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COUNTING EQUIVALENCE CLASSES FOR CO-κ-SOUSLIN EQUIVALENCE RELATIONS

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Let R be a symmetric binary relation on the reals. R is called <u>thick</u> if: there is a perfect set P of reals such that (a,b ∈ P and a ≠ b) ⇒ a $\not R$ b. R is called <u>thin</u> if it is not thick. This paper is mainly concerned with the problem of finding sufficient conditions for thickness. A particular instance of this problem is when R is an equivalence relation. This has been quite extensively studied (see [Si], [B], [H-S], [St]). The most notable results are:

(Silver [Si]). If E is a thin $\underset{\sim}{\Pi}^1_1$ equivalence relation, then E has at most $\aleph_0$-many equivalence classes.

(Burgess [Bu]). If E is a thin $\underset{\sim}{\Sigma}^1_1$ equivalence relation, then E has at most $\aleph_1$-many equivalence classes.

In this paper we prove a general theorem which subsumes both of the above, and which in addition has the following corollaries:

If E is an absolutely $\underset{\sim}{\Delta}^1_2$ thin equivalence relation, then E has at most $\aleph_1$-many equivalence classes.

If E is a thin $\underset{\sim}{\Pi}^1_2$ equivalence relation (and if $(\aleph_1)^{L[a]}$ is countable for all reals a), then E has at most $\aleph_1$-many equivalence classes.

The general method actually gives the appropriate generalization of the above to the case where E is co-κ-Souslin (see Theorem 1). Also the method applies to other relations beside equivalence relations. As an example we prove:

If R is a $\underset{\sim}{\Sigma}^1_1$ linear ordering, then there is no length $\aleph_1$ ascending chain through R.

The above results are due to the second author. Upon seeing these results, the first author noticed that the method from [H-S] can be adopted to give a proof without using the axiom of choice. Since such a choiceless proof is useful in the context of the axiom of determinacy, it is also presented here.

On the negative side:
for κ a cardinal, let B_κ be the assertion: if a (say) Borel relation R contains Y × Y, where Y is a set of reals of cardinality $\aleph_\alpha$ then R contains P × P for some

perfect set P. We produce a model of Z F C in which $B_{(\aleph_\alpha)}$ fails, for all countable ordinals α.

Defs (a) T is a tree on the set Y if: $T \subseteq Y^{<\omega}$, and $(\eta \in T,\ \tau \subseteq \eta) \Rightarrow \tau \in T$.

(b) For T a tree on Y, $[T] = \{f;\ f:\omega \to Y,\ \forall n (f \restriction n \in T)\}$.

(c) For T a tree on $\kappa \times X$, let $p[T] = \{g;\ g:\omega \to X$, and for some $h:\omega \to \kappa$, $\langle h,g \rangle \in [T]\}$. (Here we identify $\langle h,g \rangle$ with the function $f:\omega \to \kappa \times X$ where $f(n) = \langle h(n), g(n) \rangle$).

(d) A binary relation R on ω^ω is κ-Souslin (via T) if: T is a tree on $\kappa \times (\omega^2)$ and $R = p[T]$; (R is co-κ-Souslin if: $\check{R}$ is κ-Souslin (where $\check{R}$ = complement of R)).

Let R be κ-Souslin via T. As we vary through models of ZF which contain T, p[T] will always define a binary relation on ω^ω, and we will ambiguously continue to use R to denote these relations. Notice that R is absolute, i.e.: a R b holds if it holds in L[a,b,T].

Def For R κ-Souslin via T, we will call R strongly thick if: for some perfect set $P \subseteq \omega^\omega$ and for some countable $t \subseteq T$, $P^{[2]} \subseteq p[t]$, (where $P^{[2]} = \{\langle a,b \rangle;\ a,b \in P,\ a \neq b\}$).

Notice that strong thickness is an absolute property (of T).

Def For V a model of ZF, the next world after V is: V[c] where c is a Cohen real over V. (Notice, for $\bar{b}$ in V, the theory of $\bar{b}$ in $\langle V[c], \in \rangle$ does not depend on the choice of c).

Theorem 1 Let E be a co-κ-Souslin relation, via T. Assume that E is an equivalence relation, and assume that E is not strongly thick. Also assume: (*) E is an equivalence relation in the next world after L[T].

Then E has at most κ-many equivalence classes. (κ is, of course, an inf. cardinal).

