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ON THE UNIVERSALITY OF SYSTEMS OF WORDS IN PERMUTATION GROUPS

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In the classes of infinite symmetric groups, their normal subgroups, and their factor groups, we determine those groups which are equivalent in the sense that they may not be distinguished by the solvability of a system of finitely many equations in variables and parameters.

1. Introduction and results. Recently, several authors [**1, 3–5, 8, 9, 12**] studied the solvability of equations of the form $w(x_1, \dots, x_n) = y$, where w is a group word, in various kinds of groups. In [**1, 3, 12**] this problem was considered for infinite symmetric groups. Here we consider the simultaneous solvability of several equations of a similar form in infinite symmetric groups, their normal subgroups, and their factor-groups.

Let G be a group, $x_1, \dots, x_n$ variables, $y_1, \dots, y_m$ parameters, and $w_i = w_i(x_1, \dots, x_n; y_1, \dots, y_m)$ $(i \in I)$ group words in these variables and parameters. We say that $W = \{w_i \mid i \in I\}$ is G-universal if G satisfies the following property:

For all $y_1, \dots, y_m \in G$ there exist $x_1, \dots, x_n \in G$ such that for all $i \in I$, $w_i(x_1, \dots, x_n; y_1, \dots, y_m) = e$.

Two groups G and H will be called equationally equivalent, $G \equiv_{\text{eq}} H$, if for any finite set W of words w_i as above, W is G-universal iff W is H-universal.

Let S_ν denote the infinite symmetric group of all permutations of a set of cardinality $\aleph_\nu$ and, for $0 \leq \tau \leq \nu + 1$, S_ν^τ its normal subgroup comprising all permutations moving less than $\aleph_\tau$ elements of the underlying set. The problem of the elementary equivalence (definability) of the groups S_ν $(\nu \geq 0)$ was solved in Shelah [**11**]. Here we will consider the problem of the equational equivalence of the groups S_ν. A very similar problem was suggested by J. Isbell, cf. [**6**; p. 20]. Throughout this paper, let $V = \{v_1, v_2, v_3\}$ be the following set of words in parameters y_1, y_2 and variables x_1, x_2, x_3:

$$v_i = y_i^{-1} \cdot x_1^{-1} \cdot y_i \cdot x_1 \quad (i = 1, 2),$$

$$v_3 = y_1^{-1} \cdot x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1} \cdot x_1 \cdot x_3.$$

Note that $v_i = e$ means that x_1 and y_i commute $(i = 1,2)$, and $v_3 = e$ means that y_1 is a product of two conjugates of x_1. Using a result of Droste and Göbel [**2**], we will show:

THEOREM 1. (a) *Let* $\nu \geq 0$. *The following are equivalent*:
(1) *V is S_ν-universal.*
(2) *V is (S_ν/S_ν^ν)-universal.*
(3) $\aleph_\nu > 2^{\aleph_0}$.
(b) *Let* $\mu, \nu \geq 0$ *and* $\aleph_\mu > 2^{\aleph_0}$. *The following are equivalent*:
(1) $S_\nu \equiv_{\text{eq}} S_\mu$.
(2) $S_\nu/S_\nu^\nu \equiv_{\text{eq}} S_\mu/S_\mu^\mu$.
(3) $\aleph_\nu > 2^{\aleph_0}$.

We also obtain the subsequent generalization of Ehrenfeucht et al. [**3**; Theorem 3] which is partly a consequence of a result of Moran [**7**]:

THEOREM 2 *Let* $\nu, \mu \geq 0$. *The following are equivalent*:
(1) *For any finite set* $W = \{w_i \mid i \in I\}$ *of words* $w_i = w_i(x_1, \ldots, x_n; y)$ *in one parameter y and variables* $x_1, \ldots, x_n$, *W is S_ν-universal iff W is S_μ-universal.*
(2) *Either* (i) $\nu = \mu = 0$, *or* (ii) $\nu, \mu > 0$.

