CATEGORICITY OF UNCOUNTABLE THEORIES

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Introduction. This article is devoted to the proof of the following theorem.

THEOREM 0.1. *If a (first-order complete) theory T is categorical in one cardinal $\lambda > |T|$ then T is categorical in every cardinal $\mu > |T|$ where*

DEFINITION 0.1. *T is categorical in the cardinal λ if T has a model of cardinality λ, and any two models of T of cardinality λ are isomorphic.*

Łoś **[L 1]** conjectured this for countable T. Morley **[M 1]** proved the conjecture. Successive and independent approximations for uncountable T are Rowbottom **[Ro 1]**, Ressayre **[Re 1]** and **[S 1]**, **[S 2]**. A discussion on the proofs will appear at the end of the introduction.

From these investigations the notion of stability arises. A list of all results connected with stability and categoricity, with historical remarks, appears in **[S 9]**. As this list is concise and long, we shall give a more informal discussion here on stability and the number of nonisomorphic models. But a reader interested in all results, exact credits, etc., should consult **[S 9]**.

Let T be a complete theory in L, and $D_n(T)$ be the set of complete n-types consistent with T (i.e. maximal sets of formulas $\varphi(x_1, \ldots, x_n)$, $\varphi \in L$, which are consistent with T). $D(T) = \bigcup_{n<\omega} D_n(T)$. Ehrenfeucht **[E 2]** proved, in fact, that

THEOREM 0.2. *If $\mu = |D_n(T)| > |T|$ for some n, then in every $\lambda \geqq |T|$, T has at least μ nonisomorphic models.*

However, there are theories which fail to satisfy the conditions of the theorem, because of trivial reasons; for example, for the theory T_{ord} of the rational order

[1] This work was partially supported by NSF grant GP-22794.

I would like to thank M. Brown for writing the notes **[S 6]**, on which §4 is based.

$|D_n(T)| < \aleph_0$, but if the rationals are denoted by constants in the language, then for each real number, we have a different complete 1-type, so $|D_1(T)| = 2^{\aleph_0}$. This should motivate the following definition:

DEFINITION 0.2. L_λ is L when we add λ new individual constants. T is stable in λ (or λ-stable) if, for every complete extension T^* of T in L_λ, $|D_1(T^*)| \leqq \lambda$.

Another notion was suggested by **[E 1]**.

DEFINITION 0.3. T has the property (E) if T has a model M, with an infinite subset A and a formula $\varphi(x_1, \ldots, x_n)$ such that for every distinct $a_1, \ldots, a_n \in A$ there is a permutation θ such that $M \models \varphi[a_{\theta(1)}, \ldots, a_{\theta(n)}]$, and there is a permutation θ such that $M \models \neg\varphi[a_{\theta(1)}, \ldots, a_{\theta(n)}]$.

The idea was that (E) is a generalization of order, and hence implies T is complicated. He proves in **[E 1]** that for most $\lambda > |T|$, (E) implies T is not categorical in λ. Morley **[M 1]** proved that (E) implies T is not $\aleph_0$-stable, and the proof implies T is not λ-stable for every λ.

To me, this suggests the following questions.

(A) What can the stability-spectrum of a theory be? [I.e. the class $\{\lambda \mid T$ is λ-stable$\}$.]

(B) What is the connection between stability and order?

The answers are the following:

THEOREM 0.3. *For every T one of the following holds.*

(1) *For every λ, T is not stable in λ; we stipulate $\kappa(T) = \infty$.*

(2) *There is a cardinal $\kappa = \kappa(T) \leqq |T|^+$ such that T is λ-stable iff $\lambda = \lambda^{\underset{\smile}{\kappa}} + |D(T)|$ (where $\lambda^{\underset{\smile}{\kappa}} = \sum_{\mu<\kappa} \lambda^\mu$).*

(3) *There is a cardinal $\kappa = \kappa(T) \leqq |T|^+$ such that T is λ-stable iff $\lambda = \lambda^{\underset{\smile}{\kappa}} + |D(T)| + 2^{\aleph_0}$.*

See **[S 3]**, **[S 7]** for proof. If $\kappa(T) = \infty$, T is called unstable; if $\kappa(T) < \infty$, T is called stable; and if $\kappa(T) = \aleph_0$, T is superstable.

Note that if T^* is a complete extension of T in L_λ, then $\kappa(T) = \kappa(T^*)$.

THEOREM 0.4. *T is unstable iff T has a formula $\varphi(\bar{x}, \bar{y})$, a model M, and sequences $\bar{a}^n$ from M such that $M \models \varphi[\bar{a}^n, \bar{a}^m] \Leftrightarrow n < m$.*

(See **[S 9]**; and for a stronger result see **[S 5]**.)

The following results show that $\kappa(T)$ is significant for some problems.

THEOREM 0.5. (1) *T has a saturated model of cardinality λ iff $\lambda = \lambda^{\underset{\smile}{\lambda}} + |D(T)|$ or T is λ-stable*

(2) *If M_i, $i < \delta$, is an elementary increasing chain of models of T, each M_i is λ-saturated and the cofinality of δ is $\geqq \kappa(T)$ then $\bigcup_{i<\delta} M_i$ is λ-saturated.*

(See **[S 9**, §0.B] for references.)

THEOREM 0.6. *If $\lambda \geqq \kappa(T) + \aleph_1$, then among the λ-saturated models of T there is a prime one. If the cofinality of λ is $\geqq \kappa(T)$, $\lambda > \aleph_0$, this model is unique and can be characterized.*

(See [S 7], [S 10].) (This is true also for prime models over sets.)

The question of categoricity is a particular case of the question of the number of nonisomorphic models in a cardinality. The following show the relevancy of stability to this problem. For $T_1 \supset T$ let $PC(T_1, T)$ be the class of reducts of models of T_1 to L.

THEOREM 0.7. *Suppose T is not superstable. Then in every regular $\lambda > |T|$, T has 2^λ nonisomorphic models. Moreover, it has 2^λ models of cardinality λ, such that no one of them can be elementarily embedded into another. If $T_1 \supset T$, $|T_1| < \lambda$, we can demand that those models be in $PC(T_1, T)$.**

Let us introduce another important notion from Keisler [K 1].

DEFINITION 0.4. T has the f.c.p. if it has a formula $\varphi(x, \bar{y})$ such that, for arbitrarily large n,

$$(\exists \bar{y}_1 \cdots \bar{y}_n)\left[\neg (\exists x) \bigwedge_{i=1}^{n} \varphi(x, \bar{y}_i) \wedge \bigwedge_{j=1}^{n} (\exists x) \bigwedge_{i=1; i \neq j}^{n} \varphi(x, \bar{y}_i)\right] \in T.$$

DEFINITION 0.5. $I(\lambda, T_1, T)$ is the number of nonisomorphic models in $PC(T_1, T)$ of cardinality λ.

THEOREM 0.8. *If a stable T has the f.c.p., $\aleph_\alpha = \min(2^{\aleph_0}, |T|)$, $\aleph_\beta \geqq |T_1|$, then*

$$I(\aleph_\beta, T_1, T) \geqq 2^{|\beta - \alpha|}.$$

(See [S 11].)

One may ask whether the conditions in Theorems 0.7 and 0.8 are artificial. They are not.

THEOREM 0.9. *If T is countable, superstable and without the f.c.p., then there is $T_1 \supset T$, $|T_1| = 2^{\aleph_0}$, such that $I(\lambda, T_1, T) = 1$ for every $\lambda \geqq 2^{\aleph_0}$.*

(See [S 11].)

Another use of the f.c.p. is for the order $\triangleleft$ of Keisler [K 1]. For example,

THEOREM 0.10. *For countable T the following are equivalent:*

(A) *For every model M of T and regular ultrafilter D over λ, M^λ/D is λ^+-saturated.*

(B) *T does not have the f.c.p.*

(See [K 1] for $\neg$(B) $\to$ $\neg$(A) and [S 11] for the converse.)

The following answer questions of Keisler and are generalizations of our main theorem here.

THEOREM 0.11. *If every model of T of cardinality λ is $|T|^+$-universal [= embeds elementarily every model of T of power $\leqq |T|^+$], then T is categorical in λ.*

* *Added in proof* 12 *January* 1973. It has been proved that if T is unsuperstable, $\lambda \geqq |T_1| + \aleph_1$, but $I(\lambda, T_1, T) < 2^\lambda$ then (1) T is stable, (2) $T_1 \neq T$, (3) $\lambda = |T_1|$, (4) $\lambda^{\aleph_0} > \lambda$ and even more severe restrictions on λ.

THEOREM 0.12. *If every model of T of power $\lambda \geqq \aleph_1 + |T|$ is homogeneous, then every model of T of power μ, $\mu > |D(T)|$, is homogeneous; and, for every μ, $|T| + \aleph_1 \leqq \mu \leqq |D(T)|$, T has a nonhomogeneous model in μ.*

Like many other cases, the solution of a problem raises many new ones, and I would like to draw attention to some of them.

