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TOTAL ORDERS WHOSE CARRIED GROUPS SATISFY NO LAWS

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In another talk at this conference Paul Conrad has observed that the structure of a lattice ordered group is more strongly influenced by the underlying order than by the group. This is especially true of totally ordered groups. There are examples of totally ordered groups in which, except for the choice of the identity element, the group structure is completely determined by the order. These are necessarily commutative [CD]. The most obvious example is the ordered set $\mathbf{Z}$ of integers. It is not difficult to find ordered sets which carry several different groups, but only commutative groups. For example, the lexicographically ordered product $\mathbf{Z} \times \mathbf{Z}$ has this property. The ordered set of rational numbers carries (that is, is the underlying ordered set of) both commutative and non-commutative groups. More than twenty years ago, Reinhold Baer asked one of the authors if there is a totally ordered set which carries a group, but which carries only *non*-commutative groups. (See Glass [G], p. 246.) In this paper, we describe the construction of a totally ordered set which carries a free group, and whose carried groups cannot satisfy any equational group law (and so, in particular, cannot be commutative). Here, we outline the construction. The details can be found in [HMS].

We note that Baer's question is phrased in the weakest possible way, for if an ordered group satisfies some non-trivial equational law, then so does an infinite cyclic subgroup, and hence so do all abelian groups. Thus, an ordered set which carries no group satisfying a certain law "u = v" will necessarily carry no abelian group.

Any countable ordered set which carries an ordered group must have the form $\mathbb{Z}^\alpha \times D$, where $\mathbb{Z}^\alpha$ is a lexicographically ordered restricted product of copies of the integers and D is a singleton or a dense countable set. Such a set always carries some abelian ordered group as well. Hence any ordered set which carries no

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abelian group must be uncountable.

The general idea behind our construction is the following. We construct a certain uncountable ordered group $(G,\cdot,\leq)$ such that $(G,\cdot)$ is a free group on a set Y of free generators such that Y is dense in $(G,\leq)$, and such that each order-preserving permutation of $(G,\leq)$ is piecewise of the form $x \to a \cdot x \cdot b$. (That is, every non-trivial interval of $(G,\leq)$ contains a non-trivial interval on which the permutation has the given form.) Then if $(G,*,\leq)$ is any ordered group on the same carrier, the mappings $x \to x * g$ are order-preserving permutations and so must be piecewise of the form $x * g = a \cdot x \cdot b$. For any group word $w(\overline{x}) = w(x_1,\ldots,x_n)$, we wish to show that $w(\overline{g}) \neq e$ in $(G,*)$ for a suitable choice of $g_1,\ldots,g_n \in G$. But we can compute $x * w(\overline{g}) = u(\overline{a}) \cdot x \cdot v(\overline{b})$ for certain corresponding $u(\overline{a})$, $v(\overline{b})$ in $(G,\cdot)$, the computation taking place on a suitably alligned sequence of intervals. The density of the free generating set Y implies that at least one of $u(\overline{a})$, $v(\overline{b})$ acts non-trivially on x, and so $x * w(\overline{g}) \neq x$, which implies $w(\overline{g}) \neq e$. For fuller details, see [HMS] where the following theorem is proved:

<u>THEOREM</u> <u>1.1</u>. *If* $(G,\cdot,\leq)$ *is an uncountable ordered group, and* $(G,\cdot)$ *is a free group on generators* Y, *and* Y *contains a countable subset which is dense in* $(G,\leq)$, *and every order-preserving automorphism of* $(G,\leq)$ *is piecewise of the form* $x \to a \cdot x \cdot b$, *then* $(G,\leq)$ *is lawless; that is, if* $(G,*,\leq)$ *is an ordered group on the same carrier* $(G,\leq)$ *then* $(G,*)$ *satisfies no non-trivial equational group law.*