In particular, $S_0 \not\equiv_{\text{eq}} S_\nu$ whenever $\nu > 0$. Hence the following is an immediate consequence of Theorems 1 and 2:

COROLLARY 3. *Assume* (CH). *Then, for any* $\nu, \mu \geq 0$, *we have* $S_\nu \equiv_{\text{eq}} S_\mu$ *iff either* $\nu = \mu = 0$ *or* $\nu = \mu = 1$ *or* $\nu, \mu \geq 2$.

Next we just note the following

THEOREM. *It is consistent with* (ZFC + $2^{\aleph_0}$ *arbitrarily large*) *that*: (*) $S_\nu \equiv_{\text{eq}} S_\mu$ *whenever* $\aleph_0 < \aleph_\nu < \aleph_\mu \leq 2^{\aleph_0}$.

This is done by starting with a model V of set theory; then choose $\kappa = \kappa^{\aleph_0}$, and force by adding κ Cohen reals. As we do not yet know whether (*) is provable in ZFC or is independent of ZFC, this theorem is of doubtful value at present.

Concerning the permutation groups S_ν^τ, we have this result:

THEOREM 4. (a) *Let* $0 \leq \tau \leq \nu$ *and* $0 \leq \rho \leq \mu$. *Then* $S_\nu^\tau \equiv_{\text{eq}} S_\mu^\rho$ *iff either* $\tau = \rho = 0$ *or* $\tau, \rho \geq 1$.
(b) $S_\nu^\nu \not\equiv_{\text{eq}} S_\nu$ *whenever* $\aleph_\nu \leq 2^{\aleph_0}$.

Here, in (b) it remains open whether the assumption $\aleph_\nu \le 2^{\aleph_0}$ is necessary.

2. Notation. Let $\mathbf{N}$ denote the set of all positive integers and $\mathbf{N}_\infty = \mathbf{N} \cup \{\aleph_0\}$. Let $a^b = b^{-1} \cdot a \cdot b$ for $a, b \in G$ (any group), and let $A = \dot{\bigcup}_{i \in I} A_i$ mean that A is the *disjoint* union of the A_i. For a mapping f let a^f denote its value at a and $f|_A$ its restriction to A.

P_M denotes the group of all permutations of a set M and id_M (or id, if there is no ambiguity) the identity map of M. Let $p \in P_M$. An orbit of p is a minimal p-invariant subset of M. For any $n \in \mathbf{N}_\infty$, let $\bar{p}(n)$ denote the cardinality of the set of all orbits of length n of p. Let $\mathrm{supp}(p) = \{a \in M \mid a^p \neq a\}$ denote the support of p, and $|p| = |\mathrm{supp}(p)|$. For $\nu \ge 0$ and $0 \le \tau \le \nu + 1$, let $S_\nu = P_{\aleph_\nu}$, $S_\nu^\tau = \{p \in S_\nu \mid |p| < \aleph_\tau\}$ and $A_\nu = \{p \in S_\nu^0 | p|_{\mathrm{supp}(p)}$ is even$\}$. Then, as is well-known, the groups A_ν and S_ν^τ $(0 \le \tau \le \nu)$ are all non-trivial proper normal subgroups of S_ν. If $M = \dot{\bigcup}_{i \in I} M_i$, $p_i \in P_{M_i}$, and $p \in P_M$ such that $p|_{M_i} = p_i$ for each $i \in I$, then we also write $p = \bigoplus_{i \in I} p_i$.

If $w(x_1, \dots, x_n; y_1, \dots, y_m)$ is a word in parameters y_k and variables x_j, we also abbreviate it by $w(\underset{\sim}{x}_j; \underset{\sim}{y}_k)$; we include the indices j, k in this expression in order to indicate that they range over index sets J, K respectively (here $J = \{1, \dots, n\}$, $K = \{1, \dots, m\}$).

