

ON MODEL COMPLETION OF T_{aut}

E34

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ANNOTATED CONTENT

§0 Introduction

§1

[We characterize stable T for which the model completion of T_{aut} is stable (i.e., every completion is).]

§2

[We prove that “some completion is stable” is different and characterize it.]

§3

[We prove that if T is stable, T_{aut} has a model completion, T_* is an unstable complete of $T_{\text{aut}}^{\text{mc}}$, then T_* satisfies NSOP₃. Moreover, simplicity is preserved.]

§0 INTRODUCTION

On the subject, history and background see [BlSh 759]. For a complete first order T they dealt with the existence of the model completion T_{aut} of $T \cup \{\sigma \text{ is an automorphism (for } \tau_T)\}$.

We may ask:

0.1 Question: If T is stable and T_{aut} has model completion $T_{\text{aut}}^{\text{mc}}$, when is (every) completion of $T_{\text{aut}}^{\text{mc}}$ stable?

We answer in 1.6 (observation 1.7 deals with some obvious things).

Section 1 raises some question which we discuss below (assuming T stable, $T_{\text{aut}}^{\text{mc}}$ exists) some of which are answered below.

0.2 Question: 1) Can we in Claim 1.6 below replace “every completion of $T_{\text{aut}}^{\text{mc}}$ is stable” by “some completion of $T_{\text{aut}}^{\text{mc}}$ is stable”?

2) The “unstable” in 1.6 clause (a) can be replaced by “having the independence property”; but can $T_{\text{aut}}^{\text{mc}}$ be completed to a theory with the strict order property? The SOP_n 's?

3) What occurs if $T_{\text{aut}}^{\text{mc}}$ does not exist, can we still say something?

4) Point out that (a)(\equiv) (b) of 1.6 holds (for some stable T for which $T_{\text{aut}}^{\text{mc}}$ exists) and fails for others.

5) Show for stable T with $T_{\text{aut}}^{\text{mc}}$, that no completion T_* of $T_{\text{aut}}^{\text{dc}}$ has the explicit ncp (which means that for some first order $E(\bar{x}, \bar{y}, \bar{z})$, for every n for some $\bar{c} \subseteq \mathfrak{C}$, $E(\bar{x}, \bar{y}, \bar{c})$ is an equivalent relation which has $\geq n, < \aleph_0$ equivalence classes); a stronger version is

6) For such T, T_* can T_* have obstructions (see §4)?

7) What if we use σ_1, σ_2 ? What about $\sigma_1, \dots, \sigma_n$? What about pairwise commuting $\sigma_1, \dots, \sigma_n$? This is like $(T_{\text{aut}})_{\text{aut}}$ for $n = 2$.

8) Is there unstable T such that T_{aut} has model completion? (A conjecture stating that had been the starting point of Kikyo Shelah [KkSh 748]).

0.3 Discussion: We prove that:

- (A) on 0.2(1), for some T (stable with $T_{\text{aut}}^{\text{mc}}$ existing), some completion of $T_{\text{aut}}^{\text{mc}}$ are stable and some are not (still we may wonder on a general characterization, see 2.7 below)
- (B) we shall show that for no such T is any completion of $T_{\text{aut}}^{\text{mc}}$ with the strict order property and even have NSOP_3 , see 3.1
- (C) we can look at the class of existentially closed models of T_{aut} (see [ShUs 789] and references there); the results are similar.

Note

0.4 *Observation.* [Here?]

- (α) for $T =$ theory of equality, T_{aut} has a model completion and all completion of $T_{\text{aut}}^{\text{mc}}$ are stable
- (β) for T from 2.1, some completions of $T_{\text{aut}}^{\text{mc}}$ are stable and some are not
- (γ) for $T = \text{Th}(M \upharpoonright \{E, F_1, F_2, Q\})$, M from 2.1, we get that all the completions of $T_{\text{aut}}^{\text{mc}}$ are unstable.

I think

0.5 Question: What about getting (in §3) that

- (a) $T_{\text{aut}}^{\text{mc}}$ is simple in §3?
- (b) even if T is just simple, $T_{\text{aut}}^{\text{mc}} \models \text{NSOP}_3$
- (c) non elementary class (true).

See below.

§1 ON THE STABILITY OF MODEL COMPLETION
FOR T_{aut} (= $T + \sigma$ AN AUTOMORPHISM)

1.1 Hypothesis. 1) T is first order complete and for notational simplicity every formula is equivalent to a relation and τ_T having only predicates.
2) \mathfrak{C} is the monster model of T .

1.2 Definition. 1) T_{aut} is $T \cup \{\sigma \text{ is an automorphism (for } \tau_T)\}$, so σ is a new unary function symbol that is $T_{\text{aut}} = T \cup \{(\forall x_0, \dots, x_{n-1})[R(x_0, \dots, x_{n-1}) \equiv R(\sigma(x_0), \dots, \sigma(x_{n-1}))] : R \text{ an } n\text{-place predicate of } \tau_T\}$.
2) $T_{\text{aut}}^{\text{mc}}$ is the model completion, if it exists.
3) Let T_* denote any completion of $T_{\text{aut}}^{\text{mc}}$ and σ_* or σ^{N^+} is an automorphism.
4) A completion T_* of $T_{\text{aut}}^{\text{mc}}$ is cute if it has a model N^+ such that for some $M^+ \subseteq N^+$ we have $\sigma^{N^+} = \text{id}_{N^+}$.

