

Borel sets with large squares

by

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Abstract. For a cardinal μ we give a sufficient condition \oplus_μ (involving ranks measuring existence of independent sets) for:

\otimes_μ if a Borel set $B \subseteq \mathbb{R} \times \mathbb{R}$ contains a μ -square (i.e. a set of the form $A \times A$ with $|A| = \mu$) then it contains a 2^{\aleph_0} -square and even a perfect square,

and also for

\otimes'_μ if $\psi \in L_{\omega_1, \omega}$ has a model of cardinality μ then it has a model of cardinality continuum generated in a “nice”, “absolute” way.

Assuming $\text{MA} + 2^{\aleph_0} > \mu$ for transparency, those three conditions (\oplus_μ , \otimes_μ and \otimes'_μ) are equivalent, and from this we deduce that e.g. $\bigwedge_{\alpha < \omega_1} [2^{\aleph_0} \geq \aleph_\alpha \Rightarrow \neg \otimes_{\aleph_\alpha}]$, and also that $\min\{\mu : \otimes_\mu\}$, if $< 2^{\aleph_0}$, has cofinality \aleph_1 .

We also deal with Borel rectangles and related model-theoretic problems.

Annotated content

0. Introduction. We explain results and history and include a list of notation.

1. The rank and the Borel sets. We define some version of the rank for a model, and then $\lambda_\alpha(\kappa)$ is the first λ such that there is no model with universe λ , vocabulary of cardinality $\leq \kappa$ and rank $< \alpha$. Now we prove that forcing does not change some ranks of the model, can only decrease others, and c.c.c. forcing changes little. If a Borel or analytic set contains a $\lambda_{\omega_1}(\aleph_0)$ -square then it contains a perfect square (see 1.12); clearly this gives something only if the continuum is large, that is, at least $\lambda_{\omega_1}(\aleph_0)$. On the other hand (see 1.13), if $\mu = \mu^{\aleph_0} < \lambda_{\omega_1}(\aleph_0)$ then in some c.c.c. forcing extension of V : the continuum is arbitrarily large, and some Borel set contains a μ -square but no μ^+ -square. Lastly (in 1.15), assuming MA holds we prove exact results (e.g. equivalence of conditions).

2. Some model-theoretic problems. When we restrict ourselves to models of cardinality up to the continuum, $\lambda_{\omega_1}(\aleph_0)$ is the Hanf number of $L_{\omega_1, \omega}$ (see 2.1).

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Also (see 2.4) if $\psi \in L_{\omega_1, \omega}$ has a model realizing many types (say in the countable set Δ of formulas; many means $\geq \lambda_{\omega_1}(\aleph_0)$) even after c.c.c. forcing, then

$$\{\{p : p \text{ a complete } \Delta\text{-type realized in } M\} : M \models \psi\}$$

has two to the continuum members.

We then (2.5) assume $\psi \in L_{\omega_1, \omega}$ has a two-cardinal model, say for (μ, κ) , and we want to find a (μ', \aleph_0) -model; we need $\lambda_{\omega_1}(\kappa) \leq \mu$. Next, more generally, we deal with $\bar{\lambda}$ -cardinal models (i.e. we demand that P_{ζ}^M have cardinality λ_{ζ}). We define ranks (2.8), from them we can formulate sufficient conditions for a transfer theorem and compactness. We can prove that the relevant ranks are (essentially) preserved under c.c.c. forcing as in §1, and the sufficient conditions hold for \aleph_{ω_1} under GCH.

3. Finer analysis of square existence. In 3.1, 3.2 we define for a sequence $\bar{T} = \langle T_n : n < \omega \rangle$ of trees (i.e. closed sets of the plane) a rank, degsq, whose value is a bound for the size of the square it may contain. We then (3.3) deal with analytic, or more generally κ -Suslin relations, and use parallel degrees. We then prove that statements on the degrees are related to the existence of squares in κ -Suslin relations in a way parallel to what we have on Borel relations, using $\lambda_{\alpha}(\kappa)$. We then (3.7–3.11) connect it to the existence of identities for 2-place colourings. In particular, we get results of the form “there is a Borel set B which contains a μ -square iff $\mu < \lambda_{\alpha}(\aleph_0)$ ” when $\text{MA} + \lambda_{\alpha}(\aleph_0) < 2^{\aleph_0}$.

4. Rectangles. We deal with the problem of the existence of rectangles in Borel and κ -Suslin relations. The equivalence of the rank (for models), the existence of perfect rectangles and the model-theoretic statements are more delicate here.

0. Introduction. We first review the old results (from §§1, 2). The main one is

(*)₁ it is consistent that for every successor ordinal $\alpha < \omega_1$, there is a Borel subset of ${}^{\omega}2 \times {}^{\omega}2$ containing an \aleph_{α} -square but no perfect square.

In fact,

(*)₁⁺ the result above follows from $\text{MA} + 2^{\aleph_0} > \aleph_{\omega_1}$.

For this we define (Definition 1.1) for any ordinal α a property $\text{Pr}_{\alpha}(\lambda; \kappa)$ of the cardinals λ, κ . The maximal cardinal with the property of \aleph_{ω_1} (i.e. for every small cardinal, c.c.c. forcing adds an example as in (*)₁) is characterized (as $\lambda_{\omega_1}(\aleph_0)$ where $\lambda_{\alpha}(\kappa) = \min\{\lambda : \text{Pr}_{\alpha}(\lambda; \kappa)\}$); essentially it is not changed by c.c.c. forcing; so in (*)₁:

(*)₁' if in addition $\mathbf{V} = \mathbf{V}_0^{\mathbf{P}}$ where \mathbf{P} is a c.c.c. forcing then $\lambda_{\omega_1}(\aleph_0) \leq (\beth_{\omega_1})^{V_0}$.

We will generally investigate $\text{Pr}_{\alpha}(\lambda; \kappa)$, giving equivalent formulations (1.1–1.6), seeing how fast $\lambda_{\alpha}(\kappa)$ increases, e.g. $\kappa^{+\alpha} < \lambda_{\alpha}(\kappa) \leq \beth_{\omega \times \alpha}(\kappa)$ (in 1.7, 1.8). For two variants we show: $\text{Pr}_{\alpha}^2(\lambda; \kappa^+)$ ($\alpha \leq \kappa^+$) is preserved by κ^+ -c.c. forcing, $\text{Pr}_{\alpha}^l(\lambda; \kappa^+) \Rightarrow \text{Pr}_{\alpha}(\lambda; \kappa^+)$ and $\neg \text{Pr}_{\alpha}(\lambda; \kappa^+)$ is preserved by any extension of the universe of set theory. Now $\text{Pr}_{\omega_1}(\lambda; \aleph_0)$ implies that there is no Borel set as above (1.12) but if $\text{Pr}_{\omega_1}(\lambda; \aleph_0)$ fails then some c.c.c.

forcing adds a Borel set as above (1.13). We cannot in $(*)_1$ omit some set-theoretic assumption even for \aleph_2 —see 1.15, 1.16 (add many Cohen reals or many random reals to a universe satisfying e.g. $2^{\aleph_0} = \aleph_1$; then, in the new universe, every Borel set which contains an \aleph_2 -square also contains a perfect square). We can replace Borel by analytic or even κ -Suslin (using $\text{Pr}_{\kappa^+}(\kappa)$).

In §2 we deal with related model-theoretic questions with less satisfactory results. By 2.1, 2.3, giving a kind of answer to a question from [Sh 49],

- $(*)_2$ essentially $\lambda = \lambda_{\omega_1}(\aleph_0)$ is the Hanf number for models of sentences in $L_{\omega_1, \omega}$ when we restrict ourselves to models of cardinality $\leq 2^{\aleph_0}$. (What is the meaning of “essentially”? If $\lambda_{\omega_1}(\aleph_0) \geq 2^{\aleph_0}$ this fails, but if $\lambda_{\omega_1}(\aleph_0) < 2^{\aleph_0}$ it holds.)

In 2.4 we generalize it (the parallel of replacing Borel or analytic sets by κ -Suslin). We conclude (2.4(2)):

- $(*)_3$ if $\psi \in L_{\omega_1, \omega}(\tau_1)$, $\tau_0 \subseteq \tau_1$ are countable vocabularies, $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1, \omega}(\tau_0)\}$ is countable and ψ has a model which realizes $\geq \lambda_{\omega_1}(\aleph_0)$ complete $(\Delta, 1)$ -types then $|\{(M \upharpoonright \tau_0)/\cong : M \models \psi, \|M\| = \lambda\}| \geq \min\{2^\lambda, \beth_2\}$ (for any λ), as we have models as in [Sh a, VII, §4] = [Sh c, VII, §4].

If we allow parameters in the formulas of Δ , and $2^{\aleph_0} < 2^{\aleph_1}$, then $(*)_3$ holds too. However, even in the case $2^\lambda = 2^{\aleph_0}$ we prove some results in this direction (see [Sh 262] or better [Sh e, VII, §5]). We then turn to three-cardinal theorems etc., trying to continue [Sh 49] (where e.g. $(\aleph_\omega, \aleph_0) \rightarrow (2^{\aleph_0}, \aleph_0)$ was proved).

We knew those results earlier than or in 1980/1, but failed in efforts to prove the consistency of “ZFC + $\lambda_{\omega_1}(\aleph_0) > \aleph_{\omega_1}$ ” (or proving ZFC + “ $\lambda_{\omega_1}(\aleph_0) = \aleph_{\omega_1}$ ”). By the mid-seventies we knew how to get consistency of results like those in §2 (forcing with \mathbf{P} , adding many Cohen reals, i.e. in $\mathbf{V}^{\mathbf{P}}$ getting $(*)_3$ for $\lambda = (\beth_{\omega_1})^{\mathbf{V}}$). This (older proof, not the one used) is closely related to Silver’s proof of “every I_1^1 -relation with uncountably many equivalence classes has 2^{\aleph_0} ones” (a deeper one is the proof of Harrington of the Lauchli–Halpern theorem; see a generalization of the Lauchli–Halpern theorem, a partition theorem on ${}^{\kappa}2$, κ large in [Sh 288, §4]).

In fact, about 88 I wrote down for W. Hodges proofs of (a) and (b) stated below.

(a) If, for simplicity, \mathbf{V} satisfies GCH, and we add $> \aleph_{\omega_1}$ Cohen reals, then the Hanf number of $L_{\omega_1, \omega}$ below the continuum is \aleph_{ω_1} .

(b) If $\psi \in L_{\omega_1, \omega}(\tau_1)$ and some countable $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1, \omega}(\tau_0)\}$ satisfies: in every forcing extension of \mathbf{V} , ψ has a model which realizes 2^{\aleph_0} (or at least $\min\{2^{\aleph_0}, \aleph_{\omega_1}\}$) complete Δ -types, then the conclusion of $(*)_3$ above holds.

Hodges had intended to write it up. Later Hrushovski and Velickovic independently proved the statement (a).

As indicated above, the results had seemed disappointing as the main question “is $\lambda_{\omega_1}(\aleph_0) = \aleph_{\omega_1}$?” is not answered. But Hjorth asked me about (essentially) $(*)_1$ which was mentioned in [HrSh 152] and urged me to write this down.

In §3 we define a degree of Borel sets of the form $\bigcup_{n < \omega} \text{lim}(T_n) \subseteq {}^\omega 2 \times {}^\omega 2$ measuring how close they are to having perfect squares, similarly we define degrees for κ -Suslin relations, and get results similar to earlier ones under MA and nail the connection between the set of cardinalities of models of $\psi \in L_{\omega_1, \omega}$ and having squares. In §4 we deal with the existence of rectangles.

We can replace \mathbb{R}^2 by \mathbb{R}^3 without any difficulty.

In a subsequent paper [Sh 532] which we are writing, we intend to continue the present work and [Sh 202, §5] and deal with: consistency of the existence of co- κ -Suslin (and even Π_2^1 -) equivalence relations with many equivalence classes, relationship of $\lambda_{\omega_1}^1$, $\lambda_{\omega_2}^3$ etc., and also try to deal with independence (concerning 2.11 and 4.11(1)) and the existence of many disjoint sections.

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NOTATION. *Set theory:* ${}^B A = \{f : f \text{ is a function from } B \text{ to } A\}$, the set of reals is ${}^\omega 2$; $\mathcal{S}_{< \kappa}(A) = [A]^{< \kappa} = \{B \subseteq A : |B| < \kappa\}$.

By a *Borel set* B we mean the set it defines in the current universe. A μ -*square* (or a *square of size* μ) is a set of the form $A \times A$, where $A \subseteq {}^\omega 2$ and $|A| = \mu$. A (μ_1, μ_2) -*rectangle* (or *rectangle of size* (μ_1, μ_2)) is a set of the form $A_1 \times A_2$, for some $A_l \subseteq {}^\omega 2$ with $|A_l| = \mu_l$ (for $l = 0, 1$). A *perfect square* is $\mathcal{P} \times \mathcal{P}$ with $\mathcal{P} \subseteq {}^\omega 2$ perfect. A *perfect rectangle* is $\mathcal{P}_1 \times \mathcal{P}_2$ with $\mathcal{P}_l \subseteq {}^\omega 2$ perfect. Note that a perfect rectangle is a $(2^{\aleph_0}, 2^{\aleph_0})$ -rectangle and a perfect square is a 2^{\aleph_0} -square.

\mathcal{P}, \mathcal{Q} denote perfect sets; \mathbf{P}, \mathbf{Q} denote forcing notions; P, Q, R denote predicates.

A κ -*Suslin set* is $\{\eta \in {}^\omega 2 : \text{for some } \nu \text{ we have } (\eta, \nu) \in \text{lim}(T)\}$ for some $(2, \kappa)$ -tree T (see below). A κ -*Suslin relation* (say an n -place relation) is defined similarly.

For $\bar{\lambda} = \langle \lambda_\zeta : \zeta < \zeta(*) \rangle$, a $\bar{\lambda}$ -*tree* is

$$T \subseteq \bigcup_n \prod_{\zeta < \zeta(*)} {}^n(\lambda_\zeta), \quad \text{ordered by } \bar{\eta} \triangleleft \bar{\nu} \Leftrightarrow \bigwedge_{\zeta < \zeta(*)} \eta_\zeta \triangleleft \nu_\zeta.$$

We usually let $\bar{\eta} \upharpoonright l = \langle \eta_\zeta \upharpoonright l : \zeta < \zeta(*) \rangle$.

For a $\bar{\lambda}$ -tree T we define

$$\lim(T) = \left\{ \bar{\eta} \in \prod_{\zeta < \zeta(*)} \omega(\lambda_\zeta) : n < \omega \Rightarrow \bar{\eta} \upharpoonright n \in T \right\}$$

(where $\langle \eta_\zeta : \zeta < \zeta(*) \rangle \upharpoonright n = \langle \eta_\zeta \upharpoonright n : \zeta < \zeta(*) \rangle$) and

$$\lim^*(T) = \left\{ \bar{\eta} \in \prod_{\zeta < \zeta(*)} \omega(\lambda_\zeta) : (\exists \bar{\eta}' \in \lim T)(\exists k < \omega) \left(\bigwedge_{\zeta < \zeta(*)} \eta_\zeta \upharpoonright [k, \omega) = \eta'_\zeta \upharpoonright [k, \omega) \right) \right\}.$$

We will use mainly $(2, 2)$ -trees and $(2, 2, \kappa)$ -trees.

Let $\eta \sim_n \nu$ mean that η, ν are sequences of ordinals, $\lg(\eta) = \lg(\nu)$ and

$$(\forall k)[n \leq k < \lg(\eta) \Rightarrow \eta(k) = \nu(k)].$$

For a tree T as above, $u \subseteq \zeta(*)$ and $n < \omega$ let

$$T^{(\sim_n, u)} = \left\{ \bar{\eta} : (\exists k)(\exists \bar{\nu} \in \lim(T)) \left[\bar{\nu} \in \prod_{\zeta < \zeta(*)} \omega(\lambda_\zeta) \ \& \ \bar{\eta} \in \prod_{\zeta < \zeta(*)} k(\lambda_\zeta) \ \& \ (\forall \zeta \in u)(\eta_\zeta \sim_n \nu_\zeta \upharpoonright k) \right] \right\}.$$

Let $\text{Fr}_n(\lambda, \mu, \kappa)$ mean that if F_α are n -place functions from λ to λ (for $\alpha < \kappa$) then for some $A \in [\lambda]^\mu$ we have

$$a_n \neq F_\alpha(a_0, \dots, a_{n-1}) \quad \text{for distinct } a_0, \dots, a_{n-1} \in A \text{ and } \alpha < \kappa.$$

Model theory. Vocabularies are denoted by τ , so languages are denoted by e.g. $L_{\kappa, \theta}(\tau)$, models are denoted by M, N . The universe of M is $|M|$, its cardinality $\|M\|$. The vocabulary of M is $\tau(M)$ and the vocabulary of \mathbf{T} (a theory or sentence) is $\tau(\mathbf{T})$. R^M is the interpretation of R in M (for $R \in \tau(M)$). For a model M and a set $B \subseteq M$ we have: $a \in \text{cl}_{< \kappa}(B, M)$ iff for some quantifier free $\varphi = \varphi(y, x_1, \dots, x_n)$ and $b_1, \dots, b_n \in B$ we have

$$M \models \varphi[a, b_1, \dots, b_n] \ \& \ (\exists^{< \kappa} x) \varphi(x, b_1, \dots, b_n).$$

Let $\text{cl}_\kappa(B, M) = \text{cl}_{< \kappa^+}(B, M)$ and $\text{cl}(B, M) = \text{cl}_{< 2}(B, M)$. (Note that if M has Skolem functions then $\text{cl}_{< \aleph_0}(B, M) = \text{cl}_{< 2}(B, M)$ for every $B \subseteq |M|$.) If κ is an ordinal we mean $|\kappa|$ (needed just for phrasing absoluteness results; that is, if we use a cardinal κ in a universe V , and then deal with a generic extension V^P , then maybe in V^P , κ is no longer a cardinal but we like to still use it as a parameter). Let \mathbf{T} denote a theory, first order if not said otherwise.

1. The rank and the Borel sets

DEFINITION 1.1. (1) For $l < 6$, cardinals $\lambda \geq \kappa, \theta$, and an ordinal α , let $\text{Pr}_\alpha^l(\lambda; < \kappa, \theta)$ mean that for every model M with universe λ and vocabulary of cardinality $\leq \theta$, $\text{rk}^l(M; < \kappa) \geq \alpha$ (defined below), and let $\text{NPr}_\alpha^l(\lambda; < \kappa, \theta)$

be the negation. Instead of “ $< \kappa^+$ ” we may write κ (similarly below); if $\kappa = \theta^+$ we may omit it (so e.g. $\text{Pr}_\alpha^l(\lambda; \kappa)$ means $\text{Pr}_\alpha^l(\lambda; < \kappa^+, \kappa)$); if $\theta = \aleph_0$, $\kappa = \aleph_1$ we may omit them. Lastly, let $\lambda_\alpha^l(< \kappa, \theta) = \min\{\lambda : \text{Pr}_\alpha^l(\lambda; < \kappa, \theta)\}$.

(2) For a model M ,

$$\text{rk}^l(M; < \kappa) = \sup\{\text{rk}^l(w, M; < \kappa) + 1 : w \subseteq |M| \text{ finite nonempty}\}$$

where rk^l is defined below in part (3).

