THE CARDINALS OF SIMPLE MODELS FOR UNIVERSAL THEORIES

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Some results about spectra of cardinals of simple algebras in varieties and, more generally, about the cardinals of T-simple models where T is a universal theory are obtained and applications discussed. It is shown that if the language of T has power κ and if there exists a T-simple model whose power exceeds 2^κ, then T-simple models exist in all powers $\lambda \geqq \kappa$. It is further shown that if the language of T is countable, and if there exists an uncountable T-simple model, then there exists a T-simple model with the power 2^ω.

Introduction. The concept of a simple algebra is familiar and important in many branches of algebra, especially general algebra. Nonetheless, published work touching on the general problem to be discussed here is hard to find.[1] The problem (suggested by Walter Taylor) is: to characterize the class of cardinals in which a variety (or equational class) contains simple algebras.

Let κ be an infinite cardinal and V be a variety of algebras with at most κ operations. We denote by $\mathbf{SC}^\kappa(V)$ (or $\mathbf{SC}_\kappa(V)$) the class of cardinals $\beta \geqq \kappa$ (or $\beta \leqq \kappa$) such that V has a simple algebra with β elements, and we put $\mathbf{SC}(V) = \mathbf{SC}^\kappa(V) \cup \mathbf{SC}_\kappa(V)$. Our problem is to characterize the family of classes $\mathbf{SC}(V)$ that arise from varieties with at most κ operations.

We obtain results which give a complete solution of this problem, assuming the Generalized Continuum Hypothesis (GCH). Namely, $\mathbf{SC}(V)$ does not include 0 or 1, and $\mathbf{SC}^\kappa(V) = \varnothing$, $\{\kappa\}$, $[\kappa, 2^\kappa]$, or $\{\beta : \beta \geqq \kappa\}$; there are no other restrictions. Without the GCH, the above characterization still holds for $\kappa = \omega$ (the least infinite cardinal), and for every κ it is true that $\mathbf{SC}_\kappa(V)$ can be any set of cardinals $\leqq \kappa$ that

[1] The paper by Magari, [4], deserves mention. He proves that each nontrivial variety possesses a simple algebra (which has at least two elements).

excludes 0 and 1. However, for uncountable κ we can only show that when $\mathbf{SC}_\kappa(V)$ does not include all $\beta \geqq \kappa$, then it must be an interval $[\kappa, \lambda)$ where $\lambda \leqq (2^\kappa)^+$.

Our methods are model theoretic and combinatorial. In order to afford full scope to the methods, we shall deal throughout with a somewhat generalized notion of simplicity. In fact, the above results will be proved for the so-called T-simple models where T is a set of universal sentences defined in a first order language. The models may possess relations as well as operations among their fundamental terms.

By using this generalized notion of simplicity, we obtain with one sweep the main results for simple algebras; also the same results for subdirectly irreducible algebras;[2] and, finally, a new condition on the size of the "minimum compact" models defined by Taylor in **[9]**.

0. Preliminaries. By a *model* we shall here mean a system

$$\mathfrak{A} = \langle A, F_s, R_t \rangle_{s \in S; t \in T}$$

consisting of a nonempty set A (the universe of the model, usually written $|\mathfrak{A}|$), and for each $s \in S$ an operation on A of finite rank $\mu^0(s)$ and for each $t \in T$ a finitary relation over A of rank $\mu^1(t)$. The pair of functions $\mu = \langle \mu^0, \mu^1 \rangle$ is called the *type* of the model; two models are *similar* if they have the same type. If $T = \varnothing$ the model is an *algebra*. By the *language of* $\mathfrak{A}$ we mean the formal first order language appropriate for models of type μ. The *theory of* $\mathfrak{A}$, or $Th(\mathfrak{A})$, is the set of all sentences of this language that are true in $\mathfrak{A}$. A *theory* is a set T of sentences such that $T = Cn(T)$, the set of all logical consequences of T (in the same language). A theory is called *universal* (or existential, or whatever) if it is equivalent to a set of universal (or existential, or whatever) sentences.

Our considerations are framed within the Bernays-Gödel version of set theory. We use κ, λ as symbols for infinite cardinals, $i, j, \alpha, \beta, \gamma, \delta$ for ordinals, k, l, m, n for natural numbers, $|X|$ for the cardinal (or power) of the set X; a, b, c, d for elements of models, x, y, z for variables, and $\bar{x}, \bar{y}, \bar{z}$ for finite sequences of variables. We denote by ω the least infinite cardinal (aleph zero). By the cardinal of a model we mean that of its universe, written $\|\mathfrak{A}\|$; by the cardinal of a language, written $|L|$, we mean the number of formulas of L. (This is the same as the cardinality of each theory formulated in the language.) The cardinal successor to a cardinal κ is denoted by κ^+.

Concerning first order languages and model theory in general, consult **[1]**. Concerning algebras, varieties, and equational theories, see **[7]**. We must assume that the reader knows some basic concepts in these fields.

1. Relative simple models. In this section, we define the generalized concept of simplicity serving to unify the results, and we present the easier facts. We conclude with some fundamental examples. The two sections that follow contain proofs

[2] We note that Taylor **[10]** first determined the Hanf number for subdirectly irreducible algebras in a variety.

for the deepest results of this paper, which concern relative simple models for universal theories. In §4, we review what has been accomplished and give applications. In §5, we study the cardinalities of relative simple models for arbitrary elementary theories.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be similar models. By writing $f:\mathfrak{A} \to \mathfrak{B}$, we denote that f is a *homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$; that is, f is a mapping of $|\mathfrak{A}|$ into $|\mathfrak{B}|$ and it carries each of the basic operations and relations of $\mathfrak{A}$ into the corresponding operation or relation of $\mathfrak{B}$. An *embedding* is a homomorphism that constitutes an isomorphism of $\mathfrak{A}$ with a submodel of $\mathfrak{B}$. ("Submodel" has the usual meaning.) Thus, for a homomorphism we require that

$$\mathfrak{A} \models R[a_1, \ldots, a_n] \Rightarrow \mathfrak{B} \models R[f(a_1), \ldots, f(a_n)]$$

for each basic relation symbol R; for an embedding the above implication becomes an equivalence.

Let K be a class of similar models. We consider a model $\mathfrak{A}$ to be *K-simple* if $\mathfrak{A} \in K$, $\|\mathfrak{A}\| > 1$, and if every homomorphism $f:\mathfrak{A} \to \mathfrak{B}$, with $\mathfrak{B} \in K$, is either a constant map or an embedding. We are mainly interested in this notion for the case that K is an elementary class, say the class of all models satisfying the set of sentences T. In this case, we use the phrase *T-simple*. We define the classes of cardinals $\mathbf{SC}^{\kappa}(T)$ and $\mathbf{SC}_{\kappa}(T)$ just as in the introduction, assuming always that κ is a cardinal not exceeded by the cardinal of the language of T.

REMARK 1.1. Let T be an equational theory. Then T-simple models are just the simple algebras that satisfy T.

A more intrinsic characterization of T-simple models is the content of the lemma below.

DEFINITION 1.2. Let $\mathfrak{A}$ be a model, with associated first order language L.

(1) By $\pi(\mathfrak{A})$ we denote the set of formulas of L that are conjunctions of atomic formulae.

(2) By $L(\mathfrak{A})$ is meant the language derived from L by adding new constant symbols to denote the members of $\mathfrak{A}$. (We shall customarily not distinguish in writing between an element $a \in |\mathfrak{A}|$ and the constant symbol for it.)

(3) By $PD(\mathfrak{A})$, or the positive diagram of $\mathfrak{A}$, we denote the set consisting of all atomic sentences of $L(\mathfrak{A})$ that are true in the model $(\mathfrak{A}, a)_{a \in |\mathfrak{A}|}$.

LEMMA 1.3. *Let T be a theory of L and $\|\mathfrak{A}\| > 1$. An equivalent condition for $\mathfrak{A}$ to be T-simple is: $\mathfrak{A} \models T$ and, for all $a \neq b$ in $|\mathfrak{A}|$, $PD(\mathfrak{A})$ is a maximal set of atomic sentences of $L(\mathfrak{A})$ consistent with $T \cup \{\neg a \approx b\}$.*

PROOF. We note that atomic sentences are positive. The proof follows trivially from the completeness theorem for first order logic.

CONCLUSION 1.4. *$\mathfrak{A}$ is T-simple iff $\mathfrak{A} \models T$, $\|\mathfrak{A}\| > 1$, and for every atomic formula $\sigma(\bar{x})$ of L, and $a, b \in |\mathfrak{A}|$, and sequence $\bar{c}$ from $|\mathfrak{A}|$ such that $\mathfrak{A} \models \neg\sigma(\bar{c})$, there exists a formula $\varphi(y_1, y_2, \bar{x}, \bar{z}) \in \pi(\mathfrak{A})$ such that*

$$\mathfrak{A} \models (\exists \bar{z})\varphi[a, b, \bar{c}, \bar{z}] \quad \textit{and} \quad T \vdash \varphi(y_1, y_2, \bar{x}, \bar{z}) \wedge \sigma(\bar{x}) \to y_1 \approx y_2.$$

CONCLUSION 1.5. *If $\mathfrak{A}$ is T-simple and $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, then $\mathfrak{B}$ is T-simple.*

CONCLUSION 1.6. *If the class of models of T is closed under unions of chains of models, then the class of T-simple models is likewise closed under chain unions.*

From the second conclusion, using the downward Lowenheim-Skolem theorem, we infer

THEOREM 1.7. *$\mathrm{SC}^{|L|}(T)$ is a convex set of cardinals. Thus either it consists of all cardinals $\beta \geqq |L|$ or else it takes the form of an interval $[|L|, \lambda)$.*

The next two theorems give easy constructive results.

THEOREM 1.8. *For every infinite cardinal κ there is a simple algebra $\mathfrak{A}_\kappa$ with κ operations, $\|\mathfrak{A}_\kappa\| = 2^\kappa$, such that no simple algebra that satisfies all equations $\mathfrak{A}_\kappa$ satisfies has a cardinality exceeding 2^κ.*

PROOF. We define $\mathfrak{A}_\kappa$ as the algebra

$$\langle S(\kappa), +, \cdot, -, C_\delta, F_\delta \rangle_{\delta < \kappa}$$

where $\langle S(\kappa), +, \cdot, - \rangle$ is the Boolean algebra of all subsets of κ; where C_δ is the constant, or nullary operation, with value $\{\delta\}$; and where F_δ is the endomorphism of the Boolean part satisfying $F_\delta(C_\delta) = \kappa$ (the Boolean unit element). We will not prove that this algebra is simple, since it is very easy to do so.