Problem 1. *What can the function $I(\lambda, T, T)$ be?*

A detailed conjecture appears in [**S 9**, p. 283, 13]. There is a small error in it, and it can be reformulated in a simplified form as follows: For every T, either $I(\lambda, T, T) = 2^\lambda$ for every $\lambda > |T|$ *or* there are $\mu \leqq 2^{2^{|T|}}$, $0 \leqq \kappa \leqq 2^{|T|}$, $0 \leqq \gamma < |T|^+$, γ a successor or zero such that $I(\aleph_\alpha, T, T) = \beth(|\alpha|^\kappa, \gamma) + \mu$ for every $\alpha \geqq 2^{2^{|T|}}$. If T is countable $\kappa \in \{0, 1, \aleph_0, 2^{\aleph_0}\}$. Its simplest case is: If, for some λ, $I(\lambda, T, T) > 2^{2^{|T|}}$, *then*, for every $\aleph_\alpha \geqq |T|$, $I(\aleph_\alpha, T, T) \geqq |\alpha + 1|$.

Problem 2. *What can the function*

$$m(\lambda, \mu, T) = \min\{I(\lambda, T_1, T): T_1 \supset T, |T_1| = \mu\}$$

be?

[See [**S 9**, p. 288, Conjecture 19].]

Problem 3. *The same as Problem* 1, *for the class of $|T|^+$-saturated models.*

REMARK. In general $I(\lambda, T_1, T)$ depends on set theoretic considerations (e.g. on the function $2^{\aleph_\alpha}$), so there is not much hope for a characterization of $I(\lambda, T_1, T)$.

There has been some progress. E.g. Problem 1 is essentially solved for $\aleph_0$-stable (= totally transcendental) theories.**

Let us now remark, where [**Ro 1**], [**Re 1**], [**S 1**] and [**S 2**] were stuck and how here we avoid it. Let T be categorical in $\lambda > |T|$. The first step was to note the proof in [**M 1**] implies T is stable in μ, $|T| \leqq \mu < \lambda$. The second step was to prove from this that T has a saturated model in λ. This works for most λ. The third step was to show that if M is a model of T, μ-saturated, not μ^+-saturated of power $>\mu$, then T has a nonsaturated model in λ. For this it was needed to show M contains a large set of indiscernibles over a set of power μ, and the existence of a prime model among the μ-saturated models over any set.

Now in the second step, if $\lambda^{\aleph_0} > \lambda$, and $\lambda > \mu \geqq |T| \Rightarrow \mu^{\aleph_0} = \mu$, then it cannot be done. T may be stable in μ for $\lambda > \mu \geqq |T|$, but not have a saturated model in λ, if it is not superstable. In the third step, the existence of prime models among the μ-saturated ones was established by $|T|$-stability only for $2^\mu > |T|$. In fact, the existence of μ-saturated models of T over sets, for T satisfying conditions (1), (2), (3) of §4, can be shown in general only for $\mu \geqq \aleph_1$. So we are left with the case of a not $|T|$-saturated model M. The fourth step was to note $\|M\| < \lambda$, so in every $\mu \geqq \lambda$, T has only saturated models; hence T is categorical in μ. In [**Re 1**], [**S 1**] and [**S 2**] it was noted that [**M 2**] implies $\|M\|$ cannot be too large (< the Hanf number of omitting a type).

** *Added in proof* 12 *January* 1973. It seems that Problem 3 is solved, if we restrict ourselves to λ's satisfying $\lambda^{|T|} = \lambda$.

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However there was no way to go from a small non-$\aleph_1$-saturated model upward. So how do we go? In §2 we establish that T is stable, and every model of T is locally saturated. We use here **[S 9]** on Δ-types (Δ finite) and the existence of prime models over sets among the $|T^+|$-saturated models.

In §3 we define a degree on formulas, and show it is always $<\infty$. For stable T this is equivalent to the superstability of T. So we could prove that T is stable in λ, hence has a saturated model in λ; but we prefer not to do so.

Marsh **[Ma]** uses minimal formulas to simplify Morley's proof. We use a generalization of this concept—weakly minimal formulas. First we prove that in every model of cardinality $> |T|$ there is a weakly minimal formula $\theta(x, \bar{a})$ provided that the model is locally saturated. Then we prove that over every set $A \subset |M|$, $M \models T$ which satisfies the Tarski-Vaught test for formulas $(\exists x)[\theta(x, \bar{a}) \wedge \varphi(x, \bar{z})]$ there is a model N of T, $A \subset N \prec |M|$, $\{b \mid b \in N, N \models \theta[b, \bar{a}]\} \subset A$. This N is our substitute for the prime model. Those two theorems are the crux of the matter. Now we prove that if M is a nonsaturated model of T of cardinality $> |T|$, $\theta(x, \bar{a})$ a weakly minimal formula, then M omits some type which contains $\theta(x, \bar{a})$. This enables us to prove the existence of a saturated model in every $\mu > |T|$. Then if M is not saturated, we extend $\theta(x, \bar{a})$ properly and take a "prime" model, thus pushing upward the cardinality of the nonsaturated model. Remembering that every two saturated models of T of the same power are isomorphic, we conclude.[2]

The proof here is somewhat different from the one in the notes **[S 6]**, as this article is not self-contained, and the proof that Deg $[x = x] < \infty$ has been changed, and hopefully simplified. The theorems appearing here are not the best possible. We prove only the ones needed; e.g. in §4 we are interested only in models $> |T|$, but corresponding results hold without such restrictions.

1. Notation. Let λ, μ, κ denote (infinite) cardinals, α, β, γ, i, j ordinals, δ a limit ordinal, and k, l, m, n natural numbers. Let η, ν denote sequences of ordinals, $l(\eta)$ the length of η, $\eta(i)$ the ith element of η, and ${}^{\beta}\alpha = \{\eta \mid l(\eta) = \beta, (\forall i < \beta)\eta(i) < \alpha\}$, ${}^{\beta>}\alpha = \bigcup_{\gamma<\beta} {}^{\gamma}\alpha$. $|X|$ is the cardinality of X. $\bar{a}^\frown\bar{b}$ is the concatenation of the sequences $\bar{a}, \bar{b}$.

L will be a first-order language. T a complete theory in L, φ, ψ, θ, χ, ρ will be formulas of L, x, y, z variables and $\bar{x}$, $\bar{y}$, $\bar{z}$ finite sequences of variables. Let $\bar{M}$ be a $\bar{\kappa}$-saturated model of T, where $\bar{\kappa}$ is greater than the cardinalities of any other model of T we shall deal with. So we can restrict ourselves to elementary submodels of $\bar{M}$ of cardinality $<\bar{\kappa}$, and we denote them by M, N. $M \models \varphi[\bar{a}]$ means satisfaction. As $M \models \varphi[\bar{a}] \Leftrightarrow \bar{M} \models \varphi[\bar{a}]$ we omit M. Let $\varphi(M, \bar{a}) = \{c \mid c \in |M|, \models \varphi[c, \bar{a}]\}$. $|M|$ is the universe of M so $\|M\|$ is the cardinality of M. A, B, C will be subsets of $|\bar{M}|$ of cardinality $<\bar{\kappa}$. Let a, b, c, d, e be elements of $\bar{M}$. If $\bar{a} = \langle a_0 \cdots a_n\rangle$, $a_0, \ldots, a_n \in A$, we write $\bar{a} \in A$. Let φ^i be φ for $i = 0$ and $\neg\, \varphi$ for $i = 1$.

A Δ-m-type over A is a set of formulas $\varphi(x_0, \ldots, x_{m-1}, \bar{a})^i$ $[\bar{a} \in A$, $\varphi \in \Delta$, $i \in 2]$ which is finitely satisfied in $\bar{M}$. If $m = 1$ we write x instead of x_0. If $\Delta = L$ we

[2] For information on the models of categorical theory see also **[H 1]**.

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omit it, if $m = 1$ we omit it. The type $\bar{a}$ realizes over B is $\{\varphi(\bar{x}, \bar{b}) \mid \bar{b} \in B, \models \varphi[\bar{a}, \bar{b}]\}$. $S^m_\Delta(A)$ is the set of maximal Δ-m-types over A, $S^m(A) = S^m_L(A)$. $\{\bar{a}^i \mid i < \alpha\}$ is Δ-n-indiscernible over B, if the lengths of the sequences $\bar{a}^i$ are equal, and $\varphi(\bar{x}^1, \ldots, \bar{x}^n; \bar{y}) \in \Delta$, θ a permutation of $\{1, \ldots, n\}$, $i(1) < \cdots < i(n) < \alpha$, $j(1) < \cdots < j(n) < \alpha$, $\bar{b} \in B$ implies $\models \varphi[\bar{a}^{i(\theta(1))}, \ldots, \bar{a}^{i(\theta(n))}; \bar{b}] \equiv [\bar{a}^{j(\theta(1))}, \ldots, \bar{a}^{j(\theta(n))}; \bar{b}]$. If $\Delta = L$ we omit it; if it holds for every n, we omit it.

2. Stability and local saturation. The main results are: if T is categorical in $\lambda > |T|$, then

(1) T is μ-stable for $|T| \leqq \mu < \lambda$,

(2) every model of T is locally saturated.

THEOREM 2.1. *For $\lambda > |T|$, T has a model M of cardinality λ, such that for all $m < \omega$, $A \subset |M|$, $|\{p \mid p \in S^m(A)$, p realized in $M\}| \leqq |A| + |T|$.*

PROOF. Take M as an Ehrenfeucht-Mostowski model, which is the closure of a well-ordered set (see **[Mo 1]**).

CONCLUSION 2.2. If T is categorical in λ, $|T| \leqq \mu < \lambda$, *then* T is stable in μ.

REMARK. Morley **[Mo 1]** proved 2.1, 2.2 for $|T| = \mu = \aleph_0$, and the proof works as well for the other cases as noted in **[Ro 1]**, **[Re 1]**, **[S 1]**, **[S 2]**.

