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THE UNIVERSALITY SPECTRUM: CONSISTENCY FOR MORE CLASSES

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We deal with consistency results for the existence of universal models in natural classes of models (more exactly–a somewhat weaker version). We apply a result on quite general family to T_{feq} and to the class of triangle-free graphs.

0. INTRODUCTION

The existence of universal structures, for a class of structures in a given cardinality is quite natural as witnessed by having arisen in many contexts. We had wanted here to peruse it in the general context of model theory but almost all will interest a combinatorialist who is just interested in the existence of universal linear order or a triangle free graph. For a first order theory (complete for simplicity) we look at the universality spectrum $\mathrm{USP}_T = \{\lambda : T \text{ has a universal model in cardinal } \lambda\}$ (and variants). Classically we know that under GCH, every $\lambda > |T|$ is in USP_T, moreover $2^{<\lambda} = \lambda > |T| \Rightarrow \lambda \in \mathrm{USP}_T$ (i.e.–the existence of a saturated or special model, see e.g. [1]). Otherwise in general it is "hard" for a theory T to have a universal model (at least when T is unstable). For consistency see [12], [14], [15], Mekler [8] and parallel to this work Kojman-Shelah [7]; on ZFC nonexistence results see Kojman-Shelah [4], [5], [6]. We get ZFC non existence result (for T^*_{feq} under more restriction, essentially cases of SCH) in Section 2, more on linear orders (in Section 3), consistency of

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(somewhat weaker versions of) existence results abstractly (in Section 4) derived consistency results and apply them to the class of models of T_{feq} (an indexed family of independent equivalence relations) and to the class of triangle free graphs (in Section 5). The general theorem in Section 4 was intended for treating all simple theories (in the sense of [11], but this is not included as it is probably too much model theory for the expected reader here 9and for technical reasons).

1. Definition

Definition 1.1. *For a class* $\mathbf{K} = (\mathbf{K}, \leq_{\mathbf{K}})$ *of models*

1) $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$
2) $\mathrm{univ}\,(\lambda, \mathbf{K}) = \mathrm{Min}\,\{|\mathcal{P}| : \mathcal{P}$ *a set of models from* $\mathbf{K}_\lambda$ *such that for every* $N \in \mathbf{K}_\lambda$ *for some* $N \in \mathcal{P}$, M *can be* $\leq_{\mathbf{K}}$*-embedded into* $N\}$.
3) $\mathrm{Univ}\,(\lambda, \mathbf{K}) = \mathrm{Min}\,\{\|N\| : N \in \mathbf{K}$, *and every* $M \in \mathbf{K}_\lambda$ *can be* $\leq_{\mathbf{K}}$*-embedded into* $N\}$.
4) *If* $\mathbf{K}$ *is the class of models of* T, T *a complete theory, we write* T *instead* $(\mathrm{mod}\, T, \prec)$. *If* $\mathbf{K}$ *is the class of models of* T, T *a universal theory, we write* T *instead* $(\mathrm{mod}\,(T), \subseteq)$.

Claim 1.2. *1)* $\mathrm{univ}\,(\lambda, \mathbf{K}) = 1$ *iff* $\mathbf{K}$ *has a universal member of cardinality* λ.

2) Let T *be first order complete,* $|T| \leq \lambda$. *Then we have* $\mathrm{univ}\,(\lambda, T) \leq \lambda$ *implies* $\mathrm{univ}(\lambda, \mathbf{K}) = 1$ *and* $\mathrm{univ}(\lambda, T) \leq \mathrm{Univ}\,(\lambda, T) \leq \mathrm{cf}\,(\mathcal{S}_{\leq\lambda}(\mathrm{univ}\,(\lambda, T), \subseteq))$ $= \mathrm{cov}\,(\mathrm{univ}\,(\lambda, T), \lambda^+, \lambda^+, 2)$ *(see [20]; we can replace* T *with* $\mathbf{K}$ *with suitable properties).*

2. The universality Spectrum of T_{feq}

For T_{feq}, a prime example for a theory with the tree order property (but not the strict order property) we prove there are limitations on the universality spectrum; it is meaningful when SCH fails.

Definition 2.1. T^*_{feq} *is the model completion of the following theory,* T_{feq}. T_{feq} *is defined as follows:*

(a) *it has predicates* P, Q *(unary)* E *(three place, written as* $yE_xz\}$

(b) *the universe (of any model of T) is the disjoint union of P and Q, each infinite*

(c) $yE_xz \to P(x)\,\&\,Q(y)\,\&\,Q(z)$

(d) *for any fixed x, E_x is an equivalence relation on Q with infinitely many equivalence classes*

(e) *if $n < \omega$, $x_1, \dots, x_n \in P$ with no repetition and $y_1, \dots, y_n \in Q$ then for some $y \in Q$, $\bigwedge_{\ell=1}^{n} yE_{x_\ell}y_\ell$.*
(Note: T_{feq} has elimination of quantifiers).

Claim 2.2. *Assume:*

(a) $\theta < \mu < \lambda$

(b) $\mathrm{cf}\,\lambda = \lambda, \theta = \mathrm{cf}\,\theta = \mathrm{cf}\,\mu$, $\mu^+ < \lambda$

(c) $pp_{\Gamma(\theta)}(\mu) > \lambda + |i^*|$

(d) *there is $\{(a_i, b_i) : i < i^*\}$, $a_i \in [\lambda]^{<\mu}$, $b_i \in [\lambda]^\theta$ and $|\{b_i : i < i^*\}| \leq \lambda$ such that: for every $f : \lambda \to \lambda$ for some i, $f(b_i) \subseteq a_i$*

then

(1) *T_{feq} has no universal model in λ.*

(2) *Moreover,* $\mathrm{univ}\,(\lambda, T_{\text{feq}}) \geq pp_{\Gamma(\theta)}(\mu)$.

Proof. Let D be a θ-complete filter on θ, $\lambda_i = \mathrm{cf}\,\lambda_i < \mu = \sum_{i<\kappa} \lambda_i$, $\mathrm{tlim}_D \lambda_i = \mu$, $\chi =: \mathrm{tcf}\prod_{i<\theta} \lambda_i/D > i^*$ (and for (2), $\mathrm{tcf}\,(\prod_{i<\theta} \lambda_i/D) > \mathrm{univ}\,(\lambda, T_{\text{feq}})$). Also let $\langle f_\alpha : \alpha < \chi \rangle$ be $<_D$-increasing cofinal in $\prod_{i<\theta} \lambda_i/D$. Let $S = \{\delta < \lambda : \mathrm{cf}\,\delta = \theta, \delta \text{ divisible by } \mu^{\omega+1}\}$. Let $\bar{C} = \langle C_\delta : \delta \in S \rangle$ be such that: C_δ a club of δ, $\mathrm{otp}(C_\delta) = \mu$ and $[\alpha \in C_\delta \Rightarrow \alpha > 0 \text{ divisible by } \mu^\omega]$ and $\emptyset \notin \mathrm{id}^a(\bar{C})$ (i.e. for every club E of λ for stationary many $\delta \in S \cap E$, $C_\delta \subseteq E$) (exists-see [19, Section 2]).

For (1), let M^* be a candidate for being a universal model of T_{feq} of cardinality λ, for (2) let $\langle M^*_\zeta : \zeta < \kappa \rangle$ exemplify $\kappa =: \mathrm{univ}\,(\lambda, T_{\text{feq}})$; for (1) let $\kappa = 1$, $M^*_0 = M_0$. Without loss of generality $|P^{M^*_\zeta}| = |Q^{M^*_\zeta}| = \lambda$, P^{M^*} is the set of even ordinals $< \lambda$, Q^{M^*} is the set of odd ordinals $< \lambda$.

For each $i < i^*$ and $\delta \in S$ and $z \in Q^{M^*_\zeta}$ let $a'_i = \{2\alpha : \alpha \in a_i\}$ and $d[z, \delta, i, \zeta] = \{\alpha : \alpha \in \mathrm{nacc}\,C_\delta$ and for some $x \in a'_i$ there is $y < \alpha$, such that $M^*_\zeta \models yE_xz$ but there is no $y < \sup(\alpha \cap C_\delta)$ such that $M^* \models yE_xz\}$. Clearly $d[z, \delta, i, \zeta]$ is a subset of C_δ of cardinality $\leq |a_i| < \mu$.

