TO THE DECISION PROBLEM FOR BRANCHING TIME LOGIC***

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SECTION 1. INTRODUCTION

First we describe a certain branching time logic. We borrow this logic from Prior (1967) and call it BTL. It is a propositional logic whose formulas are built from propositional symbols by means of usual boolean connectives and additional unary connectives PAST, FUTURE, NECESSARY.

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Models for BTL are trees with additional unary predicates. Recall that a tree is a partially ordered set where the predecessors of any element are totally ordered. Think about elements of a given tree as possible states of the world, branches (i.e., maximal totally ordered subsets) as possible courses of history, and the partial order as the relation "later". The restriction to trees reflects the idea of one past but many possible futures.

Trees with additional unary predicates will be called colored trees. In other words, the additional unary predicates will be called colors. We suppose that any colored tree has finitely many colors. A tree with colors $P_1, \ldots, P_m$ interprets BTL formulas with propositional symbols $P_1, \ldots, P_m$. The interpretation of a formula at element t with respect to a branch B through t is defined inductively:

P_i holds at t wrt B if and only if $t \varepsilon P_i$,

NOT (φ) holds at t wrt B if and only if φ does not hold at t wrt B,

φ_1 & φ_2 holds at t wrt B if and only if both φ_1 and φ_2 hold at t wrt B,

PAST(φ) holds at t wrt B if and only if there is $u < t$ such that φ holds at u wrt B,

FUTURE(φ) holds at t wrt B if and only if there is $u > t$ such that $u \varepsilon B$ and φ holds at u wrt B,

NECESSARY(φ) holds at t wrt B if and only if for all branches C through t, φ holds at t wrt C.

A BTL formula φ is a *theorem* of BTL if for every tree T with appropriate colors, every element t of T and every branch B through t in T, φ holds at t with respect to B in T. Note that the semantics of BTL is defined in terms of a certain theory of trees.

Definition. Extend the first-order language of colored trees by an infinite list of unary predicate variables henceforth called *branch variables*. Allow quantification of branch variables. A tree T with colors P_1, ... P_m provides a standard interpretation for a portion of the extended language (i.e., for formulas with unary predicate constants among $P_1, \ldots, P_m$) under the stipulation that the branch variables range over all branches of T. The theory of these standard interpretations will be called TREE.

Remark. The first-order language of colored trees contains infinitely many unary predicate constants whereas each colored tree contains only finitely many colors. This does not constitute a contradiction. Theorems of the first-order theory of colored trees hold in those colored trees where they are defined. The same relates to TREE and other formal theories in this paper.

Definition. A formal theory T1 *reduces* to a formal theory T2 if there is an algorithm (the *reduction algorithm*) which associates a sentence $\varphi 2$ in the language of T2 with each sentence $\varphi 1$ in the language of T1 in such a way that $\varphi 1$ is a theorem of T1 if and only if $\varphi 2$ is a theorem of T2.

Strictly speaking, a reduction algorithm in the sense of the above definition reduces the decision problem for T1 to the decision problem for T2. All reduction algorithms in this paper will be systematic translations from one language to another.

Obviously, BTL reduces to TREE. Moreover, the extension of BTL by means of binary connectives SINCE and UNTIL, and many other extensions and variations of BTL reduce to TREE. This raises the question whether TREE is a decidable theory.

In our lecture in Salzburg we announced the decidability of the theory of trees with quantification over nodes and branches and sketched our decidability proof. (This theory of trees and the theory TREE readily reduce to each other: colors are easily coded.) In this paper we reduce the theory TREE to the first-order theory of binary colored trees which are bounded and well-founded. We prove the decidability of the latter theory elsewhere (Gurevich and Shelah, 1984). Thus TREE is decidable. Hence, BTL is decidable and all other formalizations of branching time logic reducible to TREE are decidable.

We believe that our interpretation of arbitrary trees in colored well-founded trees is of interest in its own right. --

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SECTION 2. MAIN THEOREM

First we define bounded trees and reduce the theory TREE to the first-order theory of bounded colored trees.

Definition. A branch of a tree is *bounded* if it has a maximal element, otherwise the branch is *unbounded*. A tree is *bounded* if all its branches are bounded.