Proof Suppose $\langle a_i \rangle_{i<\kappa^+}$ is a sequence of reals such that $i<j \Rightarrow a_i \not\!E\, a_j$. Let $\mathcal{N}$ be a transitive model of a rich fragment of set theory s.t. $T, \langle a_i \rangle_{i<\kappa^+}$ are in $\mathcal{N}$. Let N be a countable elementary substructure of n (with T, $\langle a \rangle$ still in N). Let $P = \{\Phi(x);\ \Phi$ is a formula (with parameters from N), and in N: for unboundedly many $i<\kappa^+$ $\Phi(a_i)$ holds$\}$. P is naturally ordered by inclusion. Also P is a consistency property such that: if $G \subseteq P$ is generic, then G gives rise to a real b for which: there is an elementary extension $\hat{N}$ of N s.t. $b \in \hat{N}$ and for all Φ in G $\hat{N} \models \Phi(b)$. In particular, $\hat{N} \models$ "$b = a_i$", for some i in $\hat{N}$ s.t. $i < (\kappa^+)^{\hat{N}}$, $i > j$ for all $j \in N$, $j < (\kappa^+)^N$. Also, $(\kappa)^N = (\kappa)^{\hat{N}}$ and so $(T)^{\hat{N}} = (T)^N \subseteq T$. Thus if $\hat{N}$ = "$c \not\!E\, d$" (for c,d reals in $\hat{N}$), then $c \not\!E\, d$ is actually true.

Claim If b,c are $P \times P$ generic reals, then $b \not\!E\, c$.

Notice: using the claim, the method of [ ] shows that E is strongly thick.

[Proof of Claim:

Suppose not. Let (Φ,ψ) force that $b\,E\,c$. Let δ be the term defined over N by: $\delta(a_i) = 1^{st}\, a_j\,(j>i$ and $\Phi(a_j)$ holds). Since Φ has unboundedly many solutions, δ is always defined.

Subclaim: $\delta(b)$, c are $P\times P$-generic, [prf: For $\theta(x)$ in P, let $F(\theta)$ be $\theta(\delta^{-1}(y))$. F is an order preserving bijection between P and a dense subset of P, and F maps the filter generated by b to the filter generated by $\delta(b)$. □]

Thus: by choice of Φ,ψ and of δ, $b\,E\,c$, $\delta(b)\,E\,c$. But by choice of δ;b, $\delta(b)$ are in the same $\hat{N}$, and in $\hat{N}$: $\delta(b) = a_j$ for some $j>i$ where $b = a_i$. Thus $\hat{N} \models$ "$b \not{E}\, \delta(b)$". So $b \not{E}\, \delta(b)$. So E is not transitive. E fails to be transitive in a generic extension of L[T,N] given by $P\times P$. Thus E is not an equivalence relation in the next world after L[T,N].

So it remains to show that N could be chosen inside L[T]. The only properties of N used so far are:

(i) N is a countable (possibly non-standard) model of a fragment of set theory;

(ii) $\langle N,\in,T,\langle a_i\rangle_{i<\kappa^+}\rangle$ has a certain simply describable first order properties;

(iii) $(T)^N \subseteq T$.

But clearly there is a tree $\mathcal{U}$ (on $T\times\omega$) such that paths through $\mathcal{U}$ correspond to (enumerations of) such structures N. Also $\mathcal{U}$ is in L[T]. But $\mathcal{U}$ does have a path (by making T countable, n will correspond to such a path). Thus $\mathcal{U}$ has a path in L[T].

This completes the proof of the claim □], and hence the proof of Theorem 1, □].

Notice that (*) from Theorem 1 is a consequence of: (a) There exists a real c Cohen generic over L[T]; and (a) is a consequence of either: (b) $(2^\omega)^{L[T]}$ is countable; or (c) MA + $|2^\omega|^{L[T]} < |2^\omega|$.

<u>Corollaries</u> 1. (Silver) If E is a $\underset{\sim}{\Pi}^1_1$ thin equivalence relation, then E has $\leqslant\omega$-many equivalence classes.

2. If E is an absolutely $\underset{\sim}{\Delta}^1_2$, thin, equivalence relation, then E has $\leqslant\aleph_1$-many equivalence classes.

3. If E is a thin, Π^1_2 equivalence relation, (and if $(\aleph_1)^L$ is countable), then E has $\leqslant\aleph_1$-many equivalence classes.

[Proof:

In 1. E is a co-ω-Souslin relation. In 2, 3 E is a co-$\aleph_1$-Souslin relation (via a tree T in L). In 1, 2 E is absolutely an equivalence relation (as long as $\aleph_1$ is not collapsed); thus (*) from theorem 1 holds. In 3, by assumption there is a real c Cohen generic over L = L[T], and so (*) holds. Thus 1-3 follow from theorem 1. □]

Def The field of a binary relation R is the set of reals a s.t.: a R b or b R a holds for some real b. R is called a quasi-linear order if: the relation $a E b \equiv$ ($a \not R b$ and $b \not R a$) is an equivalence relation on the field of R, and R induces a linear ordering of the E-equivalence classes (i.e.: ($b \not R a$ and $b R c$) $\Rightarrow a R c$, and: ($a R b$ and $c \not R b$) $\Rightarrow a R c$; and: $a R b \Rightarrow b \not R a$).