Define $g_{z,\delta,i,\zeta} \in \prod_{j<\theta} \lambda_j$ by: if $|a_i| < \lambda_j, \beta \in C_\delta$, $\mathrm{otp}\,(\beta \cap C_\delta) = \lambda_j$ then $g_{z,\delta,i,\zeta}(j) = \mathrm{otp}\,(\varepsilon \cap C_\delta)$ where $\varepsilon \in C_\delta \cap \beta$ is $\mathrm{Min}\,\{\varepsilon : \varepsilon \in C_\delta \cap \beta$, $\varepsilon > \sup(d[z, \delta, i, \zeta] \cap \beta)\}$ and let $g_{z,\delta,i,\zeta}(j) = 0$ if $|a_i| \geq \lambda_j$. By the choice of

$\langle f_\alpha : \alpha < \chi \rangle$ for some γ we have $g_{z,\delta,i,\zeta} <_D f_\gamma$, let $\gamma^* = \gamma^*[z,\delta,i,\zeta]$ be the first such γ. As $\mu = \text{tlim}_D \lambda_i$ clearly $\gamma^*[z,\delta,i,\zeta]$ is the first $\gamma < \chi$ such that for the D-majority of $i < \theta$, $\bigwedge_{\alpha \in d[z,\delta,i,\zeta]} \text{otp}\,(\alpha \cap C_\delta) \notin [f_\gamma(i), \lambda_i)$; clearly it is well defined. Wlog $\{b_i : i < i^*\} = \{b_i : i < i^*\}$.

As $\chi > \lambda + \kappa + |i^*$, there is $\gamma(*) < \chi$ such that: $z \in Q^{M_i^*}$, $\delta \in S$, $i < i^*$, $\zeta < \kappa \Rightarrow \gamma^*[z,\delta,i,\zeta] < \gamma(*)$. Now we can define by induction on $\alpha < \lambda$, N_α, γ_α such that:

(i) N_α is a model of T^*_{feq} with universe $\gamma_\alpha = \mu(1+\alpha)$,

(ii) all $x \in P^{N_\alpha}$ are even, all $y \in Q^{N_\alpha}$ are odd

(iii) N_α increasing continuous, $P^{N_\alpha} \neq P^{N_{\alpha+1}}$

(iv) for any $x \in P^{N_\alpha}$ there is a $y = y_{x,\alpha} \in Q^{N_{\alpha+1}} \setminus Q^{N_\alpha}$ such that $\neg(\exists z \in Q^{N_\alpha})[zE_x y]$,

(v) if $\alpha \in S, i < i^* \cap \lambda$ and $b_i \subseteq \text{Min}\,(C_\alpha)$ then there is a $z^i_\alpha \in Q^{N_{\alpha+1}} \setminus Q^{N_\alpha}$ such that $\text{Rang}\, f_{\gamma(*)} = \{\text{otp}\,(y \cap C_\alpha) :$ for some $x \in b'_i$, y is minimal such that $yE_x z^i_\alpha\}$ where $b'_i = \{2\alpha : \alpha \in b_i\}$.

[For carrying out this let $d_{\alpha,i} =^{df} \{\beta \in C_\alpha : \text{otp}\,(C_\alpha \cap \beta) = (f_{\gamma(*)}(j)+1)$ for some $j < \theta\}$, choose distinct $x_{\alpha,i,\beta} \in b'_i$ for $\beta \in d_{\alpha,i}$. Next choose $y_{\alpha,i,\beta} \in \beta \setminus \sup(C_\alpha \cap \beta)$ such that it is as in clause (iv) for $x_{\alpha,i,\beta}$ and $z^i_\alpha E_{x_{\alpha,i,\beta}} y_{\alpha,i,\beta}$.]

If $\zeta < \kappa$ and f is an embedding of $N = \bigcup_{\alpha<\lambda} N_\alpha$ into M^*_ζ, for some i we have $f(b'_i) \subseteq a'_i$. We easily get a contradiction. ∎

Remark 2.3. 1) When does (d) of 2.2 hold? (it is a condition on $\lambda > \mu > \theta$ with $i^* = \lambda$, assuming for simplicity $\theta > \aleph_0$) e.g. if

$(*)_1$ for some cardinal κ we have $\kappa^\theta \leq \lambda$, $\kappa = \text{cf}\,\kappa$, $\text{cov}\,(\lambda, \kappa^+, \kappa^+, \kappa) \leq \lambda$.

2) As for condition $(d)^-$ from claim 2.4 below, if D is the filter of co-bounded subsets of θ, it suffices to have

$(*)_2$ for some cardinal κ we have $\text{cov}\,(\lambda, \mu, \kappa^+, \kappa) \leq \lambda$, or equivalently, $\sigma \in [\mu, \lambda)$ and $\text{cf}\,(\sigma) = \kappa$ imply $\text{pp}_{\Gamma(\kappa)}(\sigma) \leq \lambda$.

3) So if $\theta = \text{cf}\,(\mu) < \beth_\omega(\theta) \leq \mu < \mu^+ < \lambda = \text{cf}\,(\lambda) < \text{pp}^+_{\Gamma(\theta)}(\mu)$ then by [23] condition $(*)_1$ holds for some $\kappa < \beth_\omega(\theta)$

4) Why have we require $\theta > \aleph_0$? as then by [20, Ch. II, 5.4] we can describe the instances of cov by instances of pp; now even without this restriction this usually holds (see there) and possibly it always hold; alternatively, we can repeat the proof of 2.2 using cov

Claim 2.4. *In 2.2 we can replace clauses (c), (d) by $(c)^+$, $(d)^-$ below and the conclusions still hold.*

(c)$^+$ $pp_D(\mu) > |i^*| + \lambda + \mathrm{univ}\,(\lambda, T_{\mathrm{feq}})$, D *a filter on* θ,

(d)$^-$ $\{(a_i, b_i) : i < i^*\}$, $a_i \in [\lambda]^{<\mu}$, $i^* \leq \lambda$ *or at least* $\{b_i : i < i^*\}$ *has cardinality* $\leq \lambda$ $b_i = \{\alpha_{i,\zeta} : \zeta < \theta\}$ *and for every* $f : \lambda \to \lambda$ *for some* i *we have* $\{\zeta < \theta : f(\alpha_{i,\zeta}) \in a_i\} \neq \emptyset \bmod D$.

The next step is:

Question 2.5. Let T be f.o. with the tree property without the strict order property; (see [16]) does 2.2 hold for it?

3. A consequence of the existence of a universal linear order

This section continues, most directly, [4].

Claim 3.1. *Assume*

(a) $_\lambda$ $\kappa < \lambda \leq 2^\kappa$ *and* $2^{<\lambda} \leq \lambda^+ < 2^\lambda$, λ *is regular.*

(b) $_\lambda$ *in* λ^+ *there is a universal linear order*

then *for* $\mu = \lambda^+$

$\otimes_{\lambda,\mu}$ *there are* $f_\alpha : \lambda \to \lambda$ *(for* $\alpha < \mu$*) such that:*

$(*)_{\lambda,\mu}$ *for no* $f : \lambda \to \lambda$ *do we have* $\bigwedge_{\alpha<\mu} f_\alpha \neq_{J^{\mathrm{bd}}_\lambda} f$.

Proof. Assume $\otimes_{\lambda,\mu}$ fails. We use κ-tuples of elements to compute invariants. Note that $2^\kappa \leq 2^{<\lambda} \leq \lambda^+$ hence $2^\kappa \in \{\lambda, \lambda^+\}$ hence $(\lambda^+)^\kappa = \lambda^+$. Let $\langle \bar{x}^\varepsilon : \varepsilon < \lambda^+\rangle$ list ${}^\kappa(\lambda^+)$. Let $\langle \eta_\alpha : \alpha < \lambda\rangle$ list λ distinct members of ${}^\kappa 2$ (not necessarily all of them). Note that as $2^{<\lambda} \leq \lambda^+$ there is a stationary $S \in I[\lambda]$, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}\,(\delta) = \lambda\}$ (see [19, Section 2] for the definition of $I[S]$)

As $S \in I[\lambda]$ by [19, Section 2] there is $\bar{C} = \langle C_\delta : \delta \in S\rangle$ an S-club system such that $\emptyset \notin \mathrm{id}_p(\bar{C})$, $\mathrm{otp}\, C_\delta = \lambda$ and

$\oplus$ for each $\alpha < \lambda$ we have $|\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc}\, C_\delta\}| \leq \lambda$.