1.3 Definition. For T as in 0.2 let:

- 1) $K_{\text{aut}}(T)$ = the class of models of T_{aut} .
- 2) $K_{\text{aut}}^{\text{ec}}(T)$ = the class of e.c. models of T_{aut} .
- 3) $K_*(T)$ is a subclass of $K_{\text{aut}}^{\text{ec}}(T)$ such that $M \cong N \in K_* \Rightarrow M \in K_*$ and if $M \subseteq N$ are from $K_{\text{aut}}^{\text{ec}}$ then $M \in K_* \Leftrightarrow N \in K_*$; there are $\leq 2^{|T|}$ such classes.
- 4) K_* is cute, etc.
- 5) $\mathfrak{C}_{\text{aut}}$ is a monster model for $K_{\text{aut}}^{\text{ec}}$, i.e., a member of $K_{\text{aut}}^{\text{ec}}$ which is $\bar{\kappa}$ -saturated of cardinality $\bar{\kappa}$; it is unique if $K_{\text{aut}}(T)$ has the JEP.
- 6) A class K_* is stable¹ if for some $\lambda < \bar{\kappa}$ there is no model $M \in K_*$, $m < \omega$, $\bar{a}_i \in {}^m M$, $i < \lambda$ and q.f. formula $\varphi(\bar{x}, \bar{y})$ which order $\{\bar{a}_i : i < \lambda\}$.
- 7) K_* is simple if there is a q.f. formula $\varphi(\bar{x}, \bar{y})$ and m such that for every λ, κ we can find $M \in K_*$, $\bar{a}_\eta \subset {}^{\ell g(\bar{y})} M$ for $\eta \in {}^\kappa > \lambda$ and $\bar{b}_\nu \in {}^{\ell g(\bar{x})} M$ for $\nu \in {}^\kappa \lambda$ such that:

- (i) $M \models \varphi(\bar{b}_\eta, \bar{a}_{\eta \upharpoonright \alpha})$ for $\alpha < \kappa, \eta \in {}^\kappa \lambda$
- (ii) no sequence in m realizes $\geq m$ of the formulas $\langle \varphi(\bar{x}, \bar{a})_{\eta \upharpoonright < 1 \rangle} : i < \lambda \rangle$.

On such models see [Sh 54], [xx], [xx].

1.4 Fact: If $T_{\text{aut}}^{\text{mc}}$ exists then $K_{\text{aut}}^{\text{ec}}(T)$ is the class of its models.

1.5 Claim. *In the claims below we can replace “ T has model completion” by dealing with the class $K_{\text{aut}}^{\text{ec}}(T)$, and replace T^* is a model completion by dealing with K_* .*

¹this is for classes as above, for general non first order classes this does not fit

1.6 Claim. *Let T be stable, $T_{\text{aut}}^{\text{mc}}$ exists. The (a) \Leftrightarrow (b) where*

- (a) $T_{\text{aut}}^{\text{mc}}$ is stable (i.e., every completion is stable)
- (b) if $M_0 \prec M_\ell \prec \mathfrak{C}$ for $\ell = 1, 2$ and $M_1 \cup M_2$ then in \mathfrak{C}^{eq} , $\text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2) = M_0$
 $\text{dcl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$
- (c) $T_{\text{aut}}^{\text{mc}}$ is dependent (i.e., every completion does not have the independence property).

Proof. (b) \Rightarrow (a)

We work in \mathfrak{C}^{eq} and use observation 1.7 below. Suppose $\mathfrak{C}_* = (\mathfrak{C}, \sigma_*)$ is an expansion of \mathfrak{C}^{eq} to a model of $T_{\text{aut}}^{\text{mc}}$ and let σ_*^{eq} be the unique extension of σ_* to an automorphism of \mathfrak{C}^{eq} . Let $\lambda = \lambda^{|T|}$, $M^+ \prec (\mathfrak{C}^{\text{eq}}, \sigma_*^{\text{eq}})$, $|M^+| = \lambda$ (note $|T| \geq \aleph_0$ here (by 1.1(1))).

Now for every $p \in \mathcal{S}(M^+, \mathfrak{C}_*^{\text{eq}})$ let $a_p \in \mathfrak{C}$ realize p in (\mathfrak{C}, σ_*) and let M_p^+, N_p^+ be such that

$$M_p^+ \prec M^+, \|M_p^+\| = |T| + \aleph_0$$

$$M_p^+ \prec N_p^+ \prec \mathfrak{C}_\sigma, \|N_p^+\| = |T|$$

$$a_p \in N_p^+$$

$$N_p^+ \upharpoonright \tau_T \quad \bigcup \quad M^+ \upharpoonright \tau_T \\ M_p^+ \upharpoonright \tau_T$$

Let $A_p = \text{acl}_{\mathfrak{C}^{\text{eq}}}(|N_p^+| \cup |M_p^+|)$. We define a two-place relation E on $\mathcal{S}(M^+, \mathfrak{C}_\sigma)$ as follows:

- \otimes pEq iff $M_p^+ = M_q^+$ and there is an isomorphism f from N_p^+ onto N_q^+ which is the identity on M_p^+ and satisfying $f_p(a_p) = a_q$.

Clearly

- \otimes_0 E is an equivalence relation on $\mathcal{S}(M^+, \mathfrak{C}_*)$
- \otimes_1 $|\mathcal{S}(M^+, \mathfrak{C}_*)/E| \leq \lambda^{|T|}$.

