not\perp \mathbf{P}$. Thus, $\overline{M}$ is $\mathbf{P}$-minimal over $\bigcup_{\eta\in I} N_\eta$.

Next, we show that if we additionally assume that $\mathbf{P} = \mathbf{P}^{\mathbf{active}}$, then we can extend the previous results to any $\aleph_\epsilon$-saturated submodel of M containing the decomposition.

Proposition 4.10 (P-NDOP, $\mathbf{P} = \mathbf{P}^{\mathbf{active}}$) *Suppose that* $\langle N_\eta, a_\eta : \eta \in I\rangle$ *is a weak* $(\overline{\lambda}, \mathbf{P})$*-decomposition of a* $(\overline{\lambda}, \mathbf{P})$*-saturated model* M*. Let* $M^* \preceq M$ *be any* $\aleph_\epsilon$*-saturated model containing* $\bigcup_{\eta\in I} N_\eta$*. Then there is no* $e \in M \setminus M^*$ *such that* $\mathrm{tp}(e/M^*) \not\perp \mathbf{P}$*.*

Proof. As both M^* and M are $\aleph_\epsilon$-saturated, it suffices to prove that there is no $e \in M \setminus M^*$ such that $\mathrm{tp}(e/M^*) \in \mathbf{P}$. Assume by way of contradiction that there were such an e. Let $M_0 \preceq M^*$ be any $\aleph_\epsilon$-prime model over $\bigcup_{\eta\in I} N_\eta$. Next, form an increasing sequence $\langle M_\alpha : \alpha \le \delta\rangle$ of $\aleph_\epsilon$-saturated models, with $M_\delta = M^*$, $M_{\alpha+1}$ is $\aleph_\epsilon$-prime over $M_\alpha \cup \{b_\alpha\}$, where $\mathrm{tp}(b_\alpha/M_\alpha)$ is regular, and for $\alpha < \delta$ a non-zero limit, M_α is $\aleph_\epsilon$-prime over $\bigcup_{\beta<\alpha} M_\beta$.

Choose $\alpha \leq \delta$ least such that there is some $e \in M \setminus M_\alpha$ such that $\mathrm{tp}(e/M_\alpha) \in \mathbf{P}$. By superstability, α cannot be a non-zero limit ordinal. Now suppose $\alpha = \beta + 1$. On one hand, if $p = \mathrm{tp}(e/M_\alpha) \in \mathbf{P}$ were non-orthogonal to M_β, then by Fact 2.3(4), there would be $e^* \in M$ such that $q = \mathrm{tp}(e^*/M_\beta)$ is regular and non-orthogonal to p, contradicting the minimality of α. On the other hand, if $p \perp M_\beta$, then as $\mathbf{P}^{\mathbf{active}} = \mathbf{P}$, $r = \mathrm{tp}(b_\beta/M_\beta) \in \mathbf{P}$, which again contradicts the minimality of α.

Thus, α must equal zero, i.e., there is $e \in M \setminus M_0$ such that $p = \mathrm{tp}(e/M_0) \in \mathbf{P}$. By Lemma 4.7, choose a $\triangleleft$-minimal $\eta \in I$ such that $p \not\perp N_\eta$.

Choose $q \in S(N_\eta)$ regular such that $p \not\perp q$ and let $q' \in S(M_0)$ be the non-forking extension of q to M_0. As both M_0 and M are $\aleph_\epsilon$-saturated, there is $c \in M \setminus M_0$ realizing q'. As $q' \in \mathbf{P}$, we have $c \in C_\eta(M)$ in the notation of Definition 4.3, which contradicts the maximality of $\{a_\nu : \nu \in Succ_I(\eta)\}$.

Corollary 4.11 (P-NDOP, $\mathbf{P} = \mathbf{P}^{\mathbf{active}}$) *Suppose that $\langle N_\eta, a_\eta : \eta \in I\rangle$ is a weak $(\overline{\lambda}, \mathbf{P})$-decomposition of M. Let $M^* \preceq M$ be any $\aleph_\epsilon$-saturated elementary submodel containing $\bigcup_{\eta\in I} N_\eta$. If $p \in \mathbf{P}$ and $p \not\perp M$, then $p \not\perp M^*$.*

Proof. As in the proof above, form an increasing sequence $\langle M_\alpha : \alpha \leq \delta\rangle$ of $\aleph_\epsilon$-saturated models, this time with $M_0 = M^*$, $M_\delta = M$, $M_{\alpha+1}$ is $\aleph_\epsilon$-prime over $M_\alpha \cup \{b_\alpha\}$, where $\mathrm{tp}(b_\alpha/M_\alpha)$ is regular, and for $\alpha < \delta$ a non-zero limit, M_α is $\aleph_\epsilon$-prime over $\bigcup_{\beta<\alpha} M_\beta$. Choose $\alpha \leq \delta$ least such that $p \not\perp M_\alpha$. We will show that $\alpha = 0$. Clearly, α cannot be a non-zero limit by superstability. Assume by way of contradiction that $\alpha = \beta + 1$. Then $p \not\perp M_\alpha$, but $p \perp M_\beta$. But, as before, this implies that $r = \mathrm{tp}(b_\beta/M_\beta) \in \mathbf{P}^{\mathbf{active}} = \mathbf{P}$. But now, M_β is an $\aleph_\epsilon$-saturated model containing $\bigcup_{\eta\in I} N_\eta$, yet there is an element of $M \setminus M_\beta$ realizing $r \in \mathbf{P}$, contradicting Proposition 4.10. Thus, $\alpha = 0$, so $p \not\perp M^*$.

Corollary 4.12 (P-NDOP, $\mathbf{P} = \mathbf{P}^{\mathbf{active}}$) *Suppose that $\langle N_\eta, a_\eta : \eta \in I\rangle$ is a weak $(\overline{\lambda}, \mathbf{P})$-decomposition of a $(\overline{\lambda}, \mathbf{P})$-saturated model M. If $p \in \mathbf{P}$ and $p \not\perp M$, then there is a unique $\triangleleft$-minimal $\eta \in I$ such that $p \not\perp N_\eta$.*

Proof. Let $M^* \preceq M$ be any $\aleph_\epsilon$-prime model over $\bigcup_{\eta\in I} N_\eta$. By Corollary 4.11 $p \not\perp M^*$, so by Lemma 4.7, $p \not\perp N_\eta$ for some $\triangleleft$-minimal $\eta \in I$. As in the proof of Lemma 4.7, the uniqueness follows from Lemma 4.2(1).

Until this point in our discussion, the submodels occurring in a decomposition could be very large, with an extreme case being that any model M has a one-element decomposition $\langle M\rangle$. The next definition limits the size of the submodels, while retaining the fact that they are at least $\aleph_\epsilon$-saturated.

Definition 4.13 A *prime* $(\overline{\lambda},\mathbf{P})$*-decomposition inside* M *(of* M*)* is a weak $(\overline{\lambda},\mathbf{P})$-decomposition inside M (of M) in which $N_{\langle\rangle}$ is $(\overline{\lambda},\mathbf{P})$-prime over $\emptyset$ and, for each $\eta\in I\setminus\{\langle\rangle\}$, N_η is $(\overline{\lambda},\mathbf{P})$-prime over $N_{\eta^-}\cup\{a_\eta\}$.

Definition 4.14 Fix a $(\overline{\lambda},\mathbf{P})$-saturated model M. A prime $(\overline{\lambda},\mathbf{P})$-decomposition $\mathfrak{d}_2=\langle N^2_\eta,a^2_\eta:\eta\in J\rangle$ *end extends* the prime $(\overline{\lambda},\mathbf{P})$-decomposition $\mathfrak{d}_1=\langle N^1_\eta,a^1_\eta:\eta\in I\rangle$ if $I\subseteq J$ and, for each $\eta\in I$, $N^2_\eta=N^1_\eta$ and $a^2_\eta=a^1_\eta$.

We say $\mathfrak{d}_2$ is a *regular end extension* of $\mathfrak{d}_1$ if, in addition $\operatorname{tp}(a_\eta/N_{\eta^-})$ is regular for each $\eta\in J\setminus I$. Furthermore, $\mathfrak{d}_2$ is a *standardly regular end extension* of $\mathfrak{d}_1$ if, $\operatorname{tp}(a_\eta/N_{\eta^-})=\operatorname{tp}(a_\nu/N_{\nu^-})$ whenever η,ν in $J\setminus I$, $\eta^-=\nu^-$, and $\operatorname{tp}(a_\eta/N_{\eta^-})\not\perp\operatorname{tp}(a_\nu/N_{\nu^-})$.

The following Lemma is straightforward, and relies on the fact that if $N\preceq M$ are both $(\overline{\lambda},\mathbf{P})$-saturated with $a\in M\setminus N$ satisfying $\operatorname{tp}(a/N)\in\mathbf{P}$, then there is $N[a]\preceq M$ that is $(\overline{\lambda},\mathbf{P})$-prime over $N\cup\{a\}$ and that $N[a]$ contains realizations of every regular type over N non-orthogonal to $\operatorname{tp}(a/N)$. Proofs of similar statements appear in Section X.3 of [4].

Lemma 4.15 *Suppose* $\mathfrak{d}=\langle N_\eta,a_\eta:\eta\in I\rangle$ *is a prime* $(\overline{\lambda},\mathbf{P})$*-decomposition inside an* $(\overline{\lambda},\mathbf{P})$*-saturated model* M*. Then:*

1. $\mathfrak{d}$ *is a prime* $(\overline{\lambda},\mathbf{P})$*-decomposition of* M *if and only if it has no proper (standardly regular) end extension; and*

2. *There is a prime* $(\overline{\lambda},\mathbf{P})$*-decomposition* $\mathfrak{d}^*$ *of* M *that is a standardly regular end extension of* $\mathfrak{d}$*.*

Similarly to the main theme of [Sh401], we wish to investigate $\mathbf{P}$-decompositions that lie above a specific triple (N,N',a), where $N\preceq N'$, with $\operatorname{tp}(a/N')\perp\mathbf{P}^\perp$ and $\operatorname{tp}(a/N')\perp N$. That is, triples where a could play the role of a_ν in some $\mathbf{P}$-decomposition with $N'=N_\eta$ and $N=N_{\eta^-}$. However, as in [Sh401], this is too much data to record at once, so we seek an ϵ-finite approximation of it.

Specifically, for M any model, let

$$\Gamma(M) := \{(A, B) : A \subseteq B \subseteq M \text{ are both } \epsilon\text{-finite}\}$$

We frequently write $\binom{B}{A}$ for elements of $\Gamma(M)$, and if A is not a subset of B, we mean $\binom{A\cup B}{A}$. Let

$$\Gamma_{\mathbf{P}}(M) := \{\binom{B}{A} \in \Gamma(M) : \mathrm{tp}(B/A) \perp \mathbf{P}^{\perp}\}$$

Definition 4.16 For $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$, a *prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition over* $\binom{B}{A}$ *inside* M*,* $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$, is a prime $(\overline{\lambda}, \mathbf{P})$-decomposition inside M in which $\langle 0\rangle$ is the unique successor of $\langle\rangle$ in I, $A \subseteq N_{\langle\rangle}$, $B \subseteq N_{\langle 0\rangle}$, and $B \subseteq \mathrm{dcl}(a_{\langle 0\rangle})$. By analogy with Definition 4.3,

- such a $\mathfrak{d}$ *is* **of M** if, for every $\eta \in I \neq \langle\rangle$, $\{a_\nu : \nu \in Succ_I(\eta)\}$ is a maximal N_η-independent subset of $C_\eta(M)$; and
- $\mathfrak{d}$ **P**-*exhausts* M *over* $\binom{B}{A}$ if, for every $\eta \in I \neq \langle\rangle$ for every regular $p \in S(N_\eta) \cap \mathbf{P}$ orthogonal to N_{η^-} (when $\eta \neq \langle\rangle$) and for every $e \in p(\mathfrak{C})$, if $e \underset{N_\eta}{\perp} \{a_\nu : \nu \in Succ_I(\eta)\}$ then $e \underset{N_\eta}{\perp} M$.

The following Lemma is straightforward. The verification of (5) is analogous to the proof of Lemma 4.4.