Let M^* be a candidate for being a universal model of T_{ord} of cardinality λ^+ wlog with universe λ^+.

For every linear order M with universe λ^+, for every $\bar{x} \in {}^\kappa M$ (a κ-tuple of members of M) and $\delta \in S$, we define a (possibly partial) function $g = g^{\bar{x}}_{M,\delta} : \mathrm{nacc}\, C_\delta \to \lambda$ as follows:

$(*)_0$ for $\alpha \in \operatorname{nacc} C_\delta$, $g(\alpha) = \beta$ *iff* for every $\zeta < \kappa$ we have:
$\eta_\beta(\zeta) = 1 \iff (\forall \gamma < \alpha)(\exists \gamma' < \sup(\alpha \cap C_\delta)\ [\gamma <_M x_\zeta \Rightarrow \gamma <_M \gamma' <_M x_\zeta]$.

Clearly $g^{\bar{x}}_{M,\delta}(\alpha)$ can have at most one value. We call $(\delta, \bar{x})$ *good* in M if for every $\alpha \in \operatorname{nacc} C_\delta$ there is $\varepsilon < \delta$ such that: $\bar{x}^\varepsilon$, $\bar{x}$ realize the same $<_M$-Dedekind cut over $\{i : i < \sup(\alpha \cap C_\delta)\}$ (necessary if $2^{<\lambda} = \lambda^+$). (The meaning is that for every $\zeta < \kappa$, x^ε_ζ, x_ζ realize the same $<_M$-Dedekind cut over $\{i : i < \sup(\alpha \cap C_\delta)\}$).

Let $h_\delta : \lambda \to \operatorname{nacc} C_\delta$ be: $h(i)$ is the $(i+1)$-th member of C_δ. We are assuming "$\otimes_{\lambda;\mu}$ fails", so $\{g^{\bar{x}}_{M^*,\delta} \circ h_\delta : \bar{x} \in {}^\kappa 2,\ \delta \in S\}$ cannot exemplify it. So we can find $h^*_{M^*} : \lambda \to \lambda$ such that:

$\otimes$ if $\bar{x} \in {}^\kappa(M^*)$, $\delta \in S$ is $(\delta, \bar{x})$ good in M^* then $(g^{\bar{x}}_{M^*,\delta} \circ h_\delta) \in {}^\lambda\lambda$ satisfies $h^* \neq_{J^{\rm bd}_\lambda} (g^{\bar{x}}_{M^*,\delta} \circ h_\delta)$.

Let $h^* = h^*_{M^*}$; let $g_\delta : \operatorname{nacc} C_\delta \to \lambda$ be $h^* \circ (h^{-1}_\delta) : \operatorname{nacc} C_\delta \to \lambda$. We now by [4] (using $S \in I[\lambda]$ i.e. $\oplus$) construct a linear order $N = M^{h^*}$ with universe λ^+, $N = \bigcup_{\alpha<\lambda} N_\alpha$, N_α increasing continuous in α with universe an ordinal $< \lambda^+$ and for each $\delta \in S$, there is a sequence $\bar{y}^\delta = \langle y^\delta_\zeta : \zeta < \kappa \rangle$ of members of $N_{\delta+1}$ such that

$(*)_1$ if $\alpha \in \operatorname{nacc} C_\delta$, $g_\delta(\alpha) = \beta$, $\zeta < \kappa$ *then*
$\eta_\beta(\zeta) = 1 \Leftrightarrow (\forall \gamma \in N_\alpha)(\exists \gamma' \in N_{\sup(\alpha \cap C_\delta)})[\gamma <_N y^\delta_\zeta \Rightarrow \gamma <_N \gamma' <_N y^\delta_\zeta]$.

Suppose $f : \lambda^+ \to \lambda^+$ is an embedding of N into M^*, let $E = \{\delta < \lambda^+ : N_\delta$ universe is δ and δ is closed under f, $f^{-1}\}$. Clearly E is a club of λ^+, hence for some $\delta \in S$ the set $A = (\operatorname{acc} E) \cap (\operatorname{nacc} C_\delta)$ is unbounded in δ (so $\delta \in \operatorname{acc}\operatorname{acc} E$). Let $\bar{x} = \langle x_\zeta : \zeta < \kappa \rangle =: \langle f(y^\delta_\zeta) : \zeta < \kappa \rangle$, so we know (similarly to [4, Section 3]) that for $\alpha \in A$ and $\zeta < \kappa$ we have $g^{\bar{x}}_{M^*,\delta}(\alpha)(\zeta) = 1 \Leftrightarrow \eta_{g_\delta(\alpha)}(\zeta) = 1$. Hence $\alpha \in A \Rightarrow g^{\bar{x}}_{M^*,\delta}(\alpha) = g_\delta(\alpha) \Rightarrow (g^{\bar{x}}_{M^*,\delta} \circ h_\delta)(\operatorname{otp}(\alpha \cap C_\delta) - 1) = h^*(\operatorname{otp}(\alpha \cap C_\delta) - 1)$ contradicting the choice of h^*. ■

Claim 3.1A. *1) In 3.1 if λ is a successor cardinal then we can get*

$\oplus^0_\lambda$ *there are $f_\alpha : \lambda \to \lambda$ for $\alpha < \lambda^+$ such that*

$(*)^\lambda$ *for every $f \in {}^\lambda\lambda$ for some $\alpha < \lambda^+$ we have $f_\alpha \neq_{D_\lambda} f$ (where D_λ is the club filter on λ).*

2) If clause (a) of 3.1 and $(b)^*$ below then $\otimes_{\lambda,\mu}$ of 3.1 holds; similarly in 3.1A, where

$(b)^*$ $\operatorname{univ}(\lambda^+, T_{\rm ord}) \leq \mu$

Proof. 1) Use [21, 3.4].

2) The same proofs. ■

Claim 3.2. *Assume λ is regular uncountable, and*

$\otimes^1_\lambda$ *there are $f_\zeta : \lambda \to \lambda$ for $\zeta < \lambda^+$ such that: for every $f : \lambda \to \lambda$ for some ζ, $\{\alpha < \lambda : f_\zeta(\alpha) = f(\alpha)\}$ is stationary.*

1) *Let $S_1, S_2 \subseteq \{\delta < \lambda^+ \colon \mathrm{cf}(\delta) = \lambda\}$ be stationary, and $\delta \in S_1 \Rightarrow \delta = \sup(\delta \cap S_2)$. We can find $\bar{C} = \langle C^\zeta_\delta : \delta \in S_1, \zeta < \lambda^+ \rangle$, such that:*

(a) *C^ζ_δ is a club of δ of order type λ.*

(b) $\mathrm{nacc}\, C^\zeta_\delta \subseteq S_2$.

(c) *for every club E of λ^+, for stationary many $\delta \in S_1$, for some $\zeta < \lambda^+$,*

(c) $\delta = \sup\Big\{\alpha : \alpha \in \mathrm{nacc}\, C^\zeta_\delta$ *and* $\sup(\alpha \cap C^\zeta_\delta) \in \mathrm{nacc}\, C^\zeta_\delta$, $\mathrm{otp}\,(\alpha \cap C_\delta)$ *is even and* $\{\alpha, \sup(\alpha \cap C^\zeta_\delta)\} \subseteq E\Big\}$.

2) *Let $\lambda = \lambda^{<\lambda}$ and $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}\,\delta = \lambda\}$ stationary. We can find $\bar{C} = \langle C^\zeta_\delta : \delta \in S, \zeta < \lambda^+ \rangle$ such that*

(a) *C^ζ_δ is a club of δ of order type λ.*

(b) *for every club E of λ^+ for stationary many $\delta \in S$, for some $\zeta < \lambda^+$, for every $\xi < \lambda$ E contains arbitrarily large (below λ) intervals of C_δ of length ξ*

3) *If λ is a successor cardinal then we can get (2) even if we omit "$\lambda = \lambda^{<\lambda}$" and weaken in $\otimes^1_\lambda$, "$f_\zeta(\alpha) = f(\alpha)$" to "$f_\zeta(\alpha) \geq f(\alpha)$".*

Remark 3.2A. 1) We can in 3.2(3) get the conclusion of 3.2(2) too if we fix ξ.