Suppose that $\mathfrak{B}$ is a simple algebra satisfying all equations $\mathfrak{A}_\kappa$ does. That means, *inter alia*, the equations defining Boolean algebras, equations expressing that each function $F_\delta^{\mathfrak{B}}$ is an endomorphism, as well as (for each $\delta < \kappa$):

$$C_\delta \cdot x \leqq F_\delta(C_\delta \cdot x),$$

$$F_\delta(C_\delta) \approx x + -x,$$

$$F_\delta(C_\gamma) \approx x \cdot -x \quad (\text{for } \delta \neq \gamma, \gamma < \kappa),$$

$$F_\delta F_\gamma(x) \approx F_\gamma(x) \qquad (\text{for all } \gamma < \kappa).$$

(To be accurate, the above should be preceded by universal quantifiers; following custom and convenience, we suppress them.)

Those equations, due to the simplicity of $\mathfrak{B}$, imply that the value of each constant C_δ is an atom of $\mathfrak{B}$. In fact, if $0 < a < C_\delta$ for some a, then the principal ideal determined by $F_\delta(a)$ is proper (since $a \leqq F_\delta(a)$, $C_\delta \cdot -a \leqq F_\delta(-a)$, and $0 = F_\delta(a) \cdot F_\delta(-a)$), and is closed under application of all the operators, so it gives a proper homomorphism from $\mathfrak{B}$. Further, they imply that the map $a \to \{\delta : C_\delta^{\mathfrak{A}} \leqq a\}$ is a homomorphism from $\mathfrak{B}$ into $\mathfrak{A}_\kappa$. Because $\mathfrak{B}$ is simple, this map embeds it into $\mathfrak{A}_\kappa$. Thus we conclude the proof that the cardinal of $\mathfrak{B}$ does not exceed 2^κ.

THEOREM 1.9. *For every infinite cardinal κ there is a simple algebra $\mathfrak{A}_\kappa$ with κ operations, $\|\mathfrak{A}_\kappa\| = \kappa$, such that every simple algebra satisfying the equations valid in $\mathfrak{A}_\kappa$ is isomorphic with $\mathfrak{A}_\kappa$.*

PROOF. Let $\mathfrak{A}_\kappa = \langle V, +, \rho_a\ (a \in |\mathfrak{F}|)\rangle$ be a one-dimensional vector space over a κ-element field $\mathfrak{F}$.

THEOREM 1.10. *For every infinite cardinal κ there is a variety V_κ with κ operations such that* $\mathrm{SC}(V_\kappa) = \{\beta : \beta \geqq \kappa\}$.

PROOF. Let $\mathfrak{A}_\kappa = \langle F, +, \cdot, c\ (c \in F)\rangle$ where $\langle F, +, \cdot\rangle$ is a κ-element field. Let V_κ be the variety of all algebras of the same type as $\mathfrak{A}_\kappa$ that satisfy all the equations valid in $\mathfrak{A}_\kappa$.

REMARK 1.11. It would be interesting to have examples like those above, for $\kappa = \omega$, manifested by some familiar kind of algebras with finitely many operations.

2. T-simple models in powers exceeding $2^{|T|}$. Our aim is

THEOREM 2.1. *Let T be a universal theory in a first order language of power κ. If there is a T-simple model whose power exceeds 2^κ then there exists, for each $\lambda \geqq \kappa$, a T-simple model of power λ.*

Throughout this section, T is a universal theory of power κ, and $\mathfrak{A}$ denotes a fixed T-simple model of power exceeding 2^κ. By Theorem 1.7, all we need is the existence of T-simple extensions of $\mathfrak{A}$ having arbitrarily large power.

We shall hold to the following conventions: a and b are distinct (fixed) elements of $\mathfrak{A}$; $L(a, b)$ is the language of the model $(\mathfrak{A}, a, b)$, a fragment of $L(\mathfrak{A})$ (Definition 1.2(2)); $\Pi = \pi[(\mathfrak{A}, a, b)]$ (Definition 1.2(1)).

DEFINITION 2.2. (1) $x, y, x_0, \ldots, x_m, \ldots$ $(m < \omega)$ are distinct variables of $L(a, b)$. $\mathscr{S}$ is the set of all sequences $\Gamma = \langle \Gamma_n : n < \omega\rangle \in {}^\omega\Pi$ such that (for all n)

(A) $T \vdash \Gamma_0 \rightarrow \bigwedge_{i<j<4;k<l<4} \{\sigma_i \approx \sigma_j \leftarrow \sigma_k \approx \sigma_l\}$, where $\sigma_0, \ldots, \sigma_3$ are the respective terms a, b, x and y.

(B_n) All variables appearing in Γ_n are among $x, y, x_0, \ldots, x_m, \ldots$.

(C_n) $T \vdash \Gamma_{n+1} \wedge a \approx b \rightarrow a \approx x_n$.

(2) y_k and $y_{k,l,m}$ $(m < \omega$ and $k < l < \omega)$ is another system of distinct variables of $L(a, b)$. If Γ belongs to $\mathscr{S}$, then Γ^* is the set constituted by the formulas listed below:

$$\Gamma_n[y_k, y_l; y_{k,l,m}\ (m < \omega)], \qquad \neg y_k \approx y_l$$

for $k < l < \omega$ and $n < \omega$. (The first formula is derived from Γ_n by substituting y_k for x, y_l for y, and $y_{k,l,m}$ for x_m.)

LEMMA 2.3. *Suppose that there exists a sequence $\Gamma \in \mathscr{S}$ such that the set Γ^* is consistent with the first order theory of the model $(\mathfrak{A}, a, b)$. For every cardinal λ, there exists a T-simple model $\mathfrak{B} \supseteq \mathfrak{A}$ that satisfies $\|\mathfrak{B}\| \geqq \lambda$.*

PROOF. Let Γ be such a sequence and λ be an infinite cardinal. Expand the language $L(\mathfrak{A})$ by adding a system of new constants, d_α and $d_{\alpha,\beta,m}$ $(\alpha < \beta < \lambda$ and $m < \omega)$; call the new language L_1. Let Γ' be the set consisting of the following sentences of L_1 (for $\alpha < \beta < \lambda$ and $n < \omega$):

(1) $\Gamma_n[d_\alpha, d_\beta; d_{\alpha,\beta,m}\ (m < \omega)]$,

(2) $\neg d_\alpha \approx d_\beta$.

Now an easy application of the compactness theorem shows that the positive diagram of $\mathfrak{A}$ is consistent with the set $T \cup \Gamma' \cup \{\neg a \approx b\}$. Let Ψ be a set of atomic sentences of L_1 maximal with respect to the conditions: $PD(\mathfrak{A}) \subseteq \Psi$; $\Psi \cup T \cup \Gamma' \cup \{\neg a \approx b\}$ is consistent (call this set Σ).

Since Σ is equivalent to a set of universal sentences, there exists a model of Σ whose every element is named by a term of the language L_1. Let $\mathfrak{B}'$ denote such a model and $\mathfrak{B}$ denote the reduct of $\mathfrak{B}'$ to the language of the theory T.

First, we note that $\mathfrak{A}$ is isomorphic to the submodel of $\mathfrak{B}$ formed by elements that have a name in $L(\mathfrak{A})$. This is so because $PD(\mathfrak{A})$ was already maximal (see Lemma 1.3). So we put $\mathfrak{A} \subseteq \mathfrak{B}$ by the natural identification.

Next we note that since $\mathfrak{B}'$ satisfies Γ', the elements $d_\alpha^{\mathfrak{B}'}$ are all distinct; so $\|\mathfrak{B}\| \geqq \lambda$. Also, $\mathfrak{B} \models T$, since $\mathfrak{B}'$ does.

To conclude, we show that $\mathfrak{B}$ is T-simple. Assume that $h:\mathfrak{B} \to \mathfrak{D}$ where $\mathfrak{D} \models T$, and that h is not an embedding. Then let $\bar{\Psi}$ be the set of all atomic sentences of L_1 that are true in the model

$$\mathfrak{D}' = (\mathfrak{D}, h(c), h(d_\alpha^{\mathfrak{B}}), h(d_{\alpha,\beta,m}^{\mathfrak{B}}))_{c \in |\mathfrak{A}|; \alpha < \beta < \lambda; m < \omega}.$$

Because h is not an embedding, and $\mathfrak{B}$ is generated by the elements named in L_1, $\bar{\Psi}$ must be strictly larger than Ψ. Therefore, by the maximality of Ψ, the above model for L_1 does not satisfy $\bar{\Psi} \cup \Sigma$. We note that this model does satisfy $\Psi \cup T$, and also the positive sentences of Γ' listed as (1). Hence we can conclude that either $h(d_\alpha) = h(d_\beta)$ for some $\alpha < \beta$ or that $h(a) = h(b)$.

From this it follows by Definition 2.2(A) and (C_n), and since $\mathfrak{D}'$ satisfies T and the instances (1) of formulae Γ_n, that $h(a) = h(b)$ and in fact

$$h(a) = h(d_\alpha) = h(d_{\alpha,\beta,m})$$

for all indices α, β, and m. Then, since $\mathfrak{A}$ is T-simple,

$$h(|\mathfrak{A}|) = \{h(a)\}.$$

Thus the set $\{h(a)\}$ is closed for all operations; and it follows that

$$h(|\mathfrak{B}|) = \{h(a)\}$$

since $\mathfrak{B}$ is generated by the elements which map into this singleton set. This concludes the proof that h is a constant map and also the proof of the lemma.

Before proceeding to prove that the hypothesis of the above lemma is satisfied, we record a combinatorial fact which has a crucial part in the argument.

DEFINITION 2.4. A sequence $\langle c_\alpha : \alpha < \gamma\rangle$ of elements of a model $\mathfrak{B}$ will be called an *almost indiscernible sequence* for $\mathfrak{B}$ provided the terms of the sequence are distinct, and whenever $\alpha < \beta < \gamma$, the elements c_α and c_β satisfy the same first order formulas (with one free variable) in the language of the model $(\mathfrak{B}, c_\delta)_{\delta < \alpha}$.

THEOREM 2.5.[3] *Suppose that $\mathfrak{B}$ is a model whose power exceeds 2^κ and whose language has power not exceeding κ. Then $\mathfrak{B}$ has an almost indiscernible sequence of length κ^+.*

PROOF. We first well order $|\mathfrak{B}|$ by a relation $<$. For each $b \in |\mathfrak{B}|$, we define a one-to-one sequence $\mathbf{u}(b)$ as follows. The length of $\mathbf{u}(b)$ will be κ^+ or some ordinal $\gamma < \kappa^+$, depending on b.

