ON CH + $2^{\aleph_1} \rightarrow (\alpha)_2^2$ FOR $\alpha < \omega_2$

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§1. Introduction.

We prove the consistency of

$$CH + 2^{\aleph_1}$$
 is arbitrarily large $+ 2^{\aleph_1} \not\rightarrow (\omega_1 \times \omega)_2^2$

(Theorem 1). In fact, we can get $2^{\aleph_1} \not\rightarrow [\omega_1 \times \omega]^2_{\aleph_0}$, see 1A. In addition to this theorem, we give generalizations to other cardinals (Theorems 2 and 3). The $\omega_1 \times \omega$ is best possible as CH implies

$$\omega_3
ightarrow (\omega imes n)_2^2.$$

We were motivated by a question of J. Baumgartner, in his talk in the MSRI meeting on set theory, October 1989, on whether $\omega_3 \rightarrow (\alpha)_2^2$ for $\alpha < \omega_2$ (if $2^{\aleph_1} = \aleph_2$, it follows from the Erdős-Rado theorem). Baumgartner proved the consistency of a positive answer with CH and 2^{\aleph_1} large. He has also proved [BH] in ZFC + CH a related polarized partition relation:

$$\binom{\aleph_3}{\aleph_2} \to \binom{\aleph_1}{\aleph_1}_{\aleph_0}^{1,1}$$

Note. The main proof here is that of Theorem 1. In that proof, in the way things are set up, the main point is proving the \aleph_2 -c.c. The main idea in the proof is using **P** (defined in the proof). It turns out that we can use as elements of \mathcal{P} (see the proof) just pairs (a, b). Not much would be changed if we used $\langle (a_n, \alpha_n) : n < \omega \rangle$, a_n a good approximation of the *n*th part of the suspected monochromatic set of order type $\omega_1 \times \omega$. In 1A, 2, and 3 we deal with generalizations and in Theorem 4 with complementary positive results.

§2. The main result.

THEOREM 1. Suppose

(a) CH (b) $\lambda^{\aleph_1} = \lambda$. Then there is an \aleph_2 -c.c., \aleph_1 -complete forcing notion P such that (i) $|\mathbf{P}| = \lambda$ (ii) $\|\mathbf{P}\| = \frac{1}{2} (\omega_1 \times \omega_2)^2$

(iii) ⊩_P CH

(iv) Forcing with P preserves cofinalities and cardinalities.

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Proof. By Erdős and Hajnal [EH] there is an algebra **B** with $2^{\aleph_0} = \aleph_1 \omega$ -place functions, closed under composition (for simplicity only), such that

$$\otimes \qquad \qquad \text{If } \alpha_n < \lambda \text{ for } n < \omega, \text{ then for some } k$$
$$\alpha_k \in \text{cl}_{\mathbf{B}} \{ \alpha_l : k < l < \omega \}.$$

(\otimes implies that for every large enough k, for every m, $\alpha_k \in cl_{\mathbf{B}}\{\alpha_l : m < l < \omega\}$.) Let

$$\mathcal{R}_{\delta} = \{ b : b \subseteq \lambda, \ \operatorname{otp}(b) = \delta, \ \alpha \in b \Rightarrow b \subseteq \operatorname{cl}_{\mathbf{B}}(b \smallsetminus \alpha) \}.$$

So by \otimes we have

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If α is a limit ordinal, $b \subseteq \lambda$, $otp(b) = \alpha$,

then for some
$$\alpha \in b$$
, $b \setminus \alpha \in \bigcup_{\delta} \mathcal{R}_{\delta}$.