2) We can replace in the assumptions and conclusions, λ^+ by μ.

Proof. 1) Let $\langle C_\delta : \delta \in S_1 \rangle$ be such that: C_δ a club of δ, $\mathrm{otp}\, C_\delta = \lambda$ and $\mathrm{nacc}\,(C_\delta) \subseteq S_2$. If $\alpha < \beta < \lambda^+$, $S_2 \cap (\alpha, \beta) \neq \emptyset$ then let $\langle (\beta^\varepsilon_{\alpha,\beta}, \gamma^\varepsilon_{\alpha,\beta}) : \varepsilon < \lambda \rangle$ list all increasing pairs from $(S_2 \cap \beta \setminus \alpha)$ (maybe with repetitions). Let $\langle f_\zeta : \zeta < \lambda^+ \rangle$ exemplify $\otimes^1_\lambda$. Let $C_\delta = \{\alpha_{\delta,\varepsilon} : \varepsilon < \lambda\}$ (increasing). Let $e = e^\zeta_\delta \subseteq \lambda$ be a club of λ such that: if $i < j$ are from e then $\gamma^{f_\zeta(i)}_{\alpha_{\delta,i},\delta} < \alpha_{\delta,j}$. Now for $\delta \in S_1$, $\zeta < \lambda^+$, we let: C^ζ_δ is $\{\alpha_{\delta,\varepsilon}, \beta^{f(\varepsilon)}_{\alpha_{\delta,\varepsilon},\alpha_{\delta,\varepsilon+1}}, \gamma^{f_\zeta(\varepsilon)}_{\alpha_{\delta,\varepsilon},\alpha_{\delta,\varepsilon+1}} : \varepsilon \in e^\zeta_\delta\}$.

Clearly C^ζ_δ is a club of δ of order type λ. Now if E is a club of λ^+, then $E \cap S_2$ is a stationary subset of λ^+ so for some $\delta \in S_1$, $\delta = \sup(E \cap S_2)$ and define $g : \lambda \to \lambda$ by: $\beta^{g(\varepsilon)}_{\alpha_{\delta,\varepsilon},\delta}$, $\gamma^{g(\varepsilon)}_{\alpha_{\delta,\varepsilon}+1,\delta}$ are the first and second members of $(E \cap S_2) \setminus (\alpha_{\delta,\varepsilon}, \delta)\}$. By the choice of $\langle f_\zeta : \zeta < \lambda^+ \rangle$ for some $\zeta < \lambda^+$, $(\exists^{\mathrm{stat}}\, \varepsilon)(g(\varepsilon) = f_\zeta(\varepsilon))$. So C^ζ_δ is as required.

2) Similar proof (and we shall not use it).
3) In the proof of (1) for $\alpha < \lambda$ let $h(\alpha, -) : \lambda^- \overset{\text{onto}}{\longrightarrow} \alpha$. We do the construction for each $\tau < \lambda^-$. The demand on $e = e^\zeta_\delta$ is changed to: if for $i < j$ are from e, then $\gamma^{h(f_\zeta(\alpha),\tau)}_{\alpha_{\delta,i},\delta} < \alpha_{\delta,j}$, and C^ζ_δ is changed accordingly. For some $\tau < \lambda$ we succeed. (really this version of $\otimes^1_\lambda$ implies the original version.) ∎

Claim 3.3. *Assume:*

(a) λ *regular,* $S \subseteq \lambda$ *stationary,* $\lambda^\kappa = \lambda$.
(b) $\bar{C} = \langle C_\delta : \delta \in S\rangle$, C_δ *a club of* δ.
(c) $\bar{\mathcal{P}} = \langle \mathcal{P}_\delta : \delta \in S\rangle, \mathcal{P}_\delta \subseteq \mathcal{P}(\text{nacc}\,(C_\delta))$ *is closed upward.*
(d) for every club E *of* λ *for some* δ, $E \cap \text{nacc}\, C_\delta \in \mathcal{P}_\delta$
(e) $\kappa < \lambda$, $T_\delta = \bigcup\{T_{\delta,\beta,\gamma} : \beta \leq \gamma, \{\beta,\gamma\} \subseteq \text{nacc}\, C_\delta\}$, *for* $\beta < \gamma \in \text{nacc}\, C_\delta$, $T_{\delta,\beta,\gamma} \subseteq {}^{\gamma \cap \text{nacc}\, C_\delta}({}^\kappa\beta)$, $|T_{\delta,\beta,\gamma}| \leq \lambda$, *and even for each* γ *the set* $\bigcup\{T_{\delta,\beta,\gamma} : \gamma \in \text{nacc}\, C_\delta,\ \beta \in \gamma \cap \text{nacc}\, C_\delta\}$ *has cardinality* $\leq \lambda$.
(f) If $A \in \mathcal{P}_\delta$, *for* $\zeta < \lambda^+$ *we have* $f_\zeta \in {}^{\text{nacc}\, C_\delta}({}^\kappa\delta)$ *and* $\left[\beta < \gamma \text{ are from } A \Rightarrow f_\zeta \restriction \beta \in T_{\delta,\beta,\gamma}\right]$ *then for some* $f^* \in {}^{\text{nacc}\, C_\delta}({}^\kappa\delta)$ *we have* $[\beta < \gamma$ *from* $A \Rightarrow f^* \restriction \beta \in T_{\delta,\beta,\gamma}]$ *and for every* $\zeta < \lambda^+$, $\{\beta \in A : f_\zeta(\beta) = f^*(\beta)\} \notin \mathcal{P}_\delta$.

Then *there is no universal linear order of cardinality* λ^+.

Proof. Similar to the previous one. ∎

Conclusion 3.5. *If* $2^\lambda > \lambda^+, \lambda = \text{cf}\,\lambda > \aleph_0$, $\bar{C} = \langle C_\delta : \delta \in S\rangle$, $S \subseteq \{\delta < \lambda^+ : \text{cf}\,\delta = \lambda\}$ *stationary,* $\lambda^+ \notin \text{id}^a(\bar{C})$ *and for each* α *we have* $|\{C_\delta \cap \alpha : \alpha \in \text{nacc}\, C_\delta\}| \leq \lambda$ then

(a) there is no universal linear order in λ^+
(b) moreover, $\text{univ}\,(\lambda^+, T_{\text{ord}}) \geq 2^\lambda$).

Discussion 3.6. (1) The condition $\otimes_\lambda$ from 3.1 holds in the models (of ZFC) constructed in [12, Section 4] where $\lambda = \aleph_0$, $2^{\aleph_0} = \aleph_2$ and there is a non meager subset of ${}^\omega 2$ of cardinality $\aleph_1$.

(2) It is clear from 3.5 that the existence of a universal graph in λ does not imply the existence of a universal linear order in λ: as by [14], [15], if $V \models \text{GCH}$, $\lambda = \lambda^{<\lambda}$, $\bar{C} = \langle C_\delta : \delta < \lambda^+, \text{cf}\,\delta = \lambda\rangle$ guesses clubs, for some λ^+-c.c. forcing notion P we have $V^P \models_P$ "there is a universal graph in λ". But in V^P the property of $\bar{C}$, guessing clubs, is preserved and it shows that there is no universal linear order.

(3) We can look at this from another point of view:
   (a) Considering the following three proofs of consistency results on the existence of universal structures: [12, Section 4] (universal linear order in $\aleph_1$), [14, Section 1] (universal graphs in λ^+, $\lambda = \lambda^{<\lambda}$ and [15] (universal graphs in other cardinals), the first result cannot be gotten by the other two proofs.
   (b) For theories with the strict order property it is "harder" to have universal models than for simple theories as the results of Section 5 on simple theories fail for the theory of linear order (by 3.5) and even all (f.o.) theories with the strict order property (as in [4])

(4) Concerning 3.5(b), note that (for any complete first order T) we have $\mathrm{Univ}(\mu, T) \leq 2^{<\mu}$ hence $\mathrm{cf}(\mathcal{S}_{\leq\mu}(2^{<\mu}), \subseteq\ ) \geq \mathrm{univ}(\mu, T)$ so under reasonable hypotheses we get in 3.5(b) equality (i.e., $\mu = \lambda^+$).