Theorem 2 If R is a Σ^1_1 quasi-linear order, then there is no length $\aleph_1$ R-increasing sequence of reals.

[Proof:

Notice: the fact that R is a quasi-linear ordering is an absolute property, hence it will remain true in generic extensions. Since R is Σ^1_1, R is ω-Souslin, say via the tree T. Assume $\langle a_i \rangle_{i<\aleph_1}$ is a sequence of reals s.t. $i<j \Rightarrow a_i R a_j$.

Let $\mathcal{N}$,N,P be as in the proof of Theorem 1. Let b,c be $P \times P$-generic reals. As observed above, R is still a quasi-linear order in L[T,b,c]. We will now obtain a contradiction.

Case 1: $b \not R c$ and $c \not R b$.

Let (Φ,ψ) force this.

Let δ be the term defined over N by $\delta(a_i) = 1^{st}\, a_j\, (j>i$ and $\Phi(a_j))$. So, as in the proof of theorem 1, we have $b \not R c$, $c \not R \delta(b)$; and $b R \delta(b)$; thus R is not a quasi-ordering.

Case 2: b R c or c R b.

By symmetry, assume b R c. Let (Φ,ψ) force b R c. Let δ be the term defined over N by: $\delta(a_i) = 1^{st}\, a_j\, (j>i$ and $\Phi(a_j))$. Let c',c be $P \times P$-generic reals s.t. the generic filters on P, which correspond to c',c, both contain $\psi(x)$.

Let $b' = \delta(c')$, $b = \delta(c)$. Thus b',c and b,c' are both $P \times P$-generic, and both correspond to filters on $P \times P$ which contain Φ,ψ. Thus b' R c, b R c' both hold. By choice of δ, c' R b' and c R b both hold. Thus c' R b' R c R b R c' holds. So c' R c' holds, and so R is not a quasi-order. □].

Theorem 2 can be strengthened in a way similar to theorem 1: If R is a quasi-order, and if R is κ-Souslin via T, and if R is a quasi-order in the next world after L[T], then there is no κ^+-ascending R-chain.

Theorem 2 answers a question raised by H. Friedman. (Friedman, previous to our results, showed that a Borel quasi-order has cofinality ω.)

A slight defect in the proof of Theorem 1 is that it used the axiom of choice. For those readers who favor some other axiom, a choiceless proof of Theorem 1 will now be given:

Let E be co-κ-Souslin via T. Assume E is a thin equivalence relation.

Consider the usual proof system for $L_{\infty,\omega}$ (see Ba). We will call a subset of $L_{\infty,\omega}$ syntactically consistent if there is no proof of a contradiction from it. Notice, consistency implies syntactical consistency, and the converse holds for countable fragments of $L_{\infty,\omega}$.

Consider the propositional version of $L_{\infty,\omega}$ (which we call $L_{\infty,0}$). $L_{\infty,0}$ will have ω-many atomic propositional sentences - by viewing the n^{th} atomic sentence as asserting "$n \in x$", sentences of $L_{\infty,0}$ can be viewed as formulas about a real x.

Let A be the first admissible set containing T as a member. For U(x) a $\underset{\sim}{\Sigma}_1$ over A subset of $L_{\infty,0} \cap A$, consider the $L_{\infty,\omega}$ theory: $U(x) \cup U(y) \cup \{x \not{E} y\}$ (here $x \not{E} y$ is the sentence: $(f: \omega \to \kappa\ (\langle f,x,y\rangle \in [T])))$, (where f is a new function symbol). If this theory is syntactically inconsistent, then there is a sentence $\theta(x)$ in $L_{\infty,0} \cap A$ s.t. $U(x) \vdash \theta(x)$, and s.t. $\{\theta(x), \theta(y), x \not{E} y\}$ is syntactically inconsistent (by Barwise compactness). By Barwise completeness, the set $S = \{\theta(x);\ \theta(x)$ is in $L_{\infty,0}$, and $\{\theta(x), \theta(y), x \not{E} y\}$ is syntactically inconsistent$\}$ is $\underset{\sim}{\Sigma}_1$ over A.

Let $W(x) = \{\neg\theta(x);\ \theta \in S\}$. So W is $\underset{\sim}{\Sigma}_1$ over A. If $U(x) \supseteq W(x)$ and if U is $\underset{\sim}{\Sigma}_1$ over A, then, by construction of W, U(x) is syntactically inconsistent if $U(x) \cup U(y) \cup \{x \not{E} y\}$ is syntactically inconsistent.