Suppose that $u_\alpha(b)$ has been defined for all $\alpha < \beta$, and $0 \leqq \beta < \kappa^+$. If $b = u_\alpha(b)$ for some $\alpha < \beta$, then $u_\beta(b)$ is not to be defined—we put $\mathbf{u}(b) = \langle u_\gamma(b) : \gamma \leqq \alpha \rangle$ in this case. If $b \neq u_\alpha(b)$ for all $\alpha < \beta$, then let $u_\beta(b)$ be the $<$-least element of $|\mathfrak{B}|$ that has the same elementary type in $(\mathfrak{B}, u_\alpha(b))_{\alpha<\beta}$ as does b. If the construction persists through all $\beta < \kappa^+$, then we put $\mathbf{u}(b) = \langle u_\beta(b) : \beta < \kappa^+ \rangle$.

Clearly, $\mathbf{u}(b)$ is almost indiscernible for each $b \in |\mathfrak{B}|$. We shall prove the existence of b such that $\mathbf{u}(b)$ has length κ^+.

Suppose that no such element exists. Then, for each $\beta < \kappa^+$, let $L(\beta)$ be the language of $\mathfrak{B}$ with distinct constants c_α, $\alpha < \beta$, adjoined. Given $b \in |\mathfrak{B}|$, we define a new sequence $\xi(b)$. The length of $\mathbf{u}(b)$ is a successor ordinal, $\gamma + 1$, where $u_\gamma(b) = b$. We put

$$\xi(b) = \langle \xi_\beta(b) : \beta \leqq \gamma \rangle,$$

in which $\xi_\beta(b)$ is the type of b in the model $(\mathfrak{B}, u_\alpha(b))_{\alpha<\beta}$. (This type is constructed as a set of formulas of $L(\beta)$.)

Now for each β there are, independently of b, at most 2^κ possible values for $\xi_\beta(b)$. The number of sequences $\xi(b)$ is thus at most

$$\sum_{\gamma<\kappa^+} (2^\kappa)^\gamma = (2^\kappa) \cdot \kappa^+ = 2^\kappa.$$

But it is easy to see that $\xi(b_1) = \xi(b_2)$ implies $b_1 = b_2$. This contradicts the assumption $\|\mathfrak{B}\| > 2^\kappa$.

LEMMA 2.6. *There exists $\Gamma \in \mathcal{S}$ such that Γ^* is consistent with the theory of $(\mathfrak{A}, a, b)$.*

PROOF. We let $\langle c_\alpha : \alpha < \kappa^+ \rangle$ be an almost indiscernible sequence for the model $(\mathfrak{A}, a, b)$ (existence is assured by Theorem 2.5). We assume, without loss of generality, that all terms of the sequence are different from a and b.

We shall construct by induction on nonnegative integers n the following: formulas Γ_n; integers $p(0) < p(1) < \cdots < p(n) < \cdots$; and for each n an increasing sequence of ordinals $\langle \alpha(n) : \alpha < \kappa^+ \rangle$. The construction will ensure that, for each n, Γ_n satisfies the applicable parts of Definition 2.2, and the variables appearing in it are among $x, y, x_0, \ldots, x_{p(n)}$; further

(S_n) each pair $\langle c_{\alpha(n)}, c_{\alpha(n)+1} \rangle$ where $\alpha < \kappa^+$ satisfies in $(\mathfrak{A}, a, b)$ the formula $\exists x_0 \cdots x_{p(n)} (\Gamma_0 \wedge \cdots \wedge \Gamma_n)(x, y)$.

[3] With our first (somewhat weaker) version of Theorem 2.1, we used the partition relation $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ (from Erdös [3]) for the proof. Theorem 2.5 is suggested by the work of Erdös and his collaborators, but we have not found it in the literature.

We note that since $\langle c_{\alpha(n)}, c_{\alpha(n)+1}\rangle$ and $\langle c_{\alpha(n)}, c_{\beta(n)}\rangle$ (for $\alpha < \beta$) have the same 2-type, condition (S_n) implies that every pair of increasing terms from the sequence $\langle c_{\alpha(n)} : \alpha < \kappa^+\rangle$ satisfies that formula. Thus, clearly, every finite subset of Γ^*, where $\Gamma = \langle \Gamma_0, \Gamma_1, \dots \rangle$, will be simultaneously satisfiable in $(\mathfrak{A}, a, b)$. This is exactly what the lemma requires.

To begin with $n = 0$, we first associate to each $\alpha < \kappa^+$ a formula φ^α in Π, say $\varphi^\alpha(x, y; x_0, \dots, x_p)$, which satisfies condition (A) for Γ_0 in Definition 2.2(1), and so that

$$(\mathfrak{A}, a, b) \models \exists x_0 \cdots x_p \varphi^\alpha[c_\alpha, c_{\alpha+1}; x_0, \dots, x_p].$$

This is done by several applications of 1.4, taking for σ there the formula $x \approx y$. There are only κ formulas altogether, consequently there exists an increasing sequence $\langle \alpha(0) : \alpha < \kappa^+\rangle$ with all formulas $\varphi^{\alpha(0)}$ identical. Let

$$\Gamma_0 = \Gamma_0(x, y; x_0, \dots, x_{p(0)})$$

be this formula. Now (S_0) is satisfied.

Suppose that $m \geqq 0$ and that Γ_n, $p(n)$, and $\langle \alpha(n) : \alpha < \kappa^+\rangle$ have been obtained for all $n \leqq m$, satisfying statement (S_n). Let $\Gamma_{(m)}$ be the conjunction of $\Gamma_0, \dots, \Gamma_m$. By (S_m), we can assign to each $\alpha < \kappa^+$ a member $d_{\alpha,m} \in |\mathfrak{A}|$ so that the formula

$$\Gamma_{(m)}[c_{\alpha(m)}, c_{\alpha(m)+1}; x_0, \dots, d_{\alpha,m}, \dots, x_{p(m)}]$$

is satisfiable in $(\mathfrak{A}, a, b)$. Using 1.4 again, we correlate with each α a formula φ^α in Π, say $\varphi^\alpha(z; z_0, \dots, z_{q-1})$, $q \geqq 1$, so that

$$T \vdash \varphi^\alpha \wedge a \approx b \rightarrow a \approx z;$$

and

$$(\mathfrak{A}, a, b) \models \exists z_0 \cdots z_{q-1} \varphi^\alpha[d_{\alpha,m}; z_0, \dots, z_q].$$

By a cardinality argument, we select a single formula $\varphi = \varphi(z; z_0, \dots, z_{q-1})$ and an increasing sequence of ordinals $\langle \beta_\delta : \delta < \kappa^+\rangle$ so that $\varphi^{\beta_\delta} = \varphi$ for all δ.

Now we put $p(m + 1) = p(m) + q$,

$$\Gamma_{m+1} \equiv \varphi(x_m; x_{p(m)+1}, \dots, x_{p(m)+q}),$$

and

$$\alpha(m + 1) = \beta_\alpha(m) \quad \text{for } \alpha < \kappa^+.$$

Statement (S_{m+1}) and clause (C_m) of Definition 2.2(1) are clearly satisfied. This completes the argument.

Lemmas 2.3 and 2.6 combined give the proof of Theorem 2.1.

REMARK. The full force of Theorem 2.5 is not needed to prove the last lemma. We only used the fact that whenever $\alpha < \beta < \delta$, the pairs $\langle c_\alpha, c_\beta\rangle$ and $\langle c_\alpha, c_\delta\rangle$ have the same type in $(\mathfrak{A}, a, b)$. This remark can be applied also to strengthen Theorem 3.2 in the next section.

3. Uncountable T-simple models, $|T| = \omega$.

We shall prove

THEOREM 3.1. *Let T be a universal theory in a denumerable first order language. If $\mathfrak{A}$ is an uncountable T-simple model, then there exists a T-simple model $\mathfrak{B} \supseteq \mathfrak{A}$ such that $\|\mathfrak{B}\| \geqq 2^\omega$.*

Throughout this section, T denotes a universal theory. We lose no generality by dealing only with models that have at least two elements that are individual constants. The first three results hold for theories of any cardinality.

We first note two results which follow respectively from the arguments of the preceding section and from Conclusion 1.6.

THEOREM 3.2. *Let $\mathfrak{A}$ be a T-simple model with two individual constants. If $\mathfrak{A}$ contains an almost indiscernible sequence of length $|T|^+$, then $\mathfrak{A}$ has T-simple extensions with unbounded cardinalities.*

PROPOSITION 3.3. *The class of T-simple models is closed under union of ascending sequences. Hence either there is no bound to the size of such models or else every T-simple model can be extended to a maximal T-simple model.*

Thus, in proving Theorem 3.1, we can limit the discussion to maximal T-simple models. We remark it follows by Theorems 2.5 and 3.2 that every such model has power at most $2^{|T|}$.

DEFINITION 3.4. (A) *A pe (positive existential) formula* is a formula of the form $\exists \bar{x} \bigvee_{i<n} \bigwedge_{j<n_i} \theta_{ij}$ where θ_{ij} are atomic formulas. Note that a conjunction of pe formulas is equivalent to a pe formula.

(B) Let $\bar{a} = \langle a_i : i < i_0 \rangle$ be a finite sequence of elements belonging to an arbitrary model $\mathfrak{A}$. The *pe type $\bar{a}$* realizes in $\mathfrak{A}$, or $pe(\bar{a})$, is the set of pe formulas $\varphi(x_0, \ldots, x_{i_0-1})$ in the language of $\mathfrak{A}$ such that $\mathfrak{A} \models \varphi[a_0, \ldots, a_{i_0-1}]$. If $A \subseteq |\mathfrak{A}|$, the *pe type $\bar{a}$ realizes over A* in $\mathfrak{A}$, or $pe(\bar{a}, A)$ is the pe type that it realizes in $(\mathfrak{A}, c)_{c\in A}$. We write a_0 in place of $\langle a_i : i < 1 \rangle$.

(C) A pe n-type for $\mathfrak{A}$ is a consistent (with $Th(\mathfrak{A})$) set of pe formulas with $x_0, \ldots, x_{n-1}$ as the only free variables. The pe n-types over a subset are defined in the natural way. A pe n-type $\mathfrak{p}$ is realized in $\mathfrak{A}$ *iff* there exists an n tuple $\bar{a}$ from $|\mathfrak{A}|$ such that $\mathfrak{p} \subseteq pe(\bar{a})$.