Hence it is enough to prove that

$$\otimes_2 pEq \Rightarrow p = q.$$

Proof of \otimes_2 . Let f witness pEq ,

Let $f^+ : A_p = \text{dcl}_{\mathfrak{C}^{\text{eq}}}(|N_p^+| \cup |M^+|) \rightarrow A_q$ extends $f \cup \text{id}_M$ and be an elementary mapping (in \mathfrak{C}^{eq}); by non forking calculus it exists and is unique. Obviously it commutes with σ_* . Also A_p (and A_q) are algebraically closed sets in \mathfrak{C}^{eq} by our hypothesis (that is, clause (b)) applied to $|M_p^+|, |N_p^+|, |M^+|$ hence by 1.7(4), 1.8(4) below, f^+ can be extended to an automorphism of \mathfrak{C}^{eq} . So by properties of model completion (and the obvious 1.8(1) below) we are done.

$\neg(b) \Rightarrow \neg(a)$:

Let M_0, M_1, M_2 form a counterexample to (b). So let $e \in \text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2) \setminus \text{dcl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$ hence we can find $\bar{a} \in {}^{\omega>}(M_1), \bar{b} \in {}^{\omega>}(M_2)$ and $n < \omega, \varphi(x, \bar{b}, \bar{a})$ such that

- $\otimes(i) \mathfrak{C}^{\text{eq}} \models \varphi[e, \bar{b}, \bar{a}]$
- $(ii) \models (\exists^{!n} x) \varphi(x, \bar{b}, \bar{a})$
- $(iii) n$ minimal under (i) + (ii).

We know $\varphi(x, \bar{b}, \bar{a}) \vdash \text{tp}(e, M_1 \cup M_2)$ and let $\{e_0, \dots, e_{n-1}\}$ list $\varphi(\mathfrak{C}^{\text{eq}}, \bar{b}, \bar{a})$.

Let $\bar{e} = \langle e_0, \dots, e_{n-1} \rangle$. Possibly increasing \bar{a}, \bar{b} for some formula $\psi = \psi(\bar{x}, \bar{b}, \bar{a})$ with $\bar{x} = \langle x_\ell : \ell < n \rangle$ we have $\models \psi(\bar{e}, \bar{b}, \bar{a})$ and $\psi(\bar{x}, \bar{b}, \bar{a}) \vdash \text{tp}(\bar{e}, M_1 \cup M_2)$.

So we can find f such that

- $\otimes f$ is an elementary mapping in \mathfrak{C}
- $\text{Dom}(f) = M_1 \cup M_2 \cup \bar{e}$
- $f \upharpoonright (M_1 \cup M_2)$ is the identity
- $f(e_0) \neq e_0$ (but of course f permutes $\{e_\ell : \ell < n-1\}$).

Let $f(\bar{e}) = \bar{e}'$. Let $\bar{e}_0 = \bar{e}, \bar{e}_1 = f(\bar{e})$.

We can find a sequence of \mathfrak{C}^{eq} -elementary mapping $\langle g_i : i < |T|^+ \rangle$ such that

$$\text{Dom}(g_i) = \text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$$

$$g_i \upharpoonright M_2^{\text{eq}} = \text{id}$$

$$\bigcup_{M_2^{\text{eq}}} \{\text{Rang}(g_i) : i < |T|^+\}.$$

Now

⊗ if $k < \omega, i_0 < \dots < i_{k-1} < \omega$ and $\eta \in {}^n 2$ then the type $p_\eta = \text{tp}(g_{i_0}(\bar{e}_{\eta(0)}) \wedge g_{i_1}(\bar{e}_{\eta(1)}) \wedge \dots \wedge g_{i_{k-1}}(\bar{e}_{\eta(k)}), \bigcup_{i < |T|} \text{Rang}(g_i))$ does not depend on η .

[Why? By induction on k , hence by transitivity of equality it is enough to prove $p_\eta = p_\nu$ when $1 = |\{\ell : \eta(\ell) \neq \nu(\ell)\}|$.

By an indiscernible sequence = indiscernible set (= symmetry of nonforking, etc.) without loss of generality $\eta(0) \neq \nu(0)$. As $\text{Rang}(\bar{e}_0) = \text{Rang}(\bar{e})$, without loss of generality $\bigwedge_{\ell < k-1} \eta(1 + \ell) = 0 = \nu(1 + \ell)$. Lastly, $\text{tp}(\bigcup_{i > 0} \text{Rang}(g_i), \text{Rang}(g_0))$ is finitely satisfiable in M_2 so by the choice of ψ we are done.]

Now for any $\eta \in ({}^{(|T|^+)} 2)$ we define the function h_η :

$$\text{Dom}(h_\eta) = M_2^{\text{eq}} \cup \bigcup \{g_i''(M_1^{\text{eq}}) : i < |T|^+\} \cup \{g_i(\bar{e}) : i < |T|^+\}$$

$$h_\eta \upharpoonright M_2^{\text{eq}} = \text{identity}$$

$$h_\eta \upharpoonright g_i''(M_1^{\text{eq}}) = \text{identity}$$

$$h_\eta(g_i(\bar{e})) = \begin{cases} g_i(\bar{e}) = g_i(\bar{e}_0) & \text{if } \eta(i) = 0 \\ g_i(\bar{e}_1) & \text{if } \eta(i) = 1 \end{cases}$$

We can find M_3, M_4, σ such that

$$\bigcup \{g_i(M_1) : i < |T|^+\} \subseteq M_3 \prec M_4 \prec \mathfrak{C}$$

$$\begin{array}{c} M_2 \cup M_4 \\ M_0 \end{array}$$

M_4 is saturated of cardinality $> \|M_3\|$

$$\sigma \in \text{Aut}(M_4), \sigma \upharpoonright M_3 = \text{identity}$$

(M_4, σ) is a model of $T_{\text{aut}}^{\text{mc}}$.