DEFINITION 2.1. M is locally saturated if for every finite Δ, and a Δ-m-type p over $|M|$ of cardinality $< \|M\|$, p is realized in M.

THEOREM 2.3. *If T is categorical in $\lambda > |T|$, then every model of T is locally saturated.*

PROOF. By 2.2, T is stable in $|T|$; so by **[S 9**, Theorem 2.13] for every finite Δ, m and for every A, $|S^m_\Delta(A)| \leqq |A| + \aleph_0$. Let M be a model of cardinality λ. Define M_i, $i \leqq \lambda$, such that $M_0 = M$, $M_\delta = \bigcup_{i<\delta} M_i$, and M_{i+1} is an (elementary) extension of M_i, in which, for every finite Δ, m, every $p \in S^m_\Delta(|M_i|)$ is realized. Now, for every $\mu < \lambda$, and finite Δ, m, M_{μ^+} is a model of T of power λ and every Δ-m-type over it of cardinality $\leqq\mu$ is realized (as it is a type over some $|M_i|$, $i < \mu^+$). As T is categorical in λ, and this holds for every $\mu < \lambda$ it follows that every model of T of cardinality λ is locally saturated.

Suppose now M is a nonlocally saturated model of T. Let Δ, m be finite, p a Δ-m-type over $|M|$ which is not realized, $|p| < \|M\|$. If $\Delta = \{\varphi_i(\bar{x}, \bar{y}^i) \mid i < |\Delta|\}$, let

$$\varphi(\bar{x}, \bar{y}) = \bigwedge_{i<|\Delta|; j\in 2} [y^{i,1}_j = y^{i,2}_j \to \varphi_i(\bar{x}, \bar{y}^i)^j].$$

Clearly for every $i < |\Delta|$, $j \in 2$ and $\bar{a} \in |M|$, there is a $\bar{b} \in |M|$ such that $\models (\forall \bar{x})[\varphi_i(\bar{x}, \bar{a})^j \equiv \varphi(\bar{x}, \bar{b})]$. So w.l.o.g. let $p = \{\varphi(\bar{x}, \bar{a}^i) \mid i < |p|\}$;

$$A = \bigcup \{\mathrm{Rang}\, \bar{a}^i \mid i < |p|\},$$

so $|A| \leqq |p| \cdot \aleph_0 < \|M\|$. So by **[S 9**, Theorem 5.8] for every finite Δ, n there is in M an infinite Δ-indiscernible set over A. Hence, by the compactness theorem (using extra predicates), T has a $|T|^+$-saturated model N, $q = \{\varphi(\bar{x}, \bar{b}_i) : i < |q|\}$ a type over $|N|$, omitted by N and $\{c_i \mid i < |T|^+\}$ an indiscernible set over $B = \bigcup \{\mathrm{Rang}\, \bar{b}_i \mid i < |q|\}$. Let $\mu \geqq \beth_{\alpha+\omega}$, $\beth_\alpha \geqq |B|$ for some α. Define c_i $|T|^+ \leqq i < \mu$ such that $\{c_i \mid i < \mu\}$ is indiscernible over B. By **[Re 1]** or **[S 2]**, there is a $|T|^+$-prime model over $B \cup \{c_i \mid i < \mu\}$, and it omits q. Using Vaught's two-cardinal theorem for cardinals far apart (see Vaught **[V 1]** or **[Mo 2]** or **[C 1]**) (adding the predicate $p^M = \{\bar{b}_i \mid i < |q|\}$) we get a non-locally-saturated model of T in λ; a contradiction.

THEOREM 2.4. *If T is categorical in $\lambda > |T|$, M, N models of T, $|M| \subset |N|$, $\bar{a} \in |M|$, and $\varphi(x, \bar{a})$ is not algebraic, then there is $c \in |N| - |M|$, $\vDash \varphi[c, \bar{a}]$.*

PROOF. If M_1 is locally saturated $\bar{a} \in |M_1|$ and $\varphi(x, \bar{a})$ is not algebraic, then

$$\{\varphi(x, \bar{a}) \wedge x \neq c \mid c \in \varphi(M_1, \bar{a})\}$$

is a type over $|M_1|$ which M_1 omits, hence its cardinality is $\|M_1\|$, so $|\varphi(M_1, \bar{a})| = \|M_1\|$.

Now if M, N, $\varphi(x, \bar{a})$ are a counterexample to 2.4, we get a contradiction as in the proof of 2.3, using Vaught's two-cardinal theorem (see **[Mo 3]** or **[Be 1]** or **[C 1]**).

LEMMA 2.5. *The following properties contradict the stability of T.*

(A) *For some $\phi(\bar{x}, \bar{y})$ there are sequences $\bar{a}^n$, $n < \omega$, such that, for all subsets w of ω, $\{\varphi(\bar{x}, \bar{a}^n) \mid n \in w\} \cup \{\neg\, \varphi(x, \bar{a}^n) \mid n \notin w\}$ is consistent.*

(B) *For some $\varphi(\bar{x}, \bar{y})$ there are sequences $\bar{a}^n$, $n < \omega$, such that for every m, $p_m = \{\varphi(\bar{x}, \bar{a}^n) \mid n < m\} \cup \{\neg\varphi(\bar{x}, \bar{a}^n) \mid n \geqq m\}$ is consistent.*

(C) *T has the property* (E) *defined in the Introduction.*

PROOF. If (A) holds then (B) holds with the same φ and $\bar{a}^n$. If (B) holds let $\bar{c}^m$ realize p_m, and let $\bar{b}^m = \bar{c}^m{}^\frown \bar{a}^m$. Let $\psi(\bar{z}, \bar{z}^1) = \psi(\bar{x}^\frown \bar{y}, \bar{x}^1{}^\frown \bar{y}^1) = \varphi(\bar{x}, \bar{y}^1)$. As $\vDash \varphi[\bar{c}^m, \bar{a}^n]$ iff $m \leqq n$, clearly $\vDash \psi[\bar{b}^m, \bar{b}^n]$ iff $\varphi[\bar{c}^m, \bar{a}^n]$ iff $m \leqq n$. So $\psi(\bar{z}, \bar{z}^1)$ shows (E) holds. If (E) holds by **[S 9**, Theorem 5.3B], T is unstable.

3. Here we define for every formula $\varphi(x, \bar{a})$ a degree which measures the complexity of $\varphi(\bar{M}, \bar{a})$. We prove that if T is categorical in $\lambda > |T|$, then the degree of every formula is $< \infty$. Lemma 3.1 and Theorem 3.2 have appeared in **[S 9**, §6].

DEFINITION 3.1. We define $\mathrm{Deg}[\varphi(x, \bar{a})]$ as an ordinal, or ∞ (stipulating $\alpha < \infty$ for any ordinal). So it suffices to define by induction on α when $\mathrm{Deg}[\varphi(x, \bar{a})] \geqq \alpha$:

(A) $\alpha = 0$: $\mathrm{Deg}[\varphi(x, \bar{a})] \geqq 0$ iff $\vDash (\exists x)\varphi(x, \bar{a})$ (otherwise the degree is not defined or treated as -1).

(B) $\alpha = \delta$: $\mathrm{Deg}[\varphi(x, \bar{a})] \geqq \delta$ *iff* for every $\beta < \delta$, $\mathrm{Deg}[\varphi(x, \bar{a})] \geqq \beta$.

(C) $\alpha = \beta + 1$: $\mathrm{Deg}[\varphi(x, \bar{a})] \geqq \alpha$ *iff* there is a formula $\psi(x, \bar{y})$ and sequences $\bar{c}^i$, $i < |T|^+$, such that

(1) $\mathrm{Deg}[\varphi(x, \bar{a}) \wedge \psi(x, \bar{c}^i)] \geqq \beta$ for every i,

(2) the $\psi(x, \bar{c}^i)$'s are almost contradictory; that is, there is $m < \omega$ such that,

for every set w of $\geqq m$ ordinals $< |T|^+$,

$$\vDash \neg(\exists x)\left[\bigwedge_{i\in w} \psi(x, \bar{c}^i)\right].$$

We say $\{\psi(x, \bar{c}^i) \mid i < |T|^+\}$ is m-almost contradictory.

LEMMA 3.1. (A) *If* $\vDash (\forall x)[\varphi(x, \bar{a}) \to \psi(x, \bar{b})]$ *then* $\mathrm{Deg}[\varphi(x, \bar{a})] \leqq \mathrm{Deg}[\psi(x, \bar{b})]$.

(B) *If $\bar{a}$ and $\bar{b}$ realize the same type then* $\mathrm{Deg}[\varphi(x, \bar{a})] = \mathrm{Deg}[\varphi(x, \bar{b})]$.

(C) *There is an ordinal* $\alpha_0 < (2^{|T|})^+$ *such that for no* $\varphi(x, \bar{a})$, $\mathrm{Deg}[\varphi(x, \bar{a})] = \alpha_0$. *Hence* $\mathrm{Deg}[\psi] \geqq \alpha_0$ *implies* $\mathrm{Deg}[\psi] > \alpha_0$.

PROOF. (A) It can be easily shown by induction on α that $\mathrm{Deg}[\varphi(x, \bar{a})] \geqq \alpha \Rightarrow \mathrm{Deg}[\psi(x, \bar{b})] \geqq \alpha$.