Proposition 1. Every tree is embeddable into a bounded tree. Hence the theory TREE reduces to the first-order theory of bounded colored trees.

Proof is clear.

We imagine our trees growing upward (in contrast to computer science trees which usually grow downward). Elements of a tree are *nodes*. Maximal nodes are *leaves*. If x, y are nodes and $x < y$ we say that x is below y and y is above x.

We use terminology of genealogical trees but we imagine our trees growing toward future (whereas genealogical trees grow toward past). If x, y are nodes and $x \leq y$ we say that x is an *ancestor* of y and y is a *descendant* of x.

If x, y are nodes of a tree, $x < y$ and there is no z with $x < z < y$ then x is the *father* of y and y is a *son* of x. Sons of the same father are *brothers*. (We could use terms "mother", "daughter" and "sister". There is no room however for both sexes. Our choice is not an expression of male chauvinism. We just prefer to make fun of our own sex.)

Definition. A tree is *binary* if no node has more than two sons.

If x_1, x_2, y are nodes of the same tree and $y = \inf\{x_1, x_2\}$ then y will be called the *meet* of x_1, x_2.

Definition. A tree is *well-founded* if the ancestors of any node are well ordered and every pair of nodes has a meet.

Remark. Usually, the first condition in the above definition defines well-founded trees. We will not be interested in trees satisfying the first but not the second condition. In the presence of the first condition the second condition ensures existence of a least element (the *root*) and ensures that every bounded, nonempty, totally ordered subset (i.e., every bounded *chain*) has an extremum.

Main Theorem. The first-order theory of bounded colored trees reduces to the first-order theory of binary, bounded, colored, well-founded trees.

The Main Theorem is proved in the sequel. It follows from Proposition 1 in Section 3, Proposition 1 in Section 4 and Proposition 1 in Section 5.

SECTION 3. COMPLETE TREES

In this section we define complete trees and reduce the first-order theory of bounded colored trees to the first-order theory of any class of bounded colored trees which contains all bounded, colored, complete trees.

A subset of a tree will be called *open* if it contains all descendants of any of its elements. This defines the tree topology.

Recall that a topological space is *connected* if it cannot be split into two disjoint nonempty open subsets. A subset of a topological space is *connected* if it is connected as a subspace. A maximal connected subset of a topological space is called a *component* or a *connected component*.

Lemma 1. Two nodes of a tree T belong to the same component of T if and only if they have a common ancestor in T.

Proof. The relation E = {(x, y): x, y have a common ancestor} is an equivalence relation on T. For it obviously is reflexive and symmetric. To check transitivity suppose $E(x_1, x_2)$ and $E(x_2, x_3)$. Let y be a common ancestor of x_1, x_2 and z be a common ancestor of x_2, x_3. Since y, z have a common descendant x_2, they are comparable. The least between them is a common ancestor of x_1, x_3. Hence, (x_1, x_3) belongs to E.

Every equivalence class of E obviously is open. It suffices to prove that every equivalence class X of E is connected.

By contradiction suppose that the subspace X splits into nonempty, disjoint, open subsets Y, Z. Pick $y \varepsilon Y$ and $z \varepsilon Z$. Let $x \varepsilon X$ be a common ancestor of y, z. Without loss of generality $x \varepsilon Y$. Hence $z \varepsilon Y$ which is impossible.□

Lemma 1 implies that components of any open subset of a tree are open.

Lemma 2. Let X be a connected, nonempty, open set of nodes of a tree. If $x \varepsilon X$, $y \notin X$ and $y < x$ then y is a lower bound for X. If Y is another connected open set which properly includes X then Y contains a proper lower bound for X.

Proof. First suppose that $x \varepsilon X$, $y \notin X$ and $y < x$. For an arbitrary $x' \varepsilon X$ let x'' be a common ancestor of x and x' in X. The nodes x'' and y have a common descendant x and are comparable. If $x'' \leq y$ then $y \varepsilon X$ which is impossible. Hence $y < x'$.

Next suppose that Y is another connected open set of nodes which properly includes X. Pick $x \varepsilon X$, $z \varepsilon Y-X$ and a common ancestor y of x, z in Y. Use the first statement of the lemma.□

Definition. Let X be an arbitrary set of nodes in a tree. L(X) (respectively PL(X)) is the set of lower (resp. proper lower) bounds for X. In other words, y belongs to L(X) (resp. to PL(X)) if $y \leq x$ (resp. $y < x$) for all x in X.