Let $\mathcal{R}_{<\omega_1} = \bigcup_{\alpha < \omega_1} \mathcal{R}_{\alpha}$. Let **P** be the set of forcing conditions

 (w,c,\mathcal{P})

where w is a countable subset of λ , $c: [w]^2 \to \{\text{red}, \text{green}\} = \{0, 1\}$ (but we write $c(\alpha, \beta)$ instead of $c(\{\alpha, \beta\})$), and \mathcal{P} is a countable family of pairs (a, b) such that

- (i) a, b are subsets of w
- (ii) $b \in \mathcal{R}_{<\omega_1}$ and a is a finite union of members of $\mathcal{R}_{<\omega_1}$
- (iii) $\sup(a) < \min(b)$
- (iv) if $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then $c(\gamma, \cdot)$ divides a or b into two infinite sets.

We use the notation

$$p = (w^p, c^p, \mathcal{P}^p)$$

for $p \in \mathbf{P}$. The ordering of the conditions is defined as follows:

$$p \leq q \iff w^p \subseteq w^q \& c^p \subseteq c^q \& \mathcal{P}^p \subseteq \mathcal{P}^q.$$

Let

$$\underline{c} = \bigcup \{ c^p : p \in \underline{G}_{\mathbf{P}} \}.$$

FACT A. **P** is \aleph_1 -complete.

Proof. Trivial—take the union. \Box

FACT B. For $\gamma < \lambda$, $\{q \in \mathbf{P} : \gamma \in w^q\}$ is open dense.

Proof. Let $p \in \mathbf{P}$. If $\gamma \in w^p$, we are done. Otherwise we define q as follows: $w^q = w^p \cup \{\gamma\}, \ \mathcal{P}^q = \mathcal{P}^p, \ c^q \upharpoonright w^p = c^p \text{ and } c^q(\gamma, \cdot) \text{ is defined so that if } (a, b) \in \mathcal{P}^q$, then $c^q(\gamma, \cdot)$ divides a and b into two infinite sets. \Box

FACT C. $\Vdash_{\mathbf{P}} "2^{\aleph_1} \ge \lambda \text{ and } \underline{c} : [\lambda]^2 \to \{\text{red, green}\}."$

Proof. The second phrase follows from Fact B. For the first phrase, define $\rho_{\alpha} \in {}^{\omega_1}2$, for $\alpha < \lambda$, by: $\rho_{\alpha}(i) = c(0, \alpha + i)$. Easily

$$\Vdash_{\mathbf{P}} ``\rho_{\alpha} \in {}^{\omega_1}2 \text{ and for } \alpha < \beta < \lambda, \ \rho_{\alpha} \neq \rho_{\beta}"$$

so $\Vdash_{\mathbf{P}} "2^{\aleph_1} \geq \lambda$." \Box

FACT D. P satisfies the \aleph_2 -c.c.

Proof. Suppose $p_i \in \mathsf{P}$ for $i < \aleph_2$. For each *i* choose a countable family \mathcal{A}^i of subsets of w^{p_i} such that $\mathcal{A}^i \subseteq \mathcal{R}_{<\omega_1}$ and $(a, b) \in \mathcal{P}^{p_i}$ implies $b \in \mathcal{A}^i$ and *a* is a finite union of members of \mathcal{A}^i . For each $\gamma \in c \in \mathcal{A}^i$ choose a function $F^i_{\gamma,c}$ (from those in the algebra **B**) such that $F^i_{\gamma,c}(c \smallsetminus (\gamma + 1)) = \gamma$. Let v_i be the closure of w_i (in the order topology).

We may assume that $\langle v_i : i < \omega_2 \rangle$ is a Δ -system (we have CH) and that $otp(v_i)$ is the same for all $i < \omega_2$. Without loss of generality (w.l.o.g.) for i < j the unique order-preserving function $h_{i,j}$ from v_i onto v_j maps p_i onto p_j , \mathcal{A}^i onto \mathcal{A}^j , $w^{p_i} \cap w^{p_j} = w^{p_0} \cap w^{p_1}$ onto itself, and

$$F_{\gamma,c}^{i} = F_{h_{i,j}(\gamma),h_{i,j},\alpha}^{j}$$

for $\gamma \in c \in \mathcal{A}^i$ (remember: **B** has $2^{\aleph_0} = \aleph_1$ functions only). Hence

$$\otimes_1$$
 $h_{i,j}$ is the identity on $v_i \cap v_j$ for $i < j$

Clearly by the definition of $\mathcal{R}_{<\omega_1}$ and the condition on $F^i_{\gamma,c}$:

We define q as follows.