Lemma 4.17 *Fix a* $(\overline{\lambda}, \mathbf{P})$*-saturated model* M *and* $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$*.*

1. *If* $N_{\langle\rangle} \preceq M$ *is* $(\overline{\lambda}, \mathbf{P})$*-prime over* $\emptyset$*, contains* A*, and* $B \underset{A}{\perp} N_{\langle\rangle}$*, and* $N_{\langle 0\rangle} \preceq M$ *is* $(\overline{\lambda}, \mathbf{P})$*-prime over* $N_{\langle\rangle} \cup B$*, then* $\langle N_{\langle\rangle}, N_{\langle 0\rangle}\rangle$ *is a prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition over* $\binom{B}{A}$ *inside* M*;*
2. *A prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition* $\mathfrak{d}$ *over* $\binom{B}{A}$ *inside* M *is a prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition over* $\binom{B}{A}$ *of* M *if and only if* $\mathfrak{d}$ *has no proper* $(\overline{\lambda}, \mathbf{P})$*-decomposition over* $\binom{B}{A}$ *end extending it;*
3. *Every prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition over* $\binom{B}{A}$ *inside* M *has a (standardly regular) end extension to a prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition over* $\binom{B}{A}$ *of* M*;*

4. *Every prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition* $\mathfrak{d}$ *over* $\binom{B}{A}$ *inside* M *is a prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition inside* M*, hence has a (standardly regular) end extension to a prime* $(\overline{\lambda}, \mathbf{P})$*-decomposition* $\mathfrak{d}^*$ *of* M*; Moreover, if* $\mathfrak{d}$ *is a decomposition over* $\binom{B}{A}$ *of* M *and is indexed by the tree* $(I, \triangleleft)$ *and* $\mathfrak{d}^*$ *is indexed by* $(J, \triangleleft)$*, then* $\neg(\langle 0\rangle \trianglelefteq \eta)$ *for all* $\eta \in J \setminus I$*.*

5. *A* $(\overline{\lambda}, \mathbf{P})$*-decomposition* $\mathfrak{d}$ *over* $\binom{B}{A}$ *inside* M *is of* M *if and only if* $\mathfrak{d}$ $\mathbf{P}$*-exhausts* M *over* $\binom{B}{A}$*.*

5 Trees of subsets of an $\aleph_\epsilon$-saturated model

Throughout this section T is superstable with P-NDOP, and $\mathbf{P}$ is closed under automorphisms of $\mathfrak{C}$, non-orthogonality, and $\mathbf{P} = \mathbf{P}^{\text{active}}$.

In addition, all models M we consider will be $\aleph_\epsilon$-saturated, and all decompositions we consider will be $(\aleph_\epsilon, \mathbf{P})$-decompositions inside/of M.

Definition 5.1 Fix an $\aleph_\epsilon$-saturated model M and $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$. Suppose $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition over $\binom{B}{A}$ of M. Then

$$\mathcal{P}_{\mathbf{P}}(\mathfrak{d}, M) = \{p \in S(M) : p \in \mathbf{P}, p \perp N_{\langle\rangle}, \text{ but } p \not\perp N_\eta \text{ for some } \eta \in I \setminus \{\langle\rangle\}\}$$

The goal for this section will be Theorem 5.12, which asserts that $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$ for any two prime, $(\aleph_\epsilon, \mathbf{P})$-decompositions $\mathfrak{d}_1, \mathfrak{d}_2$ of M above $\binom{B}{A}$. We begin by introducing another way of 'increasing' a decomposition.

Definition 5.2 A prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}_2 = \langle N^2_\eta, a^2_\eta : \eta \in J\rangle$ inside $\mathfrak{C}$ is a *blow up* of the prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}_1 = \langle N^1_\eta, a^1_\eta : \eta \in I\rangle$ inside $\mathfrak{C}$ if $J = I$, but for every $\eta \in I$, $N^1_\eta \preceq N^2_\eta$ and, when $\eta \neq \langle\rangle$, N^2_η is $(\aleph_\epsilon, \mathbf{P})$-prime over $N^1_\eta \cup N^2_{\eta^-}$.

Lemma 5.3 *Suppose that* M *is* $\aleph_\epsilon$*-saturated,* $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$*,* $\mathfrak{d}_2 = \langle N^2_\eta, a_\eta : \eta \in I\rangle$ *is a blow up of* $\mathfrak{d}_1 = \langle N^1_\eta, a_\eta : \eta \in I\rangle$*,* $A \subseteq B \subseteq N^1_{\langle 0\rangle}$*, and each* $N^2_\eta \preceq M$*. Then:*

1. *If* $\nu \in Succ_I(\eta)$*, then* $N^2_\eta \underset{N^1_\eta}{\perp} N^1_\nu$*;*

2. *If* $Y = \{\rho \in I : \neg(\eta \lhd \rho)\}$, *then* $N^2_\eta \underset{N^1_\eta}{\perp} \bigcup_{\rho \in Y} N^1_\rho$ *and* $\eta \lhd \nu$ *implies* $N^2_\nu \underset{N^1_\eta}{\perp} \bigcup_{\rho \in Y} N^1_\rho$;

3. $\mathfrak{d}_2$ *is an* $(\aleph_\epsilon, \mathbf{P})$*-decomposition inside* M *above* $\binom{B}{A}$ *if and only if* $\mathfrak{d}_1$ *is; and*

4. $\mathfrak{d}_2$ *is an* $(\aleph_\epsilon, \mathbf{P})$*-decomposition of* M *above* $\binom{B}{A}$ *if and only if* $\mathfrak{d}_1$ *is.*

Proof. This is exactly analogous to Fact 1.20 of [Sh401]. In the proof of (4), we need to appeal to $\mathbf{P}$-NDOP instead of NDOP.

Lemma 5.4 *Suppose* M *is* $\aleph_\epsilon$*-saturated and* $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$. *If* $\mathfrak{d}_2$ *is a blow up of* $\mathfrak{d}_1$ *and both* $\mathfrak{d}_1$, $\mathfrak{d}_2$ *are* $(\aleph_\epsilon, \mathbf{P})$*-decompositions of* M *over* $\binom{B}{A}$, *then* $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$.

Proof. This is very much like Fact 1.22 of [Sh401], but we give details. As notation, say $\mathfrak{d}_\ell = \langle N^\ell_\eta, a_\eta : \eta \in I\rangle$ for $\ell = 1, 2$. Fix $p \in S(M) \cap \mathbf{P}$, so in particular, p is regular. We must prove that

$$\big(p \perp N^1_{\eta^-} \text{ and } p \not\perp N^1_\eta\big) \Leftrightarrow \big(p \perp N^2_{\eta^-} \text{ and } p \not\perp N^2_\eta\big)$$

for every $\eta \neq \langle\rangle$.

First, assume $\eta \neq \langle\rangle$ and $p \perp N^1_{\eta^-}$ and $p \not\perp N^1_\eta$. As $N^1_\eta \preceq N^2_\eta$, $p \not\perp N^2_\eta$ trivially. Also, choose a regular $q \in S(N^1_\eta)$ with $p \not\perp q$. Then $q \perp N^1_{\eta^-}$ since p is, and it suffices to show that q is orthogonal to $N^2_{\eta^-}$. But this follows immediately since $N^1_\eta \underset{N^1_{\eta^-}}{\perp} N^2_{\eta^-}$.

Conversely, assume $\eta \neq \langle\rangle$ and $p \perp N^2_{\eta^-}$ and $p \not\perp N^2_\eta$. Then, since $N^1_{\eta^-} \preceq N^2_{\eta^-}$, $p \perp N^1_{\eta^-}$. As well, $(N^1_{\eta^-}, N^2_{\eta^-}, N^1_\eta)$ form an independent triple of $\aleph_\epsilon$-saturated models (see Definition 3.1) and N^2_η is $\aleph_\epsilon$-prime over their union. Thus, as $p \in \mathbf{P}$, it follows from $\mathbf{P}$-NDOP that $p \not\perp N^1_\eta$.

Lemma 5.5 *Suppose that* M *is* $\aleph_\epsilon$*-saturated,* $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$, *and for* $\ell = 1, 2$ $\mathfrak{d}_\ell = \langle N^\ell_\eta, a^\ell_\eta : \eta \in I_\ell\rangle$ *are each prime,* $\aleph_\epsilon$*-decompositions of* M *above* $\binom{B}{A}$. *If* $N^1_{\langle\rangle} = N^2_{\langle\rangle}$ *then* $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$.

Proof. First, by Lemma 4.17(4), choose a prime, $\aleph_\epsilon$-prime decomposition $\mathfrak{d}_1^* = \langle N_\eta^1, a_\eta^1 : \eta \in J_1\rangle$ of M end extending $\mathfrak{d}_1$. As notation, let $H = J_1 \setminus I_1$ and for each $\eta \in H$, let $N_\eta^2 = N_\eta^1$ and $a_\eta^2 = a_\eta^1$. It is easily checked that $\mathfrak{d}_2^* := \langle N_\eta^2, a_\eta^2 : \eta \in I_2 \cup H\rangle$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition of M.

Now, for each $p \in S(M) \cap \mathbf{P}$ with $p \perp N_{\langle\rangle}^1$ and for each $\ell = 1, 2$ there is a unique $\eta(p,\ell) \in I_\ell \cup H$ such that $p \not\perp N_{\eta(p,\ell)}$, but $p \perp N_{\eta(p,\ell)^-}$. But, as $N_\eta^2 = N_\eta^1$ for each $\eta \in H$, $\eta(p,1) \in H$ if and only if $\eta(p,2) \in H$.

Thus, for each $p \in S(M) \cap \mathbf{P}$ that is orthogonal to $N_{\langle\rangle}^1 = N_{\langle\rangle}^2$ we have $p \in \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1, M)$ if and only if $\eta(p,1) \in H$ if and only if $\eta(p,2) \in H$ if and only if $p \in \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$.

We come to the issue of the existence of blow-ups of decompositions. It is comparatively easy to blow up a decomposition inside an $\aleph_\epsilon$-saturated model M.

Lemma 5.6 *Suppose that M is $\aleph_\epsilon$-saturated and $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M. For any N^* satisfying $N_{\langle\rangle} \preceq N^* \preceq M$ that is $\aleph_\epsilon$-prime over $\emptyset$, there is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}^*$ inside M with $N_{\langle\rangle}^{\mathfrak{d}^*} = N^*$ that is a blow up of $\mathfrak{d}$.*

Proof. Choose any enumeration $\langle \eta_i : i < i^*\rangle$ of I such that $\eta_i \triangleleft \eta_j$ implies $i < j$ and so that for some $\alpha^* \le i^*$ $\eta_i \in Succ_I(\langle\rangle)$ if and only if $1 \le i < \alpha^*$. Note that $\eta_0 = \langle\rangle$ for any such enumeration. Put $N_{\langle\rangle}^* := N^*$. Then, by induction on $1 \le i < i^*$, argue that

$$N^* \underset{N_{\langle\rangle}}{\perp} \bigcup_{j<i} N_{\eta_j}^*$$

and let $N_{\eta_i}^* \preceq M$ be any $\aleph_\epsilon$-prime model over $N_{\eta_i^-}^* \cup N_{\eta_i}$. Then it is easily checked that $\mathfrak{d}^* = \langle N_\eta^*, a_\eta : \eta \in I\rangle$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M that is a blow up of $\mathfrak{d}$.

'Blowing down' a decomposition is more delicate and requires two technical Lemmas, Lemma 3.12 and Lemma 3.13 that assert the existence of $\aleph_\epsilon$-submodels of a given $\aleph_\epsilon$-saturated structure with certain properties.

Lemma 5.7 *Suppose that M is $\aleph_\epsilon$-saturated and $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M. For any $\aleph_\epsilon$-saturated $N_0 \preceq N_{\langle\rangle}$ such that for every $\eta \in Succ_I(\langle\rangle)$, either* $\mathrm{tp}(a_\eta/N_{\langle\rangle})$ *does not fork over N_0*

or $\mathrm{tp}(a_\eta/N_{\langle\rangle}$ *is regular and non-orthogonal to* N_0. *Then there is a prime* $(\aleph_\epsilon, \mathbf{P})$*-decomposition* $\mathfrak{d}_0$ *inside* M *with* $N^{\mathfrak{d}_0}_{\langle\rangle} = N_0$ *such that* $\mathfrak{d}$ *is a blow up of* $\mathfrak{d}_0$.

Proof. Choose an enumeration $\langle \eta_i : i < i^* \rangle$ of I as in the proof of Lemma 5.6. That is, $\eta_0 = \langle\rangle$, $\eta_i \lhd \eta_j$ implies $i < j$, and $\eta_i \in Succ_I(\langle\rangle)$ if and only if $1 \le i < \alpha^*$ for some $\alpha^* \le i^*$.

Put $N^0_{\eta_0} = N_0$. For $1 \le i < i^*$ we inductively construct $N^0_{\eta_i}$ to satisfy:

- $N^0_{\eta_i} \preceq N_{\eta_i}$ and $N^0_{\eta_i} \underset{N^0_{(\eta_i^-)}}{\perp} N^1_{\eta_i^-}$
- N_{η_i} is $\aleph_\epsilon$-prime over $N^0_{\eta_i} \cup N_{\eta_i^-}$
- $a_{\eta_i} \in N^0_{\eta_i}$ and $N^0_{\eta_i}$ is $\aleph_\epsilon$-prime over $N^0_{\eta_i} \cup \{a_{\eta_i}\}$.

To accomplish this, for each $1 \le i < \alpha^*$, use Lemma 3.13 to define $N^0_{\eta_i}$ (where $M_1 = N_0$, $M = N_{\eta_i}$). We can take $N^0_{\eta_i}$ to be the N there, and we can take a^* to be a_{η_i}. Similarly, for $\alpha^* \le i < i^*$ we apply Lemma 3.12, where M is taken to be $N_{\eta_i^-}$, M_1 is $N^0_{\eta_i^-}$, M_2 is $N_{\eta_i^{--}}$, a is a_{η_i}, and taking $N^0_{\eta_i}$ to be the N produced there.