(3) For a model M and $w \in [M]^* := \{u : u \subseteq |M| \text{ is finite nonempty}\}$, we shall define below the truth value of $\text{rk}^l(w, M; < \kappa) \geq \alpha$ by induction on the ordinal α (note that if $\text{cl}_{< \kappa}(w, M) = \text{cl}_2(w, M)$ for every $w \in [M]^*$ then for $l = 0, 1$, κ can be omitted). Then we can observe:

- (*)₀ $\alpha \leq \beta$ & $\text{rk}^l(w, M; < \kappa) \geq \beta \Rightarrow \text{rk}^l(w, M; < \kappa) \geq \alpha$,
 - (*)₁ $\text{rk}^l(w, M; < \kappa) \geq \delta$ (δ limit) iff $\bigwedge_{\alpha < \delta} \text{rk}^l(w, M; < \kappa) \geq \alpha$,
 - (*)₂ $\text{rk}^l(w, M; < \kappa) \geq 0$ iff $w \in [M]^*$ and no $a \in w$ is in $\text{cl}_{< \kappa}(w \setminus \{a\}, M)$.
- So we can define $\text{rk}^l(w, M; < \kappa) = \alpha$ as the maximal α such that $\text{rk}^l(w, M; < \kappa) \geq \alpha$, and ∞ if this holds for every α (and -1 whenever $\text{rk}^l(w, M; < \kappa) \not\geq 0$).

Now the inductive definition of $\text{rk}^l(w, M; < \kappa) \geq \alpha$ was already done above for $\alpha = 0$ (by (*)₂) and α limit (by (*)₁), so for $\alpha = \beta + 1$ we let

- (*)₃ $\text{rk}^l(w, M; < \kappa) \geq \beta + 1$ iff (letting $n = |w|$, $w = \{a_0, \dots, a_{n-1}\}$) for every $k < n$ and a quantifier free formula $\varphi(x_0, \dots, x_{n-1})$ (in the vocabulary of M) for which $M \models \varphi[a_0, \dots, a_{n-1}]$ we have:

CASE 1: $l = 1$. There are $a_m^i \in M$ for $m < n$, $i < 2$ such that:

- (a) $\text{rk}^l(\{a_m^i : i < 2, m < n\}, M; < \kappa) \geq \beta$,
- (b) $M \models \varphi[a_0^i, \dots, a_{n-1}^i]$ (for $i = 1, 2$), so we can assume there is no repetition in a_0^i, \dots, a_{n-1}^i ,
- (c) $a_k^0 \neq a_k^1$ but for $m \neq k$ (such that $m < n$) we have $a_m^0 = a_m^1$.

CASE 2: $l = 0$. As for $l = 1$ but in addition

- (d) $\bigwedge_m a_m = a_m^0$.

CASE 3: $l = 3$. We give to κ an additional role and the definition is like case 1 but $i < \kappa$; i.e. there are $a_m^i \in M$ for $m < n$, $i < \kappa$ such that:

- (a) for $i < j < \kappa$ we have $\text{rk}^l(\{a_m^i, a_m^j : m < n\}, M; < \kappa) \geq \beta$,
- (b) $M \models \varphi[a_0^i, \dots, a_{n-1}^i]$ (for $i < \kappa$; so we can assume there are no repetitions in a_0^i, \dots, a_{n-1}^i),
- (c) for $i < j < \kappa$, $a_k^i \neq a_k^j$ but for $m \neq k$ (such that $m < n$) we have $\bigwedge_{i, j < \kappa} a_m^i = a_m^j$.

CASE 4: $l = 2$. Like case 3 but in addition

- (d) $a_m = a_m^0$ for $m < n$.

CASE 5: $l = 5$. Like case 3 except that we replace clause (a) by

(a)⁻ for every function F with $\text{Dom}(F) = \kappa$ and $|\text{Rang}(F)| < \kappa$, for some $i < j < \kappa$ we have $F(i) = F(j)$ and

$$\text{rk}^l(\{a_m^i, a_m^j : m < n\}, M; < \kappa) \geq \beta.$$

CASE 6: $l = 4$. Like case 4 (i.e. $l = 2$) with (a)⁻ instead of (a).

We will actually use the above definition for $l = 0$ mainly. As the cardinal $\lambda_\alpha^l(< \aleph_1, \aleph_0) = \lambda_\alpha^l$ (for $l < 2$) may increase when the universe of set theory is extended (new models may be added) we will need some upper bounds which are preserved by suitable forcing. The case $l = 2$ provides one (and it is good: it does not increase when the universe is extended by a c.c.c. forcing). The case $l = 4$ shows how much we can strengthen the definition, to show for which forcing notions lower bounds for the rank for $l = 0$ are preserved. Odd cases show that variants of the definition are immaterial.

CLAIM 1.2. (1) *The truth of each of the statements $\text{Pr}_\alpha^l(\lambda; < \kappa, \theta)$, $\text{rk}^l(M; < \kappa) \geq \alpha$, $\text{rk}^l(w, M; < \kappa) \geq \alpha$ is preserved if we replace $l = 0, 2, 3, 2, 2, 2, 3, 5, 4$ by $l = 1, 3, 1, 0, 1, 4, 5, 1, 5$ respectively (i.e. $2 \rightarrow 4, 3 \rightarrow 5 \rightarrow 1, 0 \rightarrow 1, 2 \rightarrow 3, 4 \rightarrow 5, 3 \rightarrow 1, 2 \rightarrow 0, 2 \rightarrow 1$) and also if we decrease α, κ, θ or increase λ (the last two only when M is not a parameter). So the corresponding inequality on $\lambda_\alpha^l(< \kappa, \theta)$ holds.*

(2) *Also $\text{rk}^l(w_1, M; < \kappa) \geq \text{rk}^l(w_2, M; < \kappa)$ for $w_1 \subseteq w_2$ from $[M]^*$.*

(3) *Also if we expand M , the ranks (of $w \in [M]^*$, of M) can only decrease.*

(4) *If $A \subseteq M$ is defined by a quantifier free formula with parameters from a finite subset w^* of M , M^+ is M expanded by the relations defined by quantifier free formulas with parameters from w^* , $M^* = M^+ \upharpoonright A$ (for simplicity M^* has relations only) then for $w \in [A]^*$ such that $w \not\subseteq w^*$ we have $\text{rk}^l(w, M^*; < \kappa) \geq \text{rk}^l(w \cup w^*, M; < \kappa)$. Hence if $w^* = \emptyset$, then $\text{rk}^l(M^*; < \kappa) \geq \text{rk}^l(M; < \kappa)$*

(5) *In 1.1(3)(*)₂, if in the definition of $\text{cl}_{< \kappa}$ we allow any first order formula, this means just expanding M by relations for any first order formula $\varphi(\bar{x})$.*

(6) *For l odd, $\text{rk}^l(w, M; < \kappa) \geq (|\tau(M)| + \aleph_0)^+$ implies $\text{rk}^l(u, M; < \kappa) = \infty$.*

(7) *$\lambda_\alpha^l(< \kappa, \theta)$ increases (\leq) with α, θ and decreases with κ .*

(8) *There is no difference between $l = 4$ and $l = 5$.*

Proof. Check [e.g. for part (8), we can use a function F such that $(\forall \alpha < \kappa)(F(0) \neq F(1 + \alpha))$. ■_{1.2}

CLAIM 1.3. (1) For $l = 0$, if $\alpha = \text{rk}^l(M; < \kappa)$ ($< \infty$) then for some expansion M^+ of M by $\leq \aleph_0 + |\alpha|$ relations, for every $w \in [M]^*$ we have

$$\text{rk}^{l+1}(w, M^+; < \kappa) \leq \text{rk}^l(w, M; < \kappa).$$

(2) Similarly for $l = 2, 4$.

(3) If \mathbf{V}_0 is a transitive class of \mathbf{V}_1 (both models of ZFC) and $M \in \mathbf{V}_0$ is a model then:

(a) for $l < 4$:

(α) $[\text{rk}^l(w, M; < \kappa)]^{\mathbf{V}_0} \leq [\text{rk}^l(w, M; < \kappa)]^{\mathbf{V}_1}$ for $w \in [M]^*$,

(β) $[\text{rk}^l(M; < \kappa)]^{\mathbf{V}_0} \leq [\text{rk}^l(M; < \kappa)]^{\mathbf{V}_1}$,

(γ) if $l = 0, 1$ then equality holds in (α), (β),

(δ) $[\lambda_\alpha^l(\kappa)]^{\mathbf{V}_0} \leq [\lambda_\alpha^l(\kappa)]^{\mathbf{V}_1}$ if $l = 0, 1$.

(b) Assume:

(i) for every $f : \kappa \rightarrow \text{Ord}$ from \mathbf{V}_1 there is $A \in [\kappa]^\kappa$ such that $f \upharpoonright A \in \mathbf{V}_0$, or at least

(ii) every graph H on λ from \mathbf{V}_0 which in \mathbf{V}_1 has a complete subgraph of size κ , has such a subgraph in \mathbf{V}_0 , which holds if

(ii)⁺ $\mathbf{V}_1 = \mathbf{V}_0^{\mathbf{P}}$ where \mathbf{P} is a forcing notion satisfying the κ -Knaster Condition.

Then for $l = 2, 3$, in (α), (β) (of (a)) above equalities hold, and the inequality in (δ) holds.

(c) Assume $\mathbf{V}_1 = \mathbf{V}_0^{\mathbf{P}}$ where \mathbf{P} is κ -2-linked. Then for $l = 4, 5$, in clauses (α), (β) (of (a)) above we have equality, and the inequality in (δ) holds.

PROOF. (1) For $\beta < \alpha$, $n < \omega$, a quantifier free formula $\varphi = \varphi(x_0, \dots, \dots, x_{n-1})$ and $k < n$ let

$$R_\beta^n = \{\langle a_0, \dots, a_{n-1} \rangle : a_m \in M \text{ for } m < n \text{ and}$$

$$\beta = \text{rk}^l(\{a_0, \dots, a_{n-1}\}, M; < \kappa)\},$$

$$R_{\beta, \varphi}^{n, k} = \{\langle a_0, \dots, a_{n-1} \rangle \in R_\beta^n M \models \varphi[a_0, \dots, a_{n-1}] \text{ and}$$

for no $a_k^1 \in |M| \setminus \{a_0, \dots, a_{n-1}\}$ do we have

$$(\alpha) M \models \varphi[a_0, \dots, a_{k-1}, a_k^1, a_{k+1}, \dots, a_{n-1}],$$

$$(\beta) \text{rk}^l(\{a_m : m < n\} \cup \{a_k^1\}, M; < \kappa) \geq \beta\},$$

$$M^+ = (M, \dots, R_\beta^n, R_{\beta, \varphi}^{n, k}, \dots)_{\beta < \alpha, n < \omega, k < n, \varphi}.$$

Check (or see more details in the proof of 1.10 below).

(2) Similarly.

(3) The proof should be clear (for (b), look at Definition 1.1, case 3; the graph is $\{(i, j) : \text{clause (a) there holds}\}$). ■_{1.3}

REMARK 1.4. (1) In 1.3(1) we can omit “ $\alpha = \text{rk}^l(M; < \kappa)$ ” but then we must weaken the conclusion to

$$\text{rk}^{l+1}(w, M^+; < \kappa) \leq \text{rk}^l(w, M; < \kappa) \quad \text{or both are } > \alpha.$$

(2) Similarly in 1.3(2).

CONCLUSION 1.5. (1) $\text{Pr}_{\omega_1}^0(\lambda) \Leftrightarrow \text{Pr}_{\omega_1}^1(\lambda) \Leftarrow \text{Pr}_{\omega_1}^4(\lambda) \Leftrightarrow \text{Pr}_{\omega_1}^5(\lambda) \Leftarrow \text{Pr}_{\omega_1}^2(\lambda) \Leftrightarrow \text{Pr}_{\omega_1}^3(\lambda)$.

(2) If $\alpha \leq \kappa^+$ then $\text{Pr}_\alpha^0(\lambda; \kappa) \Leftrightarrow \text{Pr}_\alpha^1(\lambda; \kappa) \Leftarrow \text{Pr}_\alpha^4(\lambda; \kappa) \Leftrightarrow \text{Pr}_\alpha^5(\lambda; \kappa) \Leftarrow \text{Pr}_\alpha^2(\lambda; \kappa) \Leftrightarrow \text{Pr}_\alpha^3(\lambda; \kappa)$.

(3) For $\alpha \leq \kappa^+$, $\lambda_\alpha^l(\kappa) = \lambda_\alpha^{l+1}(\kappa)$ for $l = 0, 2, 4$, and $\lambda_\alpha^0(\kappa) \leq \lambda_\alpha^4(\kappa) \leq \lambda_\alpha^2(\kappa)$.

(4) For $\alpha \geq \kappa^+$ and $l = 0, 2, 4$ we have $\lambda_\alpha^{l+1}(\kappa) = \lambda_{\kappa^+}^{l+1}(\kappa)$.

PROOF. (1) By (2).

(2) For $\alpha = \kappa^+$ this follows from its holding for every $\alpha < \kappa^+$. For $\alpha < \kappa^+$ and $l = 0, 2, 4$ we know that $\text{NPr}_\alpha^l(\lambda; \kappa) \Rightarrow \text{NPr}_\alpha^{l+1}(\lambda; \kappa)$ by 1.3(1),(2), and $\text{Pr}_\alpha^l(\lambda; \kappa) \Rightarrow \text{Pr}_\alpha^{l+1}(\lambda; \kappa)$ by 1.2(1); together $\text{Pr}_\alpha^l(\lambda; \kappa) \Leftrightarrow \text{Pr}_\alpha^{l+1}(\lambda; \kappa)$. Now $\text{Pr}_\alpha^3(\lambda; \kappa) \Rightarrow \text{Pr}_\alpha^5(\lambda; \kappa) \Rightarrow \text{Pr}_\alpha^1(\lambda; \kappa)$ by 1.2(1); altogether we finish. (By 2.1 we know more.)

(3) Follows from part (2) and the definition.

(4) By 1.2(6). ■_{1.5}

CONVENTION 1.6. Writing $\text{Pr}_\alpha(\lambda; \kappa)$ for $\alpha \leq \kappa^+$ (omitting l) we mean $l = 0$. Similarly $\lambda_\alpha(< \kappa, \theta)$ and so $\lambda_\alpha(\kappa)$ etc.

CLAIM 1.7. Let $l \in \{0, 2, 4\}$.

(1) $\text{NPr}_{\alpha+1}^l(\kappa^{+\alpha}; \kappa)$.

(2) If α is a limit ordinal $< \kappa^+$ (in fact, $\aleph_0 \leq \text{cf}(\alpha) < \kappa^+$ suffice) and $\text{NPr}_\beta(\lambda_\beta; \kappa)$ for $\beta < \alpha$ then $\text{NPr}_{\alpha+1}^l(\sum_{\beta < \alpha} \lambda_\beta; \kappa)$.

(3) If $\text{NPr}_\alpha^l(\lambda; \kappa)$ then $\text{NPr}_{\alpha+1}^l(\lambda^+; \kappa)$.

(4) If $\text{NPr}_\alpha^l(\mu; \kappa)$ for every $\mu < \lambda$ then $\text{NPr}_{\alpha+1}^l(\lambda; \kappa)$.

PROOF. (1) Prove by induction on $\alpha < \kappa^+$, for $\alpha = 0$ use a model in which every element is definable (e.g. an individual constant) so $\text{rk}(w; M) = -1$ for $w \in [M]^*$ and hence $\text{rk}^l(M) = 0$ and consequently $\text{NPr}_1^l(\kappa; \kappa)$; for α limit use part (2) and for α successor use part (3).

(2) Let M_β witness $\text{NPr}_\beta(\lambda_\beta; \kappa)$ for $\beta < \alpha$, i.e. $\text{rk}^l(M_\beta; \kappa) < \beta$ and M_β has universe λ_β and $|\tau(M_\beta)| \leq \kappa$. Without loss of generality $\langle \tau(M_\beta) : \beta < \alpha \rangle$ are pairwise disjoint and disjoint from $\{P_\beta : \beta < \alpha\}$. Let M have universe $\lambda := \sum_{\beta < \alpha} \lambda_\beta$, $P_\beta^M = \lambda_\beta$, and $M \upharpoonright \lambda_\beta$ expand M_β and $|\tau(M)| \leq |\alpha| + \sum_{\beta < \alpha} |\tau(M_\beta)| \leq \kappa$. By 1.2(3),(4), for $w \in [\lambda_\beta]^*$, $\text{rk}^l(w, M; \kappa) \leq$

$\text{rk}^l(w, M_\beta; \kappa) < \beta \leq \alpha$. But $w \in [|M|]^*$ implies $\bigvee_{\beta < \alpha} w \in [\lambda_\beta]^*$. Clearly $\text{rk}(M; \kappa) \leq \alpha$ and hence $\text{NPr}_{\alpha+1}^l(\lambda; \kappa)$.

(3) We define M^+ such that each $\gamma \in [\lambda, \lambda^+)$ codes on $\{\zeta : \zeta < \gamma\}$ an example for $\text{NPr}_\alpha(|\gamma|; \kappa)$. More elaborately, let M be a model with universe λ such that $\text{rk}^l(M; \kappa) < \alpha$. Let $\tau(M)$ be $\{R_i : i < i^* \leq \kappa\}$, R_i an $n(i)$ -place predicate (as we replace function symbols and individual constants by predicates), R_0 is a 0-nary predicate representing “the truth”. For $\gamma \in [\lambda, \lambda^+)$ let f_γ be a one-to-one function from γ onto λ . Define $\tau^+ = \{R_i, Q_i : i < i^* \leq \kappa\}$ where R_i is $n(i)$ -place and Q_i is $(n(i)+1)$ -place. So $|\tau^+| \leq \kappa$. We define a τ^+ -model M^+ : the universe is λ^+ , $R_i^{M^+} = R_i^M$, $Q_i^{M^+} = \{\langle \alpha_0, \dots, \alpha_{n(i)} \rangle : \alpha_{n(i)} \in [\lambda, \lambda^+), \bigwedge_{l < n(i)} \alpha_l < \alpha_{n(i)}\}$ and $\langle f_{\alpha_{n(i)}}(\alpha_0), \dots, f_{\alpha_{n(i)}}(\alpha_{n(i)-1}) \rangle \in R_i^M$ (so $Q_0^{M^+} = [\lambda, \lambda^+)$). Now note that:

- (a) for $w \in [\lambda]^*$, $\text{rk}^l(w, M^+; \kappa) \leq \text{rk}^l(w, M; \kappa)$,
- (b) if $\emptyset \neq w \subseteq \gamma \in [\lambda, \lambda^+)$ then $\text{rk}^l(w \cup \{\gamma\}, M^+; \kappa) \leq \text{rk}^l(f_\gamma''[w], M; \kappa)$

(easy to check). So if $\gamma < \lambda^+$ then

- (*)₁ $\gamma < \lambda \Rightarrow \text{rk}^l(\{\gamma\}, M^+; \kappa) \leq \text{rk}^l(\{\gamma\}, M; \kappa) < \text{rk}^l(M; \kappa)$,
- (*)₂ $\gamma \in [\lambda, \lambda^+) \& \beta \geq \text{rk}^l(M; \kappa) \Rightarrow \text{rk}^l(\{\gamma\}, M^+; \kappa) \leq \beta$.

(Why (*)₂? Assume not and let $\kappa^0 = 2$, $\kappa^2 = \kappa^4 = \kappa^+$. If $\langle \gamma_i : i < \kappa^l \rangle$ strictly increasing witnesses $\text{rk}^l(\{\gamma\}, M^+) \geq \beta + 1$ for the formula $Q_0(x)$ then for some $i < j < \kappa^l$ we have $\text{rk}^l(\{\gamma_i, \gamma_j\}, M^+) \geq \beta$ and applying (b) with $\{\gamma_i\}$, γ_j here standing for w , γ there we get $\text{rk}^l(\{f_{\gamma_j}(\gamma_i)\}, M) \geq \beta$, hence $\beta + 1 \leq \text{rk}^l(M)$, a contradiction.)

Hence

- (*)₃ $\text{rk}^l(M^+; \kappa) \leq \text{rk}^l(M; \kappa) + 1$.