LEMMA 3.5. *Let $\mathfrak{A}$ be a maximal T-simple model with two individual constants. Let $\langle a_\beta : \beta < \delta \rangle$ and $\langle b_\beta : \beta < \delta \rangle$ be systems of members of $\mathfrak{A}$, where δ is an ordinal. If every pe formula satisfied by some terms of the first system is satisfied by the corresponding terms of the second, then there exists an automorphism f of $\mathfrak{A}$ such that $f(a_\beta) = b_\beta$ for all $\beta < \delta$.*

PROOF. (This lemma has a precursor in [9, Theorem 1.13(i)].) The hypothesis implies that there is an elementary extension $\mathfrak{A}_1$ of $\mathfrak{A}$ and a homomorphism f from $\mathfrak{A}$ into $\mathfrak{A}_1$ such that $f(a_\beta) = b_\beta$ for all $\beta < \delta$. Let $\mathfrak{A}_2$ be the submodel of $\mathfrak{A}_1$ generated by $|\mathfrak{A}| \cup \text{Range} f$. Choose a and b as distinct individual constants of $\mathfrak{A}$. Let Ψ be a set of atomic sentences of $L(\mathfrak{A}_2)$, $PD(\mathfrak{A}_2) \subseteq \Psi$, maximal consistent with $T \cup \{\neg a \approx b\}$.

Ψ determines a model of T, call it $\mathfrak{B}$, together with a homomorphism from $\mathfrak{A}_2$ *onto* $\mathfrak{B}$, call this g. Since $g(a) \neq g(b)$ and $\mathfrak{A}$ is T-simple, $\mathfrak{A}$ is identified with a submodel of $\mathfrak{B}$ via g. We may thus assume that $\mathfrak{A} \subseteq \mathfrak{B}$ and that g acts on $\mathfrak{A}$ as the

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identity map. Further, every homomorphism $h:\mathfrak{B} \to \mathfrak{C}$, where $\mathfrak{C} \models T$, is either an embedding or has $h(a) = h(b)$.

We claim that $\mathfrak{B}$ is T-simple. To prove this, suppose that $h:\mathfrak{B} \to \mathfrak{C}$, that $\mathfrak{C} \models T$, and that h is not an embedding. Then $h(a) = h(b)$, and consequently $h(|\mathfrak{A}|) = \{h(a)\}$ since $\mathfrak{A}$ is T-simple. Likewise, $hgf(a) = hgf(b)$ ($gf(a) = a$ since a is an individual constant; same for b); and so $hgf(|\mathfrak{A}|) = \{hgf(a)\} = \{h(a)\}$. Since $|\mathfrak{A}|$ is closed under the operations, so is $\{h(a)\}$ in $\mathfrak{C}$. Since $\mathfrak{B}$ is generated by $|\mathfrak{A}| \cup gf(|\mathfrak{A}|)$, all of $\mathfrak{B}$ maps into $\{h(a)\}$. Thus h is a constant map.

It follows from the claim (and the maximality of $\mathfrak{A}$) that $\mathfrak{A} = \mathfrak{B}$. Hence gf is an endomorphism of $\mathfrak{A}$ (mapping a_β to b_β for each β). Because of the existence of two individual constants, and the T-simplicity of $\mathfrak{A}$, gf is an embedding. Since $\mathfrak{A}$ has no proper T-simple extensions, it is not isomorphic to a proper submodel of itself, so the map is an automorphism. The proof is complete.

We deduce from the above lemma that the pe type realized in a maximal model by any finite sequence determines the (elementary) type of the same sequence, i.e., the set formed of all formulas the sequence satisfies.

In the remainder of this section, T denotes an arbitrary denumerable universal theory.

LEMMA 3.6. *Let $\mathfrak{A}$ be a maximal T-simple model with two individual constants a and b. Let $\mathfrak{p}$ be any pe 1-type for $\mathfrak{A}$. For $\mathfrak{p}$ to be realized in $\mathfrak{A}$, it is necessary and sufficient that there exists a set $\Phi \subseteq \pi(\mathfrak{A})$ [see Definition 1.2] such that Φ is closed under conjunction and contains only the variables x_n, $n < \omega$, and*

(A) *$Th(\mathfrak{A}) \cup \Phi$ is consistent and Φ implies $\mathfrak{p}$;*

(B) *for every n, there is $\varphi \in \Phi$ such that*

$$T \vdash a \approx b \wedge \varphi \to x_n \approx a.$$

PROOF. If $\mathfrak{p}$ is realized in $\mathfrak{A}$, then such a Φ can easily be constructed by a repeated use of Conclusion 1.4. Suppose now that such Φ exists. Then by (A) there is an elementary extension $\mathfrak{A}_1$ of $\mathfrak{A}$ in which, say, $\langle a_n : n < \omega\rangle$ realizes Φ. Let $\mathfrak{A}_2$ be the submodel of $\mathfrak{A}_1$ generated by $|\mathfrak{A}| \cup \{a_n : n < \omega\}$. The remainder of the argument follows almost verbatim the proof of Lemma 3.5.

LEMMA 3.7. *If $\mathfrak{p} = pe(c)$ and if $\mathfrak{p}$ and Φ are related as in the preceding lemma, then $\mathfrak{p}$ is identical with the set of positive existential consequences of Φ that have only x_0 free.*

PROOF. By Lemma 3.6(A), the set Ψ constituted by all of these formulas is a pe 1-type for $\mathfrak{A}$, and it is obviously related to Φ as in the lemma. Therefore there exists d in $\mathfrak{A}$ such that $\Psi \subseteq pe(d)$. Since $\Phi \vdash \mathfrak{p}$, so obviously $\Psi \vdash \mathfrak{p}$ and by Theorem 3.6(A), $pe(c) \subseteq pe(d)$. We conclude from this, using Lemma 3.5, that $pe(c) = pe(d)$; thus $\Psi \subseteq \mathfrak{p}$. But $\mathfrak{p} \subseteq \Psi$ by 3.6(A), so we are done.

LEMMA 3.8. *Let $\mathfrak{A}$ be a maximal T-simple model with two individual constants. Let $b', c' \in |\mathfrak{A}|$. If $\mathfrak{p}_1 = pe(b') \neq \mathfrak{p}_2 = pe(c')$, then $\Psi = Th(\mathfrak{A}) \cup \mathfrak{p}_1 \cup \mathfrak{p}_2$ is inconsistent.*

PROOF. Assume that Ψ is consistent. Let Φ_1 (Φ_2) satisfy (A) (B) of Lemma 3.6 for the realized type $\mathfrak{p}_1$ ($\mathfrak{p}_2$). Define Φ^1 as Φ_1 when, for $l > 0$, x_l is replaced by x_{2l}; and define Φ^2 as Φ_2 when, for $l > 0$, x_l is replaced by x_{2l-1}. Put $\Phi = \Phi^1 \cup \Phi^2$ and $\mathfrak{p} = \mathfrak{p}_1 \cup \mathfrak{p}_2$. Using Lemma 3.7 and consistency of Ψ we easily verify that $\mathfrak{p}$ and Φ are related as required by Lemma 3.6. Hence $\mathfrak{p}$ is realized in $\mathfrak{A}$ by an element a'. Now it follows (by two applications of Lemma 3.5) that $pe(b') = pe(a') = pe(c')$. This is a contradiction.

THEOREM 3.9. *Let $\mathfrak{A}$ be a maximal T-simple model with two individual constants. Let $A \subseteq |\mathfrak{A}|$, $|A| \leqq \omega$, and let*

$$\mathbf{S}(A) = \{pe(c, A) : c \in |\mathfrak{A}|\}.$$

Either $\mathbf{S}(A)$ is countable, or else it has the power of the continuum.

PROOF. We first note that the assumptions remain true if $\mathfrak{A}$ is replaced by $(\mathfrak{A}, c)_{c\in A}$, T is replaced by the theory in the language of the enriched model that has the sentences of T as its axioms, and A is replaced by the empty set. Thus we can assume that $A = \varnothing$ without losing generality.

We put $\mathbf{S} = \mathbf{S}(\varnothing)$, and for every $\mathfrak{p} \in \mathbf{S}$ we choose $\Phi_\mathfrak{p}$ so that $\mathfrak{p}$, $\Phi_\mathfrak{p}$ satisfy (A), (B) of Lemma 3.6. Letting $\mathbf{T} = \{\Phi_\mathfrak{p} : \mathfrak{p} \in \mathbf{S}\}$, we put $F = \bigcup \mathbf{S}$ and $G = \bigcup \mathbf{T}$, and we note that F and G are countable. Let

$$\text{for } \varphi \in F, \quad \mathbf{S}_\varphi = \{\mathfrak{p} : \mathfrak{p} \in \mathbf{S}, \varphi \in \mathfrak{p}\}, \quad F_1 = \{\varphi : \varphi \in F, |\mathbf{S}_\varphi| \leqq \omega\};$$

$$\text{for } \psi \in G, \quad \mathbf{S}^\psi = \{\mathfrak{p} : \mathfrak{p} \in \mathbf{S}, \psi \in \Phi_\mathfrak{p}\}, \quad G_1 = \{\psi : \psi \in G, |\mathbf{S}^\psi| \leqq \omega\};$$

$$\mathbf{S}_1 = \mathbf{S} - \bigcup_{\varphi \in F_1} \mathbf{S}_\varphi - \bigcup_{\psi \in G_1} \mathbf{S}^\psi.$$

Clearly, $|\mathbf{S} - \mathbf{S}_1| \leqq \omega$; also, whenever $\mathfrak{p} \in \mathbf{S}_1$ and $\varphi \in \mathfrak{p}$ and $\psi \in \Phi_\mathfrak{p}$, there are uncountably many $\mathfrak{q} \in \mathbf{S}_1$ such that $\varphi \in \mathfrak{q}$ and $\psi \in \Phi_\mathfrak{q}$. (To see this, one must recall that $\Phi_\mathfrak{p}$ is closed under conjunction and implies $\mathfrak{p}$.)