(B) It can be easily shown by induction on α that $\mathrm{Deg}[\varphi(x, \bar{a})] \geqq \alpha$ iff $\mathrm{Deg}[\varphi(x, \bar{b})] \geqq \alpha$.

(C) follows easily from (B), as the number of $\varphi(x, \bar{y})$ and complete n-types is $\leqq |T| \cdot 2^{|T|} = 2^{|T|}$.

By the way, [**S 9**, Theorem 6.4] says that in fact $\alpha_0 \leqq |T|^+$.

THEOREM 3.2. *If* $\mathrm{Deg}[x = x] > \alpha_0$, *then there are formulas* $\varphi_\eta(x, \bar{a}_\eta)$, $\eta \in {}^{\omega >}\mu$, *where* $\mu = |T|^+$ *such that*

(A) *for every* $0 < k < l < \omega$, $\eta \in {}^{l}\mu$, $\vDash (\forall x)[\varphi_\eta(x, \bar{a}_\eta) \to \varphi_{\eta|k}(x, \bar{a}_{\eta|k})]$,

(B) *for every* $\eta \in {}^{\omega >}\mu$, $\mathrm{Deg}[\varphi_\eta(x, \bar{a}_\eta)] \geqq \alpha_0$,

(C) *for every* $\eta \in {}^{\omega >}\mu$, $\{\varphi_{\eta\frown\langle i\rangle}(x, \bar{a}_{\eta\frown\langle i\rangle}) \mid i < \mu\}$ *is almost-contradictory*,

(D) *for every* $\eta \in {}^{\omega >}\mu$, $i < \mu$, $\varphi_{\eta\frown\langle i\rangle} = \varphi_{\eta\frown\langle 0\rangle}$.

PROOF. We shall define the $\varphi_\eta(x, \bar{a}_\eta)$ by induction on $\varphi(\eta)$; let, for $\eta = \langle\ \rangle =$ the empty sequence, $\varphi_{\langle\ \rangle}(x, \bar{a}_\eta) = [x = x]$. If $\varphi_\eta(x, \bar{a}_\eta)$ is defined, then, by (B), $\mathrm{Deg}[\varphi_\eta(x, \bar{a}_\eta)] \geqq \alpha_0$. Hence, by Lemma 3.1(C), $\mathrm{Deg}[\varphi_\eta(x, \bar{a}_\eta)] > \alpha_0$; hence by the definition of degree there are almost-contradictory formulas $\psi(x, \bar{c}^i)$, $i < \mu$, such that $\mathrm{Deg}[\varphi_\eta(x, \bar{a}_\eta) \wedge \psi(x, \bar{c}^i)] \geqq \alpha_0$. So define $\varphi_{\eta\frown\langle i\rangle}(x, \bar{a}_{\eta\frown\langle i\rangle}) = \varphi_\eta(x, \bar{a}_\eta) \wedge \psi(x, \bar{c}^i)$.

THEOREM 3.3. *Suppose T is categorical in* $\lambda > |T|$. *Then* $\mathrm{Deg}[x = x] < \infty$, *hence by* 3.1(A) *the degree of every formula is* $< \infty$.

PROOF. There is a theory $T_{\mathrm{sk}} \supset T$ in a language $L_{\mathrm{sk}} \supset T$ such that $|T_{\mathrm{sk}}| = |L_{\mathrm{sk}}| = |T|$ and T_{sk} has Skolem functions. Also there is a model M^* for L_{sk} such that $|M^*|$ is the closure of $\{y_i \mid i < \lambda\}$ which is an indiscernible sequence (see Ehrenfeucht and Mostowski [**E 3**] or [**Mo 1**] or [**C 1**] or [**Be 1**]). So for every $a \in |M^*|$ there is a term $\tau \in L_{\mathrm{sk}}$ and $i_1 < \cdots < i_n < \lambda$ such that $M^* \vDash a = \tau[y_{i_1}, \ldots, y_{i_n}]$. Let $\bar{\tau}$ denote a finite sequence of terms; hence, for every $\bar{a} \in |M^*|$, there is $\bar{\tau}$ and $i_1 < \cdots < i_n < \lambda$ for which $M^* \vDash \bar{a} = \bar{\tau}[y_{i_1}, \ldots, y_{i_n}]$. We say i, j realize the same cut over $X \subset \lambda = \{i \mid i < \lambda\}$, $i = j \,(\mathrm{mod}\, X)$, if for every $k \in X$, $i < k \Leftrightarrow j < k$; or equivalently $\min\{k \mid k \in X, i < k\} = \min\{k \mid k \in X, j < k\}$. Note that this is an equivalence relation with $\leqq |X| + \aleph_0$ equivalence classes.

Suppose $\mathrm{Deg}[x = x] = \infty$. Then, by Theorem 3.2, there are φ_η, a_η for $\eta \in {}^{\omega >}\mu$ which satisfy (A), (B), (C), (D) from 3.2, $\mu = |T|^+$. By adding unnecessary constants we can assume $\eta \neq \nu \Rightarrow \bar{a}_\eta \neq \bar{a}_\nu$. Let W be a subset of ${}^{\omega}\mu$ of cardinality μ

such that for every limit ordinal $\delta < \mu$ of cofinality ω there is $\eta_\delta = \eta \in W$ such that $\delta > \eta(n+1) > \eta(n)$ for every $n > 0$, and $\delta = \sup\{\eta(n) \mid n < \omega\}$. Let c_η realize $p_\eta = \{\varphi_{\eta|n}(x, \bar{a}_{\eta|n}) \mid n < \omega\}$ and $A = \bigcup \{\text{Rang}\, \bar{a}_\eta \mid \eta \in {}^{\omega>}\mu\} \cup \{c_\eta \mid \eta \in W\}$.

Clearly $|A| \leqq \mu = |T|^+ \leqq \lambda$, so there is a model M of T of cardinality λ, $A \subset |M|$. By the categoricity of T in λ, M is isomorphic to the reduct of M^* to L. So w.l.o.g., $A \subset |M^*|$.

Now we define subsets X_α of $\lambda = \{i \mid i < \lambda\}$ for $\alpha < \mu$ by induction on α such that [denote $B_\alpha = \text{cl}\{y_i \mid i \in X_\alpha\}$, cl-under the functions of L_{sk}]

(i) X_0 is empty, $X_\delta = \bigcup_{i<\delta} X_i$ and $|X_\alpha| \leqq |T|$.

(ii) For every α there is η such that $\bar{a}_\eta \in B_{\alpha+1}$, $\bar{a}_\eta \notin B_\alpha$.

(iii) If $\bar{a}_\eta \in B_\alpha$, $\beta = \sup\{\eta(n) + 1 \mid n < \omega\}$, and $\nu \in {}^{\omega>}\beta$, then $\bar{a}_\nu \in B_\alpha$.

(iv) If $\bar{a}_\eta \in B_{\alpha+1}$, and for some $i < \mu$, $\bar{a}_{\eta^\wedge\langle i\rangle} = \bar{\tau}(y_{i_1}, \ldots, y_{i_n}) \notin B_{\alpha+1}$, $i_1 < \cdots < i_n$, *then* for infinitely many $j < \mu$, there are $j_1 < \cdots < j_n$, $j_l = i_l \pmod{X_\alpha}$ for $1 \leqq l \leqq n$, such that $\bar{a}_{\eta^\wedge\langle j\rangle} = \bar{\tau}(y_{j_1}, \ldots, y_{j_n}) \in B_{\alpha+1}$.

The construction is straightforward. Note that $|X_\alpha| < \mu$ implies $|B_\alpha| < \mu$, hence $|\{\eta \mid \bar{a}_\eta \in B_\alpha\}| < \mu$ (as $\eta \neq \nu \Rightarrow \bar{a}_\eta \neq \bar{a}_\nu$).

Define $\delta = \sup\{\eta(n) \mid n < \omega, \bar{a}_\eta \in B_\omega\}$. By (ii), δ is a limit ordinal of cofinality ω (as $X_\omega = \bigcup_{n<\omega} X_n$). So by the definition of W there is $\eta = \eta_\delta \in W$, $\delta > \eta(n+1) > \eta(n)$, $\delta = \sup\{\eta(n) \mid n < \omega\}$, and so $c_\eta \in A \subset |M^*|$ realizes p_η. Let $c_\eta = \tau(y_{i(1)}, \ldots, y_{i(n)})$, $i(1) < \cdots < i(n)$. For l, $1 \leqq l \leqq n$, let

$$j(l) = \inf\{j \mid j \in X_\omega, j \geqq i(l)\}$$

and $k_0 < \omega$ be such that $j(1), \ldots, j(n) \in X_{k_0}$ and $\bar{a}_{\eta|1} \in B_{k_0}$. W.l.o.g., $k_0 = 1$, and let k be maximal such that $\bar{a}_{\eta|k} \in B_2$ (there is such k as $\sup\{\eta(n) \mid n < \omega, \bar{a}_{\eta|n} \in B_2\} < \delta$ by (ii)). Let $\nu = \eta \mid k$, $\eta(k) = i$, and $\bar{a}_{\nu^\wedge\langle i\rangle} = \bar{\tau}^*(y_{k_1}, \ldots, y_{k_m})$. By (iv) for infinitely many $\beta < \mu$, there are $j(l, \beta)$ for $1 \leqq l \leqq m$, $j(l, \beta) = k_l \pmod{X_1}$ and $\bar{a}_{\nu^\wedge\langle\beta\rangle} = \bar{\tau}^*(y_{j(1,\beta)}, \ldots, y_{j(m,\beta)}) \in B_2$. Clearly also

$$j(l, \beta) = k_l \pmod{X_1 \cup \{i(1), \ldots, i(n)\}}.$$

Hence in M^*, $\bar{a}_{\nu^\wedge\langle i\rangle}$, $\bar{a}_{\nu^\wedge\langle\beta\rangle}$ realize the same type (in L_{sk}) over $\{y_\alpha : \alpha \in X_1 \text{ or } \alpha = i(l), 1 \leqq l \leqq n\}$ hence over c_η. So for infinitely many $\beta < \mu$, $\models \varphi_{\nu^\wedge\langle i\rangle}[c_\eta, \bar{a}_{\nu^\wedge\langle\beta\rangle}]$; but, by 3.2(D), $\varphi_{\nu^\wedge\langle i\rangle} = \varphi_{\nu^\wedge\langle\beta\rangle}$ so $\models \varphi_{\nu^\wedge\langle\beta\rangle}[c_\eta, \bar{a}_{\nu^\wedge\langle\beta\rangle}]$. So c_η satisfies infinitely many formulas $\varphi_{\nu^\wedge\langle\beta\rangle}(x, \bar{a}_{\nu^\wedge\langle\beta\rangle})$, $\beta < \mu$. But they are almost-contradictory by 3.2(C); a contradiction. Hence $\text{Deg}[x = x] < \infty$.