Now for every $\eta \in ({}^{(|T|^+)} 2)$ we can find $(M_\eta^5, \sigma) \models T_{\text{aut}}$ such that $(M_4, \sigma) \subseteq (M_\eta^5, \sigma)$ and \bar{b}_η realizing $\text{tp}_{\mathfrak{C}^{\text{eq}}}(\bar{b}, M_0, \mathfrak{C})$ such that

$$\eta(i) = 0 \Leftrightarrow (\exists \bar{x})(\psi(\bar{x}, \bar{b}_\eta, g_i(\bar{a})) \wedge \sigma(\bar{x}) = x).$$

So $\{(\exists \bar{x})(\psi(\bar{x}, \bar{y}, g_i(\bar{a})) : i < |T|^+)\}$ is an independent set of formulas in (M_4, σ) hence $T_{\text{aut}}^{\text{mc}}$ is unstable.

(a) \Rightarrow (d):
Trivial.

$\neg(b) \Rightarrow \neg(c)$:
Included in the proof of $\neg(b) \Rightarrow \neg(a)$. $\square_{1.6}$

1.7 Observation. Assume $T_{\text{aut}}^{\text{mc}}$ exists, T_* any completion of it.

- 1) If \mathfrak{C} is a saturated model of T of cardinality $\bar{\kappa} = \bar{\kappa}^{<\bar{\kappa}}$, can be expanded to a model \mathfrak{C}_* of T_* .
- 2) If $M \models T, \sigma \in \text{Aut}(M)$, let σ^{eq} be the natural extension of σ to an automorphism of M^{eq} , then (it exists and is unique) $(M^{\text{eq}}, \sigma^{\text{eq}}) \models (T^{\text{eq}})_{\text{aut}}$.
- 3) $(T^{\text{eq}})_{\text{aut}}$ has a model completion T and there is a natural one to one correspondence between the completions of the model completions of $(T^{\text{eq}})_{\text{aut}}$ and $\{T_{**} : T_{**}$ a model completion of $T_{\text{aut}}^{\text{mc}}\}$ any one of the former is essentially bi-interpretable with the corresponding one of the latter (but we have the elements not in any $P_{E(\bar{x}, \bar{y})}$).
- 4) Let $\mathfrak{C}_* = (\mathfrak{C}, \sigma_*)$ be a $\bar{\kappa}$ -saturated model of T_* expanding \mathfrak{C} . If $A_\ell \subseteq \mathfrak{C}^{\text{eq}}, A_\ell = \text{acl}_{\mathfrak{C}^{\text{eq}}}(A_\ell), A_\ell$ closed under σ_* , f is an \mathfrak{C}^{eq} -elementary mapping from A_1 onto A_2 commuting with σ then f can be extended to an automorphism of $(\mathfrak{C}^{\text{eq}})_{\text{aut}}$ (it is \mathfrak{C}^{eq} expanded by σ naturally extended to σ^+).

1.8 Observation. 1) M is a model of $T, \sigma_* \in \text{Aut}(M)$ iff (M, σ_*) is a model of T_{aut} .

2) If $M \prec \mathfrak{C}$ and (M, σ_*) as a model of T_{aut} then for one and only one $\sigma_*^{\text{eq}} \in \text{Aut}(M^{\text{eq}})$ extend σ_* .

3) If $M \prec \mathfrak{C}, \sigma_*^{\text{eq}} \in \text{Aut}(M^{\text{eq}})$ then $\sigma_*^{\text{eq}} \upharpoonright M \in \text{Aut}(M)$.

4) If $A_\ell \subseteq \mathfrak{C}^{\text{eq}}$ and $A_0 = \text{acl}_{\mathfrak{C}^{\text{eq}}}(A_0)$ and f_ℓ is an \mathfrak{C}^{eq} -elementary mapping from A_ℓ onto A_ℓ for $\ell = 0, 1, 2$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ then for some automorphism F of $\mathfrak{C}^{\text{eq}}, F \upharpoonright A_0 = \text{id}_{A_0}$ and $f_2 \cup (F f_1 F^{-1})$ is an elementary mapping in \mathfrak{C}^{eq} (hence can be extended to an automorphism of \mathfrak{C}^{eq} ; if $A_1 \amalg_{A_0} A_2$ then without loss of generality $F \upharpoonright$

$(A_1 \cup A_2) = \text{id}_{A_1 \cup A_2}$).

§2

2.1 Example: There is T such that:

- (a) T is as in 1.1, stable $T_{\text{aut}}^{\text{mc}}$ exists. Moreover T is superstable, countable $I(\aleph_\alpha, T) \leq 2^{|\alpha|}$ for $\alpha \geq 2^{\aleph_0}$ (hence NDOP, NOTOP, shallow with small depths, with $\leq 2^{\aleph_0}$ dimensions)
- (b) $T_{\text{aut}}^{\text{mc}}$ exist
- (c) some completions of $T_{\text{aut}}^{\text{mc}}$ are stable and some are not.