We now assume that $\mathbf{S}$ is uncountable, that is, $\mathbf{S}_1$ is not empty. Let η denote an arbitrary sequence of ones and zeroes of length $l(\eta)$ and, if $n \leqq l(\eta)$,

$$\eta \mid n = \langle \eta(0), \ldots, \eta(n-1) \rangle.$$

We shall define $\mathbf{S}_\eta$, Φ_η for all finite η by induction on $l(\eta)$, so that

(1) for every η, there is $\mathfrak{q} \in \mathbf{S}_1$ such that $\mathfrak{p}_\eta \subset \mathfrak{q}$ and $\Phi_\eta \subset \Phi_\mathfrak{q}$;
(2) $\mathfrak{p}_\eta$, Φ_η are finite;
(3) $\Phi_\eta \vdash \mathfrak{p}_\eta$;
(4) if $n \leqq l(\eta)$, then $\Phi_{\eta|n} \subset \Phi_\eta$, $\mathfrak{p}_{\eta|n} \subset \mathfrak{p}_\eta$;
(5) $Th(\mathfrak{A}) \cup \mathfrak{p}_{\eta\frown\langle 0\rangle} \cup \mathfrak{p}_{\eta\frown\langle 1\rangle}$ is inconsistent;
(6) if $l(\eta) = n + 1$, then $T \cup \Phi_\eta \vdash a \approx b \to x_n \approx a$.

For $\eta = \langle\ \rangle$ let $\mathfrak{p}_\eta$, Φ_η be empty. Suppose that $\mathfrak{p}_\eta$, Φ_η have been defined, $l(\eta) = n$. Using (1), we choose $\mathfrak{q} \in \mathbf{S}_1$ such that $\mathfrak{p}_\eta \subset \mathfrak{q}$ and $\Phi_\eta \subset \Phi_\mathfrak{q}$. As $T \cup \Phi_\mathfrak{q}$ implies $a \approx b \to x_n \approx a$, there is a finite $\Phi^1 \subset \Phi_\mathfrak{q}$, $T \cup \Phi^1$ implies $a \approx b \to x_n \approx a$. As $\mathfrak{q}$, $\Phi_\mathfrak{q}$ are closed under conjunctions

$$\varphi \equiv \bigwedge \mathfrak{p}_\eta \in \mathfrak{q} \quad \text{and} \quad \psi \equiv \bigwedge \Phi_\eta \wedge \bigwedge \Phi^1 \in \Phi_\mathfrak{q}.$$

Since $\mathfrak{q} \in \mathbf{S}_1$, there is a $\mathfrak{p} \in \mathbf{S}_1$, different from $\mathfrak{q}$, such that $\varphi \in \mathfrak{p}$ and $\psi \in \Phi_{\mathfrak{p}}$—and consequently $\mathfrak{p}_\eta \subset \mathfrak{p}$ and $\Phi_\eta \cup \Phi^1 \subset \Phi_{\mathfrak{p}}$. By Lemma 3.8, $Th(\mathfrak{A}) \cup \mathfrak{p} \cup \mathfrak{q}$ is inconsistent. Thus, we can take finite $\mathfrak{p}_1 \subset \mathfrak{p}$, $\mathfrak{q}_1 \subset \mathfrak{q}$ such that $Th(\mathfrak{A}) \cup \mathfrak{p}_1 \cup \mathfrak{q}_1$ is inconsistent. We put

$$\mathfrak{p}_{\eta\frown\langle 0\rangle} = \mathfrak{p}_\eta \cup \mathfrak{p}_1 \quad \text{and} \quad \mathfrak{p}_{\eta\frown\langle 1\rangle} = \mathfrak{p}_\eta \cup \mathfrak{q}_1.$$

Since $\Phi_{\mathfrak{p}}$ implies $\mathfrak{p}$, we can take a finite $\Phi^2 \subset \Phi_\eta$ that implies $\mathfrak{p}_1$. Then we put

$$\Phi_{\eta\frown\langle 0\rangle} = \Phi_\eta \cup \{\psi\} \cup \Phi^2.$$

Similarly, we define $\Phi_{\eta\frown\langle 1\rangle}$ (using $\mathfrak{q}$).

It is obvious that the sets we have constructed satisfy the above-stated conditions (1) through (6).

For η of length ω we now define

$$\mathfrak{p}_\eta = \bigcup_{n<\omega} \mathfrak{p}_{\eta|n}, \qquad \Phi_\eta = \bigcup_{n<\omega} \Phi_{\eta|n}.$$

By Lemma 3.6 and conditions (1)–(6) above, each $\mathfrak{p}_\eta$ is realized by an element a_η of $\mathfrak{A}$. (The assumption that Φ be closed under conjunctions is not necessary for the conclusion.) It follows by condition (5) that if $l(\eta_1) = l(\eta_2) = \omega$ and $\eta_1 \neq \eta_2$ then $pe(a_{\eta_1}) \neq pe(a_{\eta_2})$. So $|\mathbf{S}| \geqq 2^\omega$, and this must be an equality because the language is countable. The proof is complete.

Proof of Theorem 3.1. A proof can be obtained by combining Lemma 3.5, Theorem 3.9, and [**6**, Theorem 3.4, p. 85]. But we shall prove it directly for completeness.

Assume that $\mathfrak{A}$ is a maximal T-simple model with two individual constants (and T is denumerable), and also that $\|\mathfrak{A}\| > \omega$. By Theorem 3.9, we can further assume that, for each countable $A \subset |\mathfrak{A}|$, the set $\mathbf{S}(A)$ is countable. Therefore there exists some $\mathfrak{p} \in \mathbf{S}(A)$ which is realized by uncountably many elements of $\mathfrak{A}$. We shall call such a $\mathfrak{p}$ "large."

Case I. There is countable $A \subset |\mathfrak{A}|$ and a large $\mathfrak{p} \in \mathbf{S}(A)$ such that, for every countable B, $A \subset B \subset |\mathfrak{A}|$, $\mathfrak{p}$ has at most one large extension $\mathfrak{q} \in \mathbf{S}(B)$.

In this case, we define a sequence a_α, $\alpha < \omega_1$ $(= \omega^+)$, by recursion. If, for $\alpha < \beta$, a_α is defined, let $B = A \cup \{a_\alpha : \alpha < \beta\}$. As $|\mathbf{S}(B)| \leqq \omega$, some $\mathfrak{q}_\beta \in \mathbf{S}(B)$ is realized by uncountably many elements that realize $\mathfrak{p}$. By assumption, $\mathfrak{q}_\beta$ is uniquely determined. We choose a_β to be any element realizing $\mathfrak{q}_\beta$ (and therefore realizing $\mathfrak{p}$).

The sequence $\langle a_\alpha : \alpha < \omega_1\rangle$ is almost indiscernible. In fact, it follows readily that, for $\alpha < \beta < \omega_1$, we have $pe(a_\alpha, C_\alpha) = pe(a_\beta, C_\alpha)$ where $C_\alpha = \{a_\delta : \delta < \alpha\}$. Hence by Lemma 3.5, a_α and α_β have the same elementary type over C_α.

By Theorem 3.2, $\mathfrak{A}$ is not maximal T-simple. This is a contradiction.

Case II. Not Case I.

We can obviously construct by induction on $n < \omega$ a system of sets $B_n \subset |\mathfrak{A}|$ and, for every finite 01-sequence η, a pe type $\mathfrak{p}_\eta$, so that

(A) $|B_n| \leqq \omega$, $\mathfrak{p}_\eta \in \mathbf{S}(B_{l(\eta)})$,

(B) $B_n \subset B_{n+1}$, $\mathfrak{p}_{\eta|n} \subset \mathfrak{p}_\eta$ if $l(\eta) \geqq n$,

(C) $\eta_1 \neq \eta_2$ implies $\mathfrak{p}_{\eta_1} \neq \mathfrak{p}_{\eta_2}$.

Now let $B = \bigcup_{n<\omega} B_n$, and for η of length ω let $\mathfrak{p}_\eta = \bigcup_{n<\omega} \mathfrak{p}_{\eta|n}$. Now one can show, by applying Lemmas 3.6 and 3.7 to the model $(\mathfrak{A}, b)_{b\in B}$ (which is maximal T'-simple for T' the inessential extension of T in the language of this model), that all types $\mathfrak{p}_\eta$ are realized. Since these types are all different, by (C) above, then $|S(B)| \geqq 2^\omega$; again a contradiction.

4. Simple and subdirectly irreducible algebras; Hanf numbers; minimum compact models. An algebra is *simple* if (it has more than one element and) every homomorphism from it is either an embedding or a constant map. In general algebra, a related concept is more frequently met. An algebra is called *subdirectly irreducible* if there exist in it elements a, b such that $a \neq b$ and every homomorphism f from the algebra is an embedding, or satisfies $f(a) = f(b)$. A *variety*, or equational class, of algebras is a class consisting of all models of some equational theory.

THEOREM 4.1. *Let V be a variety of algebras, κ be the cardinal of its language,* **C** *be the class of cardinals $\beta \geqq \kappa$ in which V has a simple member (or, respectively, subdirectly irreducible member). Assume that* **C** *is nonempty and does not include all $\beta > \kappa$. Then* $\mathbf{C} = [\kappa, \lambda)$, *where $\lambda \leqq (2^\kappa)^+$; if $\kappa = \omega$, then λ must be ω^+ or $(2^\omega)^+$.*

PROOF. We let T be the equational theory whose class of models is V. We enrich the language of T by two new constants a, b, and let T' be the theory whose axioms are $T \cup \{\neg a \approx b\}$. Now T and T' are universal theories. Moreover, $\mathbf{C} = \mathbf{SC}^\kappa(T)$ in the case of simple algebras; and one sees easily that $\mathbf{C} = \mathbf{SC}^\kappa(T')$ in the case of subdirectly irreducible algebras. So the theorem follows by Theorems 1.7, 2.1 and 3.1.

The analogue of Theorem 2.1 for subdirectly irreducible algebras was first proved by Taylor [**10**, Theorem 1.2], by methods similar to ours.

Now let $\mathfrak{M}$ denote any family of classes of models. We define the *Hanf number* of $\mathfrak{M}$ (if it exists) as the least cardinal η such that every member of $\mathfrak{M}$, if it contains a model in some power $\beta \geqq \eta$, must contain models of arbitrarily great power.

If the members of $\mathfrak{M}$ are the classes of all T-simple models and T is allowed to vary through all universal theories of power κ then, by Theorems 1.8 and 2.1, the Hanf number equals $(2^\kappa)^+$. Likewise, the Hanf number for simple algebras, or for subdirectly irreducible algebras in varieties of algebras with κ operations, each equals $(2^\kappa)^+$.

The concept of minimum compact model was created by Taylor to provide a fascinating connection between the better known concepts of atomic compact, and of weakly atomic compact models (see [**9**]). He proved that the cardinality of any minimum compact model with language of power κ is at most 2^κ. This is an easy corollary of our results. To show why, we shall derive a stronger, new result for models of countable type.

To avoid complications, we give a very direct definition of the concept in question, which (it follows easily from the treatment in [**9**]) is equivalent with the original definition. We call a model $\mathfrak{A}$ *minimum compact* if for each homomorphism $f: \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ satisfy precisely the same positive existential sentences, there exists $g:\mathfrak{B} \to \mathfrak{A}$ so that gf is the identity map on $\mathfrak{A}$. (This implies that f is an embedding.)