(**) $a \smallsetminus w^{p_e}$ is infinite.

This is easily done and $p_0 \leq q$, $p_1 \leq q$, provided that $q \in \mathbf{P}$. For this the problematic part is c^q and, in particular, part (iv) of the definition of \mathbf{P} . So suppose $(a, b) \in \mathcal{P}^q$, e.g., $(a, b) \in \mathcal{P}^{p_0}$. Suppose also $\gamma^* \in w^q$ so that $\sup(a) \leq \gamma^* < \sup(b)$. If $\gamma^* \in w^{p_0}$, there is no problem, as $p_0 \in \mathbf{P}$. So let us assume $\gamma^* \in w^q \setminus w^{p_0} = w^{p_1} \setminus w^{p_0}$. If $a \setminus w^{p_1}$ or $b \setminus w^{p_1}$ is infinite, we are through in view of condition (*) in the definition of c^q . Let us finally assume $a \setminus w^{p_1}$ is finite. But $a \subseteq w^{p_0} \cap w^{p_1}$. Hence $a \setminus (w^{p_0} \cap w^{p_1})$ is finite and \otimes_2 implies it is empty, i.e., $a \subseteq w^{p_0} \cap w^{p_1}$. Similarly, $b \subseteq w^{p_0} \cap w^{p_1}$. So $h_{0,1} \upharpoonright (a \cup b)$ is the identity. But $(a, b) \in \mathcal{P}^{p_0}$. But $h_{i,j}$ maps p_i onto p_j . Hence $(a, b) \in \mathcal{P}^{p_1}$. As $p_1 \in \mathbf{P}$, we get the desired conclusion. \Box

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FACT E. $\Vdash_{\mathbf{P}}$ "There is no <u>c</u>-monochromatic subset of λ of order-type $\omega_1 \times \omega$."

Proof. Let p force the existence of a counterexample. Let G be P-generic over V with $p \in G$. In V[G] we can find $A \subseteq \lambda$ of order-type $\omega_1 \times \omega$ such that $c^G \upharpoonright [A]^2$ is constant. Let $A = \bigcup_{n < \omega} A_n$ where $\operatorname{otp}(A_n) = \omega_1$ and $\sup(A_n) \leq \min(A_{n+1})$. We can replace A_n by any $A'_n \subseteq A_n$ of the same cardinality. Hence we may assume w.l.o.g.

 $(*)_1 A_n \in \mathcal{R}_{\omega_1} for \ n < \omega.$

Let $\delta_n = \sup(A_n)$ and

$$\begin{split} \beta_n &= \min\{\,\beta: \delta_n \leq \beta < \lambda, \; d(\beta, \cdot) \text{ does not} \\ & \text{ divide } \bigcup_{l \leq n} A_l \text{ into two infinite sets } \}, \end{split}$$

where $d = \underline{c}^G$. Clearly $\beta_n \leq \min(A_{n+1})$. Hence $\beta_n < \beta_{n+1}$. Let $d_n \in \{0, 1\}$ be such that $d(\beta_n, \gamma) = d_n$ for all but finitely many $\gamma \in \bigcup_{l \leq n} A_l$. Let u be an infinite subset of ω such that d_n is constant for $n \in u$ and $\{\beta_n : n \in u\} \in \mathcal{R}_{\omega}$. Let $A_l = \{\alpha_l^i : i < \omega_l\}$ in increasing order. So p forces all this on suitable names

$$\langle \beta_{z_n} : n < \omega \rangle, \ \langle \alpha_i^l : i < \omega_1 \rangle, \ \langle \delta_n : n < \omega \rangle.$$