Proof. Let us define M, I

- $|M|$ is $\{(\eta, k, n, \ell) : k, n < \omega, \ell < 2 \text{ and } \eta \in {}^\omega 2\}$ and $k = n \Rightarrow \ell = 0$
- E_n^M , a two-place relation is $\{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 \upharpoonright n = \eta_2 \upharpoonright n\}$
- E^M , a two-place relation is $\{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 = \eta_2\}$
- Q^M , a one-place relation is $\{(\eta, k, n, \ell) \in |M| : k = n\}$
- F_1^M , a one-place relation is: $F_1^M((\eta, k, n, \ell)) = (\eta, k, k, 0)$
- F_2^M , a one-place relation is: $F_2^M((\eta, k, n, \ell)) = (\eta, n, n, 0)$

Let $T = \text{Th}(M)$. Clearly it satisfies (a):

- ⊗₁ $T_{\text{aut}}^{\text{mc}}$ exists.
[Why? Check that there are no obstructions.]
- ⊗₂ $T_{\text{aut}}^{\text{mc}}$ has an unstable completion.
[Why? By 1.6, or more specifically, see below.]

We shall now prove

- ⊗₂⁺ for T_* a completion of $T_{\text{aut}}^{\text{mc}}$, T_* is unstable if:
for some $M^+ \models T_*$, for some $a \in M^+$ we have $\bigwedge_n aE_n(\sigma^{M^+}(a))$ or just
 $(\exists m) \bigwedge_{n < \omega} aE_n((\sigma^{M^+})^m(a))$, i.e., for some $m^* \in [1, \omega)$ we have $\bigwedge_n aE_n a_{M^+}$
where $a_0 = a, a_{\ell+1} = \sigma^{M^+}(a_\ell)$ for $\ell < \omega$.

Let $m^*, a, \langle a_\ell : \ell < \omega \rangle$ be as above. We define N a model of T : let $|N|$, the universe of N be

$$|M^+| \cup \{(m, k, n, \ell) : m < m^*, k, n < \omega, \ell < 2, k = n \Rightarrow \ell = 0\}$$

we assume no incidental identification.

$$E_n^N : \begin{cases} E_n^N \text{ is an equivalence relation} \\ E_n^N \upharpoonright |M^+| = E_n^{M^+} \\ \text{every } (m, k, n, \ell) \in |N| \setminus |M^+| \text{ is } E_n\text{-equivalent to } a_m \end{cases}$$

$$E^N : \begin{cases} E^N \text{ is an equivalence relation} \\ E^N \upharpoonright |M^+| = E^N \\ \{(m, k, n, \ell) \in |N| \setminus |M^+| : k, n < \omega, \ell < 2, k = n \Rightarrow \ell = 0\} \\ \text{is an } E^N\text{-equivalence class (for each } m < m^*) \end{cases}$$

$$Q^N = Q^N \cup \{(m, k, k, 0) : k < \omega\}$$

$$F_1^N \text{ extends } F_1^{M^+}, F_1^N((m, k, n, \ell)) = (m, k, k, 0)$$

$$F_2^N \text{ extends } F_2^{M^+}, F_2^N((m, k, n, \ell)) = (m, n, n, 0).$$

Easily

$$\square_1 \quad M^+ \upharpoonright \tau_T \prec N.$$

Now we define an automorphism σ^+ of N :

$$\square_2 \quad \sigma^+ \upharpoonright |M^+| = \sigma^{M^+}$$

$$\square_3 \quad \text{if } m_1, m_2 < m^*, m_2 = m_1 + 1 \pmod{m^*} \text{ then}$$

$$\sigma(m_1, n, k, \ell) \text{ is:}$$

$$(m_2, n, k, 1 - \ell) \text{ if } m_1 = m^* - 1 \ \& \ n < k$$

$$(m_2, n, k, \ell) \text{ otherwise.}$$

Easy to check that $\sigma^+ \in \text{Aut}(N)$, so $(N, \sigma) \supseteq M^+$ is a model of T_{aut} . As $T_{\text{aut}}^{\text{mc}}$ exists and $M^+ \models T_{\text{aut}}^{\text{mc}}$ there is a model $N^+ \models T_{\text{aut}}^{\text{mc}}$ such that $M^+ \prec M^+, (N, \sigma) \subseteq N^+$.

Let

$$\varphi(x, y) = Q(x) \ \& \ Q(y) \ \& \ xEy \ \& \ (\exists z)(F_1(z) \ \& \ F_2(z) = y \ \& \ (\sigma^{m^*}(z) \neq z))$$

This is a first order formula in $\mathbb{L}(\tau_{\text{Th}(M^+)}) = \mathbb{L}(\tau_{T_{\text{aut}}})$ and $N^+ \models \varphi[b_n, b_k]$ iff $n < \omega$ where $b_n = (0, n, n, 0) \in N \subseteq N^+$, so this formula has the order property in $\text{Th}(N^+) = \text{Th}(M^+)$. So $\text{Th}(M^+)$ is unstable as required in \otimes_2^+

- ⊗₃ if T_* is a completion of $T_{\text{aut}}^{\text{mc}}$ not satisfying the demand in ⊗₂⁺ then T_* is stable.
 [Why? As any model M^+ of T_* , σ^{M^+} acts as a permutation of $|M^+|/E^{M^+}$ which has no fix point and even no finite cycle. Now reflect.]
- ⊗₄ there is a completion T_* of $T_{\text{aut}}^{\text{mc}}$ which is stable.
 Why? Let f be a permutation of ${}^\omega 2$ such that
- (α) $\eta, \nu \in {}^\omega 2 \wedge \eta \upharpoonright n = \nu \upharpoonright n \Rightarrow f(\eta) \upharpoonright n = f(\nu) \upharpoonright n$
 - (β) for every $m < \omega$ (≥ 2) for some $n < \omega$ we have if $\eta \in {}^\omega 2$, then $\eta, f^m(\eta)$ are not E_n -equivalent.