4. Toward the consistency for simple theories

The aim of this proof was originally to deal with the universality spectrum of simple countable theories and as a first approximation to characterize $\{\lambda : \mathrm{univ}(\lambda^+, T) \leq \lambda^{++} < 2^\lambda\}$, but we shall do it more generally and have more consequences. On simple theories see [11].

Notation 4.0. *(1) For a set $u \in \mathcal{S}_{<\lambda}(\lambda^+) =: \{u \subseteq \lambda^+ : |u| < \lambda\}$ let $\sup_\lambda(u) = \{\alpha + \lambda : \alpha \in u\}$ also let $S^{\lambda^+}_\lambda = \{\delta < \lambda^+ : \mathrm{cf}\,\delta = \lambda\}$*

(2) If $u_1, u_2 \in \mathcal{S}_{<\lambda}(\lambda^+)$, $h : u_1 \to u_2$ is legal if it is one to one, onto, and there is a unique h^V such that: h^V is one to one order preserving from $\sup_\lambda(u_1)$ onto $\sup_\lambda(u_2)$ and for $\alpha \in u_1$, $h^V(\alpha + \lambda) = h(\alpha) + \lambda)$.

(3) We say that h is lawful if in addition h^V is the identity. We sometimes use "legal" and "lawful" for functions $h : u_1 \to u_2$ when $u_i \subseteq \lambda^+, |u_i| \geq \lambda$.

(4) Wide λ^+-trees $\mathcal{T} = (\mathcal{T}, <)$ are just subsets of ${}^{\lambda^+ >}(\lambda^+)$ of cardinality $\leq \lambda^+$ closed under initial segments with the order being initial segment. A branch is a maximal linearly ordered subset, a λ^+-branch is one of order type λ^+

Definition 4.1. *$K_{\mathrm{ap}} = (K_{\mathrm{ap}}, \leq_{K_{\mathrm{ap}}})$ is a λ-approximation family, if for some sequence $\bar\tau$ (= $\langle \tau_i : i < \lambda^+\rangle$ of vocabularies, $|\tau_i| \leq \lambda$, τ_i increasing with i, $M \restriction i$ means $(M \restriction \tau_i) \restriction i$; τ_i can have relations and functions with* *infinite arity but* $< \lambda$; *we concentrate on the case* $\tau_i = \tau$ *for all* $i < \lambda$) *the followings hold:*

(a) K_{ap} *is a set of* τ-*model with a partial order* $\leq = \leq_{K_{\mathrm{ap}}}$ (or $\tau_{\sup(M)}$-models).

(b) *if* $M \in K_{\mathrm{ap}}$ *then* $|M|$ *is a subset of* λ^+ *of cardinality* $< \lambda$ *and* $M \leq N \Rightarrow M \subseteq N$.

(c) *if* $M \in K_{\mathrm{ap}}, \delta \in S^{\lambda^+}_{\lambda}$ *then* $M \restriction \delta \in K_{\mathrm{ap}}$ *and* $M \restriction \delta \leq M$.

(d) *any* $\leq_{K_{\mathrm{ap}}}$-*increasing chain in* K_{ap} *of length* $< \lambda$ *has an upper bound.*

(e) (α) *if* $\delta \in S^{\lambda^+}_{\lambda}$, $M_0 = M_2 \restriction \delta$, $M_0 \leq_{K_{\mathrm{ap}}} M_1$, $|M_1| \subseteq \delta$ *then* M_1, M_2 *has a common* $\leq_{K_{\mathrm{ap}}}$-*upper bound* M_3, *such that* $M_3 \restriction \delta = M_1$.

(β) *if we have* $M_{1,i} (i < i^* < \lambda)$, $M_{1,i} \in K_{\mathrm{ap}}$ *increasing with* i, $|M_{1,i}| \subseteq \delta_i \in S^{\lambda^+}_{\lambda}$ *and* $M_2 \restriction \delta_i \leq_{K_{\mathrm{ap}}} M_{1,i}$, *then there is a common upper bound* M_3 *to* $\{M_2\} \cup \{M_{1,i} : i < i^*\}$

(γ) *if we have* $M_1 \in K_{\mathrm{ap}}$, $M_{2,i} \in K_{\mathrm{ap}}$ *for* $i < i^* < \lambda$ *increasing with* i, $\delta \in S^{\lambda^+}_{\lambda}$, $M_{2,i} \restriction \delta \leq M_1$ *then there is a common* $\leq_{K_{\mathrm{ap}}}$-*upper bound to* $\{M_1\} \cup \{M_{2,i} : i < i^*\}$ *such that* $M_3 \restriction \delta = M_1$.

(δ) *if (i) we have* $\langle \delta_i : i \leq i^* \rangle$ *is a strictly increasing sequence of members of* $S^{\lambda^+}_{\lambda}$, *(ii) we have* $M_{1,i} (i < i^* < \lambda)$, $M_{1,i} \in K_{\mathrm{ap}}$ *increasing with* i, *(iii)* $[i(1) < i(2) \Rightarrow M_{i(1)} = M[i(2) \restriction \delta_{i(1)}]$ *(iv)* $|M_{1,i}| \subseteq \delta_i$ *(v)* $M_{2,j} \in K_{\mathrm{ap}}$ *for* $j < j^*$ *has universe* $\subseteq \delta_{i^*}$, *and is* $<_{K_{\mathrm{ap}}}$-*increasing in* j (vi) $M_{2,j} \restriction \delta_i \leq_{K_{\mathrm{ap}}} M_{1,i}$, **then** *there is a common upper bound* M_3 *to* $\{M_{2,j} : j < j^*\} \cup \{M_{1,i} : i < i^*\}$ *such that for every* $i < i^*$ *we have* $M_3 \restriction \delta_i = M_{1,i}$

(f) *For* $\alpha < \lambda^+$, $\{M \in K_{\mathrm{ap}} : |M| \subseteq \alpha\}$ *has cardinality* $\leq \lambda$.

(g) *We call* $h : M_1 \to M_2$ *a lawful (legal)* K_{ap}-*isomorphism if* h *is an isomorphism from* M_1 *onto* M_2 *and* h *is lawful (legal). We demand:*

(α) *if* h *is a lawful* K_{ap}- *isomorphism from* $M_1 \in K_{\mathrm{ap}}$ *onto* $M_2 \in K_{\mathrm{ap}}$, *and* $M_1 \leq_{K_{\mathrm{ap}}} M'_1$ *and* h *can be extended to some lawful* h^+ *with domain* $|M'_1|$ *then for some* h', M'_2 *we have* $M_2 \leq_{K_{\mathrm{ap}}} M'_2$, $h \subseteq h'$ *and* h' *a lawful* K_{ap}-*isomorphism from* M'_1 *onto* M'_2.

(β) *if* $M_1 \in K_{\mathrm{ap}}$, $u_1 = |M_1|$, $u_2 \subseteq \lambda^+$ *and* h *a legal mapping from* u_1 *onto* u_2 *then for some* $M' \in K_{\mathrm{ap}}$, $|M'| = u_2$ *and* h *is a lawful* K_{ap}-*isomorphism from* M *onto* M'.

(γ) *lawful* K_{ap}-*isomorphisms preserve* $\leq_{K_{\mathrm{ap}}}$.

(h) *If* $M \in K_{\mathrm{ap}}$ *and* $\beta < \lambda^+$ *then for some* $M' \in K_{\mathrm{ap}}$ *we have* $M \leq_{K_{\mathrm{ap}}} M'$ *and* $\beta \in |M'|$

(i) *[Amalgamation] Assume* $M_\ell \in K_{\mathrm{ap}}$ *for* $\ell < 3$ *and* $M_0 \leq_{K_{\mathrm{ap}}} M_\ell$ *for* $\ell = 1, 2$. *Then for some* $M \in K_{\mathrm{ap}}$ *and lawful function* f *we have:*

$M_1 \le_{K_{\mathrm{ap}}} M$, the domain of f is M_2, $f \restriction |M_0|$ is the identity and f is a $\le_{K_{\mathrm{ap}}}$-embedding of M_2 into M.