3. Proof of our results. Before we can prove Theorem 1, we need a few preparations:

PROPOSITION 3.1. *Let M be a set of cardinality $\aleph_\nu$ $(\nu \ge 0)$ and $z_k \in P_M$ for $k \in K$, where K is at most countably-infinite.*

(a) *If $A \subseteq M$, let $B = \{a^z \mid a \in A, z \in Z\}$ where $Z \subseteq P_M$ is the subgroup of P_M generated by $\{z_k \mid k \in K\}$. Then B is the smallest subset of M containing A such that $z_k|_B \in P_B$ for each $k \in K$. Furthermore, $|B| = |A|$ if A is infinite.*

(b) *For each cardinality $\aleph \le \aleph_\nu$, there is a decomposition $M = \dot{\bigcup}_{l \in L} M_l$ such that $z_k | M_l \in P_{M_l}$ and $|M_l| = \aleph$ for each $k \in K$, $l \in L$.*

Proof. (a) Obvious.

(b) For $x, y \in M$, let $x \sim y$ if $x = y^z$ for some $z \in Z$. This defines an equivalence relation on M, and each equivalence class is at most countably-infinite and is invariant under any z_k $(k \in K)$. Now choose each M_l $(l \in L)$ to be an appropriate union of equivalence classes.

PROPOSITION 3.2. *Let $w_i = w_i(\underset{\sim}{x}_j; \underset{\sim}{y}_k)$ be words, where $i \in I$, $j \in J$, $k \in K$, and I, J, K are finite, and $\nu \ge 0$. If $W = \{w_i \mid i \in I\}$ is S_ν-universal, then W is also S_μ-universal for any $\mu > \nu$.*

Proof. Let M be a set of cardinality $\aleph_\mu$ and $y_k \in P_M$ $(k \in K)$. By Proposition 3.1(b), split $M = \dot{\bigcup}_{l \in L} M_l$ such that $|M_l| = \aleph_\nu$ and $y_{l,k} = y_k|_{M_l} \in P_{M_l}$ for all $k \in K$, $l \in L$. By assumption, for each $l \in L$ there are $x_{l,j} \in P_{M_l}$ $(j \in J)$ such that $w_i(\underset{\sim}{x}_{l,j}; \underset{\sim}{y}_{l,k}) = \mathrm{id}_{M_l}$ for all $i \in I$. Put $x_j = \bigoplus_{l \in L} x_{l,j} \in P_M$ for each $j \in J$. Then $w_i(\underset{\sim}{x}_j; \underset{\sim}{y}_k) = \mathrm{id}_M$ for all $i \in I$.

The following lemma is stated in a more general form than actually needed since it may be also of some independent interest. Recall that for a permutation p and $n \in \mathbf{N}_\infty$, $\bar{p}(n)$ denotes the number of orbits of length n of p.

LEMMA 3.3. *Let a, $b \in S_0$ such that a consists of precisely one infinite orbit and $\bar{b}(n) \in \mathbf{N}$ for some $n \in \mathbf{N}$. If $c \in S_0$ satisfies $a = a^c$ and $b = b^c$, then $c = \mathrm{id}$.*

Proof. As $P_{\mathbf{Z}} \cong S_0$, it suffices to prove this for elements $a, b, c \in S_{\mathbf{Z}}$ where $i^a = i + 1$ for each $i \in \mathbf{Z}$ and 0 belongs to an orbit of length n of b. Let $0^c = k \in \mathbf{Z}$. Then $a = a^c$ implies $(j \cdot k)^c = (j+1) \cdot k$ for any $j \in \mathbf{Z}$. Hence each $j \cdot k$ $(j \in \mathbf{Z})$ belongs to an orbit of length n of b. Since $\bar{b}(n) < \infty$, this shows $k = 0$. But then $c = \mathrm{id}$ by $a = a^c$.

Now suppose ϕ: $A \to B$ is a bijection from A onto B. Then ϕ induces an isomorphism ψ from P_A onto P_B defined by $b^{p^\psi} = b^{\phi^{-1} \cdot p \cdot \phi}$ for each $b \in B$, $p \in P_A$. If $x_1, \ldots, x_n \in P_A$, $y_1, \ldots, y_n \in P_B$ $(n \in \mathbf{N})$ are such that $x_i^\psi = y_i$ $(i = 1, \ldots, n)$, then we also say that ψ is an isomorphism from the algebra $\langle A, x_1, \ldots, x_n \rangle$ onto $\langle B, y_1, \ldots, y_n \rangle$, induced by ϕ.