THEOREM 4.2. *Suppose that $\mathfrak{A}$ is minimum compact and infinite, and that its language is countable. Then the cardinality of $\mathfrak{A}$ is either ω or 2^ω.*

PROOF. Let T be the theory whose axioms are all universal sentences that are valid in $\mathfrak{A}$ (the universal theory of $\mathfrak{A}$). We claim that $\mathfrak{A}$ is maximal T-simple.

In fact, if $f:\mathfrak{A} \to \mathfrak{B}$ where $\mathfrak{B} \models T$, then f is an embedding. (Since the homomorphism preserves validity of positive existential sentences, $\mathfrak{B}$ satisfies all those satisfied by $\mathfrak{A}$; since $\mathfrak{B} \models T$, it satisfies no more.) Thus $\mathfrak{A}$ is T-simple. Suppose $\mathfrak{A} \subset \mathfrak{C}$ properly and $\mathfrak{C}$ is T-simple. Then $\mathfrak{C}$ and $\mathfrak{A}$ satisfy the same existential sentences. Taking f as identity map of $\mathfrak{A}$ into $\mathfrak{C}$, there exists $g: \mathfrak{C} \to \mathfrak{A}$ (by definition of minimum compactness) so that $gf = 1$. This g can be neither constant nor an embedding. So $\mathfrak{C}$ is not T-simple—a contradiction. The contradiction implies that $\mathfrak{A}$ is maximal.

Since $\mathfrak{A}$ is maximal T-simple, the desired conclusion follows by §§2 and 3.

5. Other directions. Some related Hanf numbers have a common value that greatly exceeds those just computed.

DEFINITION 5.0. Let λ be an infinite cardinal. $\mu(\lambda)$ is the first cardinal μ such that if $\mathfrak{A}$ is a model omitting a (elementary) type $\mathfrak{p}$—i.e. $\mathfrak{p}$ is not realized in $\mathfrak{A}$—and if $\mathfrak{A}$ has at most λ operations and relations, and $\|\mathfrak{A}\| \geqq \mu$, then there are arbitrarily large models elementarily equivalent to $\mathfrak{A}$ which omit $\mathfrak{p}$.

On $\mu(\lambda)$ see Morley **[5]** or Chang **[2]**. It is identical with what Chang calls the Morley number, m_λ, and also with the Hanf number of infinitary languages $L_{\lambda^+\omega}$ having λ nonlogical symbols. In particular, $\mu(\omega) = \beth_{\omega_1}$.

In this section, we show that $\mu(\lambda)$ is the Hanf number of each of the families $\mathfrak{M}$ which we now define. (1) A member of $\mathfrak{M}$ consists of all T-simple models; T ranges over all positive theories of power λ. (2) A member of $\mathfrak{M}$ consists of all simple algebras satisfying T; T ranges over universal theories in algebraic languages of power λ. (3) A member of $\mathfrak{M}$ consists of all algebras $\mathfrak{A} = \langle A, F_s\rangle_{s\in S}$ such that $\langle A, F_s\rangle_{s\in U}$ is simple and $\mathfrak{A}$ satisfies T; T is a theory with positive universal axioms, $U \subseteq S$, and $|S|$ $(= |T|) \leqq \lambda$. (For this last family, the question of the Hanf number was asked by Chang.)

One easily verifies that the Hanf numbers for these families are bounded by $\mu(\lambda)$. In fact, each member of one of the families is a class defined by a sentence of the appropriate language $L_{\lambda^+\omega}$. (Conclusion 1.4 should give you an idea how to construct the infinitary sentences.)

We now present constructions that serve to push the numbers up to $\mu(\lambda)$.

Let $\mathfrak{A}$ be a model for a first order language L, $|L| = \lambda$, and say $\|\mathfrak{A}\| = \kappa$, such that $\mathfrak{A}$ omits the type $\mathfrak{p}$ and no model $\mathfrak{B}$ elementarily equivalent with $\mathfrak{A}$, $\|\mathfrak{B}\| > \kappa$, omits $\mathfrak{p}$. We assume that $\mathfrak{A}$ has its Skolem functions; thus $Th(\mathfrak{A})$ is implied by the set of universal sentences satisfied by $\mathfrak{A}$. Further, we assume that

$$\mathfrak{p} = \{P(x)\} \cup \{\neg x \approx a_i : i < \lambda\},$$

and that $\mathfrak{A} \models P(a_i) \wedge \neg a_i \approx a_j$ for all $i < j < \lambda$. (These things can be arranged without altering the other hypotheses.)

EXAMPLE 5.1. We construct a positive theory T and a T-simple (hence simple) algebra $\mathfrak{A}_1$ such that $\|\mathfrak{A}_1\| = \kappa$, $|T| = \lambda$ and there does not exist a T-simple algebra $\mathfrak{B}$ with $\|\mathfrak{B}\| > \kappa$.

Let us define an algebra $\mathfrak{A}_1$ of cardinality κ and with λ operations. Let $|\mathfrak{A}_1| = |\mathfrak{A}| \cup \{c_i : i < \lambda\}$ and the operations of it will be the following:

(1) For every formula $\varphi = \varphi(x_1, \ldots, x_n) \in L$ an n-place operation F_φ:

$$\begin{aligned} F_\varphi(b_1, \ldots, b_n) &= c_0 \quad \text{if } b_1, \ldots, b_n \in |\mathfrak{A}|,\ \mathfrak{A} \models \varphi[b_1, \ldots, b_n], \\ &= c_1 \quad \text{if } b_1, \ldots, b_n \in |\mathfrak{A}|,\ \mathfrak{A} \models \neg\varphi[b_1, \ldots, b_n], \\ &= c_2 \quad \text{otherwise.} \end{aligned}$$

(2) A one-place operation I:

$$\begin{aligned} I(b) &= b \quad \text{if } b \in |\mathfrak{A}|, \\ &= a_0 \quad \text{otherwise.} \end{aligned}$$

(3) For every $i < \lambda$, a two-place operation J_i:

$$\begin{aligned} J_i(b_1, b_2) &= b_2 \quad \text{if } b_1 \neq c_i, \text{ and } b_2 \in |\mathfrak{A}|, \\ &= c_i \quad \text{if } b_1 = c_i \text{ or } b_2 \notin |\mathfrak{A}|. \end{aligned}$$

(4) A one-place operation H:

$$\begin{aligned} H(b) &= a_i \quad \text{if } b = c_i,\ i < \lambda, \\ &= c_0 \quad \text{if } b \in |\mathfrak{A}|. \end{aligned}$$

(5) A two-place operation G:

$$\begin{aligned} G(b_1, b_2) &= c_0 \quad \text{if } b_1 \neq b_2;\ b_1, b_2 \in |\mathfrak{A}|, \\ &= c_1 \quad \text{otherwise.} \end{aligned}$$

(6) For any distinct $i, j, k \leqq \lambda$ a one-place function $E_{i,j,k}$:

$$\begin{aligned} E_{i,j,k}(b) &= c_j \quad \text{if } b = c_i, \\ &= c_k \quad \text{if } b = c_j, \\ &= c_i \quad \text{if } b = c_k, \\ &= b \quad \text{if otherwise,} \end{aligned}$$

where we define $c_\lambda = a_0$.

(7) For each $i \leqq \lambda$, an individual constant c_i, whose interpretation is c_i.

Let T be the set of positive sentences $\mathfrak{A}_1$ satisfies. We should prove that there is a T-simple model of power κ, but there is no T-simple model of power $> \kappa$.

As a first step, we prove $\mathfrak{A}_1$ is a T-simple model hence T has a T-simple model. Clearly $\mathfrak{A}_1 \models T$. Suppose we have a nontrivial identification $\equiv$. Then there are $x \neq y$, $x, y \in |\mathfrak{A}_1|$, $x \equiv y$.

Case I. $x = c_i$, $y = c_j$ $(i \neq j)$.

Using (6), for any $k < \lambda$, $k \neq i$, $k \neq j$,

$$c_k = E_{i,j,k}(c_j) \equiv E_{i,j,k}(c_i) = c_j.$$

So all the c_k's are identified. Now for any $z \in |\mathfrak{A}|$,

$$x = J_0(c_1, x) \equiv J_0(c_0, x) = c_0.$$

Hence all $\mathfrak{A}_1$ is identified.

Case II. $x, y \in |\mathfrak{A}|$,

$$c_0 = G(x, y) \equiv G(x, x) = c_1.$$

So we reduce this to Case I.

Case III. $x \in |\mathfrak{A}|$, $y = c_j$, $j < \lambda$. Choose $z \in |\mathfrak{A}|$, $z \neq x$,

$$z = J_j(x, z) \equiv J_j(y, z) = c_j = y \equiv x,$$

hence $z \equiv x$, $z, x \in |\mathfrak{A}|$, so we reduce it to Case II.

Case IV. $x = c_i$, $i < \lambda$, $y \in |\mathfrak{A}|$. This is just like Case III.

It remains to prove there is no T-simple model of cardinality $> \kappa$. Suppose $\mathfrak{B}$ is a T-simple model of cardinality $> \kappa$. As T is positive, the model is simple.

Let

$$A = \{a : a \in |\mathfrak{B}|, I(a) = a\},$$
$$C = \{c_i^{\mathfrak{B}} : i < \lambda\}.$$

Note that as $(\forall x)\ I(I(x)) = I(x)$, the range of $I^{\mathfrak{B}}$ is A.

Claim I. $|\mathfrak{B}| = A \cup C$.

Otherwise identify all elements of $A \cup C$; it suffices to prove that this does not imply any more identifications. For most functions their range is included in $A \cup C$. For I because $[I(a_0) = a_0] \in T$. For F_φ trivial. For J_i because $(\forall xy)[I(J_i(x, y)) = J_i(x, y) \vee J_i(x, y) = c_i] \in T$. For H because $(\forall x)[I(H(x)) = H(x) \vee H(x) = c_0] \in T$. For G, immediate.

The only functions left are $E_{i,j,k}$ which become the identity. So if $|\mathfrak{B}| \neq A \cup C$ then $\mathfrak{B}$ is not simple, or $|A \cup C| = 1$. If $\|\mathfrak{B}\| > 2$ we can identify two elements of $|\mathfrak{B}| - (A \cup C)$.

As T is a positive theory and $\mathfrak{B}$ is T-simple, $\mathfrak{B}$ is simple. Hence $\|\mathfrak{B}\| \leqq 2$, contradiction.