As P is \aleph_1 -complete, we can find $p_0 \in \mathsf{P}$ with $p \leq p_0$ so that p_0 forces $\beta_l = \beta_l$ and $\delta_n = \delta_n$ for some β_l and δ_n . We can choose inductively conditions $p_k \in \mathsf{P}$ such that $p_k \leq p_{k+1}$ and there are $i_k < j_k$ and α_i^l (for $i < j_k$) with

$$p_{k+1} \Vdash ``\alpha_{i_{k}}^{l} > \sup(w^{p_{k}} \cap \delta_{l}),$$

$$\alpha_{i}^{l} \in w^{p_{k+1}} \text{ for } i < j_{k},$$

$$\{\alpha_{i}^{l} : i < i_{k}\} \subseteq \operatorname{cl}_{\mathbf{B}}\{\alpha_{i}^{l} : i_{k} < i < j_{k}\},$$

$$\alpha_{i}^{l} = \alpha_{i}^{l} \text{ for } i < j_{k},$$

$$\underline{c}(\beta_{n}, \alpha_{i}^{l}) = d_{n} \text{ for } l \leq n, i > i_{0}, \text{ and}$$

$$\gamma \in [\delta_{m}, \beta_{m}) \cap w^{p_{k}} \text{ implies } \underline{c}(\gamma, \cdot) \text{ divides}$$

$$\{\alpha_{i}^{l} : i < j_{k}, l \leq m\} \text{ into two infinite sets}^{i}$$

(remember our choice of β_m). Let

$$l(*) = \min(u)$$

$$a = \{ \alpha_i^l : l \le l(*), \ i < \bigcup_k j_k \}$$

$$b = \{ \beta_l : l \in u \}$$

$$q = (\bigcup_k w^{p_k}, \bigcup_k c^{p_k}, \bigcup_k \mathcal{P}^{p_k} \cup \{(a, b)\})$$

Now $q \in \mathbf{P}$. To see that q satisfies condition (iv) of the definition of \mathbf{P} , let $\sup(a) \leq \gamma < \min(b)$. Then $\sup\{\alpha_{i_k}^{l(*)} : k < \omega\} \leq \gamma < \beta_{l(*)}$. But $\gamma \in w^q =$

 $\bigcup_k w^{p_k}$, so for some $k, \gamma \in w^{p_k}$. This implies

$$\gamma \notin \left(\alpha_{i_{k+1}}^{l(*)}, \delta_{l(*)}\right),$$

whence $\gamma \geq \delta_{l(*)}$ and

$$\{ \alpha_i^l : l \leq l(*), \ i < j_k \} \subseteq a,$$

which implies the needed conclusion.

Also $q \ge p_k \ge p$. But now, if $r \ge q$ forces a value to $\alpha_{\cup_k j_k}^{l(*)}$; we get a contradiction. \Box

Remark 1A. Note that the proof of Theorem 1 also gives the consistency of $\lambda \not\rightarrow [\omega_1 \times \omega]^2_{\aleph_0}$: replace " $c(\gamma, \cdot)$ divides a set x into two infinite parts" by " $c(\gamma, \cdot)$ gets all values on a set x."

$\S3.$ Generalizations to other cardinals.

How much does the proof of Theorem 1 depend on \aleph_1 ? Suppose we replace \aleph_0 by μ .

THEOREM 2. Assume $2^{\mu} = \mu^+ < \lambda = \lambda^{\mu}$ and $2 \le \kappa \le \mu$. Then for some μ^+ -complete μ^{++} -c.c. forcing notion P of cardinality 2^{μ} :

$$\Vdash_{\mathbf{P}} 2^{\mu} = \lambda, \qquad \lambda \not\to [\mu^+ \times \mu]^2_{\kappa}.$$

Proof. Let **B** and \mathcal{R}_{δ} be defined as above (for $\delta \leq \mu^+$). Clearly

 \oplus If $a \subseteq \lambda$ has no last element, then for some $\alpha \in a$, $a \smallsetminus \alpha \in \bigcup_{\delta} \mathcal{R}_{\delta}$.