Remark 4.1A. This is similar to λ^+-uniform λ forcing, see [13], [24] see also [18, AP], [22, AP].

Definition 4.1B. *We call K_{ap} homogeneous if in clause (g) of definition 4.1 we can replace "lawful" by "legal".*

Definition 4.2. *1) If K_{ap} is a λ-approximation family then*
$K_{\mathrm{md}} = \{\Gamma :$(i) Γ is a $\le_{K_{\mathrm{ap}}}$-directed subset of K_{ap}
(ii) Γ is maximal in the sense that: for every $\beta < \lambda^+$ for some
$M \in \Gamma$ we have $\beta \in |M|$
(iii) if $M \in \Gamma$, $M \le_{K_{\mathrm{ap}}} M'$, then for some $M'' \in \Gamma$, there is
a lawful K_{ap}-isomorphism h from M' onto M'' over $M\}$.

2) *K_{ap} is a simple λ-approximation if: (it is a λ-approximation family and) for every $\delta \in S^{\lambda^+}_{\lambda}$, and $\Gamma \in K_{\mathrm{md}}$ and $\{(M_i, N_i) : i < \lambda^+\}$ satisfying $M_i \in \Gamma$, $M_i \le_{K_{\mathrm{ap}}} N_i \in K_{\mathrm{ap}}$ there is a club C of λ^+ and pressing down $h : C \to \lambda^+$ such that:*
   - (*) *if $i < j$ are in $C \cap S^{\lambda^+}_{\lambda}$, $h(i) = h(j)$ and $M_i \le_{K_{\mathrm{ap}}} M \in \Gamma$, $M_j \le_{K_{\mathrm{ap}}} M \in \Gamma$ then we can find $N \in K_{\mathrm{ap}}$, $M \le_{K_{\mathrm{ap}}} N$, and a lawful $\le_{K_{\mathrm{ap}}}$-embeddings f_i, f_j of N_i, N_j into N over M_i, M_j respectively such that $f_i \restriction (N_i \restriction i) = f_j \restriction (N_j \restriction j)$.*

3) *We define K^{α}_{md} as before but $M \in \Gamma \Rightarrow |M| \subseteq \lambda\alpha$.*

4) *K_{ap} is θ-closed if $\theta = \operatorname{cf}\theta < \lambda$ and if $\langle M_i : i < \theta\rangle$ is $\le_{K_{\mathrm{ap}}}$-increasing in K_{ap} then $\bigcup_{i<\theta} M_i \in K_{\mathrm{ap}}$ is an $\le_{K_{\mathrm{ap}}}$-upper bound; moreover if $(\forall i < \theta) M_i \le_{K_{\mathrm{ap}}} N$ implies $\cup_{i<\theta} M_i \le_{K_{\mathrm{ap}}} N$ of 4.1.*

5) *K_{ap} is $(< \lambda)$-closed if it is θ-closed for every $\theta < \lambda$*

6) *K_{ap} is smooth if*
   - (α) *it is $(< \lambda)$-closed;*
   - (β) *all vocabularies τ_i are finitary;*
   - (γ) *in clauses (c),(e)(α), and (e)(γ) we can replace "$\delta \in S^{\lambda^+}_{\lambda}$" to "$\delta > 0$ is divisible by λ".*

Remark 4.2A. If $M, N \in K^{\alpha}_{\mathrm{md}}$, $\alpha < \lambda^+$, then some lawful f is an isomorphism from M onto N.

Lemma 4.3. *Suppose that*

(A) $\lambda = \lambda^{<\lambda}$;

(B) K_{ap} *is a* λ*-approximation family;*

(C) $\Gamma^*_\alpha \in K_{\mathrm{md}}$ *for* $\alpha < \alpha^*$;

(D) $\mathcal{T}$ *is a wide* λ^+*-tree,* A_α *a* λ^+*-branch of* T *for* $\alpha < \alpha^*$ *and for* $\alpha \neq \beta (< \alpha^*)$ *we have* $A_\alpha \neq A_\beta$, *and we let* $\varepsilon(\alpha, \beta) =$ *the level of the* $<_{\mathcal{T}}$*-last member of* $A_\alpha \cap A_\beta$, $\zeta(\alpha,\beta) = (\varepsilon(\alpha,\beta)+1)\lambda$.

Then *there is a forcing notion* Q *such that:*

(a) Q *is* λ*-complete of cardinality* $|\alpha^*|^{<\lambda}$

(b) Q *satisfies the version of* λ^+*-c.c. from [17, Section 1] (for simplicity - here always for* $\varepsilon = \omega$*).*

(c) For some Q*-names* $\underset{\sim}{h}_\alpha$ *and* $\underset{\sim}{\Gamma}'_\alpha$ *(for* $\alpha < \alpha^*$*) we have:* $\Vdash_Q$ *"for* $\alpha < \alpha^*$ *we have* $\underset{\sim}{\Gamma}'_\alpha \in K_{\mathrm{md}}$, $\underset{\sim}{h}_\alpha$ *is lawful , maps* λ^+ *onto* λ^+ *, and maps* Γ_α *onto* Γ'_α *such that for* $\alpha < \beta < \alpha^*$, $\underset{\sim}{\Gamma}'_\alpha \restriction \zeta(\alpha,\beta) = \underset{\sim}{\Gamma}'_\beta \restriction \zeta(\alpha,\beta)$, *so for every* $M \in \Gamma_\alpha$ *we have* $h_\alpha \restriction (|M|)$ *is lawful and is an isomorphism from* M *onto some* $M' \in \underset{\sim}{\Gamma}'_\alpha$*".*

Remark 4.3A. 1) Our freedom is in permuting $(\lambda\alpha, \lambda\alpha + \lambda)$; up to such permutation $\Gamma_\alpha \restriction (\lambda i) = \{M \in \Gamma'_\alpha : |M| \subseteq \lambda i\}$ is unique.

2) If we demand that K_{ap} be smooth the proof is somewhat simplified.

Proof. We define Q as follows:
$p \in Q$ *iff* $p = \langle (M^p_\alpha, h^p_\alpha) : \alpha \in w^p \rangle$ where

(a) $w_p \in [\alpha^*]^{<\lambda}$;

(b) $M^p_\alpha \in \Gamma_\alpha$;

(c) h^p_α a lawful mapping, Dom $h^p_\alpha = |M^p_\alpha|$;

(d) if $\alpha \neq \beta$ are in w^p, then: $h_\alpha(M^p_\alpha \restriction \zeta(\alpha,\beta))$ and $h_\beta(M^p_\beta \restriction \zeta(\alpha,\beta))$ are $\leq_{K_{\mathrm{ap}}}$-comparable;

(e) for every $\alpha \in w^p$, for some $n < \omega$, $0 = i_0 < i_1 < \ldots < i_n = \lambda^+$, we have: for $\ell \in [1,n)$, $i_\ell \in S^{\lambda^+}_\lambda$ and for every $\ell < n$

$(*)_\ell$ for every $\beta \in w$ for which $\zeta(\alpha,\beta) \in [i_\ell, i_{\ell+1})$ and $j \in [i_\ell, i_{\ell+1}) \cap S^{\lambda^+}_\lambda$ there is $\gamma \in w$ such that: $j \leq \zeta(\alpha,\gamma) \in [i_\ell, i_{\ell+1})$ and $M^p_\beta \restriction \zeta(\alpha,\beta) \leq_{K_{\mathrm{ap}}} M^p_\gamma \restriction \zeta(\alpha,\gamma)$. ∎

The order is $p \leq q$ iff: $w^p \subseteq w^q$ and for $\alpha \in w^p$: $M^p_\alpha \leq_{K_{\mathrm{ap}}} M^q_\alpha$, $h^p_\alpha \subseteq h^q_\alpha$ and $M^p_\alpha \neq M^q_\alpha \Rightarrow \bigwedge_{\beta \in w^p} h_\beta(M^p_\beta \restriction \zeta(\alpha,\beta)) \leq_{K_{\mathrm{ap}}} h_\alpha(M^q_\alpha \restriction \zeta(\alpha,\beta))$.