If X is empty then either of L(X), PL(X) is the whole tree. If X is not empty then either of L(X), PL(X) is either the empty set or a bounded chain.

Definition. Let X be an arbitrary bounded set of nodes in a tree. U(X) is the set of upper bounds for X. If $X = \{x\}$ we call U(X) a *cone* and write Cone(x) instead of $U(\{x\})$. It is easy to see that if U(X) has a minimum y then U(X) = Cone(y). If U(X) is not empty but does not have a minimum we call it a *pseudo-cone*.

Lemma 3. If Y = L(X) then Y = L(U(Y)), and if Y = U(X) then Y = U(L(Y)).

Proof. The definitions of L(X) and U(X) make sense for an arbitrary partially ordered set. It suffices to prove either of the two statements of Lemma 3 in the case of a partially ordered set.

Clearly, $Y \subseteq U(L(Y))$ for any Y. If Y = U(X) then $X \subseteq L(Y)$ and we have also $U(L(Y)) \subseteq U(X) = Y$.□

Remark. Let X be a set of nodes in a tree such that U(X) is not empty. Then X is empty or X is a chain. If X has a supremum then U(X) is a cone, otherwise U(X) is a pseudo-cone.

Lemma 4. Let G be a pseudo-cone and X be a connected, nonempty, open subset of G. Then X is a connected component of G if and only if PL(X) = L(G).

Proof. Clearly $L(G) = PL(G) \subseteq PL(X)$ and $L(G)$ is an initial segment of $PL(X)$.

If $L(G)$ is a proper initial segment of $PL(X)$ pick a node y in $PL(X)-L(G)$. By Lemma 3, G equals $U(L(G))$. Hence $y \varepsilon G$. Thus X is a proper part of the component of G which contains y.

If X is a proper part of some component Y of G then, by Lemma 2, Y contains a proper lower bound y for X. Hence $L(G) = PL(G) \neq PL(X)$.□

Definition. A tree is *complete* if every nonempty set of nodes has an infimum.

Lemma 5. A tree is complete if and only if there is a least node (the *root*) and every bounded chain has a supremum.

Proof is clear.

Theorem 1. Every tree T is embeddable into a complete tree S.

Proof. Our S consists of nonempty sets U(X) where X is a subset of T. We order S by reverse inclusion. In particular, T itself is the least member of S. Clearly, S is a partially ordered set. To embed T into S we assign Cone(x) to each x in T. It remains to prove that S is a complete tree.

Claim 1. Suppose that X_1, X_2 are S-ancestors of some Y. Then X_1, X_2 are S-comparable. Hence S is a tree.

Proof of Claim 1. Without loss of generality Y is a cone. Let Y_i be the component of X_i which includes Y. If $Y_j - Y_i$ is not empty pick an element y there. By Lemma 2, y is a proper lower bound for Y_i. If X_i is connected then $X_i = Y_i \subset Y_j \subseteq X_j$. If X_i is disconnected then, by Lemma 4, $y \varepsilon L(X_i)$ and $X_i \subset Cone(y) \subseteq Y_j \subseteq X_j$.

Suppose that $Y_1 = Y_2$. If some Y_i is connected then $X_i = Y_i = Y_j \subseteq X_j$. Suppose that both X_1 and X_2 are disconnected. By Lemma 4, $L(X_1) = L(X_2)$. By Lemma 3, $X_1 = X_2$. Claim 1 is proved.□

Claim 2. Every nonempty subset K of S has an infimum in S.

Proof of Claim 2. Let G be the union of members of K and $G^* = U(L(G))$. Then G^* is an S-lower bound for K. If X is an S-lower bound for K then $X \supseteq G$, $L(X) \subseteq L(G)$ and $X = U(L(X)) \supseteq U(L(G)) = G^*$ i.e., $X \leq G^*$. Thus G^* is the desired infimum. Claim 2 is proved. Theorem 1 is proved.□

Proposition 1. Let K be an arbitrary class of bounded colored trees which contains every bounded, colored, complete tree. The first-order theory of bounded colored trees reduces to the first-order theory of K.