Easy to construct (or use $\prod_{n < \omega} (n+1)$ instead ${}^\omega 2$) and define M^+ , a $\tau_{T_{\text{aut}}}$ -expansion of M by defining

$$\sigma^{M^+}((\eta, k, n, \ell)) = (f(\eta), k, n, \ell).$$

So if $M^+ \subseteq N^+ \models T_{\text{aut}}^{\text{mc}}$ then $T_* = \text{Th}(N^+)$ fail the demand in ⊗₂⁺ hence by ⊗₃ it is stable as required (and it is uniquely determined by M^+ , really just the action on $\text{acl}_{\mathcal{C}^{\text{eq}}}(\emptyset)$, suffice. So ⊗₄ holds. □_{2.1}

2.2 Discussion: It seems reasonable that we can characterize when this occurs thus answering fully 0.1; see below.

A closely related example is

2.3 Claim. *There is T such that:*

- (a) T is stable (complete countable first order theory) and has elimination of quantifiers for simplicity
- (b) T is superstable and small, i.e., with countable $D(T)$
- (c) T_{aut} has no model completion
- (d) some $T_{\text{aut}}(M^+)$ has a model completion where

2.4 Definition. 1) For a model $M^+ = (M, \sigma^{M^+})$ of T_{Aut} let $T_{\text{aut}}(M^+) = T_{\text{aut}} \cup \text{Th}(M, c)_{c \in M} \cup \{\sigma(c_1) = c_2 : \sigma^{M^+}(c_1) = c_1\}$.

2.5 Remark. Actually we can use any completion of $T_{\text{aut}} \cup$ (the action of σ on $\text{acl}_{\mathcal{C}^{\text{eq}}}(\emptyset, \mathcal{C}_T)$ (i.e., on the E -equivalence classes for each n).

Proof. Define M

- (a) $\tau_M = \{E_n, P_n : n < \omega\} \cup \{E, E_*\}$
- (b) $|M| = \{(\eta, k, n, \ell) : \eta \in {}^\omega 2, k < \omega, n < \omega, \ell < 2\}$
- (c) $E_n^M = \{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2) \in |M| \times |M| : \eta_1 \upharpoonright n = \eta_2 \upharpoonright n\}$
- (d) $E^M = \{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2) \in |M| \times |M| : \eta_1 = \eta_2 \text{ and } k_1 = k_2\}$
- (e) $E_*^M = \{((\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2)) \in |M| \times |M| : \eta_1 = \eta_2, k_1 = k_2, n_1 = n_2\}$
- (f) $P_n^M = \{(\eta, k, n, \ell) \in |M| : n = m\}$.

We choose σ^M such that $\sigma(\eta, k, n, \ell) = (\eta', k, n, \ell)$ and (η, η') are as in the proof of 2.1.

Remark. If we let $(d)'$ be as in 2.8 below we add $\sigma = \text{the identity}$ then $(a) + (c) + (d)'$ is impossibly by [BlSh 759].

Actually the case σ is the identity on some M is the real one because

2.6 Claim. *For any first order complete T_1 (with τ_{T_1} a set of predicates for simplicity) there is T such that:*

- (a) T is first order complete
- (b) if $a \in M, M \models T$ then we can interpret T_1 in (M, a)
- (c) $\tau_T \setminus \tau_{T_1}$ countable
- (d) some $T_{\text{aut}}(M^+)$ has a model completion.

Proof. As in 2.3 without $E_*, P_n(n < \omega)$ in any E^M -equivalence class we “plant” a model of T_1 .

2.7 Claim. *Let T_* be a completion of $T_{\text{aut}}^{\text{mc}}$.*

The following are equivalent:

Condition (a): T_* is stable.

Condition (b): If T is stable and $(\alpha) + (\beta) + (\gamma)$ below holds, then $(*)$ below holds where

- (α) $M_0^+ \prec M_\ell^+ < M_3^+$ for $\ell = 1, 2, M_0 \models T_*, M_\ell \models T_{\text{aut}}$ for $\ell = 1, 2, 3$ and
- (β) $M_\ell = M_\ell \upharpoonright \tau_T$ and $M_1 \bigcup_{M_0}^{M_3} M_2$ without loss of generality $M_3 \prec \mathfrak{C} = \mathfrak{C}_T$

- (γ) if f is an elementary mapping from $\text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$ onto itself extending $\sigma^{M_1^+} \cup \sigma^{M_2^+}$
- (*) there is an elementary mapping h from $\text{acl}_{\mathfrak{C}}(M_1 \cup M_2)$ onto itself such that $h \upharpoonright (M_1 \cup M_2) = \text{identity}_{M_1 \cup M_2}$ and $hfh^{-1} = \sigma^{M_3^+} \upharpoonright \text{acl}_{\mathfrak{C}^{\text{eq}}}(M_1 \cup M_2)$.