REMARK. In fact $\text{Deg}[x = x] < \omega$; and for every T, $\text{Deg}[\varphi] < \infty \Rightarrow \text{Deg}[\varphi] < |T|^+$. For countable T, Baldwin **[Ba 2]**, **[Ba 3]** proves that $\text{Deg}[x = x] < \omega$ provided that T is categorical in $\aleph_1$.

4. Suppose through this section that the following conditions hold:

(1) T is stable in $|T|$,

(2) every model of T is locally saturated, and $M \prec N$, $\bar{a} \in |M|$, $\theta(x, \bar{a})$ nonalgebraic implies $\theta(N, \bar{a}) \not\subseteq |M|$,

(3) $\text{Deg}[x = x] < \infty$.

We have proved in §§2 and 3 that they are satisfied if T is categorical in some $\lambda > |T|$.

DEFINITION 4.1. (A) $\psi^0(x, \bar{a}^0), \dots, \psi^n(x, \bar{a}^n)$ are a φ-partition of $\theta(x, \bar{a})$ (where $\varphi = \varphi(x, \bar{y})$) if

(1) $\vDash (\forall x)[\theta(x, \bar{a}) \leftrightarrow \bigvee_{i \leqq n} \psi^i(x, \bar{a}^i)]$.

(2) For $i \neq j$, $\vDash \neg (\exists x)[\psi^i(x, \bar{a}^i) \wedge \psi^j(x, \bar{a}^j)]$.

(3) $\psi^i(x, \bar{a}^i)$ is φ-minimal; that is,

(B) for no $\varphi(x, \bar{b})$,

$$|\{c \mid \vDash \psi^i[c, \bar{a}^i] \wedge \varphi[c, \bar{b}]\}| \geqq \aleph_0$$

and

$$|\{c \mid \vDash \psi^i[c, \bar{a}] \wedge \neg \varphi[\bar{c}, \bar{b}]\}| \geqq \aleph_0.$$

(C) If $\vDash \psi^l[c, \bar{a}^l]$, then $\psi^l[x, \bar{a}^l]$ is the φ-piece of c.

REMARK. (1) If $\theta(x, \bar{a})$ has a φ-partition, then it has a φ-partition of the form $\psi(x, \bar{a}^0), \dots, \psi(x, \bar{a}^n)$.

(2) If $\vDash \theta[c, \bar{a}]$, $\psi^l(x, \bar{a}^l)$, $0 \leqq l \leqq n$, is a φ-partition of $\theta(x, \bar{a})$, then some $\psi^l(x, \bar{a}^l)$ is the φ-piece of c.

DEFINITION 4.2. $\theta(x, \bar{a})$ is weakly minimal if, for every φ, it has a φ-partition. B partitions $\theta(x, \bar{a})$, if, for every φ, $\theta(x, \bar{a})$ has a φ-partition $\psi(x, \bar{a}^i)$, $i = 0, \dots, n$, such that $\bar{a}^i \in B$, and $\bar{a} \in B$.

REMARK. We sometimes speak as if this partition were unique and denote it by $\psi_\varphi(x, \bar{a}^i)$, $i \leqq n = n_\varphi$, $\bar{a}^i = \bar{a}^i_\varphi$.

DEFINITION 4.3. A type p is minimal if, for no $\varphi(x, \bar{a})$, both $p \cup \{\varphi(x, \bar{a})\}$ and $p \cup \{\neg \varphi(x, \bar{a})\}$ are nonalgebraic types.

LEMMA 4.1. (A) *If $\theta(x, \bar{a})$ is weakly minimal, $\bar{a} \in |M|$, then there is $A \subset |M|$, $|A| \leqq |T|$ which partitions $\theta(x, \bar{a})$.*

(B) *If A partitions $\theta(x, \bar{a})$, $\theta(x, \bar{a}) \in p \in S(A)$, then p is minimal.*

PROOF. Immediate.

THEOREM 4.2. *If $\|M\| > |T|$, and $\theta(x, \bar{a})$ is a nonalgebraic formula with minimal degree ($\geqq 0$), $\bar{a} \in |M|$ then $\theta(x, \bar{a})$ is weakly minimal.*

PROOF. Define, by induction on $l(\eta)$, $\bar{a}_\eta$ for $\eta \in {}^{\omega>}2$, such that if $\bar{a}_\eta$ is defined, $\eta \in {}^n2$, then, for $\nu = \eta\hat{}\langle 1 \rangle$, and $\nu = \eta\hat{}\langle 0 \rangle$, $p_\nu = \{\theta(x, \bar{a}) \wedge \varphi(x, \bar{a}_{\nu|l})^{\eta(l)} \mid l \leqq n\}$ is a nonalgebraic type.

If, for $l < l(\eta)$, $\bar{a}_{\eta|l}$ is defined, and there is $\bar{b}$ such that $p_\eta \cup \{\varphi(x, \bar{b})\}$ and $p_\eta \cup \{\neg \varphi(x, \bar{b})\}$ are nonalgebraic types, then we define $\bar{a}_\eta = \bar{b}$. Otherwise $\bar{a}_\eta$ is not defined. If, only for finitely many η's, $\bar{a}_\eta$ is defined, then clearly our conclusion holds.

So assume $\bar{a}_\eta$ is defined for infinitely many η's. Clearly if $\bar{a}_\eta$ is defined, $l \leqq l(\eta)$, then $\bar{a}_{\eta|l}$ is defined. Hence by König's lemma, there is an $\eta \in {}^\omega 2$ such that, for every $l < \omega$, $\bar{a}_{\eta|l}$ is defined. W.l.g. $\aleph_0 = |\{l \mid l < \omega, \eta(l) = 0\}|$ let l_k be the kth $l < \omega$ such that $\eta(l) = 0$, and $\bar{a}^n = \bar{a}_{\eta|l_k}$. So for every $n < \omega$, $\{\theta(x, \bar{a})\} \cup \{\varphi(x, \bar{a}^i) \mid i < n\} \cup \{\neg \varphi(x, \bar{a}^n)\}$ is a nonalgebraic type, and let c^n realize this type.

By Ramsey's theorem [**Ra 1**], we can assume that, for $n < i$, $\models \varphi[c^n, \bar{a}^i] \equiv \varphi[c^0, \bar{a}^1]$. As by definition $\models \varphi[c^n, \bar{a}^i]$ for $n > i$, by Lemma 2.5(B) and condition (1), clearly $i \neq n \Rightarrow \models \varphi[c^n, \bar{a}^i]$.

By Lemma 2.5(A) and compactness, there exists m' such that there are no sequences $\bar{a}_0, \ldots, \bar{a}_{m'-1}$ which make $\{\varphi(x, \bar{a}_k): k \in w\} \cup \{\neg\varphi(x, \bar{a}_k): k \notin w \text{ and } k < m'\}$ consistent for every choice of $w \subset \{0, 1, \ldots, m'-1\}$.

Another application of Ramsey's theorem shows that we can assume w.l.o.g. that, for every $w \subset \{0, 1, \ldots, m'-1\}$, *either*, for every $i_0 < \cdots < i_{m',m'-1} < \omega$,

$$(*) \qquad \models (\exists x)\left[\bigwedge_{k\in w} \varphi(x, \bar{a}^{i_k}) \wedge \bigwedge_{k\notin w; k<m'} \neg\, \varphi(x, \bar{a}^{i_k})\right]$$

or, for every $i_0 < \cdots < i_{m'-1} < \omega$,

$$(**) \qquad \models \neg\, (\exists x)\left[\bigwedge_{k\in w} \varphi(x, \bar{a}^{i_k}) \wedge \bigwedge_{k\notin w; k<m'} \neg\, \varphi(x, \bar{a}^{i_k})\right].$$

If there were $w_1 \neq w_2$ such that $|w_1| = |w_2|$ and (*) holds for w_1 and fails for w_2, then

$$(\exists x)\left[\bigwedge_{k\in w_1} \varphi(x, \bar{y}^{i_k}) \wedge \bigwedge_{k\notin w_1; k<m'} \neg\, \varphi(x, \bar{y}^{i_k})\right]$$

would be connected and antisymmetric over $\{\bar{a}^0, \ldots, \bar{a}^m, \ldots\}$ since w_2 is obtained from w_1 by a permutation. That would contradict the stability of T by Lemma 2.5(C), so (*) must hold or fail depending only on $|w|$. By our choice of m', there is a w_0 s.t. (*) fails. So (**) holds for all w s.t. $|w| = |w_0|$. Then there is no b which

(1) satisfies at least $|w_0|$ of the $\varphi(x, \bar{a}^n)$, and

(2) fails to satisfy at least $m' - |w_0|$ of the $\varphi(x, \bar{a}^n)$.