Fact 4.4. *Any increasing chain in Q of length $< \lambda$ has an upper bound.*

Proof. Let $\langle p_i : i < \delta \rangle$ be an increasing sequence in Q, $\delta < \lambda$ a limit ordinal. Let $w = \bigcup \{w^{p_i} : i < \delta\}$, and list w as $\{\alpha_j : j < j^*\}$. We now choose by induction on $j < j^*$, a member M_j of K_{ap} and a lawful mapping h_j with domain $|M_j|$ such that:

(*) (a) if $\langle (M^{p_i}_{\alpha_i}, h^{p_i}_{\alpha_i}) : i < \delta$ but $\alpha_j \in w^{p_i} \rangle$ is eventually constant, then this value is (M_j, h_j).
(b) Otherwise let $h_j(M_j) \in \Gamma_{\alpha_j}$ be a $\leq_{K_{\mathrm{ap}}}$-upper bound of $\{h^{p_i}_{\alpha_i}(M^{p_i}_{\alpha_i}) : i < \delta$ but $\alpha_j \in w^{p_i}\} \cup \{h_{j_1}(M_{j_1}) \restriction \zeta(\alpha_j, \alpha_{j_1}) : j_1 < j\}$.

If we succeed $q =^{df} \langle (M_j, h_j) : j \in w \rangle$ is a member of Q as required. Why? First we check that $q \in Q$. Clauses (a),(b),(c) are obvious; for clause (d) let $\alpha \neq \beta$ be in w, so let $\{\alpha, \beta\} = \{\alpha_{j_1}, \alpha_{j_2}\}$, $j_1 < j_2$; now if (*)(b) holds for j_2 just note that $h_{j_1}(M_{j_1}) \restriction \zeta(\alpha_{j_1}, \alpha_{j_2}) \leq h_{j_2}(M_{j_2})$ by the choice of the later; and if (*)(a) holds for j_2, then for some $i < \delta, (M_{j_2}, h_{j_2}) = (M^{p_i}_{j_2}, h^{p_i}_{j_2})$ and now check the choice of (M_{j_1}, h_{j_1}). For clause (e), clearly it is enough to prove:

(*) for every $i_1 \in (S^{\lambda^+}_\lambda \cup \{\lambda^+\})$ there is $i_0 \in i_1 \cap (S^{\lambda^+}_\lambda \cup \{0\})$ such that $(*)_\ell$ of clause (e) of the definition of Q holds with i_0, i_1 taking the role of $i_\ell, i_{\ell+1}$.

Let $i_1 \in S^{\lambda^+}_\lambda \cup \{\lambda^+\}$ be given; for each $i < i_1$ let $f(i) =^{df} \sup\{\zeta(\beta, \alpha)+1 : \beta \in w, \zeta(\beta, \alpha) \in [i, i_1)\}$ (if the supremum is on an empty set - we are in a trivial case). Clearly $[j_1 < j_2 < i_1 \Rightarrow f(j_1) < f(j_2)]$, so for some $i_0 \in i_1 \cap (S^{\lambda^+}_\lambda \cup \{0\})$ for all $i \in [i_0, i_1) \cap (S^{\lambda^+}_\lambda \cup \{0\})$ we have $f(i) = f(i_0)$. Now for each $i < i_1$ let $g(i) =^{df} \sup\{j + 1 : j < j^*, \zeta(\alpha_j, \alpha) \in [i, i_1)\}$, note: if the supremum is on the empty set then the value is zero; again it is clear that g decrease with i hence wlog for all $i \in [i_0, i_1)$ we have $g(i) = g(i_0)$
Case 1 for every $i \in [i_0, i_1)$ there is $\beta \in w$ such that: $\zeta(\beta, \alpha) \in [i_0, i_1)$ and letting $\beta = \alpha_j$ and in (*) above case (b) occurs.
Check
Case 2 not case 1
For every $\gamma \in w$ let j_γ be the first ordinal δ such that $\langle (M^{p_i}_\gamma, h^{p_i}_\gamma) : i < \delta \rangle$ is constant, and again wlog for some δ^* for every $i \in [i_0, i_1)$, $d' < \delta^*, \zeta < f(i_0)$ and $j < g(i_0)$ there is $\beta \in w$ such that $\zeta \leq \zeta(\beta, \alpha) \in [i, i_1), \beta \in \{\alpha_{j'} : j \leq j' < g(i_0)\}$ and $j_\beta \geq \delta'$, the rest should be clear.

So we have proved that $q \in Q$; now $p_i \leq_{K_{\mathrm{ap}}} q$ is straightforward. So now we have only to prove that we can carry the inductive definition from (*).

In the choice of M_j, h_j we first have chosen $h_j(M_j)$. We do it by choosing $h(M_j \restriction \zeta)$ for $\zeta \in \{\zeta(\alpha_j, \beta) : \beta \in w\}$; there we use clause (e)(δ) of Definition 4.1. Having chosen $h_j(M_j)$ we can find M_j, h_j by clauses (g)(α) + (β) of Definition 4.1. ■

Fact 4.5. *If $p \in Q$, $\alpha \in w^p$ and $N \in \Gamma_\alpha$ then for some q: $p \leq q$, $w^q = w^p$ and*
$\bigwedge_{\beta \in w^p \setminus \{\alpha\}} (M^p_\beta, h^p_\beta) = (M^q_\beta, h^q_\beta)$ and $N \leq M^q_\alpha$.

Proof. Easier than the previous one (or let $\delta = 1$, $p_0 = p$ and $\{\alpha_j : j < j^*\}$ list w^p with $\alpha = \alpha_0$, repeat the proof of 4.4, just use q to choose (M_0, h_0). ■

Note the following

Fact 4.6. *If K_{ap} is θ-closed, then the following set is dense in Q: $\{p :$ if $\alpha, \beta \in w^p$, then $h^p_\alpha(M^p_\alpha) \restriction \zeta(\alpha,\beta) = h^\ell_\alpha(M^p_\beta) \restriction \zeta(\alpha,\beta)\}$.*

Proof. Follows easily from the previous Facts. ■

Fact 4.7. *The chain condition from [17, Section 1] holds.*

Proof. Suppose $p(\delta) \in Q$ for $\delta \in S^{\lambda^+}_\lambda$. For some pressing down function $h : S^{\lambda^+}_\lambda \to \lambda^+$ we have:

(*) if $h(\delta^1) = h(\delta^2)$, $\delta^1 < \delta^2$ *then:*

(a) $\operatorname{otp} w^{p(\delta^1)} = \operatorname{otp} w^{p(\delta^2)}$

(b$_1$) $OP_{w^{p(\delta^1)}, w^{p(\delta^2)}}$ is the identity on $w^{p(\delta^1)} \cap w^{p(\delta^2)}$

(b$_2$) for $\alpha, \beta \in w^{p(\delta^1)}$ the following are equivalent: (i) $\zeta(\alpha,\beta) < \delta^1$; (ii) $\zeta(\alpha', \beta') < \delta^2$ where $\alpha' =^{df} \mathrm{OP}_{w^{p(\delta^1)}, w^{p(\delta^2)}}(\alpha)$, $\beta' =^{df} \mathrm{OP}_{w^{p(\delta^1)}, w^{p(\delta^2)}}(\beta)$;

(iii) $\zeta(\alpha', \beta') = \zeta(\alpha, \beta)$ where α', β' are as in (ii).

(c) $|M^{p(\delta^1)}_\alpha|$ is bounded in δ_2 and also $\sup\{\zeta(\alpha,\beta) : \alpha \neq \beta$ are in $w^{p(\delta^1)}\} < \delta_2$

(d) if $\alpha^2 = \mathrm{OP}_{w^{p(\delta^1)}, w^{p(\delta^2)}}(\alpha^1)$ then

(α) $\mathrm{OP}_{|M^{p(\delta^1)}_{\alpha^1}|, |M^{p(\delta^2)}_{\alpha^2}|}$ is an isomorphism from $M^{p(\delta^1)}_{\alpha^1}$ onto $M^{p(\delta^2)}_{\alpha^2}$ which is lawful.