The remaining problem, then, is to construct an ordered group satisfying the hypotheses of Theorem 1.1. First, we construct a certain ordered group $(H,\cdot,\leq)$ such that $(H,\cdot)$ is a product of free groups and $(H,\leq)$ is a dense ordered subset of the real line $(\mathbb{R},\leq)$. We then construct $(G,\cdot,\leq)$ as a certain dense ordered subgroup of $(H,\cdot,\leq)$. The order-preserving permutations of $(G,\leq)$ will arise, then, as restrictions of order-preserving permutations of $(\mathbb{R},\leq)$. We begin by listing the order-preserving permutations of $(\mathbb{R},\leq)$ as $\{\phi_\alpha\colon\ 0 \leq \alpha < 2^{\aleph_0},\ \alpha \text{ even}\}$, and then construct G as a union of a tower of subgroups G_α of H with $|G_\alpha| < 2^{\aleph_0}$. In constructing $G_{\alpha+1}$ we try to eliminate ϕ_α from being an order-preserving permutation of the eventual $G = \bigcup G_\alpha$, if possible. For this purpose, we simultaneously construct a "garbage pile" as a union of a tower of sets R_α such that $R_\alpha \cap G_\alpha$ is empty. If we have constructed G_α and R_α and wish to eliminate the permutation ϕ_α, we first try to find some free generator $b \in H$ such that $b\phi_\alpha \notin \langle G_\alpha,b\rangle$ (the subgroup generated by G_α and b) and we can add $b\phi_\alpha$ to $R_{\alpha+1}$ and let $G_{\alpha+1} = \langle G_\alpha,b\rangle$ while maintaining disjointness of $R_{\alpha+1}$ and $G_{\alpha+1}$. Then ϕ_α will not be a permutation of G since $b \in G_{\alpha+1} \subseteq G$ while

$b\phi_\alpha \notin G$. If it is not possible to find such a b, that is, for all eligible b, $b\phi_\alpha \in \langle G_\alpha, b\rangle$, we try instead to find b such that $b \notin \langle G_\alpha, b\phi_\alpha\rangle$ so that we can add b to $R_{\alpha+1}$ and let $G_{\alpha+1} = \langle G_\alpha, b\phi_\alpha\rangle$, and again ϕ_α will not be a permutation of G since $b\phi_\alpha$ G while $b \notin G$. If this is also not possible, then for all eligible b, both $b\phi_\alpha \in \langle G_\alpha, b\rangle$ and $b \in \langle G_\alpha, b\phi_\alpha\rangle$. It can then be shown that for each b, $b\phi_\alpha = g \cdot b \cdot f$ for some $g, f \in G_\alpha$ (depending on b). This shows that ϕ_α is pointwise of the desired form on a large dense subset of H. Finally, we can show, using an ultrafilter game, that ϕ_α actually has the desired form on intervals. Thus every order-preserving permutation of the resulting $(G, \leq)$ satisfies the hypotheses of Theorem 1.1, and we are done.

A refinement of the construction shows that there are $2^{2^{\aleph_0}}$ pairwise non-isomorphic separable lawless orders (contained in the real line). Moreover, it is consistent with ZFC that $2^{\aleph_0} > \aleph_1$ and for all $\aleph_1 \leq \kappa \leq 2^{\aleph_0}$ there are 2^κ pairwise non-isomorphic separable lawless orders. However, it is also consistent that there are *no* lawless orders of cardinality less than $2^{\aleph_0}$.

After [HMS] was written, the question arose whether similar techniques could be used to construct an ordered set which carries a right ordered group ($x < y \Rightarrow xz < yz$) but which does not carry an ordered group. We do this here.

Let $(\mathbb{Q}, \leq)$ be the rational numbers with the usual order. Enumerate $\mathbb{Q} = \{q_0, q_1, q_2, \ldots\}$. Let $A(\mathbb{Q}, \leq)$ denote the group of all order-preserving permutations of $(\mathbb{Q}, \leq)$, and let F be the free group on a countable set X. Note that if $r < s < t$ and $r' < s' < t'$ are elements of $\mathbb{Q}$, there exists $f \in A(\mathbb{Q})$ such that $rf = r'$, $sf = s'$, and $tf = t'$.