We can now prove Theorems 1 and 2:

Proof of Theorem 1. (a) (1) $\to$ (2): S_ν / S_ν^ν is an epimorphic image of S_ν.

(2) $\to$ (3): For contradiction, assume $\aleph_\nu \leq 2^{\aleph_0}$. Let M be a set of cardinality $\aleph_\nu$, and decompose $M = \dot{\bigcup}_{j \in J} M_j$ such that $|M_j| = \aleph_0$ for any $j \in J$, and $|J| = |M|$. For each $j \in J$, choose $y_{j,1}$, $y_{j,2} \in P_{M_j}$ such that $y_{j,1}$ consists of precisely one (infinite) orbit, $\overline{y_{j,2}}(n) \in \mathbf{N}$ for some $n \in \mathbf{N}$, and whenever j, $k \in J$, $j \neq k$, then $\overline{y_{j,2}}(m) \neq \overline{y_{k,2}}(m)$ for some $m \in \mathbf{N}$. This is possible since $|J| \leq 2^{\aleph_0} = |S_0|$. Then put $y_1 = \bigoplus_{j \in J} y_{j,1}$, $y_2 = \bigoplus_{j \in J} y_{j,2} \in P_M$. Now by (2) there are x_i, $z_i \in P_M$ such that $|z_i| < \aleph_\nu$ $(i = 1, 2, 3)$, $y_i = y_i^{x_1} \cdot z_i$ $(i = 1, 2)$, and $y_1 = x_1^{x_2} \cdot x_1^{x_3} \cdot z_3$. Let T be the smallest subset of J with $\mathrm{supp}(z_1) \cup \mathrm{supp}(z_2) \subseteq \bigcup_{j \in T} M_j^{x_1}$. Pick any

$j \in J \setminus T$. If $N = M_j^{x_1}$, then $y_1 = y_1^{x_1} \cdot z_1$ and $z_1 \mid_N = \mathrm{id}_N$ imply $y_1 \mid_N \in P_N$, and $y_1 \mid_N$ consists of precisely one (infinite) orbit on N. Hence $N = M_k$ for some $k \in J$. Now $y_2 = y_2^{x_1} \cdot z_2$ and $z_2 \mid_N = \mathrm{id}_N$ yield $y_{k,2}(m) = y_{j,2}(m)$ for any $m \in \mathbf{N}$, thus $k = j$. Hence $M_j^{x_1} = M_j$, implying $x_1 \mid_{M_j} = \mathrm{id}_{M_j}$ by Lemma 3.3. This shows $|x_1| \le |T| \cdot \aleph_0 < \aleph_\nu$ and $|y_1| < \aleph_\nu$ by our third equality, an obvious contradiction.

(3) → (1): By Proposition 3.2 it suffices to consider the case $\aleph_\nu = (2^{\aleph_0})^+$. Let $|M| = \aleph_\nu$ and $y_1, y_2 \in P_M$. By Proposition 3.1(b), decompose $M = \dot{\bigcup}_{j \in J} M_j$ with $|M_j| = \aleph_0$ and $y_{j,i} = y_i \mid_{M_j} \in P_{M_j}$ for each $j \in J$, $i = 1, 2$. We call j, $k \in J$ equivalent if there exists an isomorphism from $\langle M_j, y_{j,1}, y_{j,2} \rangle$ onto $\langle M_k, y_{k,1}, y_{k,2} \rangle$. Since $|S_0| = 2^{\aleph_0}$, there are at most $2^{\aleph_0}$ different equivalence classes on J. Hence, by $|J| = \aleph_\nu = (2^{\aleph_0})^+$, a regular cardinal, there exists an equivalence class $T \subseteq J$ of cardinality $\aleph_\nu$. Now decompose $T = \dot{\bigcup}_{i \in \mathbf{Z}} T_i$ with $|T_i| = |T|$ for each $i \in \mathbf{Z}$, and choose an element $x_1 \in P_M$ such that $x_1 \mid_{M_j} = \mathrm{id}_{M_j}$ for each $j \in J \setminus T$, and, for any $i \in \mathbf{Z}$, $j \in T_i$, we have $M_j^{x_1} = M_k$ for some $k \in T_{i+1}$ such that $x_1 \mid_{M_j}$ induces an isomorphism from $\langle M_j, y_{j,1}, y_{j,2} \rangle$ onto $\langle M_k, y_{k,1}, y_{k,2} \rangle$. Then $y_i = y_i^{x_1}$ for $i = 1, 2$, and x_1 consists of $\aleph_\nu$ infinite orbits and, possibly, fixed points. Now by Droste and Göbel [**2**; Theorem 2], any element of P_M is a product of two conjugates of x_1, in particular $y_1 = x_1^{x_2} \cdot x_1^{x_3}$ for some x_2, $x_3 \in P_M$. Thus V is S_ν-universal.