(β) $M^{p(\delta^1)}_{\alpha^1} \restriction \delta^1\lambda = M^{p(\delta^2)}_{\alpha^2} \restriction \delta^2\lambda$. ■

Now we continue as in the proof of 4.4 to prove: $h(\delta^1) = h(\delta^2) \Rightarrow p(\delta^1), p(\delta^2)$ are compatible (in the list $\{\alpha_j : j < j^*\}$ put $w^{p(\delta^1)} \cap w^{p(\delta^2)}$ an initial segment), for $j < j^*$ we use the simplicity of K_{ap}. Actually we need a stronger condition , e.g. $(*)_{\lambda^+,\omega}$ from [17, Section 1], and it is proved similarly. This finishes the proof of 4.3. ■

Claim 4.8. *Assume $K_{\mathbf{ap}}$ is a simple λ^+-approximation system. If $\Gamma_0 \subseteq K_{\mathbf{ap}}$ is directed, then for some forcing notion Q satisfying the λ^+-c.c. of [17, Section 1], $|Q| = \lambda^+$, $\Vdash_Q$ "there is a Γ and a lawful f such that $f(\Gamma_0) \subseteq \Gamma \in K_{\mathbf{md}}$".*

Proof. Natural. Without loss of generality $A \stackrel{\text{def}}{=} \cup\{|M| : M \in \Gamma_0\} = \{2\alpha : \alpha < \lambda^+\}$. $Q = \{M : M \in K_{\mathbf{ap}}$ and $M \restriction A \in K_{\mathbf{ap}}$ and $M \restriction A \leq_{K_{\mathbf{ap}}} M\}$ order by $\leq_{K_{\mathbf{ap}}}$. ∎

Conclusion 4.9. *Assume $\lambda = \lambda^{<\lambda} < 2^{\lambda^+} = \chi$, and a normal λ^+-tree $\mathcal{T}$ with $\geq \chi$ branches is given. For simplicity we assume that λ^+ is the set of members of $\mathcal{T}$, 0 is the root and $\alpha <_{\mathcal{T}} \beta \Rightarrow \alpha < \beta$ for $t \in \mathcal{T}$ and let $u_t = \{[\alpha\lambda, \alpha\lambda + \lambda) : \alpha \leq_{\mathcal{T}} t\}$. Then there is a forcing notion P such that*

(a) P is λ-complete, satisfies the λ^+-c.c. and has cardinality χ (so the cardinals in V^P are the same and cardinal arithmetic should be clear).

(b) for any λ-approximation system $K_{\mathbf{ap}}$ there are $\langle \Gamma_t^\zeta, M_t : t \in \mathcal{T}_\gamma \rangle$ for $\zeta < \lambda^{++}$ such that:

  (α) $\Gamma_t^\zeta \in K_{\mathbf{md}}^{(\ell g(t)+1)\lambda}$

  (β) $t <_T s \Rightarrow \Gamma_t^\zeta \subseteq \Gamma_s^\zeta$

  (γ) for every $\Gamma \in K_{\mathbf{md}}$ for some $\zeta < \lambda^{++}$ and λ^+-branch $B = \{t_\alpha : \alpha < \lambda^+\}$ of T and lawful onto λ^+ mapping Γ onto $\bigcup_{\alpha<\lambda^+} \Gamma_{t_\alpha}$.

(c) If $R \in V^P$ in $(< \lambda)$-complete, satisfies the [17, Section 1] λ^+-c.c., $D_i (i < \lambda^+)$ is dense, then for some directed $G \subseteq R$, $\bigwedge_i D_i \cap G \neq \emptyset$.

Proof. We use iterated forcing of length $\chi \times \lambda^{++}$, $(< \lambda)$-support, each iterand satisfying the λ^+-c.c. from [17, Section 1], $\langle P_i, \underset{\sim}{Q}_j : i \leq \chi + \lambda^{++}, j < \chi \times \lambda^{++} \rangle$ such that: for every $K_{\mathbf{ap}}$ (from V or from some intermediate universe) for unboundedly many $i < \chi \times \lambda^{++}$, we use the forcings from 4.3 or 4.8.

5. Application

Lemma 5.1. *Suppose*

(A) T is first order, complete for simplicity with elimination of quantifiers (or just inductive theory with the amalgamation and disjoint embedding property).

(*B*) *$K_{\rm ap}$ is a simple λ-approximation system such that every $M \in K_{\rm ap}$ is a model of T hence every M_Γ, where for $\Gamma \in K_{\rm md}$ we let $M_\Gamma = \bigcup\{M : M \in \Gamma\}$.*
Then

(*a*) *in 4.9 in V^P, there is a model of T of cardinality λ^{++} universal for models of T of cardinality λ^+.*

(*b*) *So in V^P,* univ $(\lambda^+, T) \leq \lambda^{++}$.

(*c*) *every model M of T of cardinality λ^+ can be embedded into M_Γ for some $\Gamma \in K_{\rm md}$.*

Proof. Straightforward. ■

Though for theories with the strict order property, the conclusion of Section 4 (and 5.1) fails, for some non simple theories we can succeed. Note that in 5.1 we have some freedom in choosing $K_{\rm ap}$ even after T is fixed.

Lemma 5.2. *Let $T = T^*_{\rm feq}$; it satisfies the assumption of 5.1 (hence its conclusions).*
In fact we can find a smooth simple λ-approximation system $K_{\rm ap}$ such that every model M of T of cardinality λ^+ is embeddable into some $M \in K^{\rm md}_{\rm ap}$.

Remark. 1) Note that there univ $(\lambda, T^*_{\rm feq})$ = univ $(\lambda, T_{\rm feq})$. Actually the λ-approximation family we get is also homogeneous.

2) The situation is similar for T_3 in 6.2.

Proof. The main point is to define $K_{\rm ap}$.

(A) $M \in K_{\rm ap}$ iff:

(i) M is a model of T

(ii) $|M| \in [\lambda^+]^{<\lambda}$

(B) $M_1 \leq_{K_{\rm ap}} M_2$ iff

(i) $M_1 \subseteq M_2$

(ii) if $\delta \in S^{\lambda^+}_\lambda$, $a \in P^{M_1} \cap \delta$, $b \in P^{M_1} \setminus \delta$ and $(\forall c \in M_1)[M_1 \models$ "bE_ac"$\Rightarrow c \notin \delta]$ *then* $(\forall c \in M_2)$ $[M_2 \models$ "bE_ac"$\Rightarrow c \notin \delta]$.

The checking of $(K_{\rm ap}, \leq_{K_{\rm ap}})$ is a λ-approximation family (see definition 4.10) as well as smoothness is straightforward.As for simplicity, quite easily we can assume that $\delta_1 < \delta_2$ are form $S^{\lambda^+}_\lambda$, $M_\ell \in K_{\rm ap}$ for $\ell = 1,2$, $|M_1| \subseteq \delta - 2$, $M_1 \restriction \delta_1 = M_1 \restriction \delta_2$and h is a legal isomorphism form M_1 onto M_2. Define a model M with universe $|M_1| \cup |M_2|$, as follows: $P^M =^{df} P^{M_1} \cup P^{M_2}$, $Q^M =^{df} Q^{M_1} \cup Q^{M_2}$, and for each $x \in P^M$, we let be the closure to an equivalence relation of the set of cases occurring in M_1 and/or M_2, now check.

The proof is straightforward. ■

Lemma 5.3. *T_3, the theory of triangle free graphs satisfies the assumption of 5.1 (hence its conclusions).*

Proof. Let xRy mean $\{x,y\}$ is an edge. The main point is to define K_{ap}

(a) $M \in K_{\mathrm{ap}}$ iff
  (i) M is a model of T
  (ii) $|M| \in [\lambda^+]^{<\lambda}$

(b) $M_1 \leq_{K_{\mathrm{ap}}} M_2$ iff
  (i) $M_1 \subseteq M_2$
  (ii) if $\delta \in S^{\lambda^+}_{\lambda}$, $a, b \in M_1$ and there is no $c \in M_1 \cap \delta$, $M_1 \models cRa \,\&\, cRb$ *then* for no $c \in M_2 \cap \delta$, $M_2 \models cRa \,\&\, cRb$. ■

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