As $\text{rk}^l(M^+; \kappa) < \alpha$, clearly M^+ witnesses $\text{NPr}_{\alpha+1}(\lambda^+; \kappa)$.

(4) Like (3). ■_{1.7}

CONCLUSION 1.8. *Remembering $\lambda_\alpha(\kappa) = \min\{\lambda : \text{Pr}_\alpha(\lambda; \kappa)\}$ we have:*

- (1) for α a limit ordinal, $\lambda_\alpha(\kappa) \leq \beth_\alpha(\kappa)$ and even $\lambda_\alpha^2(\kappa) \leq \beth_\alpha(\kappa)$,
- (2) for l even, $\langle \lambda_\alpha^l(\kappa) : 0 < \alpha < \infty \rangle$ is strictly increasing, and for a limit ordinal δ , $\lambda_\delta(\kappa) = \sup_{\alpha < \delta} \lambda_\alpha(\kappa)$,
- (3) $\lambda_0(\kappa) = \lambda_1(\kappa) = \kappa$, $\lambda_2(\kappa) = \kappa^+$, $\kappa^{+n} \leq \lambda_n(\kappa) < \kappa^{+\omega}$ and $\lambda_\omega(\kappa) = \kappa^{+\omega}$.

REMARK 1.9. $\lambda_{\omega \times \alpha}^2(\kappa) \leq \beth_{\omega \times \alpha}(\kappa)$ is proved below essentially like the Morley omitting types theorem (see [Mo], [CK] or [Sh a, VII, §5] = [Sh c, VII, §5]).

PROOF (of Conclusion 1.8). (1) We prove by induction on α that for every ordinal $\beta < \alpha$, model M with $|\tau(M)| \leq \kappa$, and $A \subseteq |M|$ with $|A| \geq \beth_{\omega \times \alpha}(\kappa)$, and $m, n < \omega$, there is $w \subseteq A$ with $|w| = n$ such that $\text{rk}^2(w, M; \kappa) \geq \omega \times \beta + m$.

For $\alpha = 0$ and α limit this is immediate. For $\alpha = \gamma + 1$ (and M, A, β, n, m as above), applying the Erdős–Rado theorem we can find distinct $a_i \in A$ for $i < \beth_{\omega \times \gamma}(\kappa)^{++}$ such that:

(a) for all $i_0 < \dots < i_{m+n}$ the quantifier free type $\langle a_{i_0}, \dots, a_{i_{m+n}} \rangle$ in M is the same,

(b) for each $k \leq m + n$ and every $i_0 < \dots < i_{n+m-k} < \beth_{\omega \times \gamma}(\kappa)$, the ordinal $\min\{\omega \times \alpha, \text{rk}^2(\{a_{i_0}, \dots, a_{i_{n+m-k}}\}, M; \kappa)\}$ is the same.

By the induction hypothesis, in clause (b) the value is $\geq \omega \times \gamma$. Hence we can prove, by induction on $k \leq m + n$, that $\text{rk}^2(\{a_{i_0}, \dots, a_{i_{n+m-k}}\}, M; \kappa) \geq \omega \times \gamma + k$ whenever $i_0 < \dots < i_{n+m-k} < \beth_{\omega \times \gamma}(\kappa)$. For $k = 0$ this holds by the previous sentence, for $k + 1$ use the definition and the induction hypothesis, for rk^2 note that by clause (b) without loss of generality $i_l + \kappa^+ < i_{l+1}$ and $a_{i_l + \zeta}$ for $\zeta < \kappa^+$ are well defined. For $k = m$ we are done.

(2) The sequence is increasing by 1.2(1), strict by 1.7(4), continuous because, for limit δ , as on the one hand $\lambda_\delta^l(\kappa) \geq \sup_{\alpha < \delta} \lambda_\alpha^l(\kappa)$ since $\lambda_\delta^l(\kappa) \geq \lambda_\alpha^l(\kappa)$ for $\alpha < \delta$, and on the other hand if M is a model with universe $\lambda := \sup_{\alpha < \delta} \lambda_\alpha(\kappa)$ and $|\tau(M)| \leq \kappa$ then $\alpha < \delta \Rightarrow \text{rk}^l(M; \kappa) \geq \text{rk}^l(M \upharpoonright \lambda_\alpha; \kappa) \geq \alpha$ hence $\text{rk}^l(M; \kappa) \geq \delta$. So $\text{Pr}_\alpha(\lambda; \kappa)$ hence $\lambda \geq \lambda_\delta^l(\kappa)$ so $\sup_{\alpha < \delta} \lambda_\alpha^l(\kappa) = \lambda \geq \lambda_\delta^l$; altogether, we are done.

(3) By [Sh 49] (for the last two clauses; the first two are trivial), will not be really used here. ■_{1.8}

CLAIM 1.10. (1) Assume \mathbf{P} is a forcing notion satisfying the κ^+ -c.c. If $\text{Pr}_\alpha^3(\lambda; \kappa)$ and $\alpha \leq \kappa^+$, then this also holds in $\mathbf{V}^{\mathbf{P}}$.

(2) If \mathbf{P} is a κ^+ -2-linked forcing notion (or just: if $p_i \in \mathbf{P}$ for $i < \kappa^+$ then for some $F : \kappa^+ \rightarrow \kappa$, $F(i) = F(j) \Rightarrow p_i, p_j$ compatible), and $\alpha \leq \kappa^+$ and $\text{Pr}_\alpha^5(\lambda; \kappa)$, then this holds in $\mathbf{V}^{\mathbf{P}}$.

REMARK 1.11. (1) $\text{NPr}_\alpha(\lambda; \kappa)$ is of course preserved by any extension as the ranks $\text{rk}^l(M; \kappa)$, $\text{rk}^l(w, M; \kappa)$ are absolute for $l = 0, 1$ (see 1.3(3)). But the forcing can add new models.

(2) So for $\alpha \leq \kappa^+$ we have $\lambda_\alpha(\kappa) \leq \lambda_\alpha^4(\kappa) \leq \lambda_\alpha^2(\kappa)$, and a κ^+ -c.c. forcing notion can only increase the first (by 1.3(3)(a)(δ)) and decrease the third by 1.10(1); a κ^+ -2-linked one fixes the second and third (as it can only decrease it by 1.10(1) and can only increase it by 1.3(3)(c)).

(3) We can deal similarly with $\text{Pr}_\alpha^l(\lambda; < \kappa, \theta)$, here and in 1.3–1.8.

PROOF (of Claim 1.10). We can concentrate on (1); anyhow let $l = \{3, 5\}$ (for part (1) we use $l = 3$, for part (2) we shall use $l = 5$, we shall return to

it later). Assume $\text{Pr}_\alpha^3(\lambda; \kappa)$ fails in $\mathbf{V}^{\mathbf{P}}$. So for some $p^* \in \mathbf{P}$ and $\alpha_0 < \alpha$ we have

$$p^* \Vdash_{\mathbf{P}} \text{“}\underline{M} \text{ is a model with universe } \lambda, \text{ vocabulary } \mathcal{T} \text{ of cardinality } \leq \kappa \\ \text{and } \text{rk}^l(\underline{M}; \kappa) = \alpha_0\text{”}.$$

Without loss of generality, every quantifier free formula $\varphi(x_0, \dots, x_{n-1})$ is equivalent to one of the form $R(x_0, \dots, x_{n-1})$, and $\mathcal{T} = \{R_{n,\zeta} : n < \omega, \zeta < \kappa\}$ with $R_{n,\zeta}$ an n -place predicate. Note that necessarily $\alpha_0 < \kappa^+$, hence $|\alpha_0| \leq \kappa$.

As we can replace \mathbf{P} by $\mathbf{P} \upharpoonright \{q \in \mathbf{P} : p^* \leq q\}$, we can assume p^* is the minimal member of \mathbf{P} . Now for nonzero $n < \omega$, $k < n$, $\zeta < \kappa$ and $\beta < \alpha_0$ (or $\beta = -1$) we define an n -place relation $R_{n,\zeta,\beta,k}$ on λ by

$$R_{n,\zeta,\beta,k} = \{ \langle a_0, \dots, a_{n-1} \rangle : a_m \in \lambda \text{ with no repetitions and for some } p \in \mathbf{P}, \\ p \Vdash_{\mathbf{P}} \text{“}\underline{M} \models R_{n,\zeta}[a_0, \dots, a_{n-1}] \ \& \ \text{rk}^l(\{a_0, \dots, a_{n-1}\}, \underline{M}; \kappa) = \beta, \\ \text{where “not } \text{rk}^3(\{a_0, \dots, a_{n-1}\}, M; \kappa) \geq \beta + 1\text{” is witnessed by} \\ \varphi = R_{n,\zeta} \text{ and } k\text{”} \}.$$

Let $M^+ = (\lambda, \dots, R_{n,\zeta,\beta,k}, \dots)_{n < \omega, \zeta < \kappa, \beta < \alpha_0, k < n}$, so M^+ is a model in \mathbf{V} with universe λ and vocabulary of cardinality $\leq \kappa$. It suffices to prove that for $\beta < \alpha_0$,

$$\otimes_\beta \quad \text{if } w = \{a_0, \dots, a_{n-1}\} \in [M^+]^*, \ M^+ \models R_{n,\zeta,\beta,k}[a_0, \dots, a_{n-1}] \\ \text{then } \text{rk}^l(\{a_0, \dots, a_{n-1}\}, M^+; \kappa) \leq \beta$$

(note that by the choice of \underline{M} and $R_{n,\zeta,\beta,k}$, if $w \in [M^+]^*$ then for some n, ζ, β, k we have $M^+ \models R_{n,\zeta,\beta,k}[a_0, \dots, a_{n-1}]$).

This we prove by induction on β , so assume the conclusion fails, that is,

$$\text{rk}^l(\{a_0, \dots, a_{n-1}\}, M^+; \kappa) \geq \beta + 1$$

(and eventually we shall get a contradiction). By the definition of rk^3 applied to $\varphi = R_{n,\zeta,\beta,k}$, β and k we know that there are a_m^i (for $m < n$, $i < \kappa^+$) as in Definition 1.1(3), case $l = 3$. In particular, $M^+ \models R_{n,\zeta,\beta,k}[a_0^i, \dots, a_{n-1}^i]$. So for each $i < \kappa^+$ by the definition of $R_{n,\zeta,\beta,k}$ necessarily there is $p_i \in \mathbf{P}$ such that

$$p_i \Vdash_{\mathbf{P}} \text{“}\underline{M} \models R_{n,\zeta}[a_0^i, \dots, a_{n-1}^i], \ \text{rk}^l(\{a_0^i, \dots, a_{n-1}^i\}, \underline{M}; \kappa) = \beta \\ \text{and } \text{rk}^l(\{a_0^i, \dots, a_{n-1}^i\}, \underline{M}; \kappa) \not\geq \beta + 1 \text{ is witnessed by} \\ \varphi = R_{n,\zeta} \text{ and } k\text{”}.$$

For part (1), as \mathbf{P} satisfies the κ^+ -c.c., for some $q \in \mathbf{P}$, $q \Vdash \text{“}\underline{Y} = \{i : p_i \in \mathcal{G}_{\mathbf{P}}\}$ has cardinality $\kappa^+\text{”}$ (in fact, p_i forces it for every large enough i). Looking at the definition of the rank in $\mathbf{V}^{\mathbf{P}}$ we see that $\langle \langle a_0^i, \dots, a_{n-1}^i \rangle : i \in \underline{Y} \rangle$ cannot be a witness for “the demand for $\text{rk}^3(\{a_0^{i_0}, \dots, a_{n-1}^{i_0}\}, \underline{M}; \kappa) > \beta$

for $R_{n,\zeta,k}$ holds” for any (or some) $i_0 \in \underline{Y}$, so for part (1),

$$(*) \quad q \Vdash_{\mathbf{P}} \text{“rk}^3(\{a_0^i, \dots, a_{n-1}^i, a_k^j\}, \underline{M}; \kappa) < \beta \text{ for some } i \neq j \text{ in } \underline{Y}\text{”}$$

(as the demand on equalities holds trivially).

As we can increase q , we can assume that q forces a value to those i, j , hence without loss of generality for some $n(*) = n + 1 < \omega$, $\zeta(*) < \kappa$ and $\beta(*) < \beta$ and for $k(*) < n + 1$ we have

$$\begin{aligned} q \Vdash_{\mathbf{P}} \text{“rk}^3(\{a_0^i, \dots, a_{n-1}^i, a_k^j\}, \underline{M}; \kappa) = \beta(*), \text{ and} \\ \text{rk}^3(\{a_0^i, \dots, a_{n-1}^i, a_k^j\}, \underline{M}; \kappa) \not\leq \beta(*) + 1 \text{ is witnessed by} \\ \varphi = R_{n(*),\zeta(*)}(x_0, \dots, x_n) \text{ and } k(*)\text{”}. \end{aligned}$$

Hence by the definition of $R_{n(*),\zeta(*),\beta(*),k(*)}$ we have

$$M^+ \Vdash R_{n(*),\zeta(*),\beta(*),k(*)}[a_0^i, \dots, a_{n-1}^i, a_k^j].$$

As $\beta(*) < \beta$, by the induction hypothesis $\otimes_{\beta(*)}$ holds and hence

$$\text{rk}^3(\{a_0^i, \dots, a_{n-1}^i, a_k^j\}, M^+; \kappa) \leq \beta(*);$$

but this contradicts the choice of a_m^i ($m < n, i < \kappa^+$) above (i.e. clause (a) of Definition 1.1(3), case $l = 3$). This contradiction finishes the induction step in the proof of \otimes_{β} , hence the proof of 1.10(1).

For part (2), we have $\langle p_i : i < \kappa^+ \rangle$ as above. In $\mathbf{V}^{\mathbf{P}}$, if $\underline{Y} = \{i : p_i \in G_{\mathbf{P}}\}$ has cardinality κ^+ , then $\langle \langle a_0^i, \dots, a_{n-1}^i \rangle : i \in \underline{Y} \rangle$ cannot witness $\text{rk}^5(\{a_0, \dots, a_{n-1}\}, M; \kappa) \geq \beta + 1$ so there is a function $\underline{F}^0 : \underline{Y} \rightarrow \kappa$ witnessing it; i.e.

$\Vdash_{\mathbf{P}}$ “if $|\underline{Y}| = \kappa^+$ then

$$\begin{aligned} i \in \underline{Y} \ \& \ j \in \underline{Y} \ \& \ i \neq j \ \& \ \underline{F}^0(i) = \underline{F}^0(j) \Rightarrow \\ \beta & > \text{rk}^5(\{a_0^i, \dots, a_{n-1}^i\} \cup \{a_0^j, \dots, a_{n-1}^j\}, M; \kappa)\text{”}. \end{aligned}$$

If $|\underline{Y}| \leq \kappa$, let $\underline{F}^0 : \underline{Y} \rightarrow \kappa$ be one-to-one. Let $p_i \leq q_i \in \mathbf{P}$, $q_i \Vdash \underline{F}^0(i) = \gamma_i$. As \mathbf{P} is κ^+ -2-linked, for some function $F^1 : \kappa^+ \rightarrow \kappa$ we have $(\forall i, j < \kappa^+)$ ($F^1(i) = F^1(j) \Rightarrow q_i, q_j$ are compatible in \mathbf{P}). We now define a function F from \underline{Y} to κ by $F(i) = \text{pr}(\gamma_i, F^1(i))$ (you can use any pairing function pr on κ). So if $i < j < \kappa^+$ and $F(i) = F(j)$ then there is $q_{i,j}$ such that $\mathbf{P} \Vdash$ “ $q_i \leq q_{i,j} \ \& \ q_j \leq q_{i,j}$ ”, hence $q_{i,j} \Vdash_{\mathbf{P}}$ “ $\text{rk}^5(\{a_0^i, \dots, a_{n-1}^i, a_k^j\}, \underline{M}; \kappa) < \beta$ ”, so possibly increasing $q_{i,j}$, for some $\beta_{i,j} < \beta$, $\zeta_{i,j} < \kappa$, and $k_{i,j} < n$ we have $q_{i,j} \Vdash$ “ $\text{rk}^5(\{a_0^i, \dots, a_{n-1}^i, a_k^j\}, \underline{M}; \kappa) = \beta_{i,j}$, and $\text{rk}^5(\{a_0^i, \dots, a_{n-1}^i, a_k^j\}) \not\leq \beta_{i,j} + 1$ is witnessed by $\varphi = R_{n+1,\zeta_{i,j}}(x_0, \dots, x_n)$ and $k_{i,j}$ ”.

Hence by the definition of $R_{n+1,\zeta_{i,j},\beta_{i,j},k_{i,j}}$ we have

$$M^+ \Vdash R_{n+1,\zeta_{i,j},\beta_{i,j},k_{i,j}}[a_0^i, \dots, a_{n-1}^i, a_k^j];$$

but $\beta_{i,j} < \beta$ and hence by the induction hypothesis

$$\text{rk}^5(\{a_0^i, \dots, a_{n-1}^j, a_k^j\}, M^+; \kappa) \leq \beta_{i,j}.$$

So F contradicts the choice of $\langle \langle a_0^i, \dots, a_{n-1}^i \rangle : i < \kappa^+ \rangle$, i.e. clause (a)⁻ of Definition 1.1, case 5. ■_{1.10}

CLAIM 1.12. *Let $B \subseteq \omega_2 \times \omega_2$ be a Borel or even analytic set and $\text{Pr}_{\omega_1}(\lambda)$.*

(1) *If B contains a λ -square then B contains a perfect square.*

(2) *If B contains a (λ, λ) -rectangle then B contains a perfect rectangle.*

(3) *We can replace analytic by κ -Suslin if $\text{Pr}_{\kappa^+}(\lambda; \kappa)$. (This applies to Σ_2^1 sets which are \aleph_1 -Suslin.)*

PROOF. You can apply the results of Section 2 to prove 1.12; specifically, 2.1(1) \Rightarrow (2) proves parts (1), (2) and 2.5(1) proves part (3); those results of §2 say more, hence their proof should be clearer.

However, we give a proof of part (1) here for the reader who is going to read this section only. Suppose that $B \subseteq \omega_2 \times \omega_2$ is a Borel or even analytic set containing a λ -square. Let T be a $(2, 2, \omega)$ -tree such that

$$B = \{(\eta_0, \eta_1) \in \omega_2 \times \omega_2 : (\exists \varrho \in {}^\omega \omega)[(\eta_0, \eta_1, \varrho) \in \text{lim}(T)]\},$$

and let $\{\eta_\alpha : \alpha < \lambda\} \subseteq \omega_2$ be such that the square determined by it is contained in B and $\alpha < \beta < \lambda \Rightarrow \eta_\alpha \neq \eta_\beta$. For $\alpha, \beta < \lambda$ let $F(\alpha, \beta) \in {}^\omega \omega$ be such that $(\eta_\alpha, \eta_\beta, F(\alpha, \beta)) \in \text{lim}(T)$. Define a model M with universe λ and vocabulary $\tau = \{R_{\nu_0, \nu_1, \nu}, Q_{\nu_0, \nu} : \nu_0, \nu_1 \in \omega^{>2} \text{ and } \nu \in \omega^{>\omega}\}$ where each $R_{\nu_0, \nu_1, \nu}$ is a binary predicate, $Q_{\nu_0, \nu}$ is a unary predicate and

$$Q_{\nu_0, \nu}^M = \{\alpha < \lambda : \nu_0 \triangleleft \eta_\alpha \ \& \ \nu \triangleleft F(\alpha, \alpha)\},$$

$$R_{\nu_0, \nu_1, \nu}^M = \{(\alpha, \beta) \in \lambda \times \lambda : \nu_0 \triangleleft \eta_\alpha \ \& \ \nu_1 \triangleleft \eta_\beta \ \& \ \nu \triangleleft F(\alpha, \beta)\}.$$

By $\text{Pr}_{\omega_1}(\lambda)$ we know that $\text{rk}^0(M) \geq \omega_1$.