(b) (1) → (3) and (2) → (3): By (a) and $\aleph_\mu > 2^{\aleph_0}$, V is S_μ- and (S_μ / S_μ^μ)-universal, hence S_ν- and (S_ν / S_ν^ν)-universal by assumption. Now (a) shows that $\aleph_\nu > 2^{\aleph_0}$.

(3) → (1) and (3) → (2): Let $w_i = w_i(\underset{\sim}{x}_j; \underset{\sim}{y}_k)$ be words ($i \in I, j \in J$; I, J finite, $k \in K = \{1, \ldots, n\}$ with $n \in \mathbf{N}$) such that $W = \{ w_i \mid i \in I \}$ is S_μ-universal ((S_μ / S_μ^μ)-universal); we claim that W is S_ν-universal ((S_ν / S_ν^ν)-universal), respectively. Let $|M| = \aleph_\nu$ and $y_k \in P_M$ ($k \in K$). By Proposition 3.1(b), there is a decomposition $M = \dot{\bigcup}_{l \in L} M_l$ such that $|M_l| = \aleph_0$ and $y_{l,k} = y_k \mid_{M_l} \in P_{M_l}$ for each $k \in K$, $l \in L$. We call l, $m \in L$ equivalent if there exists an isomorphism from $\langle M_l, y_{l,1}, \ldots, y_{l,n} \rangle$ onto $\langle M_m, y_{m,1}, \ldots, y_{m,n} \rangle$.

First let us show that W is S_ν-universal. Because of Proposition 3.2, we may assume $(2^{\aleph_0})^+ = \aleph_\nu < \aleph_\mu$. Thus there exists an equivalence class $T \subseteq L$ of cardinality $\aleph_\nu$. Fix $t \in T$, choose a set P of cardinality $\aleph_\mu$ and, for each $p \in P$, a copy $\langle M_p, y_{p,1}, \ldots, y_{p,n} \rangle$ of $\langle M_t, y_{t,1}, \ldots, y_{t,n} \rangle$, and put $A = M \,\dot{\cup} \dot{\bigcup}_{p \in P} M_p$, $y_k' = y_k \oplus \bigoplus_{p \in P} y_{p,k} \in P_A$ ($k \in K$). Since $|A| = \aleph_\mu$, by assumption there are $x_j' \in P_A$ ($j \in J$) with $w_i(\underset{\sim}{x}_j', \underset{\sim}{y}_k') = \mathrm{id}_A$ for all $i \in I$. By Proposition 3.1(a), there is a set B with $M \subseteq B \subseteq A$ and

$|B| = |M| = \aleph_\nu$ such that $y_k^+ = y_k'|_B$, $x_j^+ = x_j'|_B \in P_B$ for all $k \in K$, $j \in J$; in particular, $w_i(\underset{\sim}{x}_j^+; \underset{\sim}{y}_k^+) = \mathrm{id}_B$ for all $i \in I$. Now it is easy to see that $\langle M, y_1, \ldots, y_n \rangle$ is isomorphic to $\langle B, y_1^+, \ldots, y_n^+ \rangle$. Hence there exist $x_j \in P_M$ $(j \in J)$ such that $w_i(\underset{\sim}{x}_j; \underset{\sim}{y}_k) = \mathrm{id}_M$ for all $i \in I$, establishing (1).