Hence, if $\delta = \operatorname{otp}(a)$ is additively indecomposable, then $a \smallsetminus \alpha \in \mathcal{R}_{\delta}$ for some $\alpha \in a$.

Let P_{μ} be the set of forcing conditions

$$(w,c,\mathcal{P})$$

where $w \subseteq \lambda$, $|w| \leq \mu$, $c: [w]^2 \to \kappa$, and \mathcal{P} is a set of $\leq \mu$ pairs (a, b) such that (i) a, b are subsets of w

- (ii) $b \in \mathcal{R}_{\mu}$, and a is a finite union of members of $\bigcup_{\mu < \delta < \mu^{+}} \mathcal{R}_{\delta}$
- (iii) $\sup(a) < \min(b)$
- (iv) if $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then the function $c(\gamma, \cdot)$ gets all values $(< \kappa)$ on a or on b.

With the same proof as above we get

$$P_{\mu}$$
 satisfies the μ^{++} -c.c.,
 P_{μ} is μ^{+} -complete,

(so cardinal arithmetic is clear) and

 $\Vdash_{\mathbf{P}_{\mu}} \lambda \not\to [\mu^+ \times \mu]^2_{\kappa}.$

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What about replacing μ^+ by an inaccessible θ ? We can manage by demanding

$$\{a \cap (\alpha, \beta) : (a, b) \in \mathcal{P}, \bigcup_{n} \operatorname{otp}(a \cap (\alpha, \beta)) \times n = \operatorname{otp}(a)$$

 (α, β) maximal under these conditions }

is free (meaning there are pairwise disjoint end segments) and by taking care in defining the order. Hence the completeness drops to θ -strategical completeness. This is carried out in Theorem 3 below.

THEOREM 3. Assume $\theta = \theta^{<\theta} > \aleph_0$ and $\lambda = \lambda^{<\theta}$. Then for some θ^+ -c.c. θ -strategically complete forcing \mathbf{P} , $|\mathbf{P}| = \lambda$ and

$$\Vdash_{\mathbf{P}} 2^{\theta} = \lambda, \ \lambda \not\to (\theta \times \theta)_2^2.$$

Proof. For W a family of subsets of λ , each with no last element, let

 $Fr(W) = \{ f : f \text{ is a choice function on } W \text{ such that } \}$

 $\{a \setminus f(a) : a \in W\}$ are pairwise disjoint $\}$.

If $Fr(W) \neq \emptyset$, W is called *free*.

Let $\mathbf{P}_{<\theta}$ be the set of forcing conditions

 (w, c, \mathcal{P}, W)

where $w \subseteq \lambda$, $|w| < \theta$, $c : [w]^2 \to \{\text{red}, \text{green}\}$, W is a free family of $< \theta$ subsets of w, each of which is in $\bigcup_{\delta < \theta} \mathcal{R}_{\delta}$, and \mathcal{P} is a set of $< \theta$ pairs (a, b) such that

- (i) a, b are subsets of w
- (ii) $b \in \mathcal{R}_{\omega}$
- (iii) $\sup(a) < \min(b)$ and for some $\delta_0 < \delta_1 < \cdots < \delta_n$, $\delta_0 < \min(a)$, $\sup(a) \le \delta_n$, $a \cap [\delta_l, \delta_{l+1}) \in W$
- (iv) if $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then $c(\gamma, \cdot)$ divides a or b into two infinite sets.

We order $\mathbf{P}_{\leq \theta}$ as follows:

$$p \leq q$$
 iff $w^p \subseteq w^q$, $c^p \subseteq c^q$, $\mathcal{P}^p \subseteq \mathcal{P}^q$, $W^p \subseteq W^q$ and every
 $f \in \operatorname{Fr}(W^p)$ can be extended to a member of $\operatorname{Fr}(W^q)$.