Proof. Given a sentence φ in the first-order language of colored trees find the first unary predicate constant P which does not appear in φ. Write a sentence φ_1 in the vocabulary $\{<, P\}$ whose meaning on any bounded tree is that P is not empty and every branch of the subtree P has a maximum (for every leaf a there is $b \leq a$ such that P(b) and for every $c \leq a$, if P(c) then $c \leq b$).

Further let φ_2 be the result of restricting the quantifiers of φ by P. The desired reduction algorithm transforms φ into $\varphi_1 \rightarrow \varphi_2$.□

SECTION 4. ORIENTED WELL-FOUNDED TREES

In this section we make the crucial translation to well-founded trees. The simple technique of interpreting by embedding fails here. A non-well-founded tree is not embeddable into a well-founded tree. To overcome this difficulty we introduce oriented trees where in addition to the upward tree ordering (the vertical order) there is also a horizontal order: sons of the same father are partially ordered (from left to right). An important task of the next section will be to get rid of orientations.

The contribution of this section toward the Main Theorem is a reduction of the first-order theory of one class of bounded trees, that contains every bounded, colored, complete tree, to the first-order theory of bounded, colored, oriented, well-founded trees.

Recall that a tree is well-founded if the ancestors of any node are well-ordered and every pair of nodes has an infimum (called the meet). The *height* of a node x in a well-founded tree is the ordinal type of the sequence $\langle y: y < x \rangle$.

Lemma 1. Every well-founded tree is complete.

Proof. Let T be a well-founded tree. The unique element of T of height 0 is the root of T. If C is a bounded chain in T then a node in U(C) of minimal height is a supremum of C. Now use Lemma 5 in Section 3.□

Definition. A strict partial order H on a tree is an *orientation* of the tree if

(i) for every (x, y) in H the nodes x and y are brothers,

(ii) every node belongs to a unique maximal H-chain (in other words, the set of nodes, partially ordered by H, is a disjoint union of H-chains), and

(iii) every maximal H-chain has an H-maximal element.

A tree with an orientation is an *oriented tree*.

Lemma 2. Let H be an orientation of a tree. Then the relation "either $x < y$ or there exists z such that xHz and $z \le y$" is the transitive closure of the union of the relations H and <.

Proof. The relation in question includes the union of H and <. It is included into the transitive closure of the union. And it is transitive.□

Theorem 1. For every complete tree T there is a well-founded tree T' and an orientation H of T' such that

(i) T and T' have the same nodes, and
(ii) For all nodes x and y, x is a proper T-ancestor of y if and only if either x is a proper T' ancestor of y or there is a node z such that xHz and z is a T'-ancestor of y.

Proof. Work in a complete tree T. For every nonempty, connected, open subset X choose a point p(X) in X such that p(X) = min(X) if X is a cone. Let C(X) be the chain $\{x \in X: x \le p(X)\}$.

Construct a decreasing sequence of open subsets as follows:

$S_o = T$,
$S_\alpha = \cap\{S_\beta: \beta < \alpha\}$ if α is limit, and
$S_{\alpha+1} = \cup\{X - C(X): X \text{ is a connected component of } S_\alpha\}$.

The construction halts when the empty set is reached. If $x \in S_\alpha - S_{\alpha+1}$ we say that α is the *rank* of x. Clearly, every node is ranked and $\text{rank}(x) \le \text{rank}(y)$ if $x \le y$ (because the sets S_α are open).

Claim 1. Suppose that α is limit and S_α is not empty. Then every component X of S_α is a cone.

Proof of Claim 1. It suffices to check that x = inf(X) belongs to S_α. Let β = rank(x). By contradiction suppose that $\beta < \alpha$. By Lemma 2 in Section 3, the component of Y of $S_{\beta+1}$ which includes X contains a proper lower bound y for X. Then $y \leq x$ which is impossible. Claim 1 is proved.□

If X is a component of some nonempty $S_{\alpha+1}$ then the supremum y of PL(X) will be called the *patriarch* of X and X will be called a *clan* of y.

Claim 2. Suppose that X is a component of some nonempty $S_{\alpha+1}$ and y is the patriarch of X. Then rank(y) = α and either y = inf(X) or there is a son x of y such that X = Cone(x).