Define ψ by $\psi(x, \bar{b}^n) = \neg\varphi(x, \bar{a}^{n+m'}) \wedge \bigwedge_{k<m'} \varphi(x, \bar{a}^k) \wedge \theta(x, \bar{a})$, where $\bar{b}^n$ is $\bar{a} \frown \bar{a}^1 \frown \cdots \frown \bar{a}^{m'-1} \frown \bar{a}^{n+m'}$. If an element satisfies m' of the $\psi(x, \bar{b}^n)$'s, then it satisfies m' of the $\varphi(x, \bar{a}^k)$ and m' of the $\neg\varphi(x, \bar{a}^k)$, which is impossible. So the formulas $\psi(x, \bar{b}^n)$ are almost contradictory, as in the definition of degree; but there are only ω many of them, rather than $|T|^+$ many. We now use condition (2).

Clearly each $\psi(x, \bar{b}^n)$ is realized infinitely many times. We claim that there is an m_ψ corresponding to ψ such that for every sequence $\bar{e} \in |M|$, if $|\psi(M, \bar{e})| \geqq m_\psi$, then $|\psi(M, \bar{e})| \geqq \aleph_0$.

If it were not the case, let M be a model of T of cardinality $> 2^{\aleph_0}$. We are assuming that for all $n \in \omega$ there is an $\bar{e}^n$ such that $n < |\psi(M, \bar{e}^n)| < \aleph_0$.

Let D be an ultrafilter over ω and let $N = M^\omega/D$, and let $\bar{e} = \langle \ldots, \bar{e}^n, \ldots\rangle/D$.

Clearly $|\psi(N, \bar{e})| = \pi\, |\psi(M, \bar{e}^n)|/D = 2^{\aleph_0} < \|N\|$. But $\{\psi(x, \bar{e}) \wedge x \neq a : a \in \psi(N, \bar{e})\}$ is a type of cardinality $< \|N\|$ involving only one kind of formula, hence it is realized, by condition (2). This is a contradiction.

REMARK. This was proved, in fact, in Keisler [**K 1**]. We shall now define by induction b^α, $\alpha < \|M\|$, such that $(\exists^{\geqq m_\psi} x)\psi(x, \bar{b}_\alpha)$ and for every $\alpha_1 < \cdots < \alpha_{m'}$, we have $\neg\, (\exists x)[\bigwedge_{1\leqq k\leqq m'} \psi(x, \bar{b}^{\alpha_k})]$.

For α, suppose that, for all $\beta < \alpha$, the above holds. It is clearly sufficient to define $\bar{b}^\alpha$ so that it realizes the type

$$p = \{(\exists^{\geq m_\psi} x)\psi(x, \bar{y})\} \cup \left\{\neg(\exists x)\left[\bigwedge_{1 \leq k \leq m'-1} \psi(x, \bar{b}^{\beta_k}) \wedge \psi(x, \bar{y})\right] : \beta_1 < \beta_2 < \cdots < \alpha\right\}.$$

p is consistent: For any finite subset q of p, there is a $\bar{b}^n$ which does not appear in q. Thus $\bar{b}^n$ satisfies q. So every finite subset of p is consistent. By condition (2), p must be realized. So we can define $\bar{b}^\alpha$, $\alpha < \|M\|$, as specified. Hence each $\psi(x, \bar{b}^\alpha)$ is not algebraic.

Let $\beta = \text{Deg}[\theta(x, \bar{a})]$. We defined θ to have minimal degree among nonalgebraic formulas, so $\text{Deg}[\psi(\bar{x}, \bar{b}^\alpha)] \geq \beta$, for all α. But look at the definition of degree—we have these $\psi(x, \bar{b}^\alpha)$, $\alpha < \|M\|$, of degree $\geq \beta$, which are almost contradictory, so $\text{Deg}[\theta(\bar{x}, \bar{a})] \geq \beta + 1$. A contradiction.

THEOREM 4.3. *Suppose A partitions the weakly minimal formula $\theta(x, \bar{a})$. For every φ, $\psi_\varphi(x, \bar{a}^i)$, $i \leq n_\varphi$, $\bar{a}^i \in A$ is a φ-partition of $\theta(x, \bar{a})$. If $A \subset B$, and $|\{c \in B \mid \vDash \psi_\varphi[c, \bar{a}^i]\}| > |T|$ for every φ, i, and B satisfies*

(*) $$\vDash (\exists x)[\psi(x, \bar{b}) \wedge \theta(x, \bar{a})], \quad \bar{b} \in B,$$

*implies there is $c \in B$, $\vDash \psi[c, \bar{b}] \wedge \theta[c, \bar{a}]$ **then** there is $M \supset B$, $\theta(M, \bar{a}) \subset B$.*

PROOF. The set of $B' \supset B$ satisfying the condition (*) mentioned in the theorem such that no element of $B' - B$ realizes $\theta(x, \bar{a})$ is closed under union of increasing sequences of sets ordered by $\subset$. So, by Zorn's lemma, there is a maximal one, B^*. We shall show that B^* is the universe of an elementary submodel of $\bar{M}$ by means of the Tarski-Vaught test.

Suppose that $\vDash (\exists x)\varphi_1(x, \bar{b}_1)$ where $\bar{b}_1 \in B^*$. Pick $\varphi(x, \bar{b})$ so that $\bar{b} \in B^*$, $\vDash \varphi(x, \bar{b}) \to \varphi_1(x, \bar{b}_1)$ and $\text{Deg}[\varphi(x, \bar{b})]$ is minimal (≥ 0). Let a realize $\varphi(x, \bar{b})$. If we can show that $B^* \cup \{a\}$ satisfies (*), then by the maximality of B^*, $a \in B^*$, so that the Tarski-Vaught test will be satisfied. (If $\vDash \theta[a, \bar{a}]$, by (*), we can choose $a \in B^*$, so we can assume $a \notin \theta(\bar{M}, \bar{a})$.)

So assume there is a formula $\rho_1(x, a, \bar{c}_1)$, $\bar{c}_1 \in B^*$, such that $\vDash (\exists x)\rho(x, a, \bar{c})$ (where $\rho(x, a, \bar{c}) = \rho_1(x, a, \bar{c}_1) \wedge \theta(x, \bar{a})$) but no element of B^* satisfies $\rho(x, a, \bar{c})$. Let $d \in |\bar{M}|$ satisfy it, and $\psi_\varphi(x, \bar{a}_\varphi)$ be the ρ-piece of d. So $\rho(x, a, \bar{c}) \wedge \psi_\varphi(x, \bar{a}_\varphi)$ is consistent and either $\rho(x, a, \bar{c}) \wedge \psi_\varphi(x, \bar{a}_\varphi)$ is algebraic or $\neg\rho(x, a, \bar{c}) \wedge \psi_\varphi(x, \bar{a}_\varphi)$ is algebraic. In the second case all but finitely many elements of $\psi_\varphi(\bar{M}, \bar{a}_\varphi)$ are in $\rho(\bar{M}, a, \bar{c})$, and as $|\psi_\varphi(\bar{M}, \bar{a}_\varphi) \cap B| > |T|$, $\rho(x, a, \bar{c})$ will be realized by some element of $B \subset B^*$, a contradiction (w.l.o.g. $\vDash \rho(x, y, \bar{a}) \to \psi_\varphi$). Hence, for some $m < \omega$, $\vDash (\exists^{\leq m} x)\rho(x, a, \bar{c})$. Let

$$\chi(z, \bar{b}, \bar{c}) = (\exists x)[\varphi(x, \bar{b}) \wedge (\exists^{\leq m} y)\rho(y, x, \bar{c}) \wedge \rho(z, x, \bar{c})].$$

Clearly $\vDash \chi(d, \bar{b}, \bar{c})$ (let $x = a$), $d \notin B^*$ (by definition of ρ). Let $\psi_\chi(x, \bar{a}_\chi)$ be the χ-piece of d, and $\chi_1(x, \bar{a}^*) = \chi(x, \bar{b}, \bar{c}) \wedge \psi_\chi(x, \bar{a}_\chi)$, so $\bar{a}^* \in B^*$ and $\vDash \chi_1(d, \bar{a}^*)$.