A pair (u, h) is called an n -approximation if $u \subseteq {}^n 2$, $h : u \times u \rightarrow {}^n \omega$ and for every $\gamma < \omega_1$ there is $w \in [\lambda]^*$ such that:

$$(\oplus_1) \quad u = \{\eta_\alpha \upharpoonright n : \alpha \in w\} \text{ and } \eta_\alpha \upharpoonright n \neq \eta_\beta \upharpoonright n \text{ for distinct } \alpha, \beta \in w,$$

$$(\oplus_2) \quad \text{rk}^0(w, M) \geq \gamma,$$

$$(\oplus_3) \quad F(\alpha, \beta) \upharpoonright n = h(\eta_\alpha \upharpoonright n, \eta_\beta \upharpoonright n) \text{ for } \alpha, \beta \in w; \text{ hence}$$

$$M \models R_{\eta_\alpha \upharpoonright n, \eta_\beta \upharpoonright n, h(\eta_\alpha \upharpoonright n, \eta_\beta \upharpoonright n)}[\alpha, \beta]$$

for $\alpha, \beta \in w$.

Note that $(\{\langle \rangle\}, \{(\langle \rangle, \langle \rangle), \langle \rangle\})$ is a 0-approximation. Moreover,

(*)₀ if (u, h) is an n -approximation and $\nu^* \in u$ then there are $m > n$ and an m -approximation (u^+, h^+) such that:

$$(i) \quad \nu \in u \setminus \{\nu^*\} \Rightarrow (\exists! \nu^+)(\nu \triangleleft \nu^+ \in u^+),$$

- (ii) $(\exists^{!2}\nu^+)(\nu^* \triangleleft \nu^+ \in u^+)$ (where $\exists^{!2}x$ means “there are exactly 2 x ’s”)
- (iii) $\nu \in u^+ \Rightarrow \nu \upharpoonright n \in u$ and
- (iv) if $\nu_1, \nu_2 \in u^+$ then

$$[h(\nu_1 \upharpoonright n, \nu_2 \upharpoonright n) \triangleleft h^+(\nu_1, \nu_2) \text{ or } (\nu_1 \upharpoonright n = \nu_2 \upharpoonright n = \nu^* \ \& \ \nu_1 \neq \nu_2)].$$

[Why? For each $\gamma < \omega_1$ choose w_γ satisfying (\oplus_1) , (\oplus_2) and (\oplus_3) for $\gamma + 1$, now apply the definition of rk^0 (if $w_\gamma = \{\alpha_l^\gamma : l < |w_\gamma|\}$, $\nu^* \triangleleft \eta_{\alpha_k^\gamma}$ and $k < |w_\gamma|$ we apply it to k) to get $w_\gamma^+ = w_\gamma \cup \{\alpha_\gamma\}$ satisfying (\oplus_1) , (\oplus_2) and (\oplus_3) for γ , then choose $m_\gamma \in (n, \omega)$ such that $\langle \eta_\alpha \upharpoonright m_\gamma : \alpha \in w_\gamma^+ \rangle$ is with no repetitions. Lastly, as there are only countably many possibilities for $\langle m_\gamma, \{\eta_\alpha \upharpoonright m_\gamma : \alpha \in w_\gamma^+\}, \{(\eta_\alpha \upharpoonright m_\gamma, \eta_\beta \upharpoonright m_\gamma, F(\alpha, \beta) \upharpoonright m_\gamma) : \alpha, \beta \in w_\gamma^+\} \rangle$ for $\gamma < \omega_1$, there is a value taken for uncountably many γ ; let γ^* be one of them. Choose $m = m_{\gamma^*}$, $u^+ = \{\eta_\alpha \upharpoonright m : \alpha \in w_{\gamma^*}^+\}$ and define h^+ to satisfy (\oplus_3) .]

Repeating the procedure of $(*)_0$ $|u|$ times we get

- $(*)_1$ if $u = \{\nu_l : l < k\} \subseteq {}^n 2$ (no repetition) and (u, h) is an n -approximation then there are m , $u^+ = \{\nu_l^+ : l < 2k\}$ and h^+ such that (u^+, h^+) is an m -approximation for some $m > n$ and:
 - (i) $\nu_l \triangleleft \nu_{2l}^+, \nu_l \triangleleft \nu_{2l+1}^+, \nu_{2l}^+ \neq \nu_{2l+1}^+$,
 - (ii) if $l < k$ and $i < 2$ then $h(\nu_l, \nu_l) \triangleleft h^+(\nu_{2l+i}^+, \nu_{2l+i}^+)$ and
 - (iii) if $l_1 \neq l_2$, $l_1, l_2 < k$, $i, j < 2$ then $h(\nu_{l_1}, \nu_{l_2}) \triangleleft h^+(\nu_{2l_1+i}^+, \nu_{2l_2+j}^+)$.

Consequently, we have

- $(*)_2$ there are sequences $\langle n_i : i < \omega \rangle \subseteq \omega$ and $\langle (u_i, h_i) : i \in \omega \rangle$ such that $n_i < n_{i+1}$, (u_i, h_i) is an n_i -approximation and (u_i, h_i) , (u_{i+1}, h_{i+1}) are like (u, h) , (u^+, h^+) of $(*)_1$.

Now, let $\langle n_i : i < \omega \rangle$ and $\langle (u_i, h_i) : i \in \omega \rangle$ be as in $(*)_2$. Define

$$\mathcal{P} = \{\eta \in {}^\omega 2 : (\forall i \in \omega)(\eta \upharpoonright n_i \in u_i)\}.$$

By $(*)_1$ for (u_{i+1}, h_{i+1}) we know that \mathcal{P} is a perfect set. We claim that $\mathcal{P} \times \mathcal{P} \subseteq B$. Suppose that $\eta', \eta'' \in \mathcal{P}$ and $\eta' \neq \eta''$. Then $\eta' \upharpoonright n_{i(*)} \neq \eta'' \upharpoonright n_{i(*)}$ for some $i(*) < \omega$ and the sequence $\langle h_i(\eta' \upharpoonright n_i, \eta'' \upharpoonright n_i) : i(*) \leq i < \omega \rangle$ is \triangleleft -increasing and (as (u_i, h_i) are approximations) $(\eta' \upharpoonright n_i, \eta'' \upharpoonright n_i, h_i(\eta' \upharpoonright n_i, \eta'' \upharpoonright n_i)) \in T$ is increasing for $i \in [i(*), \omega)$. The case $\eta' = \eta'' \in \mathcal{P}$ is easier. The claim is proved. $\blacksquare_{1.12}$

THEOREM 1.13. *Assume $\text{NPr}_{\omega_1}(\lambda)$ and $\lambda \leq \mu = \mu^{\aleph_0}$. Then for some c.c.c. forcing notion \mathbf{P} , $|\mathbf{P}| = \mu$ and $\Vdash_{\mathbf{P}} “2^{\aleph_0} = \mu”$ and in $\mathbf{V}^{\mathbf{P}}$ we have*

- $(*)$ there is a Borel set $B \subseteq {}^\omega 2 \times {}^\omega 2$ such that:
 - (a) it contains a λ -square, i.e. there are pairwise distinct $\eta_\alpha \in {}^\omega 2$ for $\alpha < \lambda$ such that $(\eta_\alpha, \eta_\beta) \in B$ for $\alpha, \beta < \lambda$,

- (b) let $\mathbf{V} \models \lambda^{\aleph_0} = \lambda_1$; B contains no λ_1^+ -square, i.e. there are no $\eta_\alpha \in {}^\omega 2$ (for $\alpha < \lambda_1^+$) such that $[\alpha \neq \beta \Rightarrow \eta_\alpha \neq \eta_\beta]$ and $(\eta_\alpha, \eta_\beta) \in B$ for $\alpha, \beta < \lambda^+$,
- (c) B contains no perfect square.

Actually, B is a countable union of closed sets.

PROOF. STAGE A. Clearly for some $\alpha(*) < \omega_1$ we have $\text{NPr}_{\alpha(*)}^1(\lambda)$. Let M be a model with universe λ and a countable vocabulary such that $\text{rk}^1(M) < \alpha(*)$, say with $<^M$ the usual order. Let functions φ^M, k^M with domains

$$[\lambda]^* = \{u : u \subseteq \lambda, u \text{ is finite and } u \neq \emptyset\}$$

be such that if $u = \{\alpha_0, \dots, \alpha_{n-1}\} \in [\lambda]^*$ is increasing for definiteness and $\beta = \text{rk}^0(u, M) (< \alpha(*)$) then $\varphi^M(u)$ is a quantifier free formula in the vocabulary of M in the variables x_0, \dots, x_{n-1} for simplicity saying $x_0 < x_1 < \dots < x_{n-1}$, $k^M(u)$ is a natural number $< n = |u|$ such that $\varphi^M(u), k^M(u)$ witness $\text{rk}^1(u, M) \not\leq \beta + 1$ (the same definition makes sense even if $\beta = -1$). In particular,

$$M \models \varphi^M(u)[\dots, a, \dots]_{a \in u}.$$

We define the forcing notion \mathbf{P} . We can put the diagonal $\{(\eta, \eta) : \eta \in {}^\omega 2\}$ into B so we can ignore it. We want to produce (in $\mathbf{V}^{\mathbf{P}}$) a Borel set $B = \bigcup_{n < \omega} B_n$, with each $B_n (\subseteq {}^\omega 2 \times {}^\omega 2)$ closed (in fact perfect), so B_n is $\lim(T_n)$ for some $(2, 2)$ -tree T_n , B_0 is the diagonal, and $\bar{\eta} = \langle \eta_\alpha : \alpha < \mu \rangle$ as witnesses to $2^{\aleph_0} \geq \mu$ and such that $\{\eta_\alpha : \alpha < \lambda\}$ gives the desired square. So for some 2-place function g from λ to ω , $\alpha \neq \beta \Rightarrow (\eta_\alpha, \eta_\beta) \in \lim(T_{g(\alpha, \beta)})$, all this after we force. But we know that we shall have to use M (by 1.12). In the forcing our problem will be to prove the c.c.c., which will be resolved by using M (and rank) in the definition of the forcing. We shall have a function f which puts the information on the rank into the trees to help in not having a perfect square. Specifically, the domain of f is a subset of

$$\{(u, h) : (\exists l \in \omega)(u \in [{}^l 2]^*) \text{ and } h : u \times u \rightarrow \omega\}$$

(the functions h above are thought of as indexing the B_n 's). The function f will be such that for any distinct $\alpha_0, \dots, \alpha_{n-1} < \lambda$, if $\langle \eta_{\alpha_t} \upharpoonright l : t < n \rangle$ are pairwise distinct, $u = \{\eta_{\alpha_t} \upharpoonright l : t < n\}$, $h(\eta_{\alpha_t} \upharpoonright l, \eta_{\alpha_s} \upharpoonright l) = g(\alpha_t, \alpha_s)$ and $(u, h) \in \text{Dom}(f)$ then $\text{rk}^1(\{\alpha_l : l < n\}, M) = f_0(u, h)$, and $f_1(u, h)$ is $\eta_{\alpha_k} \upharpoonright l$, where $k = k^M(\{\alpha_t : t < n\})$, $f_2(u, h) = \varphi^M(\{\alpha_l : l < n\})$ writing the variable as $x_\nu, \nu \in u$, and $f(u, h) = (f_0(u, h), f_1(u, h), f_2(u, h))$. (Note that f is a way to say $\bigcup_n \lim(T_n)$ contains no perfect square; essentially it is equivalent to fixing an appropriate rank.) All this was to motivate the definition of the forcing notion \mathbf{P} .

A condition p (of \mathbf{P}) is an approximation to all this; it consists of:

- (1) $u^p = u[p]$, a finite subset of μ .
- (2) $n^p = n[p] < \omega$ and $\eta_\alpha^p = \eta_\alpha[p] \in {}^{n[p]}2$ for $\alpha \in u[p]$ such that $\alpha \neq \beta \Rightarrow \eta_\alpha^p \neq \eta_\beta^p$.
- (3) $\bar{m}^p = \langle m_l^p : l \leq n^p \rangle$ is a strictly increasing sequence of natural numbers with last element $m_{n^p}^p = m^p = m[p]$ and for $m < m[p]$, we have $t_m^p = t_m[p] \subseteq \bigcup_{l \leq n[p]} ({}^l 2 \times {}^l 2)$ which is downward closed (i.e., $(\nu_0, \nu_1) \in t_m^p \cap ({}^l 2 \times {}^l 2) \ \& \ k < l \Rightarrow (\nu_0 \upharpoonright k, \nu_1 \upharpoonright k) \in t_m^p$); also $(\langle \rangle, \langle \rangle) \in t_m^p$ and defining \triangleleft naturally we have:

$$\begin{aligned} & \text{if } (\eta_0, \eta_1) \in t_m^p \cap ({}^l 2 \times {}^l 2) \text{ and } l < n^p \\ & \text{then } (\exists \nu_0, \nu_1)[(\eta_0, \eta_1) \triangleleft (\nu_0, \nu_1) \in t_m^p \cap ({}^{l+1} 2 \times {}^{l+1} 2)]. \end{aligned}$$

- (4) A function $f^p = f[p]$ whose domain is a subset of

$$\begin{aligned} & \{(u, h) : \text{for some } l \leq n[p], \ u \subseteq {}^l 2, \ |u| \geq 1, \ h \text{ is a 2-place} \\ & \quad \text{function from } u \text{ to } m[p] \text{ such that } [\eta \in u \Rightarrow h(\eta, \eta) = 0] \\ & \quad \text{and } [\eta, \nu \in u \Rightarrow (\eta, \nu) \in t_{h(\eta, \nu)}^p]\} \end{aligned}$$

and f^p is such that

$$f^p(u, h) = (f_0^p(u, h), f_1^p(u, h), f_2^p(u, h)) \in [-1, \alpha(*)] \times u \times L_{\omega, \omega}(\tau(M)).$$

- (5) A function $g = g^p$ with domain $\{(\alpha, \beta) : \alpha, \beta \text{ from } u^p \cap \lambda\}$ such that $g(\alpha, \alpha) = 0$ and $\alpha \neq \beta \Rightarrow [0 < g(\alpha, \beta) < m^p \text{ and } (\eta_\alpha^p, \eta_\beta^p) \in t_{g(\alpha, \beta)}^p \cap ({}^{n^p} 2 \times {}^{n^p} 2)]$.

- (6) $t_0^p = \{(\eta, \eta) : \eta \in {}^{n^p} 2\}$.

- (7) If $u \subseteq {}^l 2$, $|u| \geq 1$, $f^p(u, h) = (\beta^*, \varrho^*, \varphi^*)$, and $l < l(*) \leq n^p$, e_i are functions with domain u (for $i = 0, 1$) such that $(\forall \varrho)[\varrho \in u \Rightarrow \varrho \triangleleft e_i(\varrho) \in {}^{l(*)} 2]$ and $(\forall \varrho \in u)[e_0(\varrho) = e_1(\varrho) \Leftrightarrow \varrho \neq \varrho^*]$, $u' = \text{Rang}(e_0 \upharpoonright u) \cup \text{Rang}(e_1 \upharpoonright u)$, and $h(\eta, \nu) = h'(e_i(\eta), e_i(\nu))$ for $\eta \neq \nu$ in u and $f^p(u', h') = (\beta', \varrho', \varphi')$ (so is well defined) then $\beta' < \beta^*$.

- (8) If $l \leq n^p$, $w \subseteq u^p \cap \lambda$ is nonempty, the sequence $\langle \eta_\alpha^p \upharpoonright l : \alpha \in w \rangle$ is with no repetitions and h is defined by $h(\eta_\alpha^p \upharpoonright l, \eta_\beta^p \upharpoonright l) = g^p(\alpha, \beta) < m_l^p$ for $\alpha \neq \beta$ from w (and $h(\eta_\alpha^p \upharpoonright l, \eta_\alpha^p \upharpoonright l) = 0$) and $u = \{\eta_\alpha^p \upharpoonright l : \alpha \in w\}$ then $f^p(u, h)$ is well defined, $f_2^p(u, h) = \varphi^M(w)$, $f_1^p(u, h) = \eta_\alpha^p \upharpoonright l$ where α is the $k^M(w)$ th member of w and $f_0^p(u, h) = \text{rk}^1(w, M)$; of course in $f_2^p(u, h) = \varphi^M(w)$ the variable x_ν in $f_2^p(u, h)$ corresponds to $x_{|\alpha \cap w|}$ if $\eta_\alpha \upharpoonright l = \nu$ (see last clause of \oplus_p below).

- (9) If $(u, h) \in \text{Dom}(f^p)$ then for some w and l , $f^p(u, h)$ is as in (8).

- (10) If $\eta_1 \neq \eta_2$ are in ${}^l 2$ with $l \leq n^p$ and $(\eta_1, \eta_2) \in t_m^p$ with $0 < m < m^p$ then for some $\alpha_1 \neq \alpha_2$ from $u^p \cap \lambda$ we have $g^p(\alpha_1, \alpha_2) = m$ and $\eta_1 \trianglelefteq \eta_{\alpha_1}^p$, $\eta_2 \trianglelefteq \eta_{\alpha_2}^p$.

The *order* is the natural one (including the following requirements: if $p \leq q$ then $n^p \leq n^q$, $m^p \leq m^q$, $\bar{m}^p = \bar{m}^q \upharpoonright (n^p + 1)$, $u^p \subseteq u^q$, $\eta_\alpha^q \upharpoonright n^p = \eta_\alpha^p$

for $\alpha \in u^p$, $t_m^p = t_m^q \cap \bigcup_{l \leq n[p]} ({}^l 2 \times {}^l 2)$ for $m < m^p$, $g^p = g^q \upharpoonright u^p$ and $f^p = f^q \upharpoonright \{(u, h) \in \text{Dom}(f^q) : u \subseteq {}^{n^p} \geq 2\}$, so if $(u, h) \notin \text{Dom}(f^p)$ with $u \subseteq {}^{n^p} \geq 2$ then $(u, h) \notin \text{Dom}(f^q)$.

Explanation. The function f^p of a condition $p \in \mathbf{P}$ carries no additional information. It is determined by the functions g^p , φ^M , k^M and the rank. Conditions (8), (9) are to say that

\oplus_p if $w_0, w_1 \subseteq \lambda \cap u^p$, $l \leq n^p$, $u = \{\eta_\alpha^p \upharpoonright l : \alpha \in w_0\} = \{\eta_\alpha^p \upharpoonright l : \alpha \in w_1\}$ (no repetitions) are nonempty and $h : u \times u \rightarrow m^p$ is such that [if either $\alpha, \beta \in w_0$ or $\alpha, \beta \in w_1$ then $h(\eta_\alpha^p \upharpoonright l, \eta_\beta^p \upharpoonright l) = g^p(\alpha, \beta)$], then $\text{rk}^1(w_0, M) = \text{rk}^1(w_1, M)$, $\varphi^M(w_0) = \varphi^M(w_1)$, $k^M(w_0) = k^M(w_1)$, and if $\alpha_i, \beta_i \in w_i$ for $i = 0, 1$ and $\eta_{\alpha_0} \upharpoonright l = \eta_{\alpha_1} \upharpoonright l$, $\eta_{\beta_0} \upharpoonright l = \eta_{\beta_1} \upharpoonright l$ then $\alpha_0 < \alpha_1 \Leftrightarrow \beta_0 < \beta_1$.