Definition 5.8 Suppose M is $\aleph_\epsilon$-saturated and $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$. We say that an ϵ-finite subset $W \subseteq M$ *has a base* $W_0 \subseteq W$ *respecting* $\binom{B}{A}$ if $A \subseteq W_0$, $W_0 \underset{A}{\perp} B$, and W is dominated by B over W_0.

Lemma 5.9 *If* $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$ *is a* $(\aleph_\epsilon, \mathbf{P})$*-decomposition inside* M *over* $\binom{B}{A}$ *and* $V \subseteq \bigcup_{\eta \in I} N_\eta$ *is* ϵ*-finite, then there is an* ϵ*-finite* W *with* $V \subseteq W$ *and* $W \setminus V \subseteq N_{\langle\rangle}$ *that has a base* $W_0 \subseteq W \cap N_{\langle\rangle}$ *respecting* $\binom{B}{A}$.

Proof. Without loss, we may assume $A \subseteq V$. It follows from the definition of an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M over $\binom{B}{A}$ that $B \underset{A}{\perp} N_{\langle\rangle}$ and that B dominates $\bigcup_{\eta \in I} N_\eta$ and hence V over $N_{\langle\rangle}$. As both B and V are ϵ-finite, it follows from superstability that there is an ϵ-finite $C \subseteq N_{\langle\rangle}$ such that $BV \underset{C}{\perp} N_{\langle\rangle}$. So B dominates V over C. Again, without loss, $A \subseteq C$. Take $W = V \cup C$. Then $W_0 := W \cap N_{\langle\rangle}$ is a base respecting $\binom{B}{A}$.

Lemma 5.10 *Suppose $W \subseteq M$ is ϵ-finite and has a base $W_0 \subseteq W$ respecting $\binom{B}{A}$ and that $N \preceq M$ is $\aleph_\epsilon$-prime over $\emptyset$, $W_0 \subseteq N$, with $N \underset{A}{\perp} B$. Then there is $N[B] \preceq M$ that is $\aleph_\epsilon$-prime over $N \cup B$ such that $W \subseteq N[B]$ and such that the two-element sequence $\langle N, N[B]\rangle$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M over $\binom{B}{A}$ (taking $a_{\langle 0\rangle}$ to be B).*

Proof. As A, B, W are all ϵ-finite, $N \underset{A}{\perp} B$, W dominated by B over W_0, and the fact that $\mathrm{tp}(W/\mathrm{acl}(W_0 \cup B))$ is stationary, it follows that $\mathrm{tp}(W/NB)$ is $\aleph_\epsilon$-isolated. Thus, $W \subseteq N[B]$ for some $\aleph_\epsilon$-prime model over $N \cup B$. Checking that the two element sequence is an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M over $\binom{B}{A}$ is routine.

Proposition 5.11 *If M is $\aleph_\epsilon$-saturated, $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$, and the ϵ-finite set $W \subseteq M$ has a base $W_0 \subseteq W$ respecting $\binom{B}{A}$, then there is an $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}$ of M over $\binom{B}{A}$ with $W_0 \subseteq N^{\mathfrak{d}}_{\langle\rangle}$ and $W \subseteq N^{\mathfrak{d}}_{\langle 0\rangle}$. Moreover, if $\mathfrak{d}_0$ is any $(\aleph_\epsilon, \mathbf{P})$-decomposition of M over $\binom{B}{A}$, then $\mathfrak{d}$ can be chosen so that $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_0, M)$.*

Proof. Suppose $\mathfrak{d}_0 = \langle N_\eta, a_\eta : \eta \in I\rangle$ is given. As $\mathrm{dcl}(a_{\langle 0\rangle}) = \mathrm{dcl}(B)$, we may assume that $a_{\langle 0\rangle} = B$. Thus, $B \underset{A}{\perp} N_{\langle\rangle}$. Choose a finite $D \subseteq N_{\langle\rangle}$ such that $A \subseteq D$ and $W \underset{DB}{\perp} N_{\langle\rangle}B$. By e.g., 1.18(9) of [Sh401] there is $N^1 \preceq N_{\langle\rangle}$ that is $\aleph_\epsilon$-prime over $\emptyset$, $N^1 \underset{A}{\perp} D$, and $N_{\langle\rangle}$ is $\aleph_\epsilon$-prime over $N^1 \cup D$.

As for non-forking, we claim that the set $\{B, W_0, N^1\}$ is independent over A. To see this, first recall that $B \underset{A}{\perp} W_0$ since W_0 is a base respecting $\binom{B}{A}$. As well, $B \underset{A}{\perp} N_{\langle\rangle}$ since $\mathfrak{d}$ is over $\binom{B}{A}$. Thus, $B \underset{D}{\perp} N_{\langle\rangle}$. Thus, by our choice of D and forking calculus, $W_0 B \underset{D}{\perp} N_{\langle\rangle}$, so $W_0 B \underset{D}{\perp} N^1$ since $N^1 \preceq N_{\langle\rangle}$. But now, as $N^1 \underset{A}{\perp} D$, we have $N^1 \underset{A}{\perp} BW_0$ which gives the independence.

By Lemma 5.7, there is an $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}_1$ inside M over $\binom{B}{A}$ with $N^{\mathfrak{d}_1}_{\langle\rangle} = N^1$. By Lemma 5.3(4) $\mathfrak{d}_1$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition of M over $\binom{B}{A}$, so by Lemma 5.4 $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_0, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1, M)$.

Next, let $N^2 \preceq M$ be $\aleph_\epsilon$-prime over $N^1 \cup W_0$. As $B \underset{N^1}{\perp} W_0$, and the $\aleph_\epsilon$-isolation of N^1 we have $N^2 \underset{N^1}{\perp} B$. Thus, by Lemma 5.6 there is a $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}_2$ inside M over $\binom{B}{A}$ with $N^{\mathfrak{d}_2}_{\langle\rangle} = N^2$. Again, by Lemma 5.3(4)

$\mathfrak{d}_2$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition of M over $\binom{B}{A}$ and by Lemma 5.4 $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$.

Put $N := N^2$. Clearly, $W_0 \subseteq N$ and we showed $B \underset{N^1}{\perp} N$. But, as B and N^1 are independent over A, $B \underset{A}{\perp} N$. So, by Lemma 5.10 there is $N[B] \preceq M$, $\aleph_\epsilon$-prime over $N \cup B$, such that $W \subseteq N[B]$ and $\langle N, N[B]\rangle$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M over $\binom{B}{A}$.

Finally, by Lemma 4.17 there is an $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}_3$ of M over $\binom{B}{A}$ end extending $\langle N, N[B]\rangle$. As $N^{\mathfrak{d}_3}_{\langle\rangle} = N = N^{\mathfrak{d}_2}_{\langle\rangle}$ we conclude by Lemma 5.5 that $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_3, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$. Thus, $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_3, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_0, M)$ and we finish.

We are finally ready to prove our main Theorem.

Theorem 5.12 *Suppose that M is $\aleph_\epsilon$-saturated and $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$. Then $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$ for any two prime $(\aleph_\epsilon, \mathbf{P})$-decompositions $\mathfrak{d}_1, \mathfrak{d}_2$ of M over $\binom{B}{A}$.*

Proof. Suppose $\mathfrak{d}_1 = \langle N_\eta, a_\eta : \eta \in I\rangle$. By symmetry, it suffices to prove that every $p \in \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1, M)$ is in $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$. Fix such a p and choose $\eta \in I \setminus \{\langle\rangle\}$ such that $p \not\perp N_\eta$ but $p \perp N_{\eta^-}$. Choose $q \in S(N_\eta)$ regular such that $p \not\perp q$ and choose a finite $V \subseteq N_\eta$ on which q is based and stationary. By Lemma 5.9 there is an ϵ-finite W such that $V \subseteq W \subseteq M$ that has a subset $W_0 = W \cap N_{\langle\rangle}$ respecting $\binom{B}{A}$. Note that since $p \perp N_{\langle\rangle}$ we have $p \perp W_0$, hence $q \perp W_0$. By applying Proposition 5.11 to W and $\mathfrak{d}_2$, we get that there is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}^*$ of M over $\binom{B}{A}$ with $\mathcal{P}_{\mathbf{P}}(\mathfrak{d}^*, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$. But, by construction, there is a type parallel to q (and hence non-orthogonal to p) in $S(N^{\mathfrak{d}_2}_{\langle 0\rangle})$. As well, since B dominates W over W_0 and $B \underset{A}{\perp} N_{\langle\rangle}$ we have $W \underset{W_0}{\perp} N_{\langle\rangle}$. As q is based on W and $q \perp W_0$, we have that q (and hence p) is orthogonal to $N_{\langle\rangle}$. Thus, $p \in \mathcal{P}_{\mathbf{P}}(\mathfrak{d}^*, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_2, M)$.

The previous Theorem inspires the following definition.

Definition 5.13 For M $\aleph_\epsilon$-saturated and $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$, $\mathcal{P}_{\mathbf{P}}(\binom{B}{A}, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}, M)$ for some (equivalently for every) prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}$ of M over $\binom{B}{A}$.

Corollary 5.14 *Suppose that M is $\aleph_\epsilon$-saturated, $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M)$, and that $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ is a prime, $(\aleph_\epsilon, \mathbf{P})$-decomposition of M satisfying (1) $N_{\langle\rangle}$*

is $\aleph_\epsilon$-prime over A; (2) $B \underset{A}{\perp} N_{\langle\rangle}$; and (3) $N_{\langle 0\rangle}$ is $\aleph_\epsilon$-prime over $N_{\langle\rangle} \cup B$. Then, for every $p \in S(M) \cap \mathbf{P}$, $p \in \mathcal{P}_{\mathbf{P}}(\binom{B}{A}, M)$ if and only if $\langle 0\rangle \trianglelefteq \eta(p)$, where $\eta(p)$ is the unique $\trianglelefteq$-minimal $\eta \in I$ satisfying $p \not\perp N_\eta$ (see Corollary 4.12).

Proof. Given $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ as above, let $X = \{\nu \in I \setminus \{\langle\rangle\} : \neg(\langle 0\rangle \trianglelefteq \nu\}$ and let $I_0 = I \setminus X$. The conditions on $\mathfrak{d}$ ensure that $\mathfrak{d}_0 := \langle N_\eta, a_\eta : \eta \in I_0\rangle$ is a prime, $(\aleph_\epsilon, \mathbf{P})$-decomposition of M above $\binom{B}{A}$. Thus, by Theorem 5.12, for any $p \in S(M) \cap \mathbf{P}$ we have

$$p \in \mathcal{P}_{\mathbf{P}}(\binom{B}{A}, M) \quad \Leftrightarrow \quad p \in \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_0, M) \quad \Leftrightarrow \quad \langle 0\rangle \trianglelefteq \eta(p)$$

The following characterization is analogous to Claim 1.24 of [Sh401].

Proposition 5.15 *Assume that $M_1 \preceq M_2$ are $\aleph_\epsilon$-saturated and $\binom{B}{A} \in \Gamma_{\mathbf{P}}(M_1)$. Then the following are equivalent:*

1. *No $p \in \mathcal{P}_{\mathbf{P}}(\binom{B}{A}, M_1)$ is realized in M_2;*
2. *There is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M_1 above $\binom{B}{A}$ that is also a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M_2 above $\binom{B}{A}$; and*
3. *Every prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M_1 above $\binom{B}{A}$ is also a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M_2 above $\binom{B}{A}$.*

Proof. (3) $\Rightarrow$ (2) is immediate since prime $(\aleph_\epsilon, \mathbf{P})$-decompositions of M_1 over $\binom{B}{A}$ exist.

(2) $\Rightarrow$ (1): Let $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ be a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M_1 above $\binom{B}{A}$ and assume that there is $e \in M_2 \setminus M_1$ such that $p = \mathrm{tp}(e/M_1) \in \mathcal{P}_{\mathbf{P}}(\mathfrak{d}, M_1)$. Choose $\eta \in I$ to be $\triangleleft$-minimal such that $p \not\perp N_\eta$. Note that $\langle 0\rangle \trianglelefteq \eta$. By Fact 2.3(4) and because N_η, M_1, M_2 are $\aleph_\epsilon$-saturated, we can replace e by the realization of a non-orthogonal regular type that satisfies $e \underset{N_\eta}{\perp} M_1$. As $e \in C_\eta(M_2)$, $\{a_\nu : \nu \in Succ_I(\eta)\}$ is not maximal N_η-independent subset of $C_\eta(M_2)$, so $\mathfrak{d}$ is not a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M_2 above $\binom{B}{A}$.

(1) $\Rightarrow$ (3): Let $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ be a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M_1 above $\binom{B}{A}$, and assume that it is not a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M_2.