Now $\chi_1(x, \bar{a}^*)$ is not algebraic, for define by induction distinct $d_n \in B^*$: d satisfies $\chi_1(x, \bar{a}^*) \wedge \bigwedge_{i<n} x \neq d_i \wedge \theta(x, \bar{a})$, so some $d_n \in B^*$ satisfies it (remember

$d \notin B^*$ and (*)). As $\psi_\chi(x, \bar{a}_\chi)$ is χ-minimal it follows $\neg\chi(x, \bar{b}, \bar{c}) \wedge \psi_\chi(x, \bar{a}_\chi)$ is algebraic. So all but finitely many of the elements of $\psi_\chi(\bar{M}, \bar{a}_\chi) \cap B^*$ realize $\chi_1(x, \bar{a}^*)$. So by the hypothesis on B, $\chi(z, \bar{b}, \bar{c})$ is realized by $\geqq |T|^+$ elements of $B \subset B^*$, say $\{b_k \mid k < |T|^+\}$. Now

$$\{\varphi(x, \bar{b}) \wedge (\exists^{\leq m} y)\rho(y, x, \bar{c}) \wedge \rho(b_k, x, \bar{c}) \mid k < |T|^+\}$$

is a set of consistent formulas (as b_k satisfies $\chi(z, \bar{b}, \bar{c})$) with parameters from B^* which is almost contradictory, because any $m + 1$ of the formulas says that only m y's satisfy $\rho(y, x, \bar{c})$, but $m + 1$ different b_k's satisfy it. By definition this implies

$$\mathrm{Deg}[\varphi(x, \bar{b})] > \inf\{\mathrm{Deg}[\varphi(x, \bar{b}) \wedge (\exists^{\leq m} y)\rho(y, x, \bar{c}) \wedge \rho(b_k, x, \bar{c})] \mid k < |T|^+\}.$$

But this violates the minimality of $\mathrm{Deg}[\varphi(x, \bar{b})]$. A contradiction.

THEOREM 4.4. *Let M be a model of T, $A \subset |M|$, $|A| \leqq |T|$, A partition the weakly minimal $\theta(x, \bar{a})$. Suppose $\|M\| > |T|$ and every nonalgebraic $p \in S(A)$ containing $\theta(x, \bar{a})$ is realized $\|M\|$ times in M. Then M is saturated.*

PROOF. Suppose that M is not saturated, and let p be a type such that $|p| < \|M\|$ and M omits p. Choose $\varphi(x, \bar{b})$ with $\bar{b} \in |M|$ such that

(1) $p \cup \{\varphi(x, \bar{b})\}$ is consistent, and

(2) among φ' satisfying (1), $\mathrm{Deg}(\varphi(x, \bar{b}))$ is minimal.

Let $p' = p \cup \{\varphi(x, \bar{b})\}$ and let a realize p', $a \notin |M|$. If $|M| \cup \{a\}$ satisfies the condition (*) of Theorem 4.3 (i.e., that for every φ and every $\bar{b} \in |M|$, if $\vDash (\exists x)(\theta(x, \bar{a}) \wedge \varphi(x, a, \bar{b}))$ then $\theta(x, \bar{a}) \wedge \varphi(x, a, \bar{b})$ is satisfied in $|M| \cup \{a\}$), then by Theorem 4.3 there is a model N such that $|M| \cup \{a\} \subset |N|$ and for all $c \in |N|$, if $\vDash \theta[c, \bar{a}]$ then $c \in |M| \cup \{a\}$. Since $M \prec N$ and $\theta(x, \bar{a})$ is satisfied by infinitely many elements of $|M|$, this is possible only if $\vDash \theta[a, \bar{a}]$, by condition (2).

Then either $\vDash \theta[a, \bar{a}]$, or there exists a formula $\bar{\psi}(y, a, \bar{c}_0)$ such that

$$\vDash (\exists y)[\theta(y, \bar{a}) \wedge \bar{\psi}(y, a, \bar{c}_0)]$$

and $\bar{c}_0 \in |M|$, but no element of M satisfies it. If $\vDash \neg\theta[a, \bar{a}]$, let $\psi(y, a, \bar{c})$ be this $(\theta(y, \bar{a}) \wedge \bar{\psi}(y, a, \bar{c}_0))$, with $\bar{c} = \bar{c}_0{}^\frown\bar{a}$. If $\vDash \theta[a, \bar{a}]$, let $\psi(y, a, \bar{c})$ be $(\theta(y, \bar{a}) \wedge y = a)$. In either case, $\psi(y, a, \bar{c})$ is satisfied by some $d \in \bar{M}$ but by nothing in M.

Let B be the union of A with the set of parameters from M occurring in p and with the range of $\bar{c}$.

Clearly $B \subset |M|$ and $|B| < |M|$. Let q be the type which d realizes over B. Now q cannot be algebraic, for if it were, everything satisfying it would be in M, since $B \subset |M|$, but $d \notin |M|$.

q is a complete type over B, so the restriction $q \mid A$ of q to only formulas with parameters in A is a complete type over A.

Now, for every formula $\chi(x, \bar{e})$ with $\bar{e} \in \bar{M}$, either $(q \mid A) \cup \{\chi(x, \bar{e})\}$ is algebraic or $(q \mid A) \cup \{\neg\chi(x, \bar{e})\}$ is algebraic, by 4.1.

For every $\chi(x, \bar{e})$ in q, $(q \mid A) \cup \{\chi(x, \bar{e})\}$ is a type over B which is satisfied by $d \notin |M|$, and hence is not algebraic. Then $(q \mid A) \cup \{\neg\chi(x, \bar{e})\}$ must be algebraic. Therefore each formula in q is realized by all but at most finitely many of the elements of M which realize $q \mid A$, so q is realized by all but at most $|q| \leqq |B|$ of the elements of M realizing $q \mid A$.

Since $(q \mid A) \in S(A)$ and $\theta(x, \bar{a}) \in q \mid A$, $q \mid A$ is realized $\|M\|$ times in M. So q is realized $\|M\| - |B| = \|M\|$ times in M. $\|M\| > |T|$, so let d_k, $k < |T|^+$, be distinct elements of M satisfying q.

If $(q \mid A) \cup \{\neg\psi(y, a, \bar{c})\}$ were algebraic, then almost every element realizing $q \mid A$ in M would realize $\psi(y, a, \bar{c})$, but nothing in M satisfies $\psi(y, a, \bar{c})$. So as before $(q \mid A) \cup \{\psi(y, a, \bar{c})\}$ must be algebraic.

Then for some finite conjunction $\rho(y, a^*)$ of formulas from $q \mid A$, $\psi'(y, a, \bar{c}') = \psi(y, a, \bar{c}) \wedge \rho(y, a^*)$ is algebraic, so $\vDash (\exists^{\leq m} y)(\psi'(y, a, \bar{c}'))$ for some $m \in \omega$.

Then $p'' = p' \cup \{(\exists^{\leq m} y)\psi'(y, x, \bar{c}') \wedge \psi'(d, x, \bar{c}')\}$ is consistent, since it is satisfied by a. Each d_k realizes exactly the same type over B that d does, so for $k < |T|^+$, $p^k = p' \cup \{(\exists^{\leq m} y)\psi'(y, x, \bar{c}')\} \cup \{\psi'(d_k, x, \bar{c}')\}$ is consistent. $p^k = p \cup \{\varphi(x, \bar{b}) \wedge (\exists^{\leq m} y)\psi'(y, x, \bar{c}') \wedge \psi'(d_k, x, \bar{c}')\}$. Since the d_k are distinct, no more than m of the formulas $\varphi(x, \bar{b}) \wedge (\exists^{\leq m} y)\psi'(y, x, \bar{c}') \wedge \psi'(d_k, x, \bar{c}')$, $k < |T|^+$, can be satisfied at one time. Then by the definition of degree,

$$\mathrm{Deg}[\varphi(x, \bar{b})] > \mathrm{Deg}[\varphi(x, \bar{b}) \wedge (\exists^{\leq m} y)\psi'(y, x, \bar{c}') \wedge \psi'(d_k, x, \bar{c}')]$$

for some k, contradicting our choice of $\varphi(x, \bar{b})$.

THEOREM 4.5. *For every $\lambda > |T|$, T has a model of power λ which is saturated.*

PROOF. Let M be any model of T of power λ. Let $\theta(x, \bar{a})$ be weakly minimal, and A partition $\theta(x, \bar{a})$; $|A| = |T|$. By condition (1), $|S(A)| = |T|$. Then the complete diagram of M together with $\{\varphi(c_i^p) : i < \lambda$ and $\varphi(x) \in p$ for some $p \in S(A)$ such that p is nonalgebraic$\} \cup \{c_i^p \neq c_j^p : i < j < \lambda\}$ is a consistent set of sentences of power λ, so it has a model N of power λ, by the downward Lowenheim-Skolem theorem.

So $N \succ M$ and $\|N\| = \lambda$ and each nonalgebraic $p \in S(A)$ is realized by λ elements of N.

$\theta(x, \bar{a})$ and A retain in N the properties we required of them, since those properties are expressible in first-order sentences.

Then N is saturated, by Theorem 4.4.

THEOREM 4.6. *Let M be a model of T, with $|T| \leqq \lambda < \|M\|$. Let $A \subset |M|$ partition $\theta(x, \bar{a})$, $|A| = |T|$, $\theta(x, \bar{a})$ weakly minimal, and let $N \succ M$. Let $B = |M| \cup \{c \in |N| : \vDash \theta[c, \bar{a}]$, and the type that c realizes over A is realized $> \lambda$ times in $M\}$.*

Then B satisfies the conditions of Theorem 4.3.

LEMMA 4.7. *Suppose b realizes some algebraic type $p \in S(A)$. Then there is $\bar{a} \in A$ and a formula φ such that*

(1) *$\vDash \varphi[b, \bar{a}]$, and*

(2) *if $\vDash \varphi[c, \bar{a}]$, then b and c realize the same type over A.*

PROOF OF LEMMA. If p is algebraic, then by compactness, some finite subtype of p is algebraic; its conjunction is an element of p, since p is complete. So for some $\psi(x, \bar{a}) \in p$, $\vDash (\exists^{\leq m} x)\psi(x, \bar{a})$.