Moreover, condition (7) gives no additional restriction unless $f_0^p(u, h) = -1$. Indeed, suppose that $u \subseteq {}^l 2$, $|u| \geq 1$, $l < l(*) \leq n^p$, $e_i : u \rightarrow {}^{l(*)} 2$, $h, \varrho^* \in u$, u' and h' are as there and $f_0^p(u, h) \geq 0$. As $f^p(u', h')$ is defined we find $w \subseteq \lambda \cap u^p$ and $\alpha_0, \alpha_1 \in w$ ($\alpha_0 \neq \alpha_1$) such that $u' = \{\eta_\alpha^p \upharpoonright l(*) : \alpha \in w\}$, $h'(\eta_\alpha^p \upharpoonright l(*), \eta_\beta^p \upharpoonright l(*)) = g^p(\alpha, \beta) < m_l^p$ and $e_i(\varrho^*) = \eta_{\alpha_i}^p \upharpoonright l(*)$ (for $i = 0, 1$). Looking at $w \setminus \{\alpha_0\}$, $w \setminus \{\alpha_1\}$ and (u, h) we see that

$$\alpha_0 = k^M(w \setminus \{\alpha_1\}), \quad \varphi^M(w \setminus \{\alpha_0\}) = \varphi^M(w \setminus \{\alpha_1\}), \\ \text{rk}^1(w \setminus \{\alpha_1\}, M) = f_0^p(u, h) \geq 0.$$

By the definition of the rank and the choice of φ^M , k^M we get $\text{rk}^1(w, M) = f_0^p(u, h)$ and hence $f_0^p(u', h') < f_0^p(u, h)$.

If $f_0^p(u, h) = -1$ then clause (7) says that there are no respective e_0, e_1 introducing a ramification.

STAGE B. \mathbf{P} satisfies the c.c.c. Let $p^i \in \mathbf{P}$ for $i < \omega_1$; let $u[p^i] = \{a_l^i : l < |u[p^i]|\}$ increasing, so with no repetition. We can assume that $|u[p^i]|$ does not depend on i , nor do $n[p^i]$, $\eta_{\alpha_i}^p$, \bar{m}^{p^i} , $\langle t_m^{p^i} : m < m^{p^i} \rangle$, $g^{p^i}(a_{l_1}^i, a_{l_2}^i)$, $f[p^i]$, and for a nonempty $v \subseteq |u[p^i]|$ such that $\bigwedge_{l \in v} a_l^i < \lambda$, $\text{rk}^1(\{a_l^i : l \in v\}, M)$, $\varphi^M(\{a_l^i : l \in v\})$, $k^M(\{a_l^i : l \in v\})$ and the truth value of $a_l^i \geq \lambda$ does not depend on i . Note that writing $\varphi[w]$ we always assume that φ carries information on the order of w .

Also by the Δ -system argument, without loss of generality,

$$a_{l_1}^{i^1} = a_{l_2}^{i^2} \ \& \ i^1 \neq i^2 \Rightarrow l_1 = l_2 \ \& \ \bigwedge_{i,j} a_{l_1}^i = a_{l_2}^j.$$

We shall show that p^0, p^1 are compatible by defining a common upper bound q :

- (i) $n^q = n[p^i] + 1$.
- (ii) $u^q = \{a_l^i : l < |u[p^i]|, i < 2\}$.

- (iii) $\eta_{a_i^q}^q$ is: $\eta_{a_0^q}^{p^0} \wedge \langle 0 \rangle$ if $i = 0$, $\eta_{a_1^q}^{p^0} \wedge \langle 1 \rangle = \eta_{a_1^q}^{p^1} \wedge \langle 1 \rangle$ if $i = 1$ and $a_i^0 \neq a_i^1$.
- (iv) $m[q] = m[p^0] + 2 \times |\lambda \cap u[p^0] \setminus u[p^1]|^2$, $\bar{m}^q = \bar{m}^{p^0} \wedge \langle m[q] \rangle$.
- (v) $g^q \supseteq g^{p^0} \cup g^{p^1}$ is such that g^q assigns new (i.e. in $[m^p, m^q]$) distinct values to “new” pairs (α, β) with $\alpha \neq \beta$, i.e. pairs from $(\lambda \times \lambda) \cap (u^q \times u^q) \setminus u^{p^0} \times u^{p^0} \setminus u^{p^1} \times u^{p^1}$.
- (vi) The trees t_m^q (for $m < m[q]$) are defined as follows. If $m = 0$ see clause (6); if $m < m[p^0]$, $m > 0$ then

$$t_m^q = t_m^{p^0} \cup \{(\eta_{a_{i_1}^q}^q, \eta_{a_{i_2}^q}^q) : \varepsilon \in \{0, 1\} \text{ and distinct } l_1, l_2 < |u[p^0]| \\ \text{satisfying } g^{p^0}(a_{l_1}^0, a_{l_2}^0) = m\};$$

and if $m \in [m[p^0], m[q]]$, $m = g^q(\alpha, \beta)$ and $\alpha \neq \beta$ then

$$t_m^q = \{(\eta_\alpha^q \upharpoonright l, \eta_\beta^q \upharpoonright l) : l \leq n^q\}.$$

- (vii) If $m \in [m[p^0], m[q]]$ then $m = g^q(\alpha, \beta)$ for one and only one pair (α, β) , and for this pair we have $\alpha \neq \beta$, $\{\alpha, \beta\} \not\subseteq u[p^0]$ and $\{\alpha, \beta\} \not\subseteq u[p^1]$.
- (viii) The function f^q is determined by g^p and clauses (8), (9) of stage A.

Of course we have to check that no contradiction appears when we define f^q (i.e. we have to check \oplus_q of the Explanation inside stage A for q). So suppose that $w_0, w_1 \subseteq \lambda \cap u[q]$, $l \leq n[q]$, u, h are as in \oplus_q . If $w_0 \subseteq u[p^i]$ (for some $i < 2$) then $g^q(\alpha, \beta) < m[p^0]$ for $\alpha, \beta \in w_0$ and therefore $g^q[w_1 \times w_1] \subseteq m[p^0]$. Consequently, either $w_1 \subseteq u[p^0]$ or $w_1 \subseteq u[p^1]$. If $l = n^q$ then necessarily $w_0 = w_1$ so we have nothing to prove. If $l < n^q$ then $(u, h) \in \text{Dom}(f^{p^0})$ (and $f^{p^0} = f^{p^1}$) and clause 8 of stage A applies.

If w_0 is contained neither in $u[p^0]$ nor in $u[p^1]$ then $g^q(\alpha, \beta) \in [m[p^0], m[q]]$ for some $\alpha, \beta \in w_0$, hence $l = n^q$, and so since $\{\eta_\alpha^q \upharpoonright l : \alpha \in w_0\} = \{\eta_\alpha^q \upharpoonright l : \alpha \in w_1\}$, clearly $w_0 = w_1$, so we are done.

Next we have to check condition (7). As we remarked (in the Explanation inside stage A) we have to consider cases of (u, h) such that $f^q(u, h) = -1$ only. Suppose that $u, l < l(*) \leq n^q$, $e_i, h, \varrho^* \in u$, u' and h' are as in (7) (and $f^q(u, h) = -1$). Let $w \subseteq u[q] \cap \lambda$ and $\alpha_0, \alpha_1 \in w$ be such that $u' = \{\eta_\alpha^q \upharpoonright l(*) : \alpha \in w\}$, $e_i(\varrho^*) = \eta_{\alpha_i}^q \upharpoonright l(*)$ (for $i = 0, 1$). If $w \subseteq u[p^i]$ for some $i < 2$ then we can apply clause (7) for p^i and get a contradiction (if $l(*) = n^q$ then $\{\eta_\alpha^q \upharpoonright n^p : \alpha \in w\}$ are already distinct). Since $\alpha \in w \setminus \{\alpha_0, \alpha_1\}$ implies $g^q(\alpha, \alpha_0) = g^q(\alpha, \alpha_1)$ (by the relation between h and h') we are left with the case $w \setminus \{\alpha_0, \alpha_1\} \subseteq u[p^0] \cap u[p^1]$, $\alpha_0 \in u[p^0] \setminus u[p^1]$, $\alpha_1 \in u[p^1] \setminus u[p^0]$ (or conversely). Then necessarily $\alpha_0 = a_{k_0}^0$ and $\alpha_1 = a_{k_1}^1$ for some $k_0, k_1 \in [0, |u[p^0]|]$. Now $k_1 = k^M(w \setminus \{\alpha_0\}) = k^M(w \setminus \{\alpha_1\}) = k_0$ by the requirements in condition (7). We see that for each $i < \omega_1$,

$$M \models \varphi^M(w \setminus \{\alpha_0\})[w \setminus \{\alpha_0, \alpha_1\} \cup \{a_{k_1}^i\}],$$

and this contradicts the fact that $\varphi^M(w \setminus \{\alpha_0\})$ and α_1 witness $\text{rk}^1(w \setminus \{\alpha_0\}, M) = -1$.

STAGE C. $|\mathbf{P}| = \mu$ hence $\Vdash_{\mathbf{P}} "2^{\aleph_0} \leq \mu"$. We shall get equality by clause (γ) at stage E below.

STAGE D. *The following subsets of \mathbf{P} are dense (for $m, n < \omega$, $\alpha < \mu$):*

$$\begin{aligned} \mathcal{I}_m^1 &= \{p \in \mathbf{P} : m[p] \geq m\}, & \mathcal{I}_n^2 &= \{p \in \mathbf{P} : n^p \geq n\}, \\ \mathcal{I}_\alpha^3 &= \{p \in \mathbf{P} : \alpha \in u[p]\}. \end{aligned}$$

Proof. Let $p \in \mathbf{P}$ and $\alpha_0 \in \mu \setminus u[p]$ be given. We shall find q with $p \leq q \in \mathcal{I}_{m[p]+1}^1 \cap \mathcal{I}_{n[p]+1}^2 \cap \mathcal{I}_{\alpha_0}^3$; this clearly suffices. We may assume that $u[p] \neq \emptyset$ and $\alpha_0 < \lambda$. We make the following definitions:

(a) $n^q = n^p + 1$, $m^q = m^p + 2 \cdot |\lambda \cap u[p]|$, $\bar{m}^q = \bar{m}^p \wedge \langle m^q \rangle$, $u^q = u^p \cup \{\alpha_0\}$.

(b) For $\alpha \in u^p$ we let $\eta_\alpha^q = \eta_\alpha^p \wedge \langle 0 \rangle$, and $\eta_{\alpha_0}^q \in {}^{(n^p+1)}2$ is the sequence constantly 1.

(c) g^q is any 2-place function from $u^q \cap \lambda$ to m^q extending g^p such that $g^q(\alpha, \alpha) = 0$, $g^q(\alpha, \beta) \neq 0$ for $\alpha \neq \beta$ and

$$\begin{aligned} g^q(\alpha, \beta) &= g^q(\alpha', \beta') \ \& \ (\alpha, \beta) \neq (\alpha', \beta') \\ &\Rightarrow (\alpha, \beta) \in u^p \times u^p \ \& \ (\alpha', \beta') \in u^p \times u^p. \end{aligned}$$

(d) t_m^q is defined as follows:

(α) if $m < m^p$, $m \neq 0$ then $t_m^q = t_m^p \cup \{(\eta_0 \wedge \langle 0 \rangle, \eta_1 \wedge \langle 0 \rangle) : (\eta_0, \eta_1) \in t_m^p \cap ({}^{n[p]}2 \times {}^{n[p]}2)\}$,

(β) if $m \in [m^p, m^q]$, $m = g^q(\alpha, \beta)$, $\alpha \neq \beta$ then $t_m^q = \{(\eta_\alpha^q \upharpoonright l, \eta_\beta^q \upharpoonright l) : l \leq n^q\}$.

(e) f^q extends f^p and satisfies (7)–(9) of stage A (note that f^q is determined by g^q).

Now check (similarly to stage B).

STAGE E. We define some \mathbf{P} -names:

(a) $\underline{\eta}_\alpha = \bigcup \{\eta_\alpha^p : p \in \mathcal{G}_{\mathbf{P}}\}$ for $\alpha < \lambda$,

(b) $\underline{T}_m = \bigcup \{t_m^p : p \in \mathcal{G}_{\mathbf{P}}\}$ for $m < \omega$,

(c) $\underline{g} = \bigcup \{g^p : p \in \mathcal{G}_{\mathbf{P}}\}$.

Clearly it is forced ($\Vdash_{\mathbf{P}}$) that:

(α) \underline{g} is a function from $\{(\alpha, \beta) : \alpha, \beta < \lambda\}$ to ω . [Why? Because \mathcal{I}_α^3 are dense subsets of \mathbf{P} and by clause (5) of stage A.]

(β) $\underline{\eta}_\alpha \in {}^\omega 2$. [Why? Because both \mathcal{I}_n^2 and \mathcal{I}_α^3 are dense subsets of \mathbf{P} .]

(γ) $\underline{\eta}_\alpha \neq \underline{\eta}_\beta$ for $\alpha \neq \beta (< \mu)$. [Why? By clause (2) of the definition of $p \in \mathbf{P}$.]

(δ) $\mathcal{T}_m \subseteq \bigcup_{l < \omega} ({}^l 2 \times {}^l 2)$ is an $(2, 2)$ -tree. [Why? By clause (3) of the definition of $p \in \mathbf{P}$ and density of $\mathcal{T}_m^1, \mathcal{T}_n^2$.]

(ε) $(\eta_\alpha, \eta_\beta) \in \lim(\mathcal{T}_{\underline{g}(\alpha, \beta)}) = \{(\nu_0, \nu_1) \in {}^\omega 2 \times {}^\omega 2 : (\forall l < \omega)((\nu_0 \upharpoonright l, \nu_1 \upharpoonright l) \in \mathcal{T}_{\underline{g}(\alpha, \beta)})\}$ (for $\alpha, \beta < \lambda$). [Why? By clause (5) of the definition of $p \in \mathbf{P}$ and (β) + (δ) above.]

(ζ) If α, β are $< \lambda$ then $(\eta_\alpha, \eta_\beta) \notin \lim(\mathcal{T}_m)$ when $m \neq g(\alpha, \beta)$ (and $m < \omega$). [Why? By clauses (2)+(10) of the definition of \mathbf{P} if $m \neq 0$, and clause (5) if $m = 0$.]

Note that by (ε) above, the Borel set $\underline{B} = \bigcup_{m < \omega} \lim(\mathcal{T}_m) \subseteq {}^\omega 2 \times {}^\omega 2$ satisfies requirement (*) (a) of the conclusion of 1.13. Moreover, by (γ) we have $\Vdash_{\mathbf{P}} "2^{\aleph_0} \geq \mu"$, completing stage C (i.e. $\Vdash_{\mathbf{P}} "2^{\aleph_0} = \mu"$).

STAGE F. We want to show (*) (c) of the conclusion of 1.13. Let $\mathbf{P}_\lambda = \{p \in \mathbf{P} : u[p] \subseteq \lambda\}$. Clearly $\mathbf{P}_\lambda \triangleleft \mathbf{P}$. Moreover, $\underline{g}, \mathcal{T}_m, \underline{B}$ are \mathbf{P}_λ -names. Since " \underline{B} contains a perfect square" is a Σ_2^1 -formula, so absolute, it is enough to prove that in $\mathbf{V}^{\mathbf{P}_\lambda}$ the set \underline{B} contains no perfect square.

Suppose that a \mathbf{P}_λ -name \mathcal{T} for a perfect tree and a condition $p \in \mathbf{P}_\lambda$ are such that

$$p \Vdash_{\mathbf{P}_\lambda} " \lim(\mathcal{T}) \times \lim(\mathcal{T}) \subseteq \underline{B} "$$

We then have (a name for) a function $\underline{m} : \lim(\mathcal{T}) \times \lim(\mathcal{T}) \rightarrow \omega$ such that

$$p \Vdash_{\mathbf{P}_\lambda} " \text{if } \eta_0, \eta_1 \in \lim(\mathcal{T}) \text{ then } (\eta_0, \eta_1) \in \mathcal{T}_{\underline{m}(\eta_0, \eta_1)} "$$

By shrinking the tree \mathcal{T} we may assume that p forces ($\Vdash_{\mathbf{P}_\lambda}$) the following:

$$\begin{aligned} & " \text{if } \eta_0, \eta_1, \eta'_0, \eta'_1 \in \lim(\mathcal{T}) \text{ and } \eta_0 \upharpoonright l = \eta'_0 \upharpoonright l \neq \eta_1 \upharpoonright l = \eta'_1 \upharpoonright l \\ & \text{then } \underline{m}(\eta_0, \eta_1) = \underline{m}(\eta'_0, \eta'_1) " \end{aligned}$$

Consequently, we may think of \underline{m} as a function from $\mathcal{T} \times \mathcal{T}$ to ω (with the convention that if $\nu_0, \nu_1 \in \mathcal{T}$ are \triangleleft -comparable then $\underline{m}(\nu_0, \nu_1) = 0$ and $\eta \wedge \langle l \rangle \triangleleft \nu_1 \in \mathcal{T} \Rightarrow \underline{m}(\eta \wedge \langle l \rangle, \eta \wedge \langle 1 - l \rangle) = \underline{m}(\nu_1, \nu_{1-l})$ and if $\text{lg}(\nu_1) = \text{lg}(\nu_2)$ then $(\nu_1, \nu_2) \in T_{\underline{m}(\nu_1, \nu_2)}$).

Choose an increasing sequence $\langle n_i : i \in \omega \rangle$ of natural numbers and sequences $\langle p_i : i \in \omega \rangle \subseteq \mathbf{P}_\lambda$, $\langle (t_i, m_i) : i \in \omega \rangle$ such that:

- (1) $p_0 \leq p_1 \leq \dots \leq p_i \leq p_{i+1} \leq \dots$,
- (2) $t_i \subseteq {}^{n_i} 2$ is a perfect tree (i.e. $[\eta \triangleleft \nu \in t_i \cap {}^{n_i} 2 \Rightarrow \eta \in t_i]$, $\langle \rangle \in t_0$, $[\eta \in {}^{n_i} 2 \cap t_i \Rightarrow \bigvee_{l < 2} \eta \wedge \langle l \rangle \in t_i]$) and $m_i : (t_i \cap {}^{n_i} 2)^2 \rightarrow \omega$,
- (3) $t_i \subseteq t_{i+1}$ is an end extension (i.e. $t_i = ({}^{n_i} 2) \cap t_{i+1}$) such that each node from $t_i \cap {}^{n_i} 2$ ramifies in t_{i+1} (i.e. has \triangleleft -incomparable extensions),
- (4) $p_i \Vdash_{\mathbf{P}_\lambda} " \mathcal{T} \cap {}^{n_i} 2 = t_i \ \& \ \underline{m} \upharpoonright (t_i \cap {}^{n_i} 2)^2 = m_i "$,
- (5) $n[p_i] > n_i$ and $m[p_i] > \max \text{Rang}(m_i)$.

Since $p_i \Vdash_{\mathbf{P}_\lambda} “(\nu_0, \nu_1) \in \underline{T}_{m_i(\nu_0, \nu_1)}”$ for $\nu_0, \nu_1 \in t_i \cap {}^{n_i}2$ we easily see (by clause (8) of the definition of \mathbf{P} , stage A) that $(t_i \cap {}^{n_i}2, m_i) \in \text{Dom}(f^{p_i})$. By clause (7) (of the definition of \mathbf{P}) (and 1.2(2) + clause (8) of the definition of \mathbf{P}) we deduce that

$$f_0^{p_{i+1}}(t_{i+1} \cap {}^{n_{i+1}}2, m_{i+1}) < f_0^{p_i}(t_i \cap {}^{n_i}2, m_i)$$

for each $i < \omega$, and this gives a contradiction (to the ordinals being well ordered).