Proof of Claim 2. Let Y be the connected component of S_α which includes X. By Lemma 2 in Section 3, Y contains a proper lower bound y' for X. If z is a proper lower bound for X of rank $\geq \alpha$ then $z \in Y$ (because y' and z are comparable) and z does not belong to any component of $S_{\alpha+1}$. Hence $z \leq p(Y)$. But y is the supremum of such nodes z, hence $y \in C(Y)$ and rank(y) = α.

Next, let x = inf(X). If $x \notin X$ then x is the patriarch of X and the claim is proved. Let $x \in X$. Then X = Cone(x) and x is a son of y. For, if $y < z < x$ then z is a proper lower bound for X above y which is impossible. Claim 2 is proved.□

Let R be the relation "x is the patriarch of a clan which contains y". Clearly, xRy implies $x < y$, and R is transitive. Hence R defines a new tree T' on the nodes of T. Clearly, every R-chain is well-ordered. The next claim completes the proof that T' is a well-founded tree.

Claim 3. For all nodes x_1 and x_2 there is a T'-meet of x_1, x_2.

Proof of Claim 3. Let x be the T-meet of x_1 and x_2. Every common T'-ancestor of x_1, x_2 is a T-ancestor of x. If x has a T'-father y then y is a common T'-ancestor of x_1, x_2. Hence either x or y is the T'-meet.

Suppose that x does not have a T'-father. Then $\alpha = \text{rank}(x)$ is limit. By Claim 1, the component S_α which contains x is a cone. By the definition of sets S_α, this component is Cone(x). By Claim 2, x is the patriarch of any component of $S_{\alpha+1}$ which meets Cone(x) (hence is included into Cone (x)). Hence x is the T'-meet of x_1, x_2. Claim 3 is proved.□

Let H be the relation "there exists α and a component Z of $S_{\alpha+1}$ such that x and y belong to Z and $x < y \leq p(Z)$". It is easy to see that H orients T'. In particular, if xHy and $\alpha+1 = \text{rank}(x)$ and Z is the component of $S_{\alpha+1}$ which contains x, y, then the patriarch of Z is a T'-father of x, y.

Finally, we prove the statement (ii) of Theorem 1. The "if" implication is obvious.

Claim 4. Suppose that x is a proper T-ancestor of y. Then either xHy holds or x is a proper T'-ancestor of y or there is z such that xHz holds and z is a T'-ancestor of y.

Proof of Claim 4. Let $\alpha = \text{rank}(x)$ and X be the component of S_α which contains x. If $x' > x$ is a node of rank α then C(X) is not singleton, X is not a cone, α is successor (by Claim 1), and therefore xHx' holds. In particular, if $\text{rank}(y) = \alpha$ then xHy holds.

Suppose that $\text{rank}(y) > \alpha$. Let Y be the component of $S_{\alpha+1}$ which contains y and let z be the patriarch of Y. If x = z then x is a proper T'-ancestor of y. Suppose that $x \neq z$. By Lemma 2 in Section 3, x is a proper lower bound for Y. Hence $x < z$. By Claim 2, rank(z) = α and z is a proper T'-ancestor of y. By the previous paragraph, xHz holds. Claim 4 is proved. Theorem 1 is proved.□

Theorem 2. Let T be a tree, T' be a well-founded tree and H be an orientation of T'. Suppose that T, T' and H satisfy the statements (i) and (ii) of Theorem 1. Then T is bounded if and only if T' is bounded.

Proof. The "only if" implication is clear. We prove the "if" implication. By contradiction suppose that T' is bounded but T has an unbounded branch B. Let A be the set of T'-heights of elements in B.

If A has a maximal element α then B has a final segment which is an H-chain. By the definition of orientations, this H-chain has an H-upper bound x. Clearly, x is a T-upper bound for B, which is impossible.

Hence A does not have a maximal element. For every $\alpha+1$ in A, all nodes in B of T'-height $\alpha+1$ are sons of the same T'-father. These T'-fathers are cofinal in B and form a T'-chain. Any T'-upper bound for this T'-chain is a T-upper bound for B which is impossible.□

Proposition 1. Let K be the class of colored trees T such that there exists a bounded well-founded tree T' and an orientation H of T' which satisfy the statements (i) and (ii) of Theorem 1. Then every colored tree in K is bounded, every bounded, colored, complete tree belongs to K, and the first-order theory of K reduces to the first-order theory of bounded, colored, oriented, well-founded trees.