Next we finish the proof of (2). For any subset $S \subseteq L$, we abbreviate

$$M_S = \dot{\bigcup_{l \in S}} M_l \quad \text{and} \quad y_k^S = y_k|_{M_S} = \bigoplus_{l \in S} y_{l,k} \in P_{M_S} \qquad (k \in K).$$

First we claim that for any infinite subset $T \subseteq L$ consisting only of pairwise equivalent elements of L such that $|T| < \aleph_\mu$, there are $x_j^T \in P_{M_T}$ $(j \in J)$ with $w_i(\underset{\sim}{x}_j^T; \underset{\sim}{y}_k^T) = \mathrm{id}_{M_T}$ for each $i \in I$.

Indeed, fix $t \in T$, choose a set P of cardinality $\aleph_\mu$ and, for each $p \in P$, a copy $\langle M_p, y_{p,1}, \ldots, y_{p,n} \rangle$ of $\langle M_t, y_{t,1}, \ldots, y_{t,n} \rangle$, and put

$$A = M_T \,\dot{\cup}\, \dot{\bigcup_{p \in P}} M_p, \; y_k' = y_k^T \oplus \bigoplus_{p \in P} y_{p,k} \in P_A \qquad (k \in K).$$

Since $|A| = \aleph_\mu$, by assumption there are x_j', $z_i \in P_A$ $(j \in J, i \in I)$ with $|z_i| < \aleph_\mu$ and $w_i(\underset{\sim}{x}_j'; \underset{\sim}{y}_k') = z_i$ for each $i \in I$. By (3.1), there exists a subset $Q \subseteq P$ of cardinality $|T|$ such that $z_i|_{M_Q} = \mathrm{id}_{M_Q}$, $x_j'|_{M_Q} \in P_{M_Q}$ for each $i \in I$, $j \in J$. Hence, since $\langle M_Q, y_1^Q, \ldots, y_n^Q \rangle$ is isomorphic to $\langle M_T, y_1^T, \ldots, y_n^T \rangle$, our claim follows.

Now we define the elements $x_j \in P_M$ $(j \in J)$ as follows. Whenever $l \in L$ belongs to a finite equivalence class, let $x_j|_{M_l} = \mathrm{id}_{M_l}$. For any infinite equivalence class $T \subseteq L$ with $|T| < \aleph_\mu$ put $x_j|_{M_T} = x_j^T$. Finally, if $T \subseteq L$ is an equivalence class with $|T| \geq \aleph_\mu$, decompose $T = \dot{\bigcup}_{t \in T} T_t$ with $|T_t| = \aleph_0$ for each $t \in T$, and put $x_j|_{M_T} = \bigoplus_{t \in T} x_j^{T_t}$. Since there are at most $|S_0| = 2^{\aleph_0} < \aleph_\nu$ equivalence classes on L, we obtain $|w_i(\underset{\sim}{x}_j; \underset{\sim}{y}_k)| < \aleph_\nu$ for each $i \in I$. This shows that W is (S_ν/S_ν^ν)-universal.

Proof of Theorem 2. (1) → (2): Put $W = \{w_1, w_2\}$ with $w_1(x_1, x_2; y) = y^{-1} \cdot x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_1$ and $w_2(x_1, x_2; y) = x_1^2$. Then W is S_ν-universal iff every $y \in S_\nu$ is a product of two conjugate involutions, i.e. iff $\nu > 0$ by Moran [**7**; Cor. 2.5].

(2) → (1): This can be shown in a similar (but here much easier) vein as the implication (3) → (1) of Theorem 1 or almost precisely as in Ehrenfeucht et al. [**3**; Proof of Theorem 3].