STAGE G. To prove (*) (b) of Theorem 1.13 we may assume that $\mathbf{V} \models “\lambda^{\aleph_0} = \lambda_1 < \mu”$. Let $\mathbf{P}_{\lambda_1} = \{p \in \mathbf{P} : u[p] \subseteq \lambda_1\} \triangleleft \mathbf{P}$. Note that the rest of the forcing (i.e. $\mathbf{P}/\mathbf{P}_{\lambda_1}$) is the forcing notion for adding μ Cohen reals so for $v \subseteq \mu \setminus \lambda_1$ the forcing notion \mathbf{P}_v is naturally defined, as also is $\mathbf{P}_{\lambda_1 \cup v}$. By stages C, E we know that $\mathbf{V}^{\mathbf{P}_{\lambda_1}} \models “2^{\aleph_0} = \lambda_1”$ and by stage F we have $\mathbf{V}^{\mathbf{P}_{\lambda_1}} \models “\text{the Borel set } \underline{B} \text{ does not contain a perfect square}”$. Suppose that after adding μ Cohen reals (over $\mathbf{V}^{\mathbf{P}_{\lambda_1}}$) we have a λ_1^+ -square contained in \underline{B} . We have λ_1^+ -branches \underline{q}_α ($\alpha < \lambda_1^+$), each is a \mathbf{P}_{v_α} -name for some countable $v_\alpha \subseteq \mu \setminus \lambda_1$. By the Δ -system lemma we can assume that $\alpha \neq \beta \Rightarrow v_\alpha \cap v_\beta = v^*$. Working in $\mathbf{V}^{\mathbf{P}_{\lambda_1 \cup v^*}}$ we see that $\mathbf{P}_{v_\alpha \setminus v^*}$ is really the Cohen forcing notion and \underline{q}_α is a $\mathbf{P}_{v_\alpha \setminus v^*}$ -name. Without loss of generality, $v^* = [\lambda_1, \lambda_1 + \omega)$, $v_\alpha = v^* \cup \{\lambda_1 + \omega + \alpha\}$ and all names \underline{q}_α are the same (under the natural isomorphism). So we have found a Cohen forcing name $\underline{\mathcal{T}} \in \mathbf{V}^{\mathbf{P}_{\lambda_1 + \omega}}$ such that

$$\begin{aligned} &\text{if } c_0, c_1 \text{ are (mutually) Cohen reals over } \mathbf{V}^{\mathbf{P}_{\lambda_1 + \omega}} \\ &\text{then } \mathbf{V}^{\mathbf{P}_{\lambda_1}}[c_0, c_1] \models (\underline{\mathcal{T}}^{c_0}, \underline{\mathcal{T}}^{c_1}) \in \underline{B} \ \& \ \underline{\mathcal{T}}^{c_0} \neq \underline{\mathcal{T}}^{c_1}. \end{aligned}$$

But the Cohen forcing adds a perfect set of (mutually) Cohen reals. By absoluteness this produces a perfect set (in $\mathbf{V}^{\mathbf{P}_{\lambda_1}}$) whose square is contained in \underline{B} . Once again by absoluteness we conclude that \underline{B} contains a perfect square in $\mathbf{V}^{\mathbf{P}_\lambda}$ already, a contradiction. $\blacksquare_{1.13}$

REMARK 1.14. Note that if B is a subset of the plane $({}^\omega\omega, {}^\omega\omega)$ which is G_δ (i.e. $\bigcap_{n < \omega} U_n$, U_n open, without loss of generality decreasing with n) and it contains an uncountable square $X \times X$ (so $X \subseteq {}^\omega\omega$ is uncountable) then it contains a perfect square. Why? Let

$$X' = \{\eta \in X : (\forall n)(\exists^{\aleph_1} \nu)(\nu \in X \ \& \ \nu \upharpoonright n = \eta \upharpoonright n)\}.$$

Let

$$\begin{aligned} K = \{ &(u, n) : \text{for some } l \text{ called } l(u, n), u \subseteq {}^l\omega, \text{ and} \\ &\eta, \nu \in u \ \& \ \eta \triangleleft \eta' \in {}^\omega\omega \ \& \ \nu \triangleleft \nu' \in {}^\omega\omega \Rightarrow (\eta', \nu') \in U_n \\ &\text{and } \eta \in u \ \& \ \eta \triangleleft \eta' \in {}^\omega\omega \Rightarrow (\eta', \eta') \in U_n\}, \end{aligned}$$

$$K' = \{(u, n) \in K : \text{for some } \bar{\nu} = \langle \nu_\varrho : \varrho \in u \rangle \text{ we have } \nu_\varrho \in X', \ \varrho \triangleleft \nu_\varrho\}.$$

Then

(a) $K' \neq \emptyset$, in fact if $\eta_1, \dots, \eta_m \in X'$ are pairwise distinct and $n < \omega$, then $(\{\eta_i \upharpoonright l : i = 1, \dots, m\}, n) \in K$ for any l large enough,

(b) if $(u, n) \in K'$ as exemplified by $\bar{\nu} = \langle \nu_\varrho : \varrho \in u \rangle$ and $\varrho^* \in u$, $\nu' \in X' \setminus \{\nu_{\varrho^*}\}$, $\nu' \upharpoonright l = \nu_{\varrho^*} \upharpoonright l$ then for any $l' \in (l, \omega)$ and $n' > n$ large enough, we have $(\{\nu_\varrho \upharpoonright l' : \varrho \in u\} \cup \{\nu' \upharpoonright l'\}, n') \in K'$.

The following depends on §3:

THEOREM 1.15. *Assume MA and $2^{\aleph_0} \geq \lambda_{\omega_1}(\aleph_0)$ or $2^{\aleph_0} > \mu$. Then: there is a Borel subset of the plane with a μ -square but with no perfect square iff $\mu < \lambda_{\omega_1}(\aleph_0)$.*

PROOF. The first clause implies the second by 1.12. If the second clause holds, let $\mu \leq \lambda_\alpha(\aleph_0)$ and $\alpha < \omega_1$. By 3.2(6) letting $\eta_i \in {}^\omega 2$ for $i < \mu$ be pairwise distinct we can find an ω -sequence of $(2, 2)$ -trees \bar{T} such that $(\eta_i, \eta_j) \in \bigcup_n \text{lim}(T_n)$ for $i, j < \mu$ and $\text{degseq}(\bar{T}) = \alpha$ (just use $A = \{(\eta_i, \eta_j) : i, j < \mu\}$ there). By 3.2(3) the set $\bigcup_n \text{lim}(T_n)$ contains no $\lambda_{\alpha+1}(\aleph_0)$ -square. ■1.15

FACT 1.16. *Assume \mathbf{P} is adding $\mu > \kappa$ Cohen reals or random reals and $\kappa > 2^{\aleph_0}$. Then in $\mathbf{V}^{\mathbf{P}}$ we have*

$(*)_\kappa$ *there is no Borel set (or analytic) $B \subseteq {}^\omega 2 \times {}^\omega 2$ such that*

- (a) *there are $\eta_\alpha \in {}^\omega 2$ for $\alpha < \kappa$ such that $[\alpha \neq \beta \Rightarrow \eta_\alpha \neq \eta_\beta]$ and $(\eta_\alpha, \eta_\beta) \in B$ for $\alpha, \beta < \kappa$,*
- (b) *B contains no perfect square.*

PROOF. Straightforward as in the (last) stage G of the proof of Theorem 1.13 (except that no relevance of (7) of stage A there).

Let \mathbf{P} be adding $\langle \mathcal{r}_\alpha : \alpha < \mu \rangle$, and assume $p \in \mathbf{P}$ forces that a Borel set \underline{B} and $\langle \eta_\alpha : \alpha < \kappa \rangle$ are as in clauses (a), (b) above. Let η_α be names in $\mathbf{P}_{v_\alpha} = \mathbf{P} \upharpoonright \{\tilde{\mathcal{r}}_\beta : \beta \in v_\alpha\}$, and \underline{B} be a name in $\mathbf{P}_v = \mathbf{P} \upharpoonright \{\tilde{\mathcal{r}}_\beta : \beta \in v\}$ where v, v_β are countable subsets of κ . Without loss of generality, $\langle v_\alpha : \alpha < (2^{\aleph_0})^+ \rangle$ is a Δ -system with heart v and $\text{otp}(v_\alpha \setminus v) = \text{otp}(v_0 \setminus v)$. In $\mathbf{V}^{\mathbf{P}_v}$ we have \underline{B} and $2^{\aleph_0} = (2^{\aleph_0})^{\mathbf{V}}$, so we can assume $v = \emptyset$ and $\text{otp}(v_\alpha)$ does not depend on α .

Without loss of generality, the order preserving function $f_{\alpha, \beta}$ from v_α onto v_β maps η_α to η_β . So for $Q = \text{Cohen}$ in the Cohen case we have a name \mathcal{T} such that $\Vdash_{\text{Cohen}} \mathcal{T} \in {}^\omega 2$ is new", $\Vdash_{\text{Cohen} \times \text{Cohen}} (\mathcal{T}(r_1); \mathcal{T}(r_2)) \in B$ ", and we can finish easily. The random case is similar. ■1.16

CONCLUSION 1.17. (1) *For $\kappa \in (\aleph_1, \aleph_{\omega_1})$ the statement $(*)_\kappa$ of 1.16 is not decided by $ZFC + 2^{\aleph_0} > \aleph_{\omega_1}$ (i.e. it and its negation are consistent with ZFC).*

(2) *1.16 applies to the forcing notion of 1.13 (with μ instead of 2^{\aleph_0}).*

Proof. (1) Starting with a universe \mathbf{V} satisfying CH, Fact 1.16 shows the consistency of “yes”. As by 1.7(1) we know that $\lambda_{\omega_1}(\aleph_0) \geq \aleph_{\omega_1}$ and $\aleph_{\omega_1} > \kappa$ (by assumption), Theorem 1.15 (with the classical consistency of $\text{MA} + 2^{\aleph_0} > \aleph_{\omega_1}$) gives the consistency of “no” (in fact, in both cases it works for all κ simultaneously).

(2) Left to the reader.

2. Some model-theoretic related problems. We turn to the model-theoretic aspect: getting Hanf numbers below the continuum, i.e. if $\psi \in L_{\omega_1, \omega}$ has a model of cardinality $\geq \lambda_{\omega_1}(\aleph_0)$ then it has a model of cardinality continuum. We show that $\text{Pr}_{\omega_1}(\lambda)$ is equivalent to a statement of the form “if $\psi \in L_{\omega_1, \omega}$ has a model of cardinality λ then it has a model generated by an “indiscernible” set indexed by ${}^\omega 2$ ” (the indiscernibility is with respect to the tree $({}^\omega 2, \triangleleft, \cap, <_{\text{lx}}, <_{\text{lg}}$), where \triangleleft is being initial segment, $\eta \cap \nu = \text{maximal } \rho \text{ with } \rho \triangleleft \eta \ \& \ \rho \triangleleft \nu$, $<_{\text{lx}}$ is lexicographic order, $\eta <_{\text{lx}} \nu$ iff $\text{lg}(\eta) < \text{lg}(\nu)$). This gives sufficient conditions for having many nonisomorphic models and also gives an alternative proof of 1.12.

We also deal with the generalization to $\bar{\lambda}$ -models, i.e. fixing the cardinalities of several unary predicates (and point to λ -like models).

CLAIM 2.1. *The following are equivalent for a cardinal λ :*

(1) $\text{Pr}_{\omega_1}(\lambda)$.

(2) *If $\psi \in L_{\omega_1, \omega}$ has a model M with $|R^M| \geq \lambda$ (R is a unary predicate) then ψ has a model of cardinality continuum; moreover, for some countable first order theory \mathbf{T}_1 with Skolem functions such that $\tau(\psi) \subseteq \tau(\mathbf{T}_1)$ and a model M_1 of \mathbf{T}_1 and $a_\eta \in R^{M_1}$ for $\eta \in {}^\omega 2$ we have:*

(*)₀ $M_1 \models \psi$,

(*)₁ M_1, a_η ($\eta \in {}^\omega 2$) are as in [Sh a, VII, §4] = [Sh c, VII, §4], i.e.:

(a) M_1 is the Skolem hull of $\{a_\eta : \eta \in {}^\omega 2\}$ and $\eta \neq \nu \Rightarrow a_\eta \neq a_\nu$,

(b) for every $n < \omega$ and a first order formula $\varphi = \varphi(x_0, \dots, x_{n-1}) \in L(\mathbf{T}_1)$ there is $n^* < \omega$ such that for every $k \in (n^*, \omega)$, $\eta_0, \dots, \eta_{n-1} \in {}^\omega 2$ and $\nu_0, \dots, \nu_{n-1} \in {}^\omega 2$ satisfying

$$\bigwedge_{m < n} \eta_m \upharpoonright k = \nu_m \upharpoonright k \quad \text{and} \quad \bigwedge_{m < l < n} \eta_m \upharpoonright k \neq \eta_l \upharpoonright k$$

we have

$$M_1 \models \text{“}\varphi[a_{\eta_0}, \dots, a_{\eta_{n-1}}] \equiv \varphi[a_{\nu_0}, \dots, a_{\nu_{n-1}}]\text{”}.$$

Note that necessarily $a_\eta \notin \text{Skolem hull}_{M_1}\{a_\nu : \nu \in {}^\omega 2 \setminus \{\eta\}\}$.

(c) $a_\eta \in R^{M_1}$.

REMARK 2.2. In (2) we can replace the assumption by “If $\psi \in L_{\omega_1, \omega}$ has, for every $\lambda' < \lambda$, a model M with $|R^M| \geq \lambda'$ then ...” (and still the new version of (2) is equivalent to (1)).

PROOF (of Claim 2.1). (1) \Rightarrow (2). Just as in [Sh 37] + [Sh 49]: we can assume $\|M\| = \lambda$ and moreover $|M| = \lambda$. Let M_1 be an expansion of M by names for subformulas of ψ , a pairing function, and then by Skolem functions. Let \mathbf{T}_1 be the first order theory of M_1 . There is (see [Ke71]) a set Γ of countably many types $p(x)$ such that M_1 omits every $p(x) \in \Gamma$ and if M'_1 is a model of \mathbf{T}_1 omitting every $p(x) \in \Gamma$ then M'_1 is a model of ψ (just for each subformula $\bigwedge_{n < \omega} \psi_n(\bar{x})$ of ψ , we have to omit a type; we can use 1-types as we have a pairing function). Let us define

$$Y = \{v \subseteq {}^\omega 2 : v \text{ is finite nonempty, its members are pairwise } \triangleleft\text{-incomparable and } v \subseteq {}^n 2 \cup {}^{n+1} 2 \text{ for some } n\},$$

$$Z = \{(v, \varphi(\dots, x_\eta, \dots)_{\eta \in v}) : v \in Y, \varphi \text{ a formula in } T_1 \text{ with the set of free variables included in } \{x_\eta : \eta \in v\} \text{ and for every } \alpha < \omega_1 \text{ there are } a_\eta^\alpha \in R^M \text{ for } \eta \in v \text{ such that: } [\eta \neq \nu \text{ from } v \Rightarrow a_\eta^\alpha \neq a_\nu^\alpha], \text{rk}^0(\{a_\eta^\alpha : \eta \in v\}, M) \geq \alpha \text{ and } M \models \varphi[\dots, a_\eta^\alpha, \dots]_{\eta \in v}\}.$$

We say for $(v_l, \varphi_l) \in Z$ ($l = 1, 2$) that $(v_2, \varphi_2) \in \text{succ}(v_1, \varphi_1)$ if for some $\eta \in v_1$ (called $\eta(v_1, v_2)$) we have $v_2 = (v_1 \setminus \{\eta\}) \cup \{\eta \wedge \langle 0 \rangle, \eta \wedge \langle 1 \rangle\}$. For $i < 2$ define the function $h_i : v_1 \rightarrow v_2$ by $h_i(\nu)$ is ν if $\nu \neq \eta$ and $h_i(\nu) = \eta \wedge \langle i \rangle$ if $\nu = \eta$. We demand, for $i = 0, 1$,

$$\varphi_2 \vdash \varphi_1(\dots, x_{h_i(\nu)}, \dots)_{\nu \in v_1}.$$

Choose inductively $\langle (v_l, \varphi_l) : l < \omega \rangle$ such that $(v_{l+1}, \varphi_{l+1}) \in \text{succ}(v_l, \varphi_l)$ is generic enough, i.e.

(\otimes)₁ if $\varphi = \varphi(x_0, \dots, x_{k-1}) \in L(\mathbf{T}_1)$ then for some $l < \omega$ and all $m \in [l, \omega)$ and $\eta_0, \dots, \eta_{k-1} \in v_m$ we have either

$$\varphi_m \vdash \varphi(x_{\eta_0}, \dots, x_{\eta_{k-1}}) \quad \text{or} \quad \varphi_m \vdash \neg \varphi(x_{\eta_0}, \dots, x_{\eta_{k-1}}),$$

(\otimes)₂ for every $p(x) \in \Gamma$ and every function symbol $f = f(x_0, \dots, x_{n-1})$ (note: in \mathbf{T}_1 a definable function is equivalent to some function symbol), for some $l < \omega$, all $m \in [l, \omega)$ and all $\eta_0, \dots, \eta_{n-1} \in v_m$ there is $\psi(x) \in p(x)$ such that

$$\varphi_m \vdash \neg \psi(f(x_{\eta_0}), \dots, f(x_{\eta_{n-1}})).$$

It is straightforward to carry out the induction (to simplify you may demand in (\otimes)₁, (\otimes)₂ just “for arbitrarily large $m \in [l, \omega)$ ”, this does not matter and the stronger version of (\otimes)₁, (\otimes)₂ can be obtained (replacing the $\omega > 2$ by a perfect subtree T and then renaming a_η for $\eta \in \text{lim}(T)$ as a_η for $\eta \in {}^\omega 2$). Then define the model by compactness.

(2) \Rightarrow (1). If not, then $\text{NPr}_{\omega_1}(\lambda)$, hence for some model M with vocabulary τ having $|\tau| \leq \aleph_0$ and of cardinality λ we have $\alpha(*) := \text{rk}^0(M) < \omega_1$. Let $\psi_{\alpha(*)} \in L_{\omega_1, \omega}(\tau)$ be as in 2.3 below, so necessarily $M \models \psi_{\alpha(*)}$. Apply to it clause (2) which holds by our present assumption (with $R^M = \lambda$), so $\psi_{\alpha(*)}$ has a model M_1 as there (so $M_1 \models \psi_{\alpha(*)}$). But $\{a_\eta : \eta \in {}^\omega 2\}$ easily witnesses $\text{rk}^0(M_1) = \infty$, and moreover, for every nonempty finite $w \subseteq \{a_\eta : \eta \in {}^\omega 2\}$ and an ordinal α we have $\text{rk}^0(w, M) \geq \alpha$. This can be easily proved by induction on α (using $(*)_1$ (b) of (2) (and $\eta \neq \nu \in {}^\omega 2 \Rightarrow a_\eta \neq a_\nu$ of $(*)_1$ (a))). $\blacksquare_{2.1}$

FACT 2.3. (1) *For every $\alpha < \kappa^+$ and vocabulary τ with $|\tau| \leq \kappa$ there is a sentence $\psi_\alpha \in L_{\kappa^+, \omega}[\tau]$ (of quantifier depth α) such that for any τ -model M ,*

$$M \models \psi_\alpha \quad \text{iff} \quad \text{rk}^0(M; < \aleph_0) = \alpha.$$

(2) *For every $\alpha < \theta^+$, $l \in \{0, 1\}$ and vocabulary τ with $|\tau| \leq \theta$ there is a sentence $\psi \in L_{\theta^+, \omega}(\exists^{\geq \kappa})[\tau]$ ($\exists^{\geq \kappa}$ is the quantifier “there are $\geq \kappa$ many”) such that for any τ -model M : $M \models \psi_\alpha^l$ iff $\text{rk}^l(M; < \kappa, \theta) = \alpha$.*

Proof. Easy to check. $\blacksquare_{2.3}$

Hence (just as in [Sh a, VIII, 1.8(2)]):

CONCLUSION 2.4. *Assume τ is a countable vocabulary. If $\psi \in L_{\omega_1, \omega}(\tau)$, R is a unary predicate, $\tau_0 \subseteq \tau$, $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1, \omega}(\tau_0)\}$ is countable and for some transitive model \mathbf{V}_1 of ZFC (maybe a generic extension of \mathbf{V} or an inner model as long as $\psi, \Delta \in \mathbf{V}_1$ and $\mathbf{V}_1 \models$ “ $\psi \in L_{\omega_1, \omega}(\tau)$, $\Delta \subseteq \{\varphi(x) : \varphi \in L_{\omega_1, \omega}(\tau_0)\}$ ”) we have*

$$\mathbf{V}_1 \models \text{“Pr}_{\omega_1}(\lambda), \text{ and } \psi \text{ has a model } M \text{ with} \\ \lambda \leq |\{\{\varphi(x) : M \models \varphi[a], \varphi(x) \in \Delta\} : a \in R^M\}|”.$$

Then:

(1) *We can find a model N of ψ with Skolem functions and $a_\alpha \in R^N$ for $\alpha < 2^{\aleph_0}$ such that for each $\alpha < 2^{\aleph_0}$ the type $p_\alpha = \{\pm\varphi(x) : N \models \pm\varphi[a_\alpha] \text{ and } \varphi(x) \in \Delta\}$ is not realized in the Skolem hull of*

$$\{a_\beta : \beta < 2^{\aleph_0} \text{ and } \beta \neq \alpha\}.$$

(2) *So $|\{M/\approx : M \models \psi, \|M\| = \lambda\}| \geq \min\{2^\lambda, \beth_2\}$; really $\{(M \upharpoonright \tau_0)/\approx : M \models \psi \text{ and } M \text{ has cardinality } \lambda\}$ has cardinality $\geq \min\{2^\lambda, \beth_2\}$. Moreover, we can find such a family of models no one of them embeddable into another by an embedding preserving $\pm\varphi(x)$ for $\varphi \in \Delta$. $\blacksquare_{2.4}$*

A natural generalization of 2.1 is

CLAIM 2.5. (1) *For cardinals $\lambda > \kappa \geq \aleph_0$ the following are equivalent:*

(a) $\text{Pr}_{\kappa^+}(\lambda; \kappa)$.