Claim II. $A \cap C = \varnothing$.

Otherwise for some $i < \lambda$, $I^{\mathfrak{B}}(c_i) = c_i$. But $[I(c_i) = a_0] \in T$, remembering $a_0 = c_\lambda$, and that, for $j \neq i, j < \lambda$,

$$[E_{i,j,\lambda}(c_i) = c_j] \wedge [E_{i,j,\lambda}(c_j) = c_\lambda] \wedge [E_{i,j,\lambda}(c_\lambda) = c_i] \in T$$

we get $\mathfrak{B} \models c_i = c_j = c_\lambda$. As j was arbitrary all the c_j's are equal, and in particular $c_0 = c_1$. For any $x \in A$,

$$x = J_0(c_1, x)$$

because

$$(\forall x)\ [J_0(c_1, I(x)) = I(x)] \in T,$$

$$J_0(c_1, x) = J_0(c_0, x) \quad \text{as } c_1 = c_0,$$
$$J_0(c_0, x) = c_0$$

because

$$(\forall x)\,[J_0(c_0, x) = x] \in T.$$

So $x = c_0$, hence all $A \cup C$ is identified with c_0, so $\|\mathfrak{B}\| = 1$, a contradiction.

Claim III. For $i < j < \lambda$, $c_i^{\mathfrak{B}} \neq c_j^{\mathfrak{B}}$.

The proof is the same as that of the previous claim, with c_j replacing c_λ.

DEFINITION. Define a model $\mathfrak{B}^*$. $|\mathfrak{B}^*| = A$ and, for any relation $R \in L$, $R^{\mathfrak{B}^*} = \{\langle b_1, \ldots, b_n\rangle : \mathfrak{B} \models F_R[b_1, \ldots, b_n] = c_0,\ b_1, \ldots, b_n \in A\}$. Define $F^{\mathfrak{B}^*}$ for the function symbol $F \in R$ similarly.

Clearly $\mathfrak{B}^*$ is an L-model.

Claim IV. For any formula $\varphi(x_1, \ldots, x_n) \in L$, and $b_1, \ldots, b_n \in A$,

$$\mathfrak{B}^* \models \varphi[b_1, \ldots, b_n] \quad \text{iff } \mathfrak{B} \models F_\varphi[b_1, \ldots, b_n] = c_0.$$

We prove this by induction on formulas

(A) φ atomic—by the definition of $\mathfrak{B}^*$. (Assume w. l. o. g. it is of the form $R(\bar{x})$ or $y = F(\bar{x})$.)

(B) $\varphi = \neg$.

Let $b_1, \ldots, b_n \in A$.

$\mathfrak{B}^* \models \varphi[b_1, \ldots, b_n]$ iff not $\mathfrak{B}^* \models \psi[b_1, \ldots, b_n]$ iff (by induction hypothesis) not $\mathfrak{B} \models F_\psi[b_1, \ldots, b_n] = c_0$ iff $\mathfrak{B} \models F_\varphi[b_1, \ldots, b_n] = c_0$.

The last iff follows from

$$(\forall x_1, \ldots, x_n)[F_\varphi(I(x_1), \ldots, I(x_n)) = c_0 \vee F_\psi(I(x_1), \ldots, I(x_n)) = c_0] \in T,$$
$$(\forall x_1, \ldots, x_n)[F_\psi(I(x_1), \ldots, I(x_n)) = c_1 \vee F_\varphi(I(x_1), \ldots, I(x_n)) = c_1] \in T,$$
$$(\forall x_1, \ldots, x_n)[F_\varphi(I(x_1), \ldots, I(x_n)) = c_0 \vee F_\varphi(I(x_1), \ldots, I(x_n)) = c_1] \in T,$$
$$(\forall x_1, \ldots, x_n)[F_\psi(I(x_1), \ldots, I(x_n)) = c_0 \vee F_\psi(I(x_1), \ldots, I(x_n)) = c_1] \in T.$$

(C) $\varphi = \varphi_1 \vee \varphi_2$—the same idea as in (B), only the sentences may have three disjuncts.

(D) $\varphi = \varphi(x_1, \ldots, x_n) = (\exists x)\psi(x, x_1, \ldots, x_n)$—the same way.

Claim V. $\mathfrak{B}^*$ is elementarily equivalent to $\mathfrak{A}$.

Apply Claim IV on sentences φ.

Claim VI. $\mathfrak{B}^*$ omits the type $p = \{P(x)\} \cup \{x \neq a_i : i < \lambda\}$.

Suppose a realizes p.

As $(\forall x)(\exists y)\,[H(y) = I(x) \vee F_P(I(x)) = c_1] \in T$ and $I^{\mathfrak{B}}(a) = a$, $F_P^{\mathfrak{B}}(I^{\mathfrak{B}}(a)) = c_0$, there is $c \in |\mathfrak{B}|$ such that $H^{\mathfrak{B}}(c) = a$. As $(\forall x)[H(I(x)) = c_0]$ clearly $c \notin A$ hence $c \in C$; so for some $i < \lambda$, $c = c_i$. But $[H(c_i) = a_i] \in T$ so $a = a_i$, a contradiction.

So $\mathfrak{B}^*$ is an L-model, elementarily equivalent to $\mathfrak{A}$, omitting p. So its power is $\leqq \kappa$. Hence $\|\mathfrak{B}\| \leqq |A \cup C| \leqq \kappa + \lambda \leqq \kappa$, a contradiction. So we prove also the second part, and finish.

EXAMPLE 5.2. We construct a simple algebra $\mathfrak{A}_2$ whose language is of power λ, such that $\|\mathfrak{A}_2\| = \kappa$, and there does not exist a *simple algebra* $\mathfrak{B}$, $\|\mathfrak{B}\| > \kappa$, satisfying all the universal sentences $\mathfrak{A}_2$ satisfies.

We put $|\mathfrak{A}_2| = |\mathfrak{A}| \cup \{c_i : i < \lambda\}$. (The c_i are new constants.) The operations of $\mathfrak{A}_2$ are the following:

(1) The functions F, F_i, $i < \lambda$, where $F(x, y, z,) = z$, if $x, y, z \in |\mathfrak{A}|$, $x \neq y$ and $F(x, y, z) = c_0$ otherwise; and $F_i(x) = a_i$ if $a_i = c_j$ for some j, and $F_i(x) = x$ otherwise.

(2) The functions from (1) in 5.1, individual constants for a_i, c_i and all functions of $\mathfrak{A}$, when we assign to them the value c_0 when they are undefined.

(3) A function h such that $h(a_i) = b_i$ for all i and $h(c) = c$ otherwise.

(4) For each ordinal α, $0 < \alpha < \lambda$, an operation F_α satisfying

$$F_\alpha(b_1, b_2) = b_2 \quad \text{iff } b_1 = c_\alpha,$$
$$F_\alpha(b_1, b_2) = c_\alpha \quad \text{otherwise.}$$

(It is essential that $0 < \alpha$ be assumed here.)

The verification is left for the reader.

EXAMPLE 5.3. We enrich $\mathfrak{A}_1$ of the first example by adding one Skolem function. Thus, the reduct of $\mathfrak{A}_3$ to the language of $\mathfrak{A}_1$ is simple. It is routine to verify that if $\mathfrak{B}$ is similar to $\mathfrak{A}_3$, $\|\mathfrak{B}\| > \kappa$, and $\mathfrak{B}$ satisfies all positive universal sentences valid in $\mathfrak{A}_3$, then the reduct of $\mathfrak{B}$ similar to $\mathfrak{A}_1$ is not simple.

6. T-simple models of small cardinality. In this section we characterize the classes $\mathbf{SC}_\kappa(T)$ and show that the two parts $\mathbf{SC}_\kappa(T)$ and $\mathbf{SC}^\kappa(T)$ are entirely unrelated, at least if T is an arbitrary universal theory.

THEOREM 6.1. *Let κ be an infinite cardinal; let T be a universal theory, $|T| \leqq \kappa$; and let $\mathbf{C} \subseteq \kappa^+$ be a set of cardinals, $0, 1 \notin \mathbf{C}$. There exists a universal theory T_1, $|T_1| \leqq \kappa$, such that $\mathbf{SC}(T_1) = \mathbf{C} \cup \mathbf{SC}^\kappa(T)$; moreover, T_1 is equational, unless $\kappa > \omega$ and $\mathbf{SC}^\kappa(T) = [\kappa, \lambda)$ with $\kappa^+ < \lambda \leqq 2^\kappa$.*

PROOF. The conclusion is obviously implied by the following three statements, which we shall prove in turn:

I. There exists an equational theory T^1 such that $|T^1| \leqq \kappa$ and $\mathbf{SC}(T^1) = \mathbf{C}$.

II. There exists a universal theory T^2 such that $\mathbf{SC}(T^2) = \mathbf{SC}^\kappa(T)$ and $|T^2| \leqq \kappa$. T^2 is equational, unless $\kappa > \omega$ and $\mathbf{SC}^\kappa(T) = [\kappa, \lambda)$, $\kappa^+ < \lambda \leqq 2^\kappa$.

III. Let T and $\bar{T}$ be universal theories. There exists a universal theory T^3 such that $\mathbf{SC}(T^3) = \mathbf{SC}(T) \cup \mathbf{SC}(\bar{T})$ and $|T^3| \leqq |T| + |\bar{T}|$. If T and $\bar{T}$ are both equational, so is T^3.

PROOF OF I. We assume that $\mathbf{C}$ is nonempty and that $\alpha \geqq 2$ is its least member. We put

$$P = \{\langle \lambda, i\rangle : \lambda \in \mathbf{C},\ i < \lambda = \alpha \text{ or } \alpha \leqq i < \lambda\}.$$

We take for the nonlogical symbols of T^1 constants $b_{\lambda,i}$ (for $\langle \lambda, i\rangle \in P$) and binary function symbols $E_{\lambda,i}$ (for $\langle \alpha, 0\rangle \neq \langle \lambda, i\rangle \in P$).

As axioms for T^1 we take the following.

For all $\langle\lambda, i\rangle, \langle\delta, j\rangle \in P - \{\langle\alpha, 0\rangle\}$,

(1) $E_{\lambda,i}(x, E_{\lambda,i}(x, y)) \approx E_{\lambda,i}(x, y)$,

(2) $E_{\lambda,i}(x, x) \approx x$,

(3) $E_{\lambda,i}(x, b_{\lambda,i}) \approx E_{\lambda,i}(b_{\lambda,i}, x) \approx b_{\lambda,i}$,

(4) $E_{\lambda,i}(x, E_{\delta,j}(y, z)) \approx E_{\lambda,i}(x, E_{\delta,j}(E_{\lambda,i}(x, y), E_{\lambda,i}(x, z)))$.