If W is syntactically inconsistent, then: for each real a, there is a $\theta \in S$, s.t. $\theta(a)$ holds. But by definition of S, for θ in $S(\theta(a)$ and $\theta(b)) \Rightarrow a E b$. Since $S \subseteq A$, S is well-orderable of length $\leqslant \kappa$. This induces a $\leqslant\kappa$-length well-ordering of the E-equivalence classes.

If W is syntactically consistent, then: by Skolem-Löwenheim, we can find in L[T] a countable $t \subseteq T$ such that: for $\overline{A}$ = first admissible containing t, and for $\overline{E}, \overline{S}, \overline{W}$ defined as above, with T,A replaced by $t, \overline{A}$, we have that $\overline{W}$ is syntactically consistent.

Let $P = \{U(x);\ U \subseteq L_{\infty,0} \cap \overline{A}$ is $\underset{\sim}{\Sigma}_1$ over $\overline{A}$, and $\overline{W} \subseteq U$, and U is consistent$\}$. P is ordered by inclusion. If G is a generic filter on P, then G gives rise to a real b s.t.: for all U in G, $b \models U$.

Let b,c be $P \times P$-generic reals.

<u>Claim</u> $b \not{E} c$.

Let $\hat{P} = \{\hat{U}(x,y): \hat{U} \subseteq L_{\infty,\omega} \cap \overline{A}$ is $\underset{\sim}{\Sigma}_1$ over $\overline{A}$, and $\hat{U}$ is a consistent theory about the pair of reals $x,y\}$. If $\hat{G}$ is $\hat{P}$-generic then $\hat{G}$ gives rise to a pair of reals $\langle b,b'\rangle$ s.t. for all $\hat{U}$ in $\hat{G}$, $\langle b,b'\rangle \models \hat{U}$.

Subclaim: If $\overline{W}(x) \cup \overline{W}(y)$ is in $\hat{G}$, then both of b,b' are P-generic.

[Proof: Let $\hat{U}_1(x,y)$ be in $\hat{P}$, $U_1(x,y) \supseteq \overline{W}(x) \cup \overline{W}(y)$. Let $U_1(x) = \{\theta(x) \in L_{\infty,0} \cap \overline{A};\ U_1(x,y) \vdash \theta(x)\}$. So $U_1(x) \supseteq \overline{W}(x)$ and U_1 is consistent. Thus U_1 is in P. Let D be a dense subset of P. Pick $U_2 \supseteq U_1$ so that $U_2 \in D$. By choice of U_1, $(U_2(x) \cup \hat{U}(x,y)) = \hat{U}_2(x,y)$ is consistent, (and so $\hat{U}_2$ is in $\hat{P}$), and $\hat{U}_2 \supseteq \hat{U}_1$. Clearly $\hat{U}_2 \Vdash (b \models U_2)$. □].

Now suppose $b E c$. Let U(x) in P force over c that $b E c$. Let $\tilde{U}(x,y) = U(x) \cup U(y) \cup \{x \overline{E} y\}$. Since $U \supseteq \overline{W}$, and since U is consistent, $\tilde{U}$ is consistent. Thus $\tilde{U}$ is in $\hat{P}$. Let $\langle b,b'\rangle$ be $\hat{P}$ generic over c, and pick $\langle b,b'\rangle$ so that $\langle b,b'\rangle \neq \tilde{U}$. By the sublemma b,c and b',c are both $P \times P$-generic. Thus by choice of U, $b E c$ and $b' E c$.

But by choice of $\widetilde{U}$, $b \widetilde{\not E} b'$, and so $b \not E b'$. Thus E is not an equivalence relation in L[T,b,b',c]. But ⟨b,b',c⟩ is $\hat{P} \times P$-generic over L[T], and $\hat{P} \times P$ is countable in L[T]. □].

Just as in the original proof of Theorem 1, the claim yields the Theorem. □].

References:

[Ba] J. Barwise, Admissible Sets and Structures; Springer-Verlag (Ω-series), 1975.

[Bu] J. Burgess, Infinitary languages and descriptive set theory, Ph. D. thesis, Univ. of California, Berkeley, 1974.

[H-S] L. Harrington; R. Sami; Equivalence relations, projective and beyond; Logic Colloquium 78; ed. M. Boffa, D. van Dalen, K. McAloon. North-Holland (1979); 247-264.

[Si] J. Silver, Π^1_1 equivalence relations (to appear).

[St] J. Stern, Relations d'equivalence coanalytique; in: Seminair de Theorie des Ensembles; Publ. Math. de l'Universite Paris VII.