§4. A provable partition relation.

CLAIM 4. Suppose $\theta > \aleph_0$, $n, r < \omega$, and $\lambda = \lambda^{<\theta}$. Then $(\lambda^+)^r \times n \to (\theta \times n, \theta \times r)_2^2$.

Proof. We prove this by induction on r. Clearly the claim holds for r = 0, 1. So w.l.o.g. we assume $r \ge 2$. Let c be a 2-place function from $(\lambda^+)^r \times n$ to {red, green}. Let $\chi = \beth_2(\lambda)^+$. Choose by induction on l a model N_l such that

$$N_l \prec (H(\chi), \in, <^*),$$

 $|N_l| = \lambda, \ \lambda + 1 \subseteq N_l, \ N_l^{<\theta} \subseteq N_l, \ c \in N_l \text{ and } N_l \in N_{l+1}.$ Here <* is a well-ordering of $H(\chi)$. Let

$$A_{l} = \left[(\lambda^{+})^{r} \times l, \ (\lambda^{+})^{r} \times (l+1) \right],$$

and let $\delta_l \in A_l \setminus N_l$ be such that $\delta_l \notin x$ whenever $x \in N_l$ is a subset of A_l and $otp(x) < (\lambda^+)^r$. W.l.o.g. we have $\delta_l \in N_{l+1}$. Now we shall show

(*) If
$$Y \in N_0$$
, $Y \subseteq A_m$, $|Y| = \lambda^+$ and $\delta_m \in Y$,
then we can find $\beta \in Y$ such that $c(\beta, \delta_l) = \text{red for all } l < n$.

Why does (*) suffice? Assume (*) holds. We can construct by induction on $i < \theta$ and for each *i* by induction on l < n an ordinal $\alpha_{i,l}$ such that

(a) $\alpha_{i,l} \in A_l$ and $j < i \Rightarrow \alpha_{j,l} < \alpha_{i,l}$

(b) $\alpha_{i,l} \in N_0$

- (c) $c(\alpha_{i,l}, \delta_m) = \text{red}$ for m < n
- (d) $c(\alpha_{i,l}, \alpha_{i_1,l_1}) = \text{red}$ when $i_1 < i$ or $i_1 = i \& l_1 < l$.

Accomplishing this suffices as $\alpha_{i,l} \in A_l$ and

$$l < m \Rightarrow \sup A_l \le \min A_m.$$

Arriving in the inductive process at (i, l), let

$$Y = \{ \beta \in A_l : c(\beta, \alpha_{j,m}) = \text{red} \quad \text{if } j < i, \ m < n, \ \text{or} \ j = i, \ m < l \}.$$

Now clearly $Y \subseteq A_l$. Also $Y \in N_0$ as all parameters are from N_0 , their number is $< \theta$ and $N_0^{<\theta} \subseteq N_0$. Also $\delta_l \in Y$ by the induction hypothesis (and $\delta_l \in A_l$). So by (*) we can find $\alpha_{i,l}$ as required.

Proof of (*). $Y \not\subseteq N_0$, because $\delta_m \in Y$ and $Y \in N_0$. As $|Y| = \lambda^+$, we have $\operatorname{otp}(Y) \geq \lambda^+$. But $\lambda^+ \to (\lambda^+, \theta)^2$, so there is $B \subseteq Y$ such that $|B| = \lambda^+$ and $c \upharpoonright B \times B$ is constantly red or there is $B \subseteq Y$ such that $|B| = \theta$ and $c \upharpoonright B \times B$ is constantly green. In the former case we get the conclusion of the claim. In the latter case we may assume $B \in N_0$, hence $B \subseteq N_0$, and let $k \leq n$ be maximal such that

$$B' = \{ \xi \in B : \bigwedge_{l < k} c(\delta_l, \xi) = \operatorname{red} \}$$