Proof. $(b) \Rightarrow (a)$:

As in the proof of 1.6.

$\neg(b) \Rightarrow \neg(a)$:

We can use compactness to replace $\neg(b)$ by a finite failure, and continue as in the proof of 1.6.

2.8 Remark. We can make $\neg(b)$ more explicit as in the proof of 2.7.

§3 NSOP₃

As by [KkSh 748], if $T_{\text{aut}}^{\text{mc}}$ exists, then T fails the strict order property. It seems reasonable to ask if any $T_{\text{aut}}^{\text{mc}}$, which exists, can have the strict order property. As we understand the stable case, it seems reasonable to deal with it. In fact, more turn out to hold.

3.1 Claim. *[T as in 1.1.] If T is stable, any completion T_* of $T_{\text{aut}}^{\text{mc}}$ satisfies NSOP₃ (see [Sh 500, §2] and [ShUs 789]).*

Proof. 1) Clause (a):

Let T_* be completion of $T_{\text{aut}}^{\text{mc}}$ and $\varphi(\bar{x}, \bar{y})(\ell g(\bar{x}) = \ell g(\bar{y}) = n^* < \omega)$ a first order formula in $\mathbb{L}(\tau_{T_*})$ and for some $M \models T_*$ we have $M \models \varphi(\bar{a}_n, \bar{a}_m)^{\text{if}(n < m)}$. Hence we can find an E.M.-template Φ such that $\tau_\Phi \supseteq \tau_{T_*} = \tau_T \cup \{\sigma\}$ and for linear orders $I \subseteq J$, $\text{EM}(I, \Phi) \prec \text{EM}(J, \Phi) \neq T_*$, with skeleton $\langle \bar{a}_t : t \in J \rangle$ such that $\text{EM}(J, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t]^{\text{if}(s < Jt)}$ for $s, t \in J$ (so $\bar{a}_t \in \text{EM}(\{t\}, \Phi)$ (see, e.g., [Sh:c, VII] or [Sh:e, III]). Now (recalling that $\text{EM}_\tau(I, \Phi) = \text{EM}(I, \Phi) \upharpoonright \tau$) without loss of generality

- ⊗₁ if $I_1, I_2 \subseteq J, I_0 = I_1 \cap I_2$ and if $t \in I_1 \setminus I_0$ then there is $s \in I_0$ such that $s < t$ & $]s, t[_J \cap I_2 \subseteq I_0$ or $t < s$ & $[t, s]_J \cap I_2 \subseteq I_0$ then $\text{tp}_{L(\tau_{T_*})}(\text{EM}_{\tau_{T_*}}(I_1, \Phi), \text{EM}_{\tau_{T_*}}(I_2, \Phi))$ is f.s. (finitely satisfiable) in $\text{EM}_{\tau_{T_*}}(I_0, \Phi)$
 [Why? Let $I \times \mathbb{Z}$ be ordered lexicographically, choose Φ' such that $\text{EM}(I, \Phi') = \text{EM}(I \times \mathbb{Z}, \Phi)$, with skeleton $\bar{a}'_t = \bar{a}_{(t,0)}$; can look at [Sh 394].]

For $u \subseteq \{0, 1, 2\}$ let $M_u^2 = \text{EM}(u, \Phi)$ and if $|u| = |v|$ both subsets of $\{0, 1, 2\}$ let $f_{v,u}$ be the canonical isomorphism from M_u onto M_v . Let $M_u^1 = M_u^2 \upharpoonright \tau_{T_*}, M_u^0 = M_u^2 \upharpoonright \tau_T$. Let N be such that $M_{\{0,1,2\}}^0 \prec N, N$ is $\|M_{\{0,1,2\}}^0\|^+$ -saturated

- ⊗₂ in $N, \bigcup_{M_\emptyset^0} \{M_{\{0\}}^0, M_{\{1\}}^0, M_{\{2\}}^0\}$
 [Why? By ⊗₁ and nonforking calculus.]

Let $g_0 =: f_{\{0\}, \{2\}} \cup f_{\{2\}, \{0\}}$

- ⊗₃ g_0 is an elementary mapping (inside N)
 [Why? Nonforking calculus.]

Let g_1 be an elementary mapping inside N extending g_0 with domain $M_{\{0,2\}}^0$.

Let $M_{\{0,2\}}^{0,*} = g(M_{\{0,1\}}^0)$.

Let $M_{\{0,2\}}^{1,*}$ be an expansion of $M_{\{0,2\}}^{0,*}$ by an automorphism $\sigma^{M_{\{0,2\}}^{1,*}}$ such that g_1 is an isomorphism from $M_{\{0,2\}}^1$ onto $M_{\{0,2\}}^{1,*}$, clearly exists.

As N is a model of the stable theory T without loss of generality $\text{tp}_{L^*(\tau_T)}(|M_{\{0,2\}}^{1,*}|, |M_{\{0,1,2\}}^0|)$ does not fork over $|M_{\{0\}}^0| \cup |M_{\{2\}}^0|$.

Now the point is that

⊙ $h = \sigma^{M_{\{0,1\}}^1} \cup \sigma^{M_{\{0,2\}}^{1,*}} \cup \sigma^{M_{\{1,2\}}^1}$ is a permutation of $|M_{\{0,1\}}^{1,*}| \cup |M_{\{0,1\}}^1| \cup |M_{\{1,2\}}^1|$ and is an elementary mapping.