Then, by Definition 4.16, there is $\eta \in I \setminus \{\langle\rangle\}$ such that $\{a_\nu : \nu \in Succ_I(\eta)\}$ is not a maximal, N_η-independent subset of $C_\eta(M_2)$. As both N_η and M_2 are $\aleph_\epsilon$-saturated, this implies that there is $e \in C_\eta(M_2)$ such that $\text{tp}(e/N_\eta) \in \mathbf{P}$, but $e \underset{N_\eta}{\not\perp} \{a_\nu : \nu \in Succ_I(\eta)\}$. By Lemma 4.17(5), $\mathfrak{d}$ $\mathbf{P}$-exhausts M_1 over $\binom{B}{A}$ so $e \underset{N_\eta}{\not\perp} M_1$. Thus, $p = \text{tp}(e/M_1)$ is an element of $\mathcal{P}_{\mathbf{P}}(\binom{B}{A}, M_1)$ that is realized in M_2.

For pairs $\binom{B_1}{A_1}$ and $\binom{B_2}{A_2}$ from $\Gamma(M)$, we consider two ways in which $\binom{B_2}{A_2}$ can extend $\binom{B_1}{A_1}$, corresponding to the former having 'more information' or 'appearing higher up in a $\mathbf{P}$-decomposition.'

First, write $\binom{B_1}{A_1} \le_a \binom{B_2}{A_2}$ if both are from $\Gamma(M)$, $A_1 \subseteq A_2$, $B_1 \subseteq B_2$, $B_1 \underset{A_1}{\perp} A_2$, and B_2 dominated by B_1 over A_2. Intuitively, think of $\binom{B_2}{A_2}$ as being a 'better approximation' of (N, N', a).

The next approximation, which should be thought of as 'stepping up in the tree' is given by $\binom{B_1}{A_1} \le_b \binom{B_2}{A_2}$ if and only if $A_2 = B_1$, and $\text{tp}(B_2/A_2)$ is regular and is orthogonal to A_1.

Finally, let $\le^*$ be the transitive closure of $\le_a \cup \le_b$.

Proposition 5.16 *Fix an $\aleph_\epsilon$-saturated model M and $\binom{B_1}{A_1}$, $\binom{B_2}{A_2}$ from $\Gamma_{\mathbf{P}}(M)$.*

1. *If $\binom{B_1}{A_1} \le_a \binom{B_2}{A_2}$, then $\mathcal{P}_{\mathbf{P}}(\binom{B_1}{A_1}, M) = \mathcal{P}_{\mathbf{P}}(\binom{B_2}{A_2}, M)$;*
2. *If $\binom{B_1}{A_1} \le_b \binom{B_2}{A_2}$, then $\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A_2}, M)$ is a proper subset of $\mathcal{P}_{\mathbf{P}}(\binom{B_1}{A_1}, M)$*
3. *If $\binom{B_1}{A_1} \le^* \binom{B_2}{A_2}$ then $\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A_2}, M) \subseteq \mathcal{P}_{\mathbf{P}}(\binom{B_1}{A_1}, M)$;*
4. *If $A_1 = A_2$ (whose common value we denote by A) $\text{tp}(B_1/A)$, $\text{tp}(B_2, A)$ are both regular, and $B_1 \underset{A}{\not\perp} B_2$, then $\mathcal{P}_{\mathbf{P}}(\binom{B_1}{A}, M) = \mathcal{P}_{\mathbf{P}}(\binom{B_2}{A}, M)$.*
5. *If $A_1 = A_2 = A$ and $B_1 \underset{A}{\perp} B_2$, then the sets $\mathcal{P}_{\mathbf{P}}(\binom{B_1}{A}, M)$ and $\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A}, M)$ are disjoint.*

Proof. (1) Let $N_{\langle\rangle} \preceq M$ be $\aleph_\epsilon$-prime over $\emptyset$ with $A_2 \subseteq N_{\langle\rangle}$ and $B_2 \underset{A_2}{\perp} N_{\langle\rangle}$. Let $N_{\langle\rangle}$ be $\aleph_\epsilon$-prime over $N_{\langle\rangle} \cup B_2$, let $a_{\langle 0\rangle} = B_2$, and let $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ be a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M over $\binom{B_2}{A_2}$ end extending $\langle N_{\langle\rangle}, N_{\langle 0\rangle}\rangle$. It follows easily by the forking calculus that $\mathfrak{d}$ is also a prime

$(\aleph_\epsilon, \mathbf{P})$-decomposition of M over $\binom{B_1}{A_1}$. Thus, two applications of Theorem 5.12 yield

$$\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A_2}, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}, M) = \mathcal{P}_{\mathbf{P}}(\binom{B_1}{A_1}, M)$$

(2) Given $A_1 \subseteq B_1 = A_2 \subseteq B_2 \subseteq M$ with $\mathrm{tp}(B_2/A_2) \perp A_1$, first choose an $\aleph_\epsilon$-prime $N_{\langle\rangle} \preceq M$ containing A_1 with $B_2 \underset{A_1}{\perp} N_{\langle\rangle}$. Note that $\mathrm{tp}(B_2/A_2) \perp N_{\langle\rangle}$. Let $a_{\langle\rangle}$ be an arbitrary element of $N_{\langle\rangle}$, let $a_{\langle 0\rangle} := A_2$, and choose $N_{\langle 0\rangle} \preceq M$ to be $\aleph_\epsilon$-prime over $N_{\langle\rangle} \cup A_2$, with $N_{\langle 0\rangle} \underset{N_{\langle\rangle} A_2}{\perp} B_2$. Also, choose $N_{\langle 0,0\rangle} \preceq M$ to be $\aleph_\epsilon$-prime over $N_{\langle 0\rangle} \cup B_2$ and let $a_{\langle 0,0\rangle} := B_2$.

Let $\mathfrak{d}_0$ be the three-element prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\langle N_\eta, a_\eta : \eta \in \{\langle\rangle, \langle 0\rangle, \langle 0,0\rangle\}\rangle$ inside M above $\binom{B_1}{A_1}$. Next, by 'collapsing', let $\mathfrak{d}'_0 = \langle N'_\eta, a'_\eta : \eta \in \{\langle\rangle, \langle 0\rangle\}\rangle$ be the two-element prime $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M above $\binom{B_2}{A_2}$, where $N'_{\langle\rangle} := N_{\langle 0\rangle}$, $a'_{\langle\rangle} := a_{\langle 0\rangle}$, $N'_{\langle 0\rangle} := N_{\langle 0,0\rangle}$, and $a'_{\langle 0\rangle} := a_{\langle 0,0\rangle}$.

Next, choose a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}' = \langle N'_\eta, a'_\eta : \eta \in I'\rangle$ of M above $\binom{B_2}{A_2}$ end extending $\mathfrak{d}'_0$. It follows immediately from Theorem 5.12 that $\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A_2}, M) = \mathcal{P}_{\mathbf{P}}(\mathfrak{d}', M)$, so to obtain the inclusion $\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A_2}, M) \subseteq \mathcal{P}_{\mathbf{P}}(\binom{B_1}{A_1}, M)$ it suffices to construct a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in J\rangle$ inside M over $\binom{B_1}{A_1}$ such that, for any $p \in S(M) \cap \mathbf{P}$, if $p \not\perp N'_\eta$ but $p \perp N'_{\eta^-}$ for some $\eta \in I'$ with $\langle 0\rangle \trianglelefteq \eta'$, there is $\eta \in J$ such that $\langle 0\rangle \trianglelefteq \eta$, $p \not\perp N_\eta$, but $p \perp N_{\eta^-}$.

We accomplish this as follows: Recall that N_η, a_η were defined for $\eta \in \{\langle\rangle, \langle 0\rangle, \langle 0,0\rangle\}$ above. Let $J' \subseteq I'$ be $\{\langle\rangle\} \cup \{\eta \in I' : \langle 0\rangle \trianglelefteq \eta\}$, and define a function h with domain J' by $h(\eta) := \langle 0\rangle\hat{\ }\eta$ if $\eta \neq \langle\rangle$. That is, the function h is 'undoing' the collapse given above. Let $J = \{\langle\rangle, \langle 0\rangle\} \cup \{h(\eta : \eta \in J'\})$, and for each $\eta \in J'$, put $N_{h(\eta)} := N'_\eta$ and $a_{h(\eta)} := a'_\eta$. Then $\mathfrak{d} := \langle N_\eta, a_\eta : \eta \in J\rangle$ is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M above $\binom{B_1}{A_1}$, and for any $p \in \mathcal{P}_{\mathbf{P}}(\mathfrak{d}', M)$, if $p \not\perp N'_\eta$ for some $\eta \in J'$, then $p \not\perp N_{h(\eta)}$. Thus, $\mathfrak{d}$ is as required.

To show that the inclusion is strict, choose any regular type $q \in S(N_{\langle 0\rangle})$ that is non-orthogonal to $\mathrm{tp}(B_2/N_{\langle 0\rangle})$. It is easy to check that the non-forking extension of q to $S(M)$ is an element of $\mathcal{P}_{\mathbf{P}}(\binom{B_1}{A_1}, M) \setminus \mathcal{P}_{\mathbf{P}}(\binom{B_2}{A_2}, M)$.

(3) follows immediately from (1) and (2).

(4) By symmetry, it suffices to show that $\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A}, M) \subseteq \mathcal{P}_{\mathbf{P}}(\binom{B_1}{A}, M)$, so fix a regular type $p \in S(M) \cap \mathbf{P} \setminus \mathcal{P}_{\mathbf{P}}(\binom{B_1}{A}, M)$. We will eventually produce

a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}_2$ inside M over $\binom{B_2}{A}$ with the property that $p \not\perp N_\eta$ for some η satisfying $\neg(\langle 0\rangle \trianglelefteq \eta)$, which suffices by Lemma 4.17(4) and Theorem 5.12.

We begin by choosing an $\aleph_\epsilon$-prime (over $\emptyset$) $N_{\langle\rangle} \preceq M$ that contains A, but $B_1B_2 \underset{A}{\perp} N_{\langle\rangle}$. Note that B_1 and B_2 are domination equivalent over $N_{\langle\rangle}$.

Let $a_{\langle\rangle} \in N_{\langle\rangle}$ be arbitrary, let N^1 be $\aleph_\epsilon$-prime over $N_{\langle\rangle} \cup B_1$, and let $a_{\langle 0\rangle} := B_1$. Then $\mathfrak{d}_1 := \langle N_\eta, a_\eta : \eta \in \{\langle\rangle, \langle 0\rangle\}\rangle$ is a two-element prime $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M over $\binom{B_1}{A}$. Let $\mathfrak{d}_1' = \langle N_\eta, a_\eta : \eta \in I\rangle$ be a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M over $\binom{B_1}{A}$ end extending $\mathfrak{d}_1$. Next, let $\mathfrak{d}_1^* = \langle N_\eta, a_\eta : \eta \in J\rangle$ be a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M end extending $\mathfrak{d}_1'$. Let $H = \{\eta \in J : \neg(\langle 0\rangle \trianglelefteq \eta)\}$. Then H is a subtree of J, whose intersection with I is $\{\langle\rangle\}$. Furthermore, as $p \in \mathbf{P}$, it follows from Corollary 4.12 that $p \not\perp N_\eta$ for some $\eta \in J$. However, since $p \notin \mathcal{P}_{\mathbf{P}}(\binom{B_1}{A}, M)$, it follows from Theorem 5.12 that $p \notin \mathcal{P}_{\mathbf{P}}(\mathfrak{d}_1', M)$, hence $p \not\perp N_\eta$ for some $\eta \in H$.

But now, choose $N^2 \preceq M$ to be $\aleph_\epsilon$-prime over $N_{\langle\rangle} \cup B_2$. Let $\mathfrak{d}_2 := \langle N_\eta, a_\eta : \eta \in H\rangle\hat{\ }(N^2, B_2)$. As B_1 and B_2 are domination equivalent over $N_{\langle\rangle}$, it is easily checked that $\mathfrak{d}_2$ is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M over $\binom{B_2}{A}$. Let $\mathfrak{d}_2^* = \langle N_\eta, a_\eta : \eta \in I_2\rangle$ be any prime $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M end extending $\mathfrak{d}_2$. But, as $p \not\perp N_\eta$ for some $\eta \in H$, it follows from independence that $p \perp N_\nu$ for any $\nu \in I_2$ satisfying $\langle 0\rangle \trianglelefteq \nu$. Thus, $p \notin \mathcal{P}_{\mathbf{P}}(\binom{B_2}{A}, M)$ by Theorem 5.12 again.

(5) Let $N_{\langle\rangle} \preceq M$ be $\aleph_\epsilon$-prime over A with $N_{\langle\rangle} \underset{A}{\perp} B_1B_2$ and choose an ϵ-finite $B_0 \in N_{\langle\rangle}$ arbitrarily. For $\ell = 1, 2$, choose $N_{\langle\ell\rangle}$ to be $\aleph_\epsilon$-prime over $N_{\langle\rangle} \cup B_\ell$. Clearly,

$$\mathfrak{d}' := \{(N_{\langle\rangle}, B_0), (N_{\langle 0\rangle}, B_1), (N_{\langle 1\rangle}, B_1)\}$$

is a three element, prime $(\aleph_\epsilon, \mathbf{P})$-decomposition inside M. By Lemma 4.15(2) there is a prime, $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ of M end extending $\mathfrak{d}'$. It is easily checked that $\mathfrak{d}$ satisfies the hypotheses of Corollary 5.14, as does the modification formed by exchanging the roles of $\langle 0\rangle$ and $\langle 1\rangle$. Thus, for any $p \in S(M) \cap \mathbf{P}$, we have $p \in \mathcal{P}_{\mathbf{P}}(\binom{B_1}{A}, M)$ if and only if $\langle 0\rangle \trianglelefteq \eta(p)$, and that $p \in \mathcal{P}_{\mathbf{P}}(\binom{B_2}{A}, M)$ if and only if $\langle 1\rangle \trianglelefteq \eta(p)$. As the elements $\langle 0\rangle$ and $\langle 1\rangle$ are incompatible, it follows that the sets $\mathcal{P}_{\mathbf{P}}(\binom{B_1}{A}, M)$ and $\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A}, M)$ are disjoint.