Proof. The first statement follows from Theorem 2. The second statement follows from Theorems 1 and 2. The third statement is obvious.□

SECTION 5. BINARY TREES

In this section we reduce the first-order theory of bounded, colored, oriented, well-founded trees to the first-order theory of binary, bounded, colored, well-founded trees.

An orientation of H of a binary tree looks especially simple. Every maximal H-chain consists of at most two nodes. If $(x, y) \varepsilon H$ and z is the father of x, y then we say that x is the *left son* of z and y is the *right son* of z. H defines the following *lexicographic order* on the tree: x *lexicographically precedes* y if the meet of x, y has two sons, the sons are H-ordered, x descends from the left son and y descends from the right son.

Lemma 1. Every linear order C is isomorphic to the lexicographic order of leaves (i.e., maximal nodes) of some binary, bounded, oriented, well-founded tree.

Proof. The desired tree T will be a set of non-empty segments of C ordered by reverse inclusion. If A is a segment of C with at least two points we split A into a nonempty initial segment (called the *left part* of A) and the remaining nonempty final segment (the *right part* of A). If A will be a node of T then the left part of A and the right part of A will be the left and the right sons of A in T.

We construct a decreasing sequence of equivalence relations on C whose equivalence classes are segments of C. We start with $E_o = CxC$. If α is limit then E_α is the intersection of all E_α with $\beta < \alpha$. For every α, $E_{\alpha+1}$ is the relation "x and y belong to the same part of the same equivalence class of E_α". The construction is completed when the equality relation is reached. The desired tree T consists of all equivalence classes of all these equivalence relations.□

Theorem 1. For every well-founded tree T with an orientation H there is a binary, bounded, oriented, well-founded tree T' such that

(i) the universe of T' extends the universe of T,
(ii) for all x, y in T, $x < y$ in T iff $x < y$ in T', and
(iii) for every pair (x, y) of T-brothers, $(x, y) \varepsilon H$ iff x lexicographically precedes y in T'.

Proof. By Lemma 1, for every maximal H-chain C there is a binary, bounded, oriented, well-founded tree Tr(c) such that C is isomorphic to the set of leaves of Tr(C) with the lexicographical order. Let I(C) be an isomorphism from C to the chain of leaves of Tr(C).

For every nonsingleton maximal H-chain C we add some auxiliary nodes to T. Let x be the T-father of nodes in C. Graft a copy of Tr(C) at x in such a way that the root of Tr(C) becomes a son of x, and identify nodes of C with leaves of Tr(C) with respect to I(C). The resulting tree is the desired binary tree. We orient this binary tree with respect to the grafts. In other words, the union of the orientations of the grafts is the orientation of our binary tree.□

Proposition 1. The first-order theory of bounded, colored, oriented, well-founded trees reduces to the first-order theory of binary, bounded, colored, well-founded trees.

Proof. Let φ be a first-order sentence in a vocabulary comprising the binary predicate symbols <, H and a number of unary predicate constants. Find first unary predicate constants L, P, which do not appear in φ.

Write a first-order sentence φ_1 and a first-order formula $L^*(u, v)$ in the vocabulary $\{<, L\}$ such that if T is a binary tree with a color L which satisfies φ_1 then for every pair of brother nodes in T exactly one brother belongs to L, and an arbitrary pair (x, y) of nodes satisfies $L^*(x, y)$ in T if and only if there exist nodes x', y', z such that $x' \leq x$, $y' \leq y$, x' and y' are different sons of z and $x' \varepsilon L$.

Write a first-order sentence φ_2 in the vocabulary {<, L, P} such that if T is a bounded well-founded tree with colors L, P which satisfies φ_2, then P forms a bounded well-founded subtree of T and L^* is an orientation of the subtree P. Let φ_3 be the result of the following changes in φ: first restrict all quantifiers in φ by P, then replace H by L^*. It is easy to see that

$$\varphi \mapsto (\varphi_1 \ \& \ \varphi_2 \rightarrow \varphi_3)$$

is the desired reduction.□

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