For all $\lambda, \delta \neq \alpha$,

(5) $E_{\alpha,i}(b_{\alpha,j}, y) \approx y$ (*for* $j < \alpha$, $1 \leqq i < \alpha$ *and* $i \neq j$),

(6) $E_{\lambda,i}(b_{\alpha,j}, y) \approx y$ (*for* $j \neq 1$),

(7) $E_{\alpha,1}(b_{\lambda,i}, y) \approx E_{\alpha,1}(b_{\lambda,j}, y)$ (*for* $\alpha \leqq i, j < \lambda$),

(8) $E_{\alpha,1}(b_{\lambda,\alpha}, E_{\alpha,1}(b_{\delta,\alpha}, y)) \approx b_{\alpha,1}$ (*for* $\lambda \neq \delta$),

(9) $E_{\lambda,i}(b_{\lambda,j}, b_{\alpha,1}) \approx b_{\alpha,1}$ (*for* $i \neq j$).

To see that $\mathbf{C} \subseteq \mathbf{SC}(T^1)$ we must define some algebras. Let $A = \{b_{\lambda,i} : \langle\lambda, i\rangle \in P\}$, and for each $\langle\lambda, i\rangle \neq \langle\alpha, 0\rangle$, let $F_{\lambda,i}$ be the operation on A defined by $F_{\lambda,i}(x, y) = y$ if $x \neq b_{\lambda,i}$; $F_{\lambda,i}(x, y) = b_{\lambda,i}$ if $x = b_{\lambda,i}$. Then for each $\gamma \in \mathbf{C}$ we put

$$\mathfrak{A}^\gamma = \langle A^\gamma; b^\gamma_{\lambda,i}(\langle\lambda, i\rangle \in P); F^\gamma_{\delta,j}(\langle\delta, j\rangle \in P - \{\langle\alpha, 0\rangle\})\rangle,$$

where

$$\begin{aligned} A^\gamma &= \{b_{\lambda,i} \in A : \lambda \in \{\alpha, \gamma\}\}; \\ b^\gamma_{\lambda,i} &= b_{\lambda,i} \quad \text{if } \lambda \in \{\alpha, \gamma\}, \\ &= b_{\alpha,1} \quad \text{if not;} \end{aligned}$$

and

$$\begin{aligned} F^\gamma_{\delta,j}(x, y) &= F_{\delta,j}(x, y) \quad \text{if } \delta \in \{\alpha, \gamma\}, \\ &= F_{\alpha,1}(x, y) \quad \text{if not.} \end{aligned}$$

It is easy to check the definitions and the axioms above to see that $\mathfrak{A}^\gamma$ has power γ and that $\mathfrak{A}^\gamma \models T^1$. We claim that $\mathfrak{A}^\gamma$ is simple. In fact, the claim is obvious on the grounds that, for any two distinct elements $u, v \in A^\gamma$, one of the basic operations satisfies $F(u, y) = y$ and $F(v, y) = v$ for all y (or the same condition with u and v interchanged).

To prove that $\mathbf{SC}(T^1) \subseteq \mathbf{C}$ is just a little harder. Each simple model of T^1 is isomorphic with one of the $\mathfrak{A}^\gamma$ constructed above. For let $\mathfrak{B}$ be a simple algebra, $\mathfrak{B} \models T^1$. Then for any λ, i, and $x \in B$, the map $h(y) = E^{\mathfrak{B}}_{\lambda,i}(x, y)$ is a projection (by equation (1)), and its kernel is a congruence relation on $\mathfrak{B}$ (by equation (4)); thus h is constant or one-to-one. If h is one-to-one, then $h(y) = h(h(y))$ so $h(y) = y$ for all y. From this we conclude (with the help of equations (2) and (3)) that

$$\begin{aligned} E^{\mathfrak{B}}_{\lambda,i}(x, y) &= y \quad \text{if } x \neq b^{\mathfrak{B}}_{\lambda,i}, \\ &= b^{\mathfrak{B}}_{\lambda,i} \quad \text{if } x = b^{\mathfrak{B}}_{\lambda,i}. \end{aligned} \tag{10}$$

Now we have by equations (5) and (6) that $b^{\mathfrak{B}}_{\alpha,0} \neq b^{\mathfrak{B}}_{\alpha,1}$, and that for all $y \in B - \{b^{\mathfrak{B}}_{\lambda,i} : \langle\lambda, i\rangle \in P\}$, and for all fundamental operations E,

$$E(y, z) = z = E(b^{\mathfrak{B}}_{\alpha,0}, z) \quad (\text{for all } z).$$

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So the map that collapses all nonconstant elements into $b^{\mathfrak{B}}_{\alpha,0}$ is a nonconstant homomorphism of $\mathfrak{B}$. This implies that $B = \{b^{\mathfrak{B}}_{\lambda,i}\}$. The remainder of the proof that $\mathfrak{B} \cong \mathfrak{A}^{\gamma}$ for some $\gamma \in \mathbf{C}$ is straightforward. Use (5) through (10).

PROOF OF II. We note that by Theorems 1.7, 2.1, and 3.1, if this statement requires T^2 to be equational, then either Theorem 1.8, 1.9, or 1.10 will give the desired result. (The theories produced there have no small simple models.) We only need show here the existence of a *universal* theory with the prescribed properties. To minimize triviality, we assume that $\mathbf{SC}^{\kappa}(T) \neq \varnothing$.

Let the operation and relation symbols of T be $\{F_u\}_{u\in U}$ and $\{R_v\}_{v\in V}$. We can describe T^2 by its models. A model of T^2 will be any system

$$\mathfrak{A} = \langle A, F_u, R_v, c, H_\delta\rangle_{u\in U;v\in V;\delta<\kappa}$$

in which H_δ is a projection for all $\delta < \kappa$ (i.e., $H_\delta H_\delta(x) = H_\delta(x)$); $H_\delta H_\lambda = H_\delta$ for all δ, λ; $H_\delta(x) = c \Leftrightarrow x = c$;

$$F_u(x_1, \ldots, x_n) = H_0F_u(x_1, \ldots, x_n) = F_u(H_0x_1, \ldots, H_0x_n);$$

$$R_v(x_1, \ldots, x_m) \Rightarrow H_0x_1 = x_1 \;\&\cdots\&\; H_0x_m = x_m;$$

the model

$$\langle H_0(A), F_u \restriction H_0(A), R_v \restriction H_0(A)\rangle \models T;$$

and

$$H_\delta(x) = H_\lambda(x) \Rightarrow x = c \quad \text{if } \delta \neq \lambda.$$

The proof in detail is trivial.

PROOF OF III. We can assume that T and $\bar{T}$ have disjoint sets of nonlogical symbols, $\tau(T)$ and $\tau(\bar{T})$ respectively. We can also assume that each theory has among its models all 1-element models of the same similarity type. We take for the nonlogical symbols of T^3 those in the set $\tau(T) \cup \tau(\bar{T}) \cup \{P\}$ where P is a "new" binary operation symbol. We shall not write out axioms for T^3, but their significance is that a model $\mathfrak{C}$ satisfies T^3 just in case, for some $\mathfrak{A} \models T$ and $\mathfrak{B} \models \bar{T}$,

$$\mathfrak{C} \cong \mathfrak{A} \oplus \mathfrak{B} = \langle A \times B, P, \ldots\rangle$$

in which $P(\langle x, y\rangle, \langle u, v\rangle) = \langle x, v\rangle$; and

$$F(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \langle F^{\mathfrak{A}}(x_1, \ldots, x_n), y_1\rangle,$$

$$G(\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle) = \langle x_1, G^{\mathfrak{B}}(y_1, \ldots, y_m)\rangle$$

if $F \in \tau(T)$ and $G \in \tau(\bar{T})$; and the relations are also defined by their values on the proper coordinates.

The details are straightforward.

7. Remarks and problems.

Problem 1. *Characterize the classes* $\mathbf{SC}(T)$, *for* T *a universal theory of power* $\kappa > \omega$.

Problem 2. *Study T-simple algebras for languages with finitely many operation symbols, and for finitely axiomatizable T. In particular*

(A) *Does there exist an equational theory T of finite type such that* $\mathbf{SC}(T) = [\omega, 2^\omega]$?

(B) *For finitely axiomatizable universal (or positive universal) theories T, what can* $\mathbf{SC}(T)$ *or* $\mathbf{SC}_\omega(T)$ *be?*

7.1. If we omit from the assumption of Lemma 3.5 the maximality of $\mathfrak{A}$, the proof can yet be used to show the existence of "homogeneous" T-simple models; more exactly, let $\mathfrak{A}$ be T-simple with two individual constants and $\lambda = \lambda^{\underline{\mu}} \geqq \|\mathfrak{A}\|$ $[\lambda^{\underline{\mu}} = \sum_{\kappa<\mu} \lambda^\kappa]$. Then $\mathfrak{A}$ has an extension $\mathfrak{B}$ which is T-simple, $\|\mathfrak{B}\| \leqq \lambda$ and

> if $\langle a_\beta : \beta < \delta < \mu \rangle$, $\langle b_\beta : \beta < \delta < \mu \rangle$ are systems of members of $\mathfrak{B}$ and every pe formula satisfied by some terms of the first system is satisfied by the corresponding terms of the second, *then* there is an automorphism f of $\mathfrak{B}$, such that $f(a_\beta) = b_\beta$ for all $\beta < \delta$.

7.2. Let L be, for simplicity, countable, T a universal theory and $\psi(x)$ a Σ_2 positive formula in $L_{\omega_1\omega}$. Let K be the class of models $\mathfrak{A}$ of T for which $\{a \in |\mathfrak{A}| : \mathfrak{A} \vDash \psi(a)\}$ is a set of generators. The theorems of §§2 and 3 can be generalized to K-simple models without difficulties.

Definition 7.3. $\mathfrak{A}$ is (T, λ)-simple, if $\mathfrak{A}$ is a model of T and every homomorphism from $\mathfrak{A}$ either is an embedding or has range of cardinality $<\lambda$.

Let T be universal. By our method we can prove that if T has a (T, λ)-simple model of cardinality $> 2^{\lambda+|T|}$, then T has $(T, |T|^+)$-simple models of arbitrarily large cardinalities.

Problem 3. *Characterize the classes*

$$\{\langle \mu, \lambda \rangle : \textit{there is a } (T, \lambda)\textit{-simple } \mathfrak{A},\ \|\mathfrak{A}\| = \mu\}.$$

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