Corollary 5.17 *Suppose that M is $\aleph_\epsilon$-saturated and that $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I\rangle$ is any weak* **P***-decomposition inside M. Choose any incomparable nodes $\eta_1, \eta_2 \in I$. If, for each $\ell = 1, 2$, $A_\ell \subseteq N^-_{\eta_\ell}$ is ϵ-finite on which $\mathrm{tp}(a_{\eta_\ell}/N^-_{\eta_\ell})$ is based and stationary and $B_\ell = \mathrm{acl}(A_\ell \cup \{a_{\eta_\ell}\})$, then the sets $\mathcal{P}_{\mathbf{P}}(\binom{B_1}{A_1}, M)$ and $\mathcal{P}_{\mathbf{P}}(\binom{B_2}{A_2}, M)$ are disjoint.*

Proof. As η_1 and η_2 are incomparable, neither one is $\langle\rangle$, so let μ denote the meet $\eta_1^- \wedge \eta_2^-$. By incomparability again, there are distinct ordinals $\alpha_1 \neq \alpha_2$ such that $\mu\hat{\ }\langle\alpha_1\rangle \trianglelefteq \eta_1$, while $\mu\hat{\ }\langle\alpha_2\rangle \trianglelefteq \eta_2$. Choose an ϵ-finite $E \subseteq M_\mu$ over which both types $\mathrm{tp}(a_{\mu\hat{\ }\langle\alpha_\ell\rangle}/M_\mu)$ are based and stationary, and let $C_\ell = \mathrm{acl}(a_{\mu\hat{\ }\langle\alpha_\ell\rangle} \cup E)$ for each ℓ. As $C_1 \underset{E}{\perp} C_2$ it follows from Proposition 5.16(5) that the sets $\mathcal{P}_{\mathbf{P}}(\binom{C_1}{E}, M)$ and $\mathcal{P}_{\mathbf{P}}(\binom{C_2}{E}, M)$ are disjoint. But, by Proposition 5.16(3) $\mathcal{P}_{\mathbf{P}}(\binom{B_\ell}{A_\ell}, M) \subseteq \mathcal{P}_{\mathbf{P}}(\binom{C_1}{E}, M)$ for each ℓ and the result follows.

Proposition 5.18 *Suppose that M is $\aleph_\epsilon$-saturated and $p_1 \in S(A_1)$, $p_2 \in S(A_2)$ are non-orthogonal, trivial, regular types over ϵ-finite subsets of M. If, for $\ell = 1, 2$, I_ℓ is a maximal, A_ℓ-independent subset of $p_\ell(M)$, then there are cofinite subsets $J_\ell \subseteq I_\ell$ and a bijection $h : J_1 \to J_2$ such that*

$$\mathcal{P}_{\mathbf{P}}(\begin{pmatrix} c \\ A_1 \end{pmatrix}, M) = \mathcal{P}_{\mathbf{P}}(\begin{pmatrix} h(c) \\ A_2 \end{pmatrix}, M)$$

for every $c \in J_1$.

Proof. Let $D = A_1 \cup A_2$. For $\ell = 1, 2$, let $J_\ell := \{c \in I_1 : c \underset{A_\ell}{\perp} D\}$ and let q_ℓ denote the non-forking extension of p_ℓ to $S(D)$. Then J_ℓ is a cofinite subset of I_ℓ and is a maximal, D-independent subset of $q_\ell(M)$. As the regular types are trivial and non-orthogonal, p_1 and p_2 are not almost orthogonal, so as M is $\aleph_\epsilon$-saturated, we have that for every $c \in q_1(M)$, there is $c' \in q_2(M)$ such that $c_1 \underset{D}{\not\perp} c_2$. It follows that there is a unique bijection $h : J_1 \to J_2$ satisfying $c \underset{D}{\not\perp} h(c)$ for each $c \in J_1$. Thus, $\mathcal{P}_{\mathbf{P}}(\binom{c}{A_1}, M) = \mathcal{P}_{\mathbf{P}}(\binom{h(c)}{A_2}, M)$ by Clauses (1) and (4) of Proposition 5.16.

6 Decompositions and non-saturated models

Until this point, we have been looking at various flavors of decompositions of $\aleph_\epsilon$-saturated models. It would be desirable to see what effect these results have on understanding decompositions of arbitrary models. In the first subsection, given an arbitrary model M and a sufficiently saturated elementary extension M^*, one can produce an $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ of M^* that 'enumerates M as slowly as possible.' In particular, given any ϵ-finite $A \subseteq M$, there is a finite subtree $J \subseteq I$, an elementary submodel $M^J \preceq M^*$ that is $\aleph_\epsilon$-prime over $\bigcup_{\eta\in J} M_\eta$, and an ϵ-finite B, $A \subseteq B \subseteq M$ that satisfy $M \underset{B}{\perp} M_J$.

In the second subsection, we obtain a weak uniqueness result for $\mathbf{P}$-decompositions of unsaturated models M satisfying certain constraints. Whereas these conditions seem contrived, Theorem 6.19 plays a major role in [3].

6.1 Large extensions of weak decompositions

As usual, we assume that $\mathbf{P}$ is a set of stationary, regular types closed under isomorphism and non-orthogonality, and we assume that our theory T is superstable with $\mathbf{P}$-NDOP.

Definition 6.1 Suppose that $M \preceq M^*$ are given, with M arbitrary, but M^* sufficiently saturated. A prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}^* = \langle N_\eta, a_\eta : \eta \in I\rangle$ of M^* *respects* M if there is a continuous, elementary chain $\langle M_\alpha : \alpha \leq \alpha^*\rangle$ of $\aleph_\epsilon$-saturated elementary substructures of M^* with $\bigcup_{\alpha\leq\alpha^*} M_\alpha = M^*$; a sequence $\langle \mathfrak{d}_\alpha : \alpha \leq \alpha^*\rangle$ of prime $(\aleph_\epsilon, \mathbf{P})$-decompositions of M_α with $\mathfrak{d}_{\alpha^*} = \mathfrak{d}^*$; and a sequence $\langle a_\alpha : \alpha \leq \alpha^*\rangle$ of elements from M^* that satisfy the following constraints:

1. $M_0 = N_{\langle\rangle}$ and the sets M and M_0 are independent;
2. If $\beta \leq \alpha$ then $\mathfrak{d}_\alpha$ end extends $\mathfrak{d}_\beta$ with $\mathfrak{d}_\gamma = \bigcup \mathfrak{d}_\alpha$ for γ a limit ordinal;
3. The trees I_α indexing the decompositions $\mathfrak{d}_\alpha$ satisfy $|I_{\alpha+1} \setminus I_\alpha| \leq 1$ for each $\alpha < \alpha^*$;
4. If $I_{\alpha+1}\setminus I_\alpha = \{\eta\}$, then N_η is $\aleph_\epsilon$-prime over $N_{\eta^-} \cup \{a_\alpha\}$ and $N_\eta \underset{N_{\eta^-} a_\alpha}{\perp} M M_\alpha$.

Lemma 6.2 *Suppose that $M \preceq M^*$, where M^* is saturated and $||M^*|| > ||M|| + 2^{|T|}$. Then a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}^*$ of M^* respecting M exists.*

Proof. We recursively construct sequences $\langle M_\alpha \rangle$, $\langle \mathfrak{d}_\alpha \rangle$ and $\langle a_\alpha \rangle$ with the additional constraint of $||M_\alpha|| < ||M^*||$ for each $\alpha < \alpha^*$ as follows. First, choose $N_{\langle\rangle} \preceq M^*$ to be $\aleph_\epsilon$ saturated with $N_{\langle\rangle} \perp M$, let $M_0 = N_{\langle\rangle}$, $I_0 = \{\langle\rangle\}$, and $\mathfrak{d}_0 = \langle N_{\langle\rangle} \rangle$. For $\alpha \le \alpha^*$ a limit ordinal, simply take unions.

Next, fix an enumeration $\langle c_i : i < \lambda \rangle$ of M^* with $\lambda = ||M^*||$ and the elements of M forming an initial segment and assume that M_β and $\mathfrak{d}_\beta$ have been defined. Let c^* be the least element of M^* that is not an element of M_β. There are now two cases, depending on $\mathrm{tp}(c^*/M_\beta)$.

Case 1: $\mathrm{tp}(c^*/M_\beta) \perp \mathbf{P}^{\mathbf{active}}$.

In this case, choose a regular type $q \in S(M_\beta)$ non-orthogonal to $\mathrm{tp}(c^*/M_\beta)$. As M^* is saturated, choose an element $a_\beta \in M^*$ realizing q with $a_\beta \mathop{\not\perp}_{M_\beta} c^*$. Let $\mathfrak{d}_{\beta+1} = \mathfrak{d}_\beta$, and let $M_{\beta+1} \preceq M^*$ be $\aleph_\epsilon$-prime over $M_\beta \cup \{a_\beta\}$ and satisfying $M_{\beta+1} \mathop{\perp}_{M_\beta a_\beta} M$.

Case 2: $\mathrm{tp}(c^*/M_\beta) \not\perp \mathbf{P}^{\mathbf{active}}$.

In this case, choose a regular type $q \in S(M_\beta) \cap \mathbf{P}^{\mathbf{active}}$ non-orthogonal to $\mathrm{tp}(c^*/M_\beta)$. By Corollary 4.12, there is a unique $\eta \in I_\beta$ such that $q \not\perp N_\eta$, but $q \perp N_{\eta^-}$ (if $\eta \neq \langle\rangle$). Without loss, we may assume that q does not fork over N_η. As M^* is saturated, we can choose an element $a_\beta \in M^*$ realizing q with $a_\beta \mathop{\not\perp}_{M_\beta} c^*$. Let γ be the least ordinal such that $\nu := \eta\hat{\ }\langle\gamma\rangle \notin I_\beta$. Choose $N_\nu \preceq M^*$ to be $\aleph_\epsilon$-prime over $N_\eta \cup \{a_\beta\}$ and satisfying $N_\nu \mathop{\perp}_{N_\eta \cup \{a_\beta\}} M M_\beta$. As N_η is $\aleph_\epsilon$-saturated, it follows by Fact 2.3(2) that $N_\nu \mathop{\perp}_{N_\eta} M_\beta$. Choose $M_{\beta+1} \preceq M^*$ to be $\aleph_\epsilon$-prime over $M_\beta \cup N_\nu$ and satisfying $M_{\beta+1} \mathop{\perp}_{M_\beta N_\nu} M$. Let $I_{\beta+1} = I_\beta \cup \{\nu\}$, let Let $\mathfrak{d}_{\beta+1} = \mathfrak{d}_\beta\hat{\ }\langle N_\nu \rangle$, and let $M_{\beta+1} \preceq M^*$ be $\aleph_\epsilon$-prime over $M_\beta \cup N_\nu$.

Note that in either case, $R^\infty(c^*/M_{\beta+1}) < R^\infty(c^*/M_\beta)$, so by continuing in this fashion, c^* will be contained in $M_{\beta+k}$ for some finite k.

Suppose that $M \preceq M^*$, and that $\mathfrak{d}^*$ is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M^* respecting M, as witnessed by the sequences $\langle M_\alpha \rangle$, $\langle \mathfrak{d}_\alpha \rangle$, $\langle a_\alpha \rangle$. Let $(\star)_\alpha$ denote the statement:

> For all finite sets $A \subseteq M$, $B \subseteq M_\alpha$, and finite subtree $t \subseteq I_\alpha$, there is a finite set $A^* \subseteq M$ containing A and a finite subtree $t^* \subseteq I_\alpha$ containing t such that $\operatorname{tp}(B/\bigcup\{N_\rho : \rho \in t^*\})$ is $\aleph_\epsilon$-isolated and $M \underset{A^*}{\perp} \{N_\rho : \rho \in t^*\}B$.

Lemma 6.3 *$(\star)_\alpha$ holds for all $\alpha \leq \alpha^*$.*

Proof. We prove this by induction on α. For $\alpha = 0$, this is immediate since $M_0 = N_{\langle\rangle}$ and is independent from M over $\emptyset$, hence over any finite subset of M. For α a non-zero limit ordinal, this follows easily from superstability.