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Let $b_1, \dots, b_m$ be the elements realizing $\psi(x, \bar{a})$, with $b = b_1$. For each k, $1 \leqq k \leqq m$, if b_1 and b_k do not realize the same type over A, let $\psi_k(x, \bar{a}^k)$ be an element of p such that $\vDash \neg\psi_k[b_k, \bar{a}^k]$, if there is one. Otherwise let $\psi_k(x, \bar{a}^k)$ be $x = x$. Then $(\bigwedge_{1 \leqq k \leqq m} \psi_k(x, \bar{a}^k) \wedge \psi(x, \bar{a}))$ has properties (1) and (2).

PROOF OF THEOREM 4.6. If this set B did not satisfy the hypothesis of Theorem 4.3, then there would exist a φ' and $\bar{b}' \in |M|$ and $\bar{c} \in (|N| - |M|) \cap B$ such that $\vDash (\exists x)(\varphi'(x, \bar{b}', \bar{c}) \wedge \theta(x, \bar{a}))$, but there is no such x in B.

Suppose d realizes $\varphi'(x, \bar{b}', \bar{c}) \wedge \theta(x, \bar{a})$. Then since $\vDash \theta[d, \bar{a}]$, let $\psi_{\varphi'}(x, \bar{a})$ be the φ'-piece of d.

Let $\varphi(x, \bar{b}, \bar{c}) = \varphi'(x, \bar{b}', \bar{c}) \wedge \theta(x, \bar{a}) \wedge \psi_{\varphi'}(x, \bar{a}')$.

If $\varphi(x, \bar{b}, \bar{c})$ were not algebraic, then, by the definition of $\psi_{\varphi'}$, $\psi_{\varphi'}(x, \bar{a}) \wedge \neg\varphi'(x, \bar{b}', \bar{c})$ would be algebraic, so that all but finitely many of the elements satisfying $\psi_{\varphi'}(x, \bar{a})$ would satisfy $\varphi'(x, \bar{b}', \bar{c})$. Since $\psi_{\varphi'}(x, \bar{a})$ has parameters from M and is satisfied by $d \notin M$, it is satisfied by infinitely many elements of M, so we would have $\varphi'(x, \bar{b}', \bar{c})$ satisfied by elements of M, contrary to our choice of φ'. Hence $\varphi(x, \bar{b}, \bar{c})$ is algebraic.

Now let $\bar{c} = \langle c_0, \dots, c_n \rangle$. We will define, by induction on $k \leqq n$, a sequence $\bar{c}' \in |M|$ such that, for each $k \leqq n$, $\langle c_0, \dots, c_k \rangle$ and $\langle c_0', \dots, c_k' \rangle$ realize the same $(k + 1)$-type over the set $A^* = A \cup$ Range of $\bar{b} \cup$ {all elements of M realizing p, for each p, such that $p \in S(A)$ and $\theta(x, \bar{a}) \in p$ and p is realized $\leqq \lambda$ times in M}. Suppose that we have $c_0', \dots, c_{k-1}'$ as specified. Here is how we get c_k'.

Case 1. c_k realizes an algebraic type over $A^* \cup \{c_0, \dots, c_{k-1}\}$. By Lemma 4.7, there is a $\rho(x, c_0, \dots, c_{k-1}, \bar{e})$ with $\bar{e} \in A^*$ such that $\vDash \rho[c_k, c_0, \dots, c_{k-1}, \bar{e}]$ and when $\vDash \rho[c, c_0, \dots, c_{k-1}, \bar{e}]$, then c and c_k realize the same type over $A^* \cup \{c_0, \dots, c_{k-1}\}$.

By the induction hypothesis, $\vDash (\exists x)\rho(x, c_0', \dots, c_{k-1}', \bar{e})$. Since all parameters are from M, we can find some $c_k' \in |M|$ such that $\vDash \rho[c_k', c_0', \dots, c_{k-1}', \bar{e}]$. So $\langle c_0, \dots, c_k \rangle$ and $\langle c_0', \dots, c_k' \rangle$ realize the same $(k + 1)$-type over A^*.

Case 2. c_k is not algebraic over $A^* \cup \{c_0, \dots, c_{k-1}\}$. Let p_k be the type c_k realizes over A. Since $c_k \in B - |M|$, p_k is realized $>\lambda$ times in M. Let ρ be any formula such that $\vDash \rho[c_k, c_0, \dots, c_{k-1}, \bar{e}]$, with $\bar{e} \in A^*$.

Now since p_k is a complete type over A, and $\vDash \theta[c_k, \bar{a}]$, there is some $\bar{a}^* \in A$ such that $\psi_\rho(x, \bar{a}^*) \in p_k$, so either $p_k \cup \{\rho(x, c_0, \dots, c_{k-1}, \bar{e})\}$ is algebraic or $p_k \cup \{\neg\rho(x, c_0, \dots, c_{k-1}, \bar{e})\}$ is algebraic. Since $p_k \cup \{\rho(x, c_0, \dots, c_{k-1}, \bar{e})\}$ is a subtype of the type of c_k over $A^* \cup \{c_0, \dots, c_{k-1}\}$, it is not algebraic, so $p_k \cup \{\neg\rho(x, c_0, \dots, c_{k-1}, \bar{e})\}$ is. Then $p_k \cup \{\neg\rho(x, c_0', \dots, c_{k-1}', \bar{e})\}$ is algebraic. So all but finitely many of the elements realizing p_k realize $\rho(x, c_0', \dots, c_{k-1}', \bar{e})$. There are at most λ such formulas $\rho(x, c_0, \dots, c_{k-1}, \bar{e})$ to consider, since $|A^*| \leqq |A| + |S(A)| \cdot \lambda = \lambda$ and $|T| \leqq \lambda$, and there are more than λ elements satisfying p_k, so there must be elements in M satisfying p_k and such that for all formulas such that $\vDash \rho[c_k, c_0, \dots, c_{k-1}, \bar{e}]$, $\vDash \rho[c_k', c_0', \dots, c_{k-1}', \bar{e}]$, as required.

Now since d satisfies an algebraic formula over $A^* \cup \{c_0, \dots, c_n\}$, we can, by Case 1, find $d' \in M$ such that $\langle c_0, \dots, c_n, d \rangle$ and $\langle c_0', \dots, c_n', d' \rangle$ satisfy the same type over A^*. If d' were in A^*, then $d = d'$, but $d \notin M$. So $d' \notin A^*$. Then, by definition of 4^*, the type that d', and hence d, realizes over A is realized in M more than λ times. So $\{c \in |N| : c$ realizes the same type as $d\} \subset B$. d is algebraic over $B \subset |N|$, so $d \in |N|$. Then $d \in B$. But we assumed that $d \notin B$ to begin with. Then B must satisfy the conditions of Theorem 4.3 after all.

THEOREM 4.8. *If T is categorical in power λ for some $\lambda > |T|$, then it is categorical in every power $\mu > |T|$.*

PROOF. By Theorem 4.5, T has a saturated model of power μ for each $\mu > |T|$. Since any two saturated models of T of the same power are isomorphic, it suffices to show T has no nonsaturated models of power $> |T|$.

Suppose M is a nonsaturated model of T and $\|M\| > |T|$. Let $\theta(x, \bar{a})$ and A be as before. By Theorem 4.4, since M is not saturated, there is some $p_0 \in S(A)$ such that $\theta(x, \bar{a}) \in p_0$ and p_0 is realized $\leqq \kappa$ times in M, with $\aleph_0 \leqq \kappa < \|M\|$. Pick any $\lambda_1 > \|M\|$ and let $N \succ M$ be a saturated model of T of power λ_1.

Let $B = |M| \cup \{c \in |N| : \theta[c, \bar{a}]$ and the type of c over A is realized $> \kappa$ times in $M\}$.

$|B| = \lambda_1$, for if $p \in S(A)$ and $\theta(x, \bar{a}) \in p$ and p is realized $> \kappa$ times in M, then p is realized λ_1 times in N, since N is saturated. There exist $p \in S(A)$ with $\theta(x, \bar{a}) \in p$ that are realized more than κ times, for otherwise we would have

$$|\theta(M, \bar{a})| \leqq \sum_{p \in S(A), \theta(x, \bar{a}) \in p} |\{c \in M : c \text{ realizes } p\}| \leqq \kappa \cdot |S(A)| < \|M\|;$$

by condition (2), applied to subsets of $\{(\theta(x, \bar{a}) \wedge x \neq c) : c \in |M|\}$, we have $|\theta(M, \bar{a})| = \|M\|$. So $|B| = \lambda_1$.

By Theorem 4.6, B satisfies the hypotheses of Theorem 4.3, so there is a model M' of T such that $B \subset |M'|$ and $\theta(M', \bar{a}) \subset B$. So if $c \in M'$ realizes p_0, then $\vDash \theta[c, \bar{a}]$, hence $c \in B$, hence $c \in M$ (since p_0 is realized $\leqq \kappa$ times by assumption). Also, $\|M'\| \geqq |B| = \lambda_1$.

Since for fixed κ this holds for all $\lambda_1 \geqq \|M\|$, by the method of Morley **[Mo 2]**, T has a model N' of power λ such that p_0 is realized $\leqq |T|$ times in N'. But T has a saturated model of power λ and T is categorical in λ. A contradiction. (We do not really need to use something as strong as Morley's results here.)

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