(b) If M is a model, $\tau(M)$ countable, $R, R_0 \in \tau(M)$ unary predicates, $|R_0^M| \leq \kappa$, $\lambda \leq |R^M|$ then we can find M_0, M_1, a_η ($\eta \in {}^\omega 2$) such that:

- (i) M_1 is a model of the (first order) universal theory of M (and is a $\tau(M)$ -model),
- (ii) $a_\eta \in R^{M_1}$ for $\eta \in {}^\omega 2$ are pairwise distinct,
- (iii) M_1 is the closure of $\{a_\eta : \eta \in {}^\omega 2\} \cup M_0$ under the functions of M_1 , so
 - (α) M_1 also includes the individual constants of M ; in general $\|M_1\| = 2^{\aleph_0}$,
 - (β) if $\tau(M)$ has predicates only then $|M_1| = \{a_\eta : \eta \in {}^\omega 2\} \cup |M_0|$,
- (iv) M_0 is countable, $M_0 \subseteq M$, $M_0 \subseteq M_1$, $M_0 = \text{cl}_M(M_0 \cap R_0^M)$, $R_0^{M_1} = R_0^{M_0} (\subseteq R_0^M)$; in fact, we can have:
 - (*) $(M_1, c)_{c \in M_0}$ is a model of the universal theory of $(M, c)_{c \in M_0}$,
- (v) for every $n < \omega$ and a quantifier free first order formula $\varphi = \varphi(x_0, \dots, x_{n-1}) \in L(\tau(M))$ there is $n^* < \omega$ such that for every $k \in (n^*, \omega)$ and $\eta_0, \dots, \eta_{n-1} \in {}^\omega 2$, $\nu_0, \dots, \nu_{n-1} \in {}^\omega 2$ satisfying

$$\bigwedge_{m < n} \eta_m \upharpoonright k = \nu_m \upharpoonright k \quad \text{and} \quad \bigwedge_{m < l < n} \eta_m \upharpoonright k \neq \eta_l \upharpoonright k$$

we have

$$M_1 \models \text{“}\varphi[a_{\eta_0}, \dots, a_{\eta_{n-1}}] \equiv \varphi[a_{\nu_0}, \dots, a_{\nu_{n-1}}]\text{”};$$

we can even allow parameters from M_0 in φ (but k depends on them).

(2) For cardinals $\lambda > \kappa \geq \aleph_0$ the following are equivalent:

- (a)' $\text{Pr}_{\omega_1}(\lambda; \kappa)$.
- (b)' Like (b) above, but we omit “ $M_0 \subseteq M$ ”.

REMARK 2.6. (1) See 4.6, 4.7 how to use Claim 2.5.

(2) In (b), if M has Skolem functions then we automatically get also:

- (i)⁺ M_1 is a model of the first order theory of M ,
- (iii)⁺ M_1 is the Skolem hull of $\{a_\eta : \eta \in {}^\omega 2\} \cup M_0$,
- (iv)⁺ $M_0 \prec M$, $M_0 \prec M_1$, M_0 countable (and $R_0^{M_1} = R_0^{M_0} \subseteq R_0^M$),
- (v)⁺ clause (v) above holds even for φ any (first order) formula of $L_{\omega, \omega}(\tau(M))$.

Proof. (1)(a) \Rightarrow (b). Like the proof of 2.1(1) \Rightarrow (2), applied to $(M, c)_{c \in R_0^M}$, but the set $M_0 \cap R_0^M$ is chosen by finite approximation, i.e. (letting Y be as there and $\tau = \tau(M)$) we let

$$Z = \{(v, \varphi(\dots, x_\eta, \dots)_{\eta \in v}, A) : v \in Y, \varphi \text{ a quantifier free formula} \\ \text{in } L_{\omega, \omega}(\tau) \text{ with set of free variables included in } \{x_\eta : \eta \in v\} \\ \text{and parameters from } A, A \text{ is a finite subset of } R_0^M, \\ \text{and for every ordinal } \alpha < \kappa^+ \text{ there are } a_\eta^\alpha \in R^M \text{ for } \eta \in v \\ \text{such that } [\eta \neq \nu \text{ from } v \Rightarrow a_\eta^\alpha \neq a_\nu^\alpha] \text{ and} \\ \text{rk}(\{a_\eta^\alpha : \eta \in v\}, M) \geq \alpha \text{ and } M \models \varphi[\dots, a_\eta^\alpha, \dots]_{\eta \in v}\}.$$

We need the “for every $\alpha < \kappa^+$ ” here because we want to fix elements of R_0^M , and there are κ possible choices.

\neg (a) \Rightarrow \neg (b). Like the proof of 2.3; assume $\text{NPr}_\alpha(\lambda; \kappa)$, $\alpha < \kappa^+$, let M witness it, choose $R_0 = \alpha + 1$, $R = \lambda$; without loss of generality, $\tau(M) = \{R_{n, \zeta} : n < \omega, \zeta < \kappa\}$, $R_{n, \zeta}$ is n -place, in M every quantifier free formula is equivalent to some $R_{n, \zeta}$. Let $R_{n, k}^* := \{(i_0, \dots, i_{n-1}, \beta, \zeta) : M \models R_{n, \zeta}(i_0, \dots, i_{n-1}), \{i_0, \dots, i_{n-1}\} \text{ is with no repetition, increasing for simplicity, and } \text{rk}(\{i_0, \dots, i_{n-1}\}, M; \kappa) = \beta, \text{ with } \text{rk}(\{i_0, \dots, i_{n-1}\}, M; \kappa) \not\geq \beta + 1 \text{ being witnessed by } \varphi(\{i_0, \dots, i_{n-1}\}) = R_{n, \zeta}, k(\{i_0, \dots, i_{n-1}\}) = k\}$ where the functions φ, k are as in the proof of 1.13. Let M be $(\lambda, <, R, R_0, R_{n, k}^*)_{n \in (0, \omega), k < n}$ expanded by Skolem functions.

So assume toward a contradiction that (b) holds, hence for this model M there are models M_0, M_1 and $a_\eta \in M_1$ for $\eta \in {}^\omega 2$ as required in clauses (i)–(v) of (b) of claim 2.5. Choose a nonempty finite subset w of ${}^\omega 2$ and β and ζ such that letting $w = \{\eta_0, \dots, \eta_{m-1}\}$ with $a_{\eta_l} <^{M_1} a_{\eta_{l+1}}$, we have

- (α) $M_1 \models R_{m, k}^*(a_{\eta_0}, \dots, a_{\eta_{m-1}}, \beta, \zeta)$,
- (β) $\beta \in R_0^{M_0} (\subseteq \alpha)$,
- (γ) β minimal under those constraints.

Note that there are $m, \eta_0, \dots, \eta_{m-1}, \beta$ and ζ such that (α) holds: for every nonempty $w \subseteq {}^\omega 2$, as M_1 is elementarily equivalent to M there are β, ζ as required in (α). Now (α) implies $\zeta \in R_0^{M_1}$, but $R_0^{M_1} = R_0^{M_0}$, so clause (β) holds too, and so we can satisfy (γ) too as the ordinals are well ordered. Let $\varphi' = \varphi(x_0, \dots, x_{m-1}, \beta, \zeta)$; note that the parameters are from $R_0^{M_1}$ (as M_1 is elementarily equivalent to M) and hence from $R_0^{M_0} \subseteq M$, and clause (v) (of (b) of 2.5) applies to φ' and $\langle \eta_0, \dots, \eta_{m-1} \rangle$, giving $n^* < \omega$. We can find $\eta'_k \in {}^\omega 2$, $\eta'_k \neq \eta_k$, $\eta'_k \upharpoonright n^* = \eta_k \upharpoonright n^*$, and easily for $w' = \{\eta, \dots, \eta_{m-1}, \eta'_k\}$ we can find $\beta' < \beta$, $\zeta' < \kappa$ and k such that $M_1 \models R_{m+1, k'}(a_{\eta_0}, \dots, a_{\eta_{m-1}}, a_{\eta'_k}, \beta', \zeta')$; then if $\beta \geq 0$ we get a contradiction to clause (γ) above. If $\beta = 0$ we use clause (i) to copy the situation to M and get a contradiction.

(2) Similar proof. $\blacksquare_{2.5}$

NOTATION 2.7. Let $\bar{\lambda}$ denote a finite (or countable) sequence of pairs of infinite cardinals $\langle (\lambda_\zeta; \kappa_\zeta) : \zeta < \zeta(*) \rangle$ such that κ_ζ increases with ζ , so e.g. $\bar{\lambda}^\oplus = \langle (\lambda_\zeta^\oplus; \kappa_\zeta^\oplus) : \zeta < \zeta^\oplus(*) \rangle$. We shall identify a strictly increasing $\bar{\kappa} = \langle \kappa_\zeta : \zeta \leq \zeta(*) \rangle$ with $\langle (\kappa_{\zeta+1}; \kappa_\zeta) : \zeta < \zeta(*) \rangle$.

Let $R, R_0, Q_0, \dots, R_{\zeta(*)-1}, Q_{\zeta(*)-1}$ be fixed unary predicates and $\bar{R} = \langle (R_\zeta, Q_\zeta) : \zeta < \zeta(*) \rangle$.

A $\bar{\lambda}$ -model M is a model M such that $R, R_\zeta, Q_\zeta \in \tau(M)$ are all unary predicates, $|R_\zeta^M| = \lambda_\zeta$, $|Q_\zeta^M| = \kappa_\zeta$ for $\zeta < \zeta(*)$, $Q_\zeta^M \subseteq R_\zeta^M$ and $\langle R_\zeta^M : \zeta < \zeta(*) \rangle$ are pairwise disjoint, and $R^M = \bigcup_{\zeta < \zeta(*)} R_\zeta^M$.

For $a \in R^M$ let $\zeta(a)$ be the ζ such that $a \in R_\zeta^M$ (e.g. $R_\zeta^M = \lambda_\zeta \setminus \bigcup_{\xi < \zeta} \lambda_\xi$, $\kappa_{\zeta+1} = \lambda_\zeta$). For a $\bar{\lambda}$ -model M we say that $a \in \text{cl}_\kappa(A, M)$ if $A \cup \{a\} \subseteq M$ and for some $n < \omega$, and quantifier free formula $\varphi(x_1, y_1, \dots, y_n)$ and $b_1, \dots, b_n \in A$ we have

$$M \models \varphi(a, b_1, \dots, b_n) \ \& \ (\exists^{\leq \kappa} x) \varphi(x, b_1, \dots, b_n).$$

DEFINITION 2.8. (1) For $l < 6$, $\bar{\lambda}$ as in Notation 2.7, and an ordinal α let $\text{Pr}_\alpha^l(\bar{\lambda}; \theta)$ mean that for every $\bar{\lambda}$ -model M with $|\tau(M)| \leq \theta$ we have $\text{rk}^l(M, \bar{\lambda}) \geq \alpha$ (and $\text{NPr}_\alpha^l(\bar{\lambda}, \theta)$ is the negation; if θ is omitted it means κ_0 ; remember $\bar{\lambda} = \langle (\lambda_\zeta, \kappa_\zeta) : \zeta < \zeta(*) \rangle$) where the rank is defined in part (2) below.

(2) For a $\bar{\lambda}$ -model M , $\text{rk}^l(M, \bar{\lambda}) = \sup\{\text{rk}^l(w, M, \bar{\lambda}) + 1 : w \in [\bar{R}^M]^*\}$ where the rank is defined in part (3) below and:

(α) if $\zeta(*)$ is finite, then

$$[\bar{R}^M]^* = \{w : w \text{ a finite subset of } R^M \text{ not disjoint from any } R_\zeta^M\},$$

(β) if $\zeta(*)$ is infinite, then

$$[\bar{R}^M]^* = \{w : w \text{ a finite nonempty subset of } R^M\}.$$

(3) For a $\bar{\lambda}$ -model M and $w \in [R^M]^*$ we define the truth value of “ $\text{rk}^l(w, M; \bar{\lambda}) \geq \alpha$ ” by induction on α :

• CASE A, $\alpha = 0$:

$$\text{rk}^l(w, M; \bar{\lambda}) \geq \alpha \quad \text{iff} \quad \text{no } a \in w \cap R^M \text{ belongs to } \text{cl}_{\kappa_{\zeta(a)}}(w \setminus \{a\}, M).$$

• CASE B, α is a limit ordinal:

$$\text{rk}^l(w, M; \bar{\lambda}) \geq \alpha \quad \text{iff} \quad \text{rk}^l(w, M; \bar{\lambda}) \geq \beta \text{ for every ordinal } \beta < \alpha.$$

• CASE C, $\alpha = \beta + 1$. We demand two conditions:

(α) exactly as in Definition 1.1(3)(*)₃ except that if $l = 2, 3, 4, 5$ we use $\kappa = \kappa_{\zeta(a_k)}^+$,

(β) if $\zeta < \zeta(*)$ and $w \cap R_\zeta^M = \emptyset$ then $\text{rk}^l(w \cup \{a\}, M; \theta) \geq \beta$ for some $a \in R_\zeta^M$.

CLAIM 2.9. *The parallel of the following holds: 1.2 (+ statements in 1.1), also 1.3 (use $\alpha < \kappa_0^+$), 1.5(2) (for $\alpha \leq \kappa_0^+$), 1.10 (satisfying κ_0^+ -c.c.) and we adopt 1.6.*

CLAIM 2.10. *If α is a limit ordinal and $\lambda_\xi \geq \beth_\alpha(\kappa_\xi)$ for every $\xi < \zeta(*)$ then $\text{Pr}_\alpha(\bar{\lambda})$.*

PROOF. Use indiscernibility and Erdős–Rado as in the proof of 1.8(1).

In more details, the induction hypothesis on α is, assuming $\zeta(*) < \omega$: if $A \subseteq R^M$ and $\bigwedge_{\zeta < \zeta(*)} |A \cap R_\zeta^M| \geq \beth_{\omega \times \alpha}$ then for every $\beta < \alpha$, $k < \omega$ and every $\bar{m} = \langle m_\zeta : \zeta < \zeta(*) \rangle$ where $m_\zeta \in (0, \omega)$ for some $w \subseteq A$ we have $\bigwedge_\zeta |w \cap R_\zeta^M| = m_\zeta$ and $\text{rk}(w; M, \bar{\lambda}) \geq \omega \times \beta + k$. Then for $\alpha = \gamma + 1$, choose distinct $a_i^\zeta \in A \cap R_\zeta^M$ ($i < \beth_{\omega \times \gamma + m + m_\zeta}$) and use polarized partition (see Erdős, Hajnal, Máté, Rado [EHMR]) on $\langle \langle a_i^\zeta : i < \beth_{\omega \times \alpha} \rangle : \zeta < \zeta(*) \rangle$. For $\zeta(*)$ infinite use $A \subseteq R^M$ such that $w_A = \{\zeta : A \cap R_\zeta^M \neq \emptyset\}$ is finite nonempty and $\zeta \in w_A \Rightarrow |A \cap R_\zeta^M| \geq \beth_{\omega \times \alpha}$, and proceed as above. ■_{2.10}

CLAIM 2.11. *Let $\zeta(*) < \omega$, $\kappa_0^\varepsilon < \dots < \kappa_{\zeta(*)}^\varepsilon$ and $\bar{\lambda}^\varepsilon = \langle (\kappa_{\xi+1}^\varepsilon, \kappa_\xi^\varepsilon) : \xi < \zeta(*) \rangle$ (for $\varepsilon \leq \omega$).*

(1) *If $\text{Pr}_n(\bar{\lambda}^n)$ for $n < \omega$, and for some $\theta \leq \kappa_0^\omega$ there is a tree $\mathcal{T} \in {}^\theta > (\kappa_0^\omega)$ of cardinality $\leq \kappa_0^\omega$ with $\geq \kappa_{\zeta(*)}^\omega$ θ -branches, then:*

- ⊗ *every first order sentence which has a $\bar{\lambda}^n$ -model for each n , also has a $\bar{\lambda}^\omega$ -model,*
- ⊗' *if \mathbf{T} is a first order theory of cardinality $\leq \kappa_0^\omega$ and every finite $\mathbf{T}' \subseteq \mathbf{T}$ has a $\bar{\lambda}^n$ -model for each n then \mathbf{T} has a $\bar{\lambda}^\omega$ -model.*

(2) *So if $\bar{\lambda}^\varepsilon = \bar{\lambda}$ for $\varepsilon \leq \omega$ are as above then we have κ_0^ω -compactness for the class of $\bar{\lambda}^\omega$ -models, where*

- (⊕) *a class \mathfrak{K} of models is κ -compact whenever for every set \mathbf{T} of $\leq \kappa$ first order sentences, if every finite subset of \mathbf{T} has a model in \mathfrak{K} then \mathbf{T} has a model in \mathfrak{K} .*

(3) *In part (1) we can use $\bar{\lambda}^n$ with domain w_n if $w_n \subseteq w_{n+1}$ and $\zeta(*) = \bigcup \{w_n : n < \omega\}$.*

PROOF. Straightforward if you have read [Sh 8], [Sh 18], [Sh 37] or will read the proof of 2.12 below (only that now the theory is not necessarily countable, no types omitted, and by compactness it is enough to deal with the case of $\zeta(*)$ finite). ■_{2.11}

CLAIM 2.12. *Let $\zeta(*) < \omega_1$ and $\bar{\lambda}^\varepsilon = \langle (\kappa_{\xi+1}^\varepsilon, \kappa_\xi^\varepsilon) : \xi < \zeta(*) \rangle$ for each $\varepsilon \leq \omega_1$ (and κ_ξ^ε strictly increasing with ξ). If $\text{Pr}_\varepsilon(\bar{\lambda}^\varepsilon)$ for every $\varepsilon < \omega_1$ and $\kappa_\xi^{\omega_1} \leq 2^{\aleph_0}$, and $\psi \in L_{\omega_1, \omega}$, and for each $\varepsilon < \omega_1$ there is a $\bar{\lambda}^\varepsilon$ -model satisfying ψ , then there is a $\bar{\lambda}^{\omega_1}$ -model satisfying ψ .*

PROOF. For simplicity, again as in [Sh 8], [Sh 18], [Sh 37], let M_ε be a $\bar{\lambda}^\varepsilon$ -model of ψ for $\varepsilon < \omega_1$. By expanding the M_ε 's by a pairing function and giving names of subformulas of ψ we have a countable first order theory \mathbf{T} with Skolem functions, a countable set Γ of 1-types and M_ε^+ such that:

- (a) M_ε^+ is a $\bar{\lambda}^\varepsilon$ -model of \mathbf{T} omitting each $p \in \Gamma$,
- (b) if M is a model of \mathbf{T} omitting every $p \in \Gamma$ then M is a model of ψ .