For the successor case, fix $\alpha = \beta + 1$ and assume that $(\star)_\beta$ holds. The verification of $(\star)_\alpha$ splits into two cases, depending on whether or not I_β is extended. Here, we discuss the case where $I_\alpha = I_\beta \cup \{\nu\}$ and leave the other (easier) case to the reader. So N_ν is $\aleph_\epsilon$-prime over $N_{\nu^-} \cup \{a_\beta\}$, $N_\nu \underset{N_{\nu^-}}{\perp} M_\beta$, and M_α is $\aleph_\epsilon$-prime over both sets $\bigcup\{N_\rho : \rho \in I_\alpha\}$ and $M_\beta \cup N_\nu$.

Towards verifying $(\star)_\alpha$, fix finite sets $A \subseteq M$, $B \subseteq M_\alpha$, and a finite subtree $t \subseteq I_\alpha$. Begin by choosing finite sets $C_\nu \subseteq N_\nu$ and $C_\beta \subseteq M_\beta$ such that

$$\operatorname{stp}(B/C_\beta C_\nu) \vdash \operatorname{stp}(B/M_\beta N_\nu)$$

Without loss, we may assume $a_\beta \in C_\nu$ and $C_\eta \cup C_\beta \subseteq B$.

Next, by superstability choose finite sets $D \subseteq \bigcup\{N_\rho : \rho \in I_\beta\}$ and $A' \subseteq M$ containing A such that

$$C_\nu \underset{DA'}{\perp} \bigcup\{N_\rho : \rho \in I_\beta\}M$$

. Similarly, choose finite sets $E_\beta \subseteq M_\beta$ and $A'' \subseteq M$ containing A' such that

$$B \underset{E_\beta A''}{\perp} M_\beta M$$

Without loss, we may assume $D \subseteq E_\beta$, $\nu \in t$, and $D \subseteq \bigcup\{N_\rho : \rho \in s\}$, where $s := t \setminus \{\nu\}$.

Now apply $(\star)_\beta$ to the triple (A'', E_β, s) and get a finite set $A^* \subseteq M$ and a finite tree $s^* \subseteq I_\beta$. Let $t^* := s^* \cup \{\nu\}$. We claim that (A^*, t^*) are as desired in the statement of $(\star)_\alpha$.

Claim 1: $B/\bigcup\{N_\rho : \rho \in t^*\}$ is $\aleph_\epsilon$-isolated.

To see this, first note that $C_\beta \subseteq M_\beta$ is $\aleph_\epsilon$-isolated over $\bigcup\{N_\rho : \rho \in s^*\}$. Since $M_\beta \underset{N_{\nu^-}}{\perp} N_\nu$ and N_{ν^-} is $\aleph_\epsilon$-saturated, it follows that C_β is $\aleph_\epsilon$-isolated over $\bigcup\{N_\rho : \rho \in t^*\}$ as well. Also, $C_\nu \subseteq N_\nu$, so it follows immediately that $C_\beta C_\nu / \bigcup\{N_\rho : \rho \in t^*\}$ is $\aleph_\epsilon$-isolated as well. But, as $\mathrm{stp}(B/C_\beta C_\nu) \vdash \mathrm{stp}(B/\bigcup\{N_\rho : \rho \in t^*\}$, the result follows.

Claim 2: $M \underset{A^*}{\perp} N_0 N_\nu B$, where $N_0 := \bigcup\{N_\rho : \rho \in s^*\}$.

First, it follows from our application of $(\star)_\beta$ that $M \underset{A^*}{\perp} N_0 E_\beta$. We next consider C_ν. By the definition of E_β and A'' we have $C_\nu \underset{E_\beta A''}{\perp} M_\beta M$. So, by monotonicity, we have $C_\nu \underset{E_\beta A^*}{\perp} N_0 M$, hence $C_\nu \underset{E_\beta A^* N_0}{\perp} M$. Thus, the transitivity of non-forking yields

$$M \underset{A^*}{\perp} N_0 E_\beta C_\nu$$

Finally, our choice of N_ν gives $N_\nu \underset{N_{\nu^-} a_\beta}{\perp} M_\beta M$. But $a_\beta \in C_\nu \subseteq N_\nu$, so $N_\nu \underset{N_{\nu^-} C_\nu}{\perp} N_0 E_\beta A^* M$. As $N_{\nu^-} \subseteq N_0$, monotonicity yields

$$M \underset{N_0 E_\beta A^*}{\perp} N_\nu$$

and we finish by quoting the transitivity of non-forking.

Proposition 6.4 *Suppose that $M \preceq M^*$ with M^* saturated and $||M^*|| > ||M|| + 2^{|T|}$. If $\mathfrak{d}^*$ is a prime $(\aleph_\epsilon, \mathbf{P})$-decomposition of M^* respecting M, then for every finite $A \subseteq M$ and every finite subtree $t \subseteq I_{\mathfrak{d}^*}$, there is a finite set $A^* \subseteq M$ containing A, a finite subtree $t^* \subseteq I_{\mathfrak{d}^*}$ extending t, and $M_{t^*} \preceq M^*$ that is $\aleph_\epsilon$-prime over $\bigcup\{N_\rho : \rho \in t^*\}$ such that $A \subseteq M_{t^*}$, but $M \underset{A^*}{\perp} M_{t^*}$.*

Proof. Fix finite $A \subseteq M$ and $t \subseteq I_{\mathfrak{d}^*}$. If $M^* = M_{\alpha^*}$, then applying $(\star)_{\alpha^*}$ to the triple (A, A, t) yields a finite set $A^* \subseteq M$ containing A and t^* such that $\mathrm{tp}(A/\bigcup\{N_\rho : \rho \in t^*\})$ is $\aleph_\epsilon$-isolated and $M \underset{A^*}{\perp} \{N_\rho : \rho \in t^*\}$. Thus, as M^* is saturated, we can find $M_{t^*} \preceq M^*$ containing A that is both $\aleph_\epsilon$-prime over $\bigcup\{N_\rho : \rho \in t^*\}$ and is independent from M over A^*.

6.2 A weak uniqueness theorem for P-decompositions

The goal of this subsection is Theorem 6.19, which is used in [3]. As we only seek a sufficient condition, the statements and assumptions in Theorem 6.19 are inelegant at best. Additionally, throughout this subsection we assume

T **is totally transcendental with P-NDOP and** $\mathbf{P} = \mathbf{P}^{\mathbf{active}}$

The assumption of the theory T being totally transcendental is only used in Lemma 6.7, and one could easily imagine it being replaced by much weaker assumptions. We begin with a standard fact about superstable theories.

Lemma 6.5 *Suppose that $p \in S(A)$ is stationary and that J is an infinite, A-independent set of realizations of p. Let $B \supseteq A \cup J$, let $p' \in S(B)$ denote the non-forking extension of p, and let $C \supseteq B$ be constructible over B. Then p' has a unique extension to $S(C)$.*

Definition 6.6 Given any model M, a $\mathbf{P}^{\mathbf{r}}$*-decomposition* $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ *inside* M is a weak $\mathbf{P}$-decomposition inside M with the additional property that $\mathrm{tp}(a_\nu/M_{\nu^-}) \in \mathbf{P}$ (hence is regular) for every $\nu \in I \setminus \{\langle\rangle\}$. $\mathfrak{d}$ is a $\mathbf{P}^{\mathbf{r}}$*-decomposition of* M if, in addition, for every $\eta \in I$, $\{a_\nu : \nu \in Succ(\eta)\}$ is a maximal M_η-independent set of realizations of types in $\mathbf{P}$. A $\mathbf{P}^{\mathbf{r}}$-decomposition of M is $\mathbf{P}$*-finitely saturated* if, for every ϵ-finite $A \subseteq M$ and $b \in M$ such that $\mathrm{tp}(b/A) \in \mathbf{P}$, there is some $\eta \in I$ such that $\mathrm{tp}(b/A) \not\perp M_\eta$.

As notation, given a $\mathbf{P}^{\mathbf{r}}$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ of M, let $I' = I \setminus \{\langle\rangle\}$. For each $\eta \in I'$, let $p_\eta = \mathrm{tp}(a_\eta/M_{\eta^-})$ and fix an ϵ-finite $A_\eta \subseteq M_{\eta^-}$ over which p_η is based and stationary. We let $\mathcal{P}_{\mathbf{P}}\binom{a_\eta}{A_\eta}$ abbreviate $\mathcal{P}_{\mathbf{P}}(\binom{\mathrm{acl}(A_\eta a_\eta)}{A_\eta}, \mathfrak{C})$. Note that by Proposition 5.16(1), $\mathcal{P}_{\mathbf{P}}\binom{a_\eta}{A_\eta} = \mathcal{P}_{\mathbf{P}}\binom{a_\eta}{A'_\eta}$ for any ϵ-finite $A'_\eta \subseteq M_{\eta^-}$ on which p_η is based and stationary.

Let $C_\eta := \{\rho \in I' : \rho^- = \eta^- \text{ and } p_\rho = p_\eta\}$ and let $J_\eta := \{a_\rho : \rho \in C_\eta\}$.

Lemma 6.7 *Fix any $\mathbf{P}^{\mathbf{r}}$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ of M and choose any $\eta \in I'$ for which C_η is infinite. Denote $p_\eta, A_\eta, C_\eta, J_\eta$ by p, A, C, J, respectively. For any $b \in \mathfrak{C}$ realizing $p|_A$, if $b \underset{A}{\perp} M_{\eta^-}J$ then $b \underset{A}{\perp} M$.*

Proof. Fix any element b such that $b \underset{A}{\perp} M_{\eta^-}J$. Let $D := \bigcup\{M_\rho : \rho \in C\}$ and let $E := \bigcup\{M_\nu : \nu \in I\}$. First, as J is infinite,

$$\mathrm{tp}(b/M_{\eta^-}J) \vdash \mathrm{tp}(b/D)$$

by Lemma 6.5. Next, $\mathrm{tp}(b/D) \vdash \mathrm{tp}(b/E)$ by the independence of the tree, orthogonality, and the non-forking calculus. Next, form a maximal, continuous elementary chain of submodels $\langle M_\alpha : \alpha < \beta \rangle$ of M such that M_0 is constructible over E, and given M_α, $M_{\alpha+1}$ is constructible over $M_\alpha \cup \{b_\alpha\}$ for some b_α such that $\mathrm{tp}(b_\alpha/M_\alpha)$ is regular. (Here is where we use the assumption that T is totally transcendental.) Clearly, the maximality of the sequence implies that the union is all of M. However, by Lemma 6.5 and the fact that $\mathrm{tp}(b_\alpha/M_\alpha) \perp \mathbf{P}$ (which follows from $\mathbf{P} = \mathbf{P}^{\mathbf{active}}$) we conclude that

$$\mathrm{tp}(b/E) \vdash \mathrm{tp}(b/M)$$

That $b \underset{A}{\perp} M$ follows by the transitivity of non-forking.

Lemma 6.8 *Suppose that $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ is a $\mathbf{P}^{\mathbf{r}}$-decomposition of M and there is $q \in \mathbf{P}$ and $\eta \in \max(I')$ such that $q \not\perp M_\eta$, but $q \perp M_{\eta^-}$. Then, for any $\nu \in I$,*

$$\nu \lhd \eta \qquad \textit{if and only if} \qquad q \in \mathcal{P}_{\mathbf{P}}\binom{a_\nu}{A_\nu}$$

Proof. First, assume that $\nu \lhd \eta$. Let $\mathfrak{d}_0 := \langle M_\delta, a_\delta : \nu^- \trianglelefteq \delta \trianglelefteq \eta \rangle$. As in the proof of Lemma 5.6, we can blow up $\mathfrak{d}_0$ to a sequence $\mathfrak{d}_0^* := \langle M_\delta^*, a_\delta : \nu^- \trianglelefteq \delta \trianglelefteq \eta \rangle$, where $\mathfrak{d}_0^*$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside $\mathfrak{C}$, with $q \not\perp M_\eta^*$, but $q \perp M_{\eta^-}^*$. Thus, $q \in \mathcal{P}_{\mathbf{P}}\binom{a_\nu}{A_\nu}$ by its definition and Lemma 4.17(3).

Conversely, assume by way of contradiction that $q \in \mathcal{P}_{\mathbf{P}}\binom{a_\nu}{A_\nu}$ but $\neg(\nu \lhd \eta)$. As $\nu \neq \eta$ and $\eta \in \max(I')$, ν and η are incomparable. However, since $q \in \mathcal{P}_{\mathbf{P}}\binom{a_{\eta^-}}{A_{\eta^-}}$ from above, it follows from Corollary 5.17 that ν and η^- are comparable. Thus, $\eta^- \lhd \nu$. But then, as $q \perp M_{\eta^-}$ and $M_\eta \underset{M_{\eta^-}}{\perp} a_\nu M_{\nu^-}$, it follows that q is orthogonal to any chain starting with M_{ν^-} and a_ν.

Definition 6.9 Suppose $S \subseteq \mathbf{P}$. A $\mathbf{P}^{\mathbf{r}}$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ (inside $\mathfrak{C}$) *supports* S if, for every $q \in S$, there is a (unique) $\eta(q) \in \max(I')$ such that $q \not\perp M_{\eta(q)}$, but $q \perp M_{\eta(q)^-}$. If $\mathfrak{d}$ supports S, we let

- $\mathrm{Field}(S) := \{\eta(q) \in \max(I') : q \in S\}$; and
- $I^S := \{\nu \in I : \nu \lhd \eta \text{ for some } \eta \in \mathrm{Field}(S)\}$.