[Why? Let $B_0 = |M_{\{0\}}^0| \cup |M_{\{2\}}^0|$, $B_1 = |M_{\{0,1\}}^0| \cup |M_{\{2,2\}}^0|$.

By [Sh:c, XII], the pair (B_0, B_1) satisfies the T.V. condition inside N (i.e., if $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$, $N \models \varphi[\bar{a}, \bar{b}]$, $\bar{a} \subseteq B_1$, $\bar{b} \subseteq B_0$ then for some $\bar{a}' \subseteq B_0$, $N \models \varphi[\bar{a}', \bar{b}]$. Moreover, we can allow $\bar{b} \subseteq |M_{\{0,2\}}^{0,*}|$ then this follows.]

So for some N' , $N \prec N' \models T$ and there is an automorphism h' of N' extending h and we can extend (N', h') to a model (N'', h'') of T_* . By this model clearly

$$(N'', h'') \models \varphi[\bar{a}_0, a_1] \text{ using } M_{\{0,1\}}^1$$

$$(N'', h'') \models \varphi[\bar{a}_1, \bar{a}_2] \text{ using } M_{\{1,2\}}^1$$

$$(N'', h'') \models \varphi[\bar{a}_2, \bar{a}_0] \text{ using } M_{\{0,2\}}^{1,*} \text{ and}$$

$$g_1 \text{ being an isomorphism from } M_{\{0,2\}}^1 \text{ onto } M_{\{0,2\}}^{1,*}.$$

This is enough to show $T_* \models \text{NDOP}_3$.

3.2 Claim. *T is stable or just simple then any T_* (assuming it exists, K_* in general) is simple.*

Proof. We write it for K_* . Choose $\kappa = \text{cf}(\kappa) > |T|$ and μ a strong limit singular cardinal of cofinality κ . Let $\langle \lambda_i : i < \kappa \rangle$ be increasing with limit μ , $\lambda_0 > \kappa$, $\lambda_\kappa = \mu$, $\langle {}_*(M_i^+ : i < \kappa) \rangle$ be an increasing sequence of elementary submodels of \mathfrak{C}_{K_*} (check notation), $\|{}_*(M_i^+ \| = 2^{\lambda_i}$, ${}_*(M_i^+$ is λ_i^+ -homo universal (in $K_{\text{aut}}^{\text{ec}}(T)$), $M^+ = \cup \{ {}_*(M_i^+ : i < \kappa \}$. Let $\langle p_i^+ : i < \mu^+ \rangle$ be a sequence of existential types in $\mathbb{L}(\tau \cup \{\sigma\})$ each of cardinality $\leq \kappa$ with domain $\subseteq M$, and we shall prove that for some $\alpha < \beta < \mu^+$, $p_\alpha^+ \cup p_\beta^+$ is realized in \mathfrak{C}_{K_*} , this suffices.

For each $\alpha < \mu^+$, we can find $a_\alpha \in \mathfrak{C}_{K_*}$ realizing p_i and $N_{3,\alpha}^* \prec \mathfrak{C}_{K_*}$ of cardinality κ to which a_i belongs and $N_{2,\alpha}^+ = N_{3,\alpha}^+ \cap M^+ \prec M^+$ and $\text{tp}_{\mathfrak{C}}(|N_{3,\alpha}^+|, |M^+|)$ does not fork over $|N_{2,\alpha}^+|$. Let $N_{1,\alpha}^+ \prec N_{3,\alpha}^+$ be of cardinality $|T|$ such that $a_i \in N_{1,\alpha}^+$, $\text{tp}_{\mathfrak{C}}(|N_{1,\alpha}^+|, |M^+|)$ does not fork over $|N_{0,i}^+|$ where $N_{0,i}^+ = N_{1,i}^+ \upharpoonright M^+ \prec M^+$. Without loss of generality $\alpha < \mu^+ \Rightarrow N_{0,\alpha}^+ = N_0^+$ and for every $\alpha, \beta < \mu^+$ there is an isomorphism $h_{\beta,\alpha}$ from $N_{3,\alpha}^+$ onto $N_{3,\beta}^+$ mapping $a_\alpha, N_{1,\alpha}^+, N_{2,\alpha}^+$ to $a_\beta, N_{1,\beta}^+, N_{2,\beta}^+$ respectively and $h_{\beta,\alpha} \upharpoonright N_0^+ = \text{id}_{N_0^+}$. Moreover, without loss of generality for some well ordering $<^*$ all $h_{\beta,\alpha}$ are order preserving.

Let $\kappa > \bar{\kappa}$, \mathfrak{B} be an elementary submodel of $(\mathcal{H}(\chi), \in)$ of cardinality 2^κ such that $T, \kappa, \mu, \mathfrak{C}, \mathfrak{C}_{K_*}, M^+, \langle N_i^+ : i < \mu^+ \rangle$ belongs and such that $[\mathfrak{B}] \leq \kappa \subseteq \mathfrak{B}$. Now choose $\alpha(2) \in \mu^+ \setminus \mathfrak{B}$, and let $M_0^+ = N_{1,\alpha}^+ \upharpoonright \mathfrak{B}$. Clearly $M_0^+ \prec M^+$ and there is $\alpha(1) \in \mu^+ \cap \mathfrak{B}$ such that $h_{\alpha(1),\alpha(2)}$ is the identity on M_0^+ .

[FILL?]

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