<u>LEMMA</u> <u>1</u>. *There is an embedding of* F *into* $A(\mathbb{Q}, \leq)$ *such that*

1. F *is transitive on* $\mathbb{Q}$
2. $e \neq f \in F$ *implies* f *moves some point of* $\mathbb{Q}$ *down*

and

3. *For any pair* q_n, q_m *with* $n < m$, *there exists* $f \in F$ *such that* $q_k f = q_k$ *for all* $k < n$, $q_n f > q_n$ *and* $q_m f < q_m$.

Proof: Enumerate $F = \{w_0, w_1, w_2, \ldots\}$ and enumerate the set of all pairs of rationals (q_n, q_m), $n < m$. By induction, for each pair (q_n, q_m) choose $x \in X$ not previously used and let $\overline{x} \in A(\mathbb{Q}, \leq)$ such that $q_n\overline{x} = q_m$ and for some $b_i \in \mathbb{Q}$ greater than all previously considered q_j, $b_i\overline{x} = b_i$. We consider $\overline{x}$ defined now on $(-\infty, b_i]$ but reserve the definition of $\overline{x}$ on (b_i, ∞) until later. Next we choose some $y \in X$ not previously used and let $\overline{y} \in A(\mathbb{Q}, \leq)$ such that for all q_k with $k < n$, $q_k\overline{y} = q_k$, $q_n\overline{y} > q_n$, and $q_m\overline{y} < q_m$, and for some $c_i \in \mathbb{Q}$ greater than all previously considered q_j, $c_i\overline{y} = c_i$. We consider $\overline{y}$ defined on $(-\infty, c_i]$ but reserve the definition of $\overline{y}$ on (c_i, ∞) until later. Since any free group admits a total order, the right regular representation of F shows that F is isomorphic to a subgroup of $A(\mathbf{Q},\leq)$. It follows that F is isomorphic to a subgroup of $A(I,\leq)$ for any non-trivial interval I of $\mathbf{Q}$, and so if we choose an interval I lying above all previously considered b_k, c_k, for the element $e \neq w_i \in F$ involving letters $x_{i,1},\ldots,x_{i,r}$, we can choose $\overline{x}_{i,1},\ldots,\overline{x}_{i,r} \in A(I)$ such that $\overline{w}_i = w_i(\overline{x}_{i,1},\ldots,\overline{x}_{i,r}) \neq e$. These $\overline{x}_{i,j}$ may be spliced to any previously defined $\overline{x}_{i,j}$ so that the resulting $\overline{x}_{i,j} \in A(\mathbf{Q},\leq)$. We then repeat the process, and the lemma is proved.

<u>LEMMA 2</u>. *The free group* F *can be right ordered in such a way that*

1. *For all* $e \neq f \in F$ *there exist* $x,y \in F$ *such that* $x < y$ *and* $fy < fx$

and

2. *For all* $x,y \in F$ *such that* $x < y$ *there exists* $f \in F$ *such that* $fy < fx$.

Proof: Embed F in $A(\mathbf{Q},\leq)$ as in Lemma 1. Define $x < y$ to mean that $q_i\overline{y}\,\overline{x}^{-1} > q_i$ where i is the smallest integer j such that $q_j\overline{y}\,\overline{x}^{-1} \neq q_j$. It is straight forward to verify that $<$ is a right order of F with the required properties.