Lemma 6.10 *Suppose $S \subseteq \mathbf{P}$ and fix a $\mathbf{P^r}$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ (inside $\mathfrak{C}$) that supports S. Then:*

1. *If $\nu \in I^S$, then $\operatorname{tp}(a_\nu/M_{\nu^-})$ is trivial;*
2. *for $\nu \in I'$, $\nu \in I^S$ if and only if $\mathcal{P}_{\mathbf{P}}\binom{a_\nu}{A_\nu} \cap S \neq \emptyset$; and*
3. *if, for all $\delta \in I^S$, there is a single ϵ-finite $A^* \subseteq M_\delta$ such that $\operatorname{tp}(a_\nu/M_\delta)$ is based and stationary on A^* for every $\nu \in Succ_{I^S}(\delta)$, then for any $\nu \in Succ_{I^S}(\delta)$ and any $b \in \mathfrak{C}$ realizing $\operatorname{tp}(a_\nu/A^*)$, if $\mathcal{P}_{\mathbf{P}}\binom{b}{A^*} \cap S \neq \emptyset$, then $b \underset{A^*}{\perp} M_\delta$.*

Proof. (1) It follows immediately from the definition of $\mathbf{P^r}$-decompositions and I^S that $\operatorname{tp}(a_\nu/M_{\nu^-}) \in \mathbf{P}$ and has positive $\mathbf{P}$-depth. Hence, the type is trivial by Lemma 3.11.

(2) This is immediate from unpacking the definitions and Lemma 6.10.

(3) Choose A^*, δ, ν, and b as required. Choose $r \in \mathcal{P}_{\mathbf{P}}\binom{b}{A^*} \cap S$ and look at $\eta(r) \in \max(I')$. By Lemma 6.10, $\delta \triangleleft \eta(r)$. Choose $\mu \in Succ_{I^S}(\delta)$ satisfying $\mu \triangleleft \eta(r)$. By our choice of A^* and Lemma 6.10 again, $r \in \mathcal{P}_{\mathbf{P}}\binom{a_\mu}{A^*}$, so by Proposition 5.16(5), $b \underset{A^*}{\not\perp} a_\mu$ But then, as $\operatorname{tp}(b/A^*)$ is a trivial regular type, b is domination equivalent to a_μ over A^*. Since $a_\mu \underset{A^*}{\perp} M_\delta$, we conclude that the same holds for b.

Definition 6.11 Fix $S \subseteq \mathbf{P}$ and a model M. A $\mathbf{P^r}$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ of M is *S-reasonable* if

1. $\mathfrak{d}$ is $\mathbf{P}$-finitely saturated and supports S;
2. for each $\eta \in I'$:
   (a) $C_\eta \cap I_S$ is infinite;
   (b) $p_\rho = p_\eta$ iff $p_\rho \not\perp p_\eta$ for every $\rho \in I'$ such that $\rho^- = \eta^-$; and
   (c) If $b \in \mathfrak{C}$ and $\operatorname{tp}(b/A_\eta) = p_\eta|_{A_\eta}$ and $\mathcal{P}_{\mathbf{P}}\binom{b}{A_\eta} \cap S \neq \emptyset$, then $b \underset{A_\eta}{\perp} M_{\eta^-}$.

Definition 6.12 A *weak bijection* between two infinite sets I and J is a bijection $h : I' \to J'$, where I', J' are cofinite subsets of I, J, respectively.

As notation, for $\eta \in I^S \setminus \{\langle\rangle\}$, let $J^S_\eta = \{a_\rho : \rho \in C_\eta \cap I^S\}$.

Proposition 6.13 *Fix a set $S \subseteq \mathbf{P}$ and a model M. For $\ell = 1, 2$, let $\mathfrak{d}_\ell = \langle M_{\eta_\ell}, a_{\eta_\ell} : \eta_\ell \in I_\ell\rangle$ be two S-reasonable $\mathbf{P}^{\mathbf{r}}$-decompositions of M. For any $\eta_\ell \in I^S_\ell$, choose $\eta_{3-\ell} \in I_{3-\ell}$ such that $p_{\eta_1} \not\perp p_{\eta_2}$. There is a weak bijection $h : J^S_{\eta_1} \to J^S_{\eta_2}$ satisfying $\mathcal{P}_{\mathbf{P}}\binom{a}{A_{\eta_1}} = \mathcal{P}_{\mathbf{P}}\binom{h(a)}{A_{\eta_2}}$ for each $a \in \mathrm{dom}(J_{\eta_1})$.*

Proof. For definiteness, assume we have that $\eta_1 \in I^S_1$. Let $E = A_{\eta_1} \cup A_{\eta_2}$. For $\ell = 1, 2$, let $p_\ell \in S(E)$ be parallel to p_{η_ℓ}, let $J'_\ell = \{a \in J_{\eta_\ell} : a \underset{A_{\eta_\ell}}{\perp} E\}$, and let

$$J^S_\ell = \{a \in J'_\ell : \mathcal{P}_{\mathbf{P}}\begin{pmatrix} a \\ A_{\eta_\ell}\end{pmatrix} \cap S \neq \emptyset\},$$

which is a cofinite subset of $J^S_{\eta_\ell}$. In particular, $J^S_1 \neq \emptyset$ since $C_{\eta_1} \cap I^S$ is infinite. As well, choose a maximal E-independent set J^*_ℓ of realizations of p_ℓ in $\mathfrak{C}$ extending J_ℓ. As p_1 and p_2 are non-orthogonal trivial regular types, it follows from Proposition 5.18 that there is a unique bijection $h : J^*_1 \to J^*_2$ satisfying $h(a) \underset{E}{\not\perp} a$ for each $a \in J^*_1$.

As $a_\ell \underset{A_{\eta_\ell}}{\perp} E$ for $\ell = 1, 2$ and every $a_\ell \in J^*_\ell$, by Proposition 5.16(1) we have that

$$\mathcal{P}_{\mathbf{P}}\begin{pmatrix} a \\ A_{\eta_1}\end{pmatrix} = \mathcal{P}_{\mathbf{P}}\begin{pmatrix} a \\ E\end{pmatrix} = \mathcal{P}_{\mathbf{P}}\begin{pmatrix} h(a) \\ E\end{pmatrix} = \mathcal{P}_{\mathbf{P}}\begin{pmatrix} h(a) \\ A_{\eta_2}\end{pmatrix}$$

for each $a \in J^*_\ell$.

Claim. For every $a \in J^S_1$, $h(a) \in J^S_2$.

Proof of Claim: Choose any $a \in J^S_1$. We first find an element $b \in J_{\eta_2}$ such that $h(a) \underset{E}{\not\perp} b$. Since $a = a_\rho$ for some $\rho \in I^S_1$ satisfying $\rho^- = \eta_1^-$, $\mathcal{P}_{\mathbf{P}}\binom{a}{A_{\eta_1}} \cap S \neq \emptyset$. As the two sets are equal, $\mathcal{P}_{\mathbf{P}}\binom{h(a)}{A_{\eta_2}} \cap S \neq \emptyset$ as well. As $\mathfrak{d}_2$ is S-reasonable, this implies $h(a) \underset{A_{\eta_2}}{\perp} M_{\eta_2^-}$ Next, we argue that $h(a)$ must fork with J_{η_2} over $M_{\eta_2^-}$, because if this were not the case, then by Lemma 6.7 we would have $h(a) \underset{E}{\perp} M$. But, as $a \underset{E}{\not\perp} h(a)$, the fact that p_{η_2} has weight one would imply that $a \underset{E}{\perp} M$, which is absurd since $a \in M$.

Thus, $h(a)$ forks with J_{η_2} over $M_{\eta_2^-}$. By triviality, there is a unique $b \in J_{\eta_2}$ such that $h(a) \underset{M_{\eta_2^-}}{\not\perp} b$. However, as both $h(a)$ and b are free from $M_{\eta_2^-}$ over

A_{η_2}, it follows that $h(a)$ and b fork over A_{η_2}, completing the first part of our argument.

Next, since $h(a)$ realizes p_2, it is free from E over A_{η_2}. As p_{η_2} has weight one, the last two statements imply that b is free from E over A_{η_2} as well. Thus, $b \in J_2'$. As well, we have that $\mathcal{P}_{\mathbf{P}}\binom{a}{E} = \mathcal{P}_{\mathbf{P}}\binom{b}{E}$, so the latter has non-empty intersection with S. Thus, $b \in J_2^S$.

Finally, note that both $h(a)$ and b are elements of J_2^* that fork with each other over E. Thus, $h(a) = b$ by the E-independence of J_2^*. So $h(a) \in J_2^S$, completing the proof of the Claim.

It follows from the Claim that J_2^S is non-empty. Once we know this, the situation becomes symmetric, so by running the Claim backwards, h^{-1} maps J_2^S into J_1^S. That is, the restriction of h to J_1^S is a bijection with J_2^S, which completes the proof of the Proposition.

We set some notation about partial maps between trees. Given a tree I, a *large subtree* of I is a non-empty (downward closed) subtree J such that for every $\eta \in J$, $Succ_I(\eta) \setminus J$ is finite. Given two trees J and K, an *almost embedding h from J to K* has $\mathrm{dom}(h)$ a large subtree of J, $range(h) \subseteq K$, $h(\langle\rangle_J) = \langle\rangle_K$, and for all $\eta, \nu \in \mathrm{dom}(h)$,

$$\eta \triangleleft \nu \quad \text{if and only if} \quad h(\eta) \triangleleft h(\nu)$$

The trees J and K are *almost isomorphic* if there is an almost embedding h from J to K in which $range(h)$ is a large subtree of K.

For J any tree and $\nu \in J$, let $J_{\trianglerighteq \nu}$ be the tree with root ν and universe $\{\eta \in J : \eta \trianglerighteq \nu\}$. Given two trees J and K and $\nu \in J$, $\mu \in K$, an *almost embedding h from J to K over (ν, μ)* is an almost embedding from $J_{\trianglerighteq \nu}$ to $K_{\trianglerighteq \mu}$.

Finally, if J and K are trees indexing decompositions, we call a pair $(\eta, \nu) \in J \times K$ *$\mathcal{P}_{\mathbf{P}}$-equivalent* if either $\eta = \langle\rangle = \nu$, or both $\eta, \nu \neq \langle\rangle$ and $\mathcal{P}_{\mathbf{P}}\binom{a_\eta}{A_\eta} = \mathcal{P}_{\mathbf{P}}\binom{a_\nu}{A_\nu}$. An *almost $\mathcal{P}_{\mathbf{P}}$-embedding from J to K* is an almost embedding h from J to K with the pair $(\eta, h(\eta))$ $\mathcal{P}_{\mathbf{P}}$-equivalent for each $\eta \in \mathrm{dom}(h)$. Note that if h is an almost $\mathcal{P}_{\mathbf{P}}$-embedding and $h(\eta) = \nu$, then the restriction of h to $J_{\trianglerighteq \eta} := \{\delta \in \mathrm{dom}(h) : \delta \trianglerighteq \eta\}$ is an almost $\mathcal{P}_{\mathbf{P}}$-embedding over (η, ν).

Given all of this notation, the proof of the following Corollary simply involves successively iterating Proposition 6.13, using the fact that each decomposition is $\mathbf{P}$-finitely saturated.

Corollary 6.14 *Fix a set $S \subseteq \mathbf{P}$ and a model M. For $\ell = 1, 2$, suppose that $\mathfrak{d}_\ell = \langle M_{\eta_\ell}, a_{\eta_\ell} : \eta_\ell \in I_\ell \rangle$ are S-reasonable $\mathbf{P}^{\mathbf{r}}$-decompositions of M with the additional property that for each ℓ and $\nu_\ell \in I_\ell$,*

$$\{p : \textit{there is } \eta_\ell \in Succ(\nu_\ell) \textit{ such that } p_{\eta_\ell} = p \wedge \mathcal{P}_{\mathbf{P}}\binom{a_{\eta_\ell}}{A_{\eta_\ell}} \cap S \neq \emptyset\}$$

is finite. Then:

1. *For $\ell = 1, 2$, there is an almost $\mathcal{P}_{\mathbf{P}}$-embedding h from I_ℓ^S to $I_{3-\ell}^S$; and*
2. *For $\ell = 1, 2$ and any $\mathcal{P}_{\mathbf{P}}$-equivalent pair $(\eta_\ell, \eta_{3-\ell}) \in I_\ell^S \times I_{3-\ell}^S$ there is an almost $\mathbf{P}$-embedding from I_ℓ^S to $I_{3-\ell}^S$ over $(\eta_\ell, \eta_{3-\ell})$.*

If we wish to conclude more, namely that the trees I_1^S and I_2^S are almost isomorphic, then we need show that the almost embeddings given above preserve lengths, i.e., that $\lg(h(\eta)) = \lg(\eta)$ for every $\eta \in \mathrm{dom}(h)$. To accomplish this, we need to put additional constraints on the shapes of the trees I^S. The conditions we require are severe, but will be easily satisfied in our construction in [3].