Now as in the proof of 2.1 we can find a model M^+ and a_η^ζ for $\zeta < \zeta(*)$ and $\eta \in {}^\omega 2$ such that:

- (α) M^+ is a model of \mathbf{T} .
- (β) $a_\eta^\zeta \in R_\zeta^M$ and $\eta \neq \nu \Rightarrow a_\eta^\zeta \neq a_\nu^\zeta$.
- (γ) For every first order formula $\varphi(x_0, \dots, x_{n-1}) \in L_{\omega, \omega}(\tau(\mathbf{T}))$ and ordinals $\zeta(0), \dots, \zeta(n-1) < \zeta(*)$, there is $k < \omega$ such that if $\eta_0, \dots, \eta_{n-1} \in {}^\omega 2$, $\nu_0, \dots, \nu_{n-1} \in {}^\omega 2$, $\langle \eta_l \upharpoonright k : l < n \rangle$ is with no repetitions and $\eta_l \upharpoonright k = \nu_l \upharpoonright k$ then $M^+ \models \varphi(a_{\eta_0}^{\zeta(0)}, \dots, a_{\eta_{n-1}}^{\zeta(n-1)}) \equiv \varphi(a_{\nu_0}^{\zeta(0)}, \dots, a_{\nu_{n-1}}^{\zeta(n-1)})$.
- (δ) If $\sigma(x_0, \dots, x_{n-1})$ is a term of $\tau(\mathbf{T})$ and $\zeta(0), \dots, \zeta(n-1) < \zeta(*)$, and $p \in \Gamma$ then for some $k < \omega$ and any $\varrho_0, \dots, \varrho_{n-1} \in {}^{k 2}$ pairwise distinct there is $\varphi(x) \in p(x)$ such that

$$(*) \quad \varrho_l \triangleleft \eta_l \in {}^\omega 2 \Rightarrow M^+ \models \neg \varphi(a_{\eta_0}^{\zeta(0)}, \dots, a_{\eta_{n-1}}^{\zeta(n-1)})$$

(i.e. this is our way to omit the types in Γ).

(ε) If $\zeta(0), \dots, \zeta(n-1) < \zeta(*)$, $\sigma(x_0, \dots, x_{n-1})$ is a term of $\tau(\mathbf{T})$, and $m < n$, then for some $k < \omega$, we have

- (*) if $\eta_0, \dots, \eta_{n-1} \in {}^\omega 2$, $\nu_0, \dots, \nu_{n-1} \in {}^\omega 2$, $\eta_l \upharpoonright k = \nu_l \upharpoonright k$, $\langle \eta_l \upharpoonright k : l < \omega \rangle$ is without repetitions and $\zeta(l) < \zeta(m) \Rightarrow \eta_l = \nu_l$ then

$$\begin{aligned} M^+ \models Q_{\zeta(m)}(\sigma(a_{\eta_0}^{\zeta(0)}, \dots, a_{\eta_{n-1}}^{\zeta(n-1)})) \ \& \ Q_{\zeta(m)}(\sigma(a_{\nu_0}^{\zeta(0)}, \dots, a_{\nu_{n-1}}^{\zeta(n-1)})) \\ \Rightarrow \sigma(a_{\eta_0}^{\zeta(0)}, \dots, a_{\eta_{n-1}}^{\zeta(n-1)}) &= \sigma(a_{\nu_0}^{\zeta(0)}, \dots, a_{\nu_{n-1}}^{\zeta(n-1)}). \end{aligned}$$

Now choose $Y_\zeta \subseteq {}^\omega 2$ of cardinality $\lambda_\zeta^{\omega_1}$ and let M^* be the $\tau(M)$ -reduct of the Skolem hull in M^+ of $\{a_\eta^\zeta : \zeta < \zeta(*) \text{ and } \eta \in Y_\zeta\}$. This is a model as required. ■_{2.12}

CONCLUSION 2.13. If $\mathbf{V}_0 \models GCH$, $\mathbf{V} = \mathbf{V}_0^{\mathbf{P}}$ for some c.c.c. forcing notion \mathbf{P} then e.g.

- (*) if $\aleph_{\omega \times 3} < 2^{\aleph_0}$, $\langle \aleph_0, \aleph_\omega, \aleph_{\omega+\omega} \rangle \rightarrow \langle \aleph_\omega, \aleph_{\omega+\omega}, \aleph_{\omega+\omega+\omega} \rangle$ (see [Sh 8]), i.e., letting

$$\bar{\lambda}^0 = \langle (\aleph_0, \aleph_\omega), (\aleph_\omega, \aleph_{\omega+\omega}) \rangle, \quad \bar{\lambda}^1 = \langle (\aleph_\omega, \aleph_{\omega+\omega}), (\aleph_{\omega+\omega}, \aleph_{\omega+\omega+\omega}) \rangle,$$

for any countable first order \mathbf{T} , if every finite $\mathbf{T}' \subseteq \mathbf{T}$ has a $\bar{\lambda}^0$ -model then \mathbf{T} has a $\bar{\lambda}^1$ -model.

(**) $\langle \kappa_\xi : \xi \leq \xi(*) \rangle \rightarrow \langle \kappa'_\xi : \xi \leq \xi(*) \rangle$ if $\bigwedge_\xi \kappa_\xi^{+\omega} \leq \kappa_{\xi+1}$ and $\kappa'_0 \leq \kappa'_1 \leq \dots \leq \kappa'_{\xi(*)} \leq 2^{\aleph_0}$ (and versions like 2.11(1)).

PROOF. By 2.10 if $\bar{\lambda} = \langle (\lambda_\xi, \kappa_\xi) : \xi < \zeta(*) \rangle$ with $\lambda_\xi \geq \beth_\omega(\kappa_\xi)$ then $\text{Pr}_n(\bar{\lambda})$ (really $\lambda_\xi \geq \beth_k(\kappa_\xi)$ for k depending on n only suffices, see [EHMR]). Now c.c.c. forcing preserves this and we apply 2.11. Similarly we can use θ^+ -c.c. forcing \mathbf{P} and deal with cardinals in the interval $(\theta, 2^\theta)$ in $\mathbf{V}^{\mathbf{P}}$. ■_{2.13}

REMARK 2.14. We can say parallel things for the compactness of $(\exists^{\geq \lambda})$, for λ singular $\leq 2^{\aleph_0}$ (or $\theta + |\mathcal{T}| < \lambda \leq$ number of θ -branches of \mathcal{T}), e.g. we get the parallel of 2.13.

In more details, if $\mathbf{V}_0 = \mathbf{V}^{\mathbf{P}}$ where \mathbf{P} satisfies the θ^+ -c.c. then

(*) in $\mathbf{V}_0^{\mathbf{P}}$, for any singular $\lambda \in (\theta, 2^\theta)$ such that $\mathbf{V}_0 \models$ “ λ is strong limit” we have

(*) the class $\{(\lambda, <, R_\zeta \dots)_{\zeta < \theta} : R_\zeta \text{ an } n_\zeta\text{-relation, } (\lambda, <) \text{ is } \lambda\text{-like}\}$ of models is θ -compact, and we can axiomatize it.

There are also consistent counterexamples, see [Sh 532].

The point of proving (*) is

⊗ for a vocabulary τ of cardinality $\leq \theta$, if we let $\mathbf{T}_\tau^{\text{sk}}$ be a first order theory with Skolem functions $\tau(T_\tau^{\text{sk}})$ (but the Skolem functions are new), then the following are equivalent for a first order $\mathbf{T} \subseteq L_{\omega, \omega}(\tau)$:

(a) \mathbf{T} has a λ -like model.

(b) The following is consistent: $\mathbf{T} \cup \mathbf{T}_\tau^{\text{sk}} \cup \{\sigma(\dots, x_{\eta_l}^{n(l)}, y_{n(l)}, \dots)_{l < k} = \sigma(\dots, x_{\nu_l}^{n(l)}, y_{n(l)}, \dots)_{l < k} \vee \sigma(\dots, x_{\eta_l}^{n(l)}, y_{n(l)}, \dots) > y_n : n(l) < \omega, \eta_l \in {}^\omega 2, \text{ and for some } j < \omega, \langle \eta_l \upharpoonright j : l < k \rangle \text{ is with no repetition, } \eta_l \upharpoonright j = \nu_l \upharpoonright j, n(l) < n \Rightarrow \eta_l = \nu_l\} \cup \{\sigma(\dots, x_{\eta_l}^{n(l)}, y_{n(l)}, \dots) < y_n : n(l) < n \text{ and } \eta_l \in {}^\omega 2\}$.

3. Finer analysis of square existence

DEFINITION 3.1. (1) For an ω -sequence $\bar{T} = \langle T_n : n < \omega \rangle$ of $(2, 2)$ -trees, we define a function degseq (square degree). Its domain is

$$\text{pfap} = \text{pfap}_{\bar{T}} = \{(u, g) : (\exists n)(u \in [{}^n 2]^*, g \text{ is a 2-place function from } u \text{ to } \omega)\}$$

and its values are ordinals (or ∞ or -1). For this we define the truth value of “ $\text{degseq}_{\bar{T}}(u, g) \geq \alpha$ ” by induction on the ordinal α .

CASE 1, $\alpha = -1$:

$$\text{degseq}_{\bar{T}}(u, g) \geq -1 \quad \text{iff} \quad (u, g) \in \text{pfap}_{\bar{T}} \text{ and } \eta, \nu \in u \Rightarrow (\eta, \nu) \in T_{g(\eta, \nu)}.$$

CASE 2, α is limit:

$$\text{degseq}_{\bar{T}}(u, g) \geq \alpha \quad \text{iff} \quad \text{degseq}_{\bar{T}}(u, g) \geq \beta \text{ for every } \beta < \alpha.$$

CASE 3, $\alpha = \beta + 1$: $\text{degsq}_{\bar{T}}(u, g) \geq \alpha$ iff for every $\varrho^* \in u$, for some m , $u^* \subseteq {}^m 2$, g^* and functions h_0, h_1 , we have:

- $h_i : u \rightarrow u^*$,
- $(\forall \eta \in u) \eta \triangleleft h_i(\eta)$,
- $(\forall \eta \in u)[h_0(\eta) = h_1(\eta) \Leftrightarrow \eta \neq \varrho^*]$,
- $u^* = \text{Rang}(h_0) \cup \text{Rang}(h_1)$,
- g^* is a 2-place function from u^* to ω ,
- $g^*(h_i(\eta), h_i(\nu)) = g(\eta, \nu)$ for $i < 2$ and $\eta, \nu \in u$,
- $\text{degsq}_{\bar{T}}(u^*, g^*) \geq \beta$ (so $(u^*, g^*) \in \text{pfap}$).

(2) We define $\text{degsq}_{\bar{T}}(u, g) = \alpha$ iff

$$\text{for every ordinal } \beta, \text{degsq}(u, g) \geq \beta \Leftrightarrow \beta \leq \alpha$$

(so $\alpha = -1, \alpha = \infty$ are legal values).

(3) We define $\text{degsq}(\bar{T}) = \bigcup \{\text{degsq}_{\bar{T}}(u, g) + 1 : (u, g) \in \text{pfap}_{\bar{T}}\}$.

CLAIM 3.2. Assume \bar{T} is an ω -sequence of $(2, 2)$ -trees.

(1) For every $(u, g) \in \text{pfap}_{\bar{T}}$, $\text{degsq}_{\bar{T}}(u, g)$ is an ordinal, ∞ or -1 . Any automorphism F of $({}^\omega 2, \triangleleft)$ preserves this (it acts on \bar{T} too, i.e.

$$\text{degsq}_{\bar{T}}(u, g) = \text{degsq}_{(F(T_n):n<\omega)}(F(u), g \circ F^{-1}).$$

(2) $\text{degsq}(\bar{T}) = \infty$ iff $\text{degsq}(\bar{T}) \geq \omega_1$ iff there is a perfect square contained in $\bigcup_{n<\omega} \lim(T_n)$ iff for some c.c.c. forcing notion \mathbf{P} , $\Vdash_{\mathbf{P}} \text{“}\bigcup_{n<\omega} \lim(T_n) \text{ contains a } \lambda_{\omega_1}(\aleph_0)\text{-square”}$ (so those properties are absolute).

(3) If $\text{degsq}(\bar{T}) = \alpha(*) < \omega_1$ then $\bigcup_{n<\omega} \lim(T_n)$ contains no $\lambda_{\alpha(*)+1}(\aleph_0)$ -square.

(4) For each $\alpha(*) < \omega_1$ there is an ω -sequence $\bar{T} = \langle T_n : n < \omega \rangle$ of $(2, 2)$ -trees with $\text{degsq}(\bar{T}) = \alpha(*)$.

(5) If $\bar{T} = \langle T_n : n < \omega \rangle$ is a sequence of $(2, 2)$ -trees then the existence of an \aleph_1 -square in $\bigcup_{n<\omega} \lim(T_n)$ is absolute.

(6) Moreover, for $\alpha(*) < \omega_1$ we have: if $\mu < \lambda_{\alpha(*)}(\aleph_0)$ and A, B are disjoint subsets of ${}^\omega 2 \times {}^\omega 2$ of cardinality $\leq \mu$, then some c.c.c. forcing notion \mathbf{P} adds \bar{T} as in (4) (i.e. an ω -sequence $\bar{T} = \langle T_n : n < \omega \rangle$ of $(2, 2)$ -trees with $\text{degsq}_{\bar{T}}(\bar{T}) = \alpha(*)$) such that

$$A \subseteq \bigcup_{n<\omega} \lim[f(T_n)], \quad B \cap \bigcup_{n<\omega} \lim[f(T_n)] = \emptyset.$$

Proof. Easy.

E.g. (3) Let $\lambda = \lambda_{\alpha(*)+1}(\aleph_0)$, assume $\{\eta_i : i < \lambda\} \subseteq {}^\omega 2$, $[i < j \Rightarrow \eta_i \neq \eta_j]$ and $(\eta_i, \eta_j) \in \bigcup_n \lim(T_n)$, and let $(\eta_i, \eta_j) \in \lim(T_{g(\eta_i, \eta_j)})$. For $(u, f) \in \text{pfap}_{\bar{T}}$ where $u = \{\nu_0, \dots, \nu_{k-1}\}$ (with no repetition, $<_{\text{lx}}$ -increasing) let $R_{(u, f)} = \{(\alpha_0, \dots, \alpha_{k-1}) : \alpha_l < \lambda \text{ and } \nu_l \triangleleft \eta_{\alpha_l} \text{ for } l < k \text{ and } f(\nu_l, \nu_m) = g(\eta_{\alpha_l}, \eta_{\alpha_m}) \text{ for } l, m < k\}$. Let $M = (\lambda, R_{(u, f)})_{(u, f) \in \text{pfap}_{\bar{T}}}$. Clearly if we have

$\alpha_0, \dots, \alpha_{k-1} < \lambda$ and n such that $\langle \eta_{\alpha_l} \upharpoonright n : l < k \rangle$ is with no repetition, and $g(\eta_{\alpha_{l(1)}}, \eta_{\alpha_{l(2)}}) = f(\eta_{\alpha_{l(1)}} \upharpoonright n, \eta_{\alpha_{l(2)}} \upharpoonright n)$ then $R_{(u,f)}(\alpha_0, \alpha_1, \dots, \alpha_{k-1})$ and we can then prove

$$\text{rk}(\{\alpha_0, \dots, \alpha_{k-1}\}, M) \leq \text{degsq}_{\bar{T}}(u, f)$$

(by induction on the left ordinal). But M is a model with countable vocabulary and cardinality $\lambda = \lambda_{\alpha(*)+1}(\aleph_0)$. Hence by the definition of $\lambda_{\alpha(*)+1}$ we have $\text{rk}(M) \geq \alpha(*) + 1$, so $\alpha(*) + 1 \leq \text{rk}(M) \leq \text{degsq}(\bar{T}) \leq \alpha(*)$ (by the previous sentence, an earlier sentence and a hypothesis respectively). Contradiction.

(4) Let $w_i \subseteq [-1, \alpha(*))$ be finite $\neq \emptyset$, $w_i \subseteq w_{i+1}$ and $[-1, \alpha(*)) = \bigcup_{i < \omega} w_i$ and let

$W = \{\eta : \eta \text{ is a (strictly) decreasing sequence of ordinals, possibly empty}\}$.

We choose, by induction on $i < \omega$, n_i and an indexed set $\langle (u_x^i, f_x^i, \alpha_x^i) : x \in X_i \rangle$ such that:

- (a) $n_i < \omega$, $n_0 = 0$, $n_i < n_{i+1}$,
- (b) X_i is finite including $\bigcup_{j < i} X_j$,
- (c) for $x \in X_i$, $u_x^i \subseteq {}^{n_i}2$, f_x^i is a 2-place function from u_x^i to ω and $\alpha_x^i \in w_i$,
- (d) for some $x \in X_i$, $u_x^i = \{0_{n_i}\}$,
- (e) h_i is a function from X_i into W and $h_i \subseteq h_{i+1}$,
- (f) $|X_0| = 1$, and h_0 is constantly $\langle \rangle$,
- (g) if $x \in X_i$ then $\alpha_x^{i+1} = \alpha_x^i$, the function $\nu \mapsto \nu \upharpoonright n_i$ is one-to-one from u_x^{i+1} onto u_x^i and $\nu \in u_x^{i+1} \Rightarrow \nu \upharpoonright [n_i, n_{i+1}) = 0_{[n_i, n_{i+1})}$ and $\eta, \nu \in u_x^{i+1} \Rightarrow f_x^{i+1}(\eta, \nu) = f_x^i(\eta \upharpoonright n_i, \nu \upharpoonright n_i)$,
- (h) for some $x = x_i \in X_i$, $\beta = \beta_i \in w_{i+1} \cap \alpha_x^i$ and $\varrho^* = \varrho_i^* \in u_x^i$ and $\mathcal{Y}_i \in W$ such that $h_i(x_i) \triangleleft \mathcal{Y}_i$ we can find $y = y_i$ such that:

- (α) $X_{i+1} = X_i \cup \{y_i\}$, $y_i \notin X_i$,
- (β) $\alpha_y^{i+1} = \beta$,
- (γ) the function $\nu \mapsto \nu \upharpoonright n_i$ is a function from u_y^{i+1} onto u_x^i , almost one-to-one: ϱ^* has exactly two predecessors, say ϱ_1^y, ϱ_2^y , and any other $\varrho \in u_x^i \setminus \{\varrho^*\}$ has exactly one predecessor in u_y^{i+1} ,
- (δ) for $\nu, \eta \in u_y^{i+1}$ if $(\nu, \eta) \neq (\varrho_1^y, \varrho_2^y)$ and $(\nu, \eta) \neq (\varrho_2^y, \varrho_1^y)$ then $f_y^{i+1}(\eta, \nu) = f_x^i(\eta \upharpoonright n_i, \nu \upharpoonright n_