Now we want to construct a right ordered group $(G,\cdot,\leq)$ such that there is no ordered group $(G,*,\leq)$ with the same carrier. The construction is exactly as in [HMS] except that we begin with the free group F right ordered as in Lemma 2. Let $H = \prod_{i=0}^{\infty} F_i$ with each $F_i \approx F$ and order H lexicographically. We then construct G as an uncountable subdirect product containing the restricted product and a suborder of H just as in [HMS], where it is proved that every order-preserving permutation ϕ of $(G,\cdot,\leq)$ has the property that for each non-trivial interval I, there is a non-trivial subinterval I' such that on I' either ϕ has the form $x\phi = a \cdot x \cdot b$ or the form $x\phi = a \cdot x^{-1} \cdot b$. The latter cannot possibly preserve order. For example, if $e < a$ then $a \cdot e^{-1} = a > e = a \cdot a^{-1}$ and so $a \cdot e^{-1} \cdot b > a \cdot a^{-1} \cdot b$. We now claim that, in fact, $x\phi = x \cdot b$. Let us introduce the following notation. For $g \in G \subseteq H$, $g = (g(0),g(1),g(2),\ldots)$, $g(i) \in F$. For any finite sequence $s = (s(0),s(1),\ldots,s(n-1))$, $s(i) \in F$, let $I_s = \{(s(0),s(1),\ldots,s(n-1),s(n),\ldots) \in G\colon\ s(i)$ arbitrary for $i \geq n\}$. Now if ϕ has the form $x\phi = a \cdot x \cdot b$ on an interval I we can choose a finite sequence s of length n such that $I_s \subseteq I$ and $a(n) \neq e$. Then choose $x,y \in G$ such that for $i < n$, $x(i) = y(i) = s(i)$, but $x(n) < y(n)$ while $a(n)x(n) > a(n)y(n)$ (possible because of 2 in Lemma 2). It follows that $a(n)x(n)b(n) > a(n)y(n)b(n)$ and so $a \cdot x \cdot b > a \cdot y \cdot b$, and ϕ does not preserve order. Hence $a = e$, and $x\phi = x \cdot b$.

It remains to show that $(G,\leq)$ does not carry any ordered group. Suppose $(G,*,\leq)$ is a right ordered group. We will show that $(G,*,\leq)$ is not an ordered group. For every $g \in G$ the mapping $x \to x * g$ is an order preserving permutation of $(G,\leq)$ and so has the form $x \to x \cdot b$ on some interval I_s. Because G is uncountable and there are only countably many I_s, it follows that for some one sequence of length n, and an uncountable set $\{x_\alpha\} \subseteq G$, on I_s $x * x_\alpha = x \cdot b_\alpha$ for some $b_\alpha \in G$. Since $\{x_\alpha\}$ is uncountable but F^n is countable, there exist $\alpha \neq \beta$ such that $b_\alpha(i) = b_\beta(i)$ for all $i < n$. Since $x_\alpha \neq x_\beta$, for all $g \in G$, $g * x_\alpha \neq g * x_\beta$ and so $b_\alpha \neq b_\beta$. Let m be the least integer such that $b_\alpha(m) \neq b_\beta(m)$. Suppose $b_\alpha(m) < b_\beta(m)$.

Case 1. $x_\beta < x_\alpha$. We can choose $a \in G$ such that $a(i) = s(i)$ for all $i < n$ and $a(m) = e$. Then $a \in I_s$ and $a * x_\beta = a \cdot b_\beta > a \cdot b_\alpha = a * x_\alpha$ since $a(i)b_\beta(i) = a(i)b_\alpha(i)$ for $i < m$ while $a(m)b_\beta(m) = b_\beta(m) > b_\alpha(m) = a(m)b_\alpha(m)$. Hence order is not preserved by left multiplication $(*)$ by a.

Case 2. $x_\alpha < x_\beta$. We can choose $a \in G$ such that $a(i) = s(i)$ for all $i < n$ and $a(m)b_\alpha(m) > a(m)b_\beta(m)$ by 2. of Lemma 2. Then $a \in I_s$ and $a * x_\alpha = a \cdot b_\alpha > a \cdot b_\beta = a * x_\beta$, and again, order is not preserved.

We have proved:

THEOREM. *There exists a totally ordered set* $(G,\leq)$ *which carries a right ordered group* $(G,\cdot,\leq)$ *but which does not carry any ordered group.*

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