Definition 6.15 A *two-coloring* of a tree I is a sequence $\langle E_\eta : \eta \in I \rangle$ where each E_η is an equivalence relation on $Succ(\eta)$ with at most two classes, each of which is infinite. (If $Succ(\eta) = \emptyset$, then of course E_η is empty as well.) A node $\eta \in I$ has *uniform depth n* if every branch of the tree $I_{\trianglerighteq \eta}$ has length exactly n. A node η *often has unbounded depth* if every large subtree $J \subseteq I_{\trianglerighteq \eta}$ has an infinite branch. A node η is an (m, n)-*cusp* if there are infinite sets $A_m, A_n, B \subseteq Succ(\eta)$ such that

1. the set $A_m \cup A_n$ is pairwise E_η-equivalent;
2. each $\delta \in A_m$ has uniform depth m;
3. each $\rho \in A_n$ has uniform depth n; and
4. each $\gamma \in B$ is often unbounded.

A *cusp* is an (m, n)-cusp for some $m \neq n$.

Fix any function $\Phi : \omega \to \omega$. We say the two-colored tree I is Φ-*proper* if, for every node $\eta \in I$,

1. either η has uniform depth n for some n, or else η often has unbounded depth;

2. if η is an (m, n)-cusp, then $\lg(\eta) = \Phi(m - n)$;

3. if E_η has two classes, then η is a cusp;

4. if J is a large subtree of I, $\eta \in J$ is often unbounded, then there is a cusp $\nu \in J$ with $\nu \trianglerighteq \eta$.

Note that if I is a two-colored tree satisfying the conditions above, then for every $\gamma \in I$ that is of any uniform depth k, there are a unique η, δ satisfying $\delta \trianglelefteq \gamma$, $\eta = \delta^-$, η is a cusp, and δ has uniform depth n for some $n \geq k$.

Lemma 6.16 *Suppose that $M, S, \mathfrak{d}_1, \mathfrak{d}_2$ satisfy the assumptions of Corollary 6.14 and additionally assume that both I_1^S, I_2^S, when two-colored by the relations E_η defined by $E_\eta(\delta, \rho)$ iff $\delta^- = \eta = \rho^-$ and $p_\delta = p_\rho$, are Φ-proper for the same function Φ. Then for every $\mathcal{P}_{\mathbf{P}}$-equivalent pair $(\eta, \nu) \in I_1^S \times I_2^S$,*

1. *η is often unbounded in I_1^S if and only if ν is often unbounded in I_2^S;*

2. *for any n, η has uniform depth n if and only if ν has uniform depth n;*

3. *if $\lg(\eta) = \lg(\nu)$ and η has uniform depth n for some n, then any almost $\mathcal{P}_{\mathbf{P}}$-embedding over (η, ν) preserves lengths; and*

4. *if $\lg(\eta) \leq \lg(\nu)$ and η is an (m, n)-cusp, then ν is also an (m, n)-cusp, $\lg(\eta) = \lg(\nu)$, and for any almost $\mathcal{P}_{\mathbf{P}}$-embedding h over (η, ν), $\lg(h(\delta)) = \lg(\delta)$ for all $\delta \in \mathrm{dom}(h) \cap Succ(\eta)$ of uniform depth m or n;*

5. *if $\lg(\eta) = \lg(\nu)$ then every almost $\mathcal{P}_{\mathbf{P}}$-embedding over (η, ν) preserves lengths; and*

6. *if $\lg(\eta) = \lg(\nu)$, then the number of E_η-classes in I_1^S equals the number of E_ν-classes in I_2^S.*

Proof. (1) First assume that η is often unbounded. By Corollary 6.14(2), choose an almost $\mathcal{P}_{\mathbf{P}}$-embedding h from I_1^S to I_2^S over (η, ν). Choose a strictly $\triangleleft$-increasing sequence $\langle \eta_n : n \in \omega \rangle$ from $\mathrm{dom}(h)$ with $\eta_0 = \eta$. Then

$\langle h(\eta_n) : n \in \omega \rangle$ is a strictly $\triangleleft$-increasing sequence in I_2^S with $h(\eta_0) = \nu$. Thus, ν cannot have any finite uniform depth, so it must be often unbounded by properness. The converse is symmetric.

(2) Suppose that ν has uniform depth n. Then by (1), η has uniform depth m for some m. Arguing as in (1), $m \leq n$, since if we choose any almost $\mathcal{P}_{\mathbf{P}}$-embedding h from I_1^S to I_2^S over (η, ν), then the image of any strictly $\triangleleft$-increasing sequence $\langle \eta_i : i < m \rangle$ with $\eta_0 = \eta$ would be a strictly $\triangleleft$-increasing sequence of length m over ν. But then, by symmetry, we would also have $n \leq m$, so $n = m$. The converse is symmetric.

(3) Suppose that h is any almost $\mathcal{P}_{\mathbf{P}}$-embedding over (η, ν), where $\lg(\eta) = \lg(\nu)$, η has uniform depth n. Then ν also has uniform depth n. So, every maximal $\triangleleft$-increasing sequence extending η has length n, the image of any such sequence under h is also a strictly $\triangleleft$-increasing sequence of length n, but there is no strictly $\triangleleft$-increasing sequence of length more than n extending ν. Thus, h must map immediate successors to immediate successors, and consequently preserve lengths.

(4) Suppose that η is an (m, n)-cusp and $\lg(\eta) \leq \lg(\nu)$. Choose an almost $\mathcal{P}_{\mathbf{P}}$-embedding h from I_1^S to I_2^S over (η, ν). Choose E_η-equivalent $\delta \in Succ(\eta) \cap \mathrm{dom}(h)$ of uniform depth m and $\rho \in Succ(\eta) \cap \mathrm{dom}(h)$ of uniform depth n. Choose $\mu \in I_2^S$ and $q \in S(M_\mu^2)$ such that p_δ (which $= p_\rho$) is non-orthogonal to q. By the definition of h, both $h(\delta), h(\rho) \in Succ(\mu)$. We argue that $\mu = h(\eta)$. To see this, first note that since h is $\triangleleft$-preserving, $h(\eta) \triangleleft h(\delta)$ and $h(\eta) \triangleleft h(\rho)$, so $h(\eta) \trianglelefteq \mu$. But, it follows from (2) that $h(\delta)$ is uniformly of depth m and $h(\rho)$ is uniformly of depth n. Thus, μ is an (m, n)-cusp and hence $\lg(\mu) = \Phi(m-n) = \lg(\eta)$. As we assumed that $\lg(\eta) \leq \lg(\nu)$ and $h(\eta) = \nu$, we have that $\lg(\mu) = \lg(h(\eta))$, hence $\mu = h(\eta) = \nu$. This yields $\lg(\nu) = \lg(\eta)$. Finally, the argument above showed that $h(\delta) \in Succ(\nu)$ whenever $\delta \in \mathrm{dom}(h) \cap Succ(\eta)$ has uniform depth m or n.

(5) Assume that $\lg(\eta) = \lg(\nu)$ and fix any almost $\mathcal{P}_{\mathbf{P}}$-embedding h from I_1^S to I_2^S over (η, ν). Note that $\lg(h(\mu)) \geq \lg(\mu)$ for any $\mu \in \mathrm{dom}(h)$ simply because h is $\triangleleft$-preserving. We first consider the often unbounded nodes $\mu \in \mathrm{dom}(h)$. Specifically, we argue by induction on k that $\lg(h(\mu)) = \lg(\mu)$ for every often unbounded node $\mu \in \mathrm{dom}(h)$ for which there is a cusp $\zeta \trianglerighteq \mu$ with $\zeta \in \mathrm{dom}(h)$ and $\lg(\zeta) = \lg(\mu) + k$.

When $k = 0$, this means that any such μ is itself a cusp, so $\lg(h(\mu)) = \lg(\mu)$ by (4). Next, assume that the statement holds for k, and choose $\mu \in \mathrm{dom}(h)$ with some cusp $\zeta \in \mathrm{dom}(h)$ with $\mu \trianglelefteq \zeta$ and $\lg(\zeta) = \lg(\mu) + k + 1$.

Choose $\rho \in Succ(\mu)$ with $\mu \trianglelefteq \rho \trianglelefteq \zeta$. Then $\lg(h(\rho)) = \lg(\rho)$ by our inductive assumption, so $h(\rho) \in Succ(h(\mu))$, hence $\lg(h(\mu)) = \lg(\mu)$ as well. Thus, we have shown that lengths are preserved for all often unbounded nodes $\mu \in \mathrm{dom}(h)$.

Next, assume that $\gamma \in \mathrm{dom}(h)$ has uniform depth. By the remark following Definition 6.15, choose μ and δ such that μ is a cusp, $\mu = \delta^-$, $\delta \trianglelefteq \gamma$, and δ has uniform depth n for some $n \geq k$. The last sentence of (4) implies that $\lg(h(\delta)) = \lg(\delta)$. Thus, $\lg(h(\gamma)) = \lg(\gamma)$ follows from (3). So h is length-preserving.

(6) As the hypotheses are symmetric, it suffices to prove that the number of E_η-classes is at most the number of E_ν-classes. Using Corollary 6.14, choose an almost $\mathcal{P}_{\mathbf{P}}$-embedding h over (η, ν). By (5), h maps immediate successors of η to immediate successors of ν. As well, for each $\delta \in \mathrm{dom}(h) \cap Succ(\eta)$, $p_\delta \not\perp p_{h(\delta)}$. As non-orthogonality is an equivalence relation on regular types, this implies that h maps E_η-classes to E_ν-classes, and maps distinct E_η-classes to distinct E_ν-classes. As there are at most two E_η-classes, the inequality follows.

Theorem 6.17 *Fix a set $S \subseteq \mathbf{P}$ and a model M. For $\ell = 1, 2$, suppose that $\mathfrak{d}_\ell = \langle M_{\eta\ell}, a_{\eta\ell} : \eta \in I_\ell \rangle$ satisfy the hypotheses of Lemma 6.16. Then there is an almost $\mathcal{P}_{\mathbf{P}}$-isomorphism h from I_1^S to I_2^S.*

Proof. Using Corollary 6.14, choose any almost $\mathcal{P}_{\mathbf{P}}$-embedding h of I_1^S to I_2^S such that, for any $\delta \in \mathrm{dom}(h)$, $\mathrm{dom}(h) \cap C_\delta$ is a cofinite subset of C_δ and $\mathrm{range}(h) \cap C_{h(\delta)}$ is a cofinite subset of $C_{h(\delta)}$. From Lemma 6.16 we know that h preserves levels and, for each node $\eta \in \mathrm{dom}(h)$, the number of $E_{h(\eta)}$-classes is equal to the number of E_η-classes. It follows that $\mathrm{range}(h)$ is a large subtree of I_2^S, so h is an almost $\mathcal{P}_{\mathbf{P}}$-isomorphism between I_1^S and I_2^S.

Finally, we exhibit an extreme case, whose hypotheses are satisfied in [3].

Definition 6.18 Fix $S \subseteq \mathbf{P}$, a model M, and a function $\Phi : \omega \to \omega$. A $\mathbf{P}^{\mathbf{r}}$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ of M is (S, Φ)-*simple* if

1. $\mathfrak{d}$ supports S and $\mathbf{P}$-finitely saturates M;
2. for every $\eta \in I^S$

(a) $Succ_{IS}(\eta)$ is empty or infinite, but E_η is trivial, i.e., $p_\nu = p_\mu$ for all $\nu, \mu \in Succ_{IS}(\eta)$;

(b) η is either of some finite uniform depth or is a cusp; and

(c) if η is an (m,n)-cusp, then $\Phi(m-n) = \lg(\eta)$.

Theorem 6.19 *Fix a set $S \subseteq \mathbf{P}$ and a model M, and a function $\Phi : \omega \to \omega$. If $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are both (S,Φ)-simple $\mathbf{P}^{\mathbf{r}}$-decompositions of M, then the trees I_1^S and I_2^S are almost $\mathcal{P}_{\mathbf{P}}$-isomorphic.*

Proof. Because of Theorem 6.17, we only need to verify that the hypotheses of Lemma 6.16 are satisfied for each of the decompositions. But this is routine, once one notes that Clause 2(b) is satisfied because of the triviality of E_η and Lemma 6.10(3).

References

[1] M. Makkai, A survey of basic stability theory with particular emphasis on orthogonality and regular types, *Israel Journal of Mathematics* **49** (1984), 181-238.

[2] M.C. Laskowski and S. Shelah, Decompositions of saturated models of stable theories, *Fundamenta Mathematicae* **191** (2006) 95-124.

[3] M.C. Laskowski and S. Shelah, Borel completeness of some $\aleph_0$-stable theories (preprint).

[4] S. Shelah, *Classification Theory*, (revised edition) North Holland, Amsterdam, 1990.

[Sh401] S. Shelah, Characterizing an $\aleph_\epsilon$-saturated model of superstable NDOP theories by its $\mathbb{L}_{\infty,\aleph_\epsilon}$-theory, *Israel Journal of Math* **140** (2004) 61-111.