Before we prove Theorem 4, let us note the following useful observation:

REMARK 3.4. *Let $w_i = w_i(\underset{\sim}{x}_j; \underset{\sim}{y}_k)$ be words, where $i \in I$, $j \in J$, $k \in K$, and I, J, K are finite. Let $1 \leq \tau \leq \nu$ and $y_k \in S_\nu^\tau$ $(k \in K)$. If there are $x_j' \in S_\nu$ $(j \in J)$ such that $w_i(\underset{\sim}{x}_j'; \underset{\sim}{y}_k) = \mathrm{id}$ for all $i \in I$, then there are also*

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$x_j \in S_\nu^\tau$ $(j \in J)$ *such that* $w_i(\underset{\sim}{x_j}; \underset{\sim}{y_k}) = \mathrm{id}$ *for all* $i \in I$. *In particular, if* $W = \{w_i \mid i \in I\}$ *is* S_ν*-universal, then* W *is also* S_ν^τ*-universal.*

Proof. Let M be the underlying set of cardinality $\aleph_\nu$ and

$$Y = \bigcup_{k \in K} \operatorname{supp}(y_k).$$

By Proposition 3.1(a), there is a set $Z \subseteq M$ with $Y \subseteq Z$ and $|Z| < \aleph_\tau$ such that $x_j^+ = x_j' \restriction_Z \in P_Z$ for all $j \in J$. Putting $x_j = x_j^+ \oplus \mathrm{id}_{M \setminus Z} \in S_\nu^\tau$ $(j \in J)$, we obtain $w_i(\underset{\sim}{x_j}; \underset{\sim}{y_k}) = \mathrm{id}_M$ for all $i \in I$.

Proof of Theorem 4. (a) First assume $S_\nu^\tau \equiv_{\text{eq}} S_\mu^\rho$. In S_ν^0 no element of $S_\nu^0 \setminus A_\nu$ is a commutator, but if $\rho \geq 1$, each element of S_ν^ρ is a commutator in S_ν^ρ by Ore [**10**; pp. 313, 314]. This shows the assertion.

The converse can be established by showing that $S_\nu^\tau \equiv_{\text{eq}} S_\nu^1$ whenever $1 < \tau \leq \nu$, and that $S_\nu^1 \equiv_{\text{eq}} S_\mu^1$ $(S_\nu^0 \equiv_{\text{eq}} S_\mu^0)$ whenever $\nu, \mu > 0$ $(\nu, \mu \geq 0)$. Since the methods applied are similar to (and easier than) the ones used for the proof of the implication (3) → (1) of Theorem 1, we leave the details to the reader.

(b) Clearly, $S_0^0 \not\equiv_{\text{eq}} S_0$ since any element of S_0 is a commutator by Ore's Theorem [**10**]. Hence assume $\nu > 0$ now. By Theorem 1, it suffices to show that V is S_ν^ν-universal. We present two arguments for this.

Proof I. By Theorem 1, V is S_μ-universal for some $\mu > 0$ with $\aleph_\mu > 2^{\aleph_0}$. By Remark 3.4, V is S_μ^μ-universal. Hence by (a), V is S_ν^ν-universal.

Proof II. Let $|M| = \aleph_\nu$ and $y_i \in P_M$ with $|y_i| < \aleph_\nu$ $(i = 1, 2)$. Decompose $M = A \,\dot{\cup}\, B$ with $A = Y \,\dot{\cup}\, C$ where $Y = \operatorname{supp}(y_1) \cup \operatorname{supp}(y_2)$ and $|C| = |Y| + \aleph_0$. Define $x_1' \in P_A$ such that $x_1' \restriction_Y = \mathrm{id}_Y$ and $x_1' \restriction_C$ consists precisely of $|C|$ infinite orbits. By Droste and Göbel [**2**; Theorem 2], $y_1 \restriction_A = x_1'^{x_2'} \cdot x_1'^{x_3'}$ for some x_2', $x_3' \in P_A$. Hence $x_j = x_j' \oplus \mathrm{id}_B \in P_M$ satisfies $|x_j| < \aleph_\nu$ $(j = 1, 2, 3)$ and $v_i(\underset{\sim}{x_j}; \underset{\sim}{y_k}) = \mathrm{id}_M$ for $i = 1, 2, 3$.

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