has cardinality θ . If k = n, any member of B' is as required in (*). So assume k < n. Now $B' \in N_k$, since $B \in N_0 \prec N_k$ and $\{N_l, A_l\} \in N_k$ and $\delta_l \in N_k$ for l < k. Also

$$\{\xi \in B' : c(\delta_k, \xi) = \operatorname{red}\}$$

is a subset of B' of cardinality $\langle \theta$ by the choice of k. So for some $B'' \in N_0$, $c \upharpoonright \{\delta_k\} \times (B' \smallsetminus B'')$ is constantly green (e.g., as $B' \subseteq N_0$, and $N_0^{\langle \theta} \subseteq N_0$). Let

$$Z = \{ \delta \in A_k : c \upharpoonright \{ \delta \} imes (B' \smallsetminus B'') ext{ is constantly green } \}$$

and

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$$Z' = \{ \delta \in Z : (\forall \alpha \in B' \smallsetminus B'') (\delta < \alpha \Leftrightarrow \delta_k < \alpha) \}.$$

So $Z \subseteq A_k$, $Z \in N_k$, $\delta_k \in Z$ and therefore $\operatorname{otp}(Z) = \operatorname{otp}(A_k) = (\lambda^+)^r$. Note that $k \neq m \Rightarrow Z' = Z$ and $k = m \Rightarrow Z' = Z \setminus \sup(B' \setminus B'')$, so Z' has the same properties. Now we apply the induction hypothesis; one of the following holds (note that we can interchange the colours): (a) There is $Z'' \subseteq Z'$, $\operatorname{otp}(Z'') = \theta \times n$, $c \upharpoonright Z'' \times Z''$ is constantly red, w.l.o.g. $Z'' \in N_k$, or (b) there is $Z'' \subseteq Z'$, $\operatorname{otp}(Z'') = \theta \times (r-1)$, $c \upharpoonright Z'' \times Z''$ green and w.l.o.g. $Z'' \in N_k$. If (a), we are done; if (b), $Z'' \cup (B' \setminus B'')$ is as required. \Box

Remark 4A. So
$$(\lambda^+)^{n+1} \to (\theta \times n)^2$$
 for $\lambda = \lambda^{<\theta}$, $\theta = cf(\theta) > \aleph_0$ (e.g., $\lambda = 2^{<\theta}$).

Remark 4B. Suppose $\lambda = \lambda^{<\theta}$, $\theta > \aleph_0$. If c is a 2-colouring of $(\lambda^{+r})^s \times n$ by k colours and every subset of it of order type $(\lambda^{+(r-1)})^s \times n$ has a monochromatic subset of order type θ for each of the colours, one of the colours being red, then by the last proof we get

- (a) There is a monochromatic subset of order type $\theta \times n$ and of colour red or
- (b) There is a colour d and a set Z of order type $(\lambda^{+r})^s$ and a set B of order type θ such that B < Z or Z < B and

$$\{ (\alpha, \beta) : \alpha \in B, \ \beta \in Z \text{ or } \alpha \neq \beta \in B \}$$

are all coloured with d.

So we can prove that for 2-colourings by k colours c

$$(\lambda^{+r})^s \times n \to (\theta \times n_1, \dots, \theta \times n_k)^2$$

when r, s, n are sufficiently large (e.g., $n \ge \min\{n_l : l = 1, ..., k, s \ge \sum_{l=1}^k n_l\})$ by induction on $\sum_{l=1}^k n_l$.

Note that if c is a 2-colouring of λ^{+2k} , then for some l < k and $A \subseteq \lambda^{+2k}$ of order type $\lambda^{+(2l+2)}$ we have

(*) If $A' \subseteq A$, $\operatorname{otp}(A') = \lambda^{+2l}$, and d is a colour which appears in A, then there is $B \subseteq A'$ of order type θ such that B is monochromatic of colour d.

We can conclude $\lambda^{+2k} \to (\theta \times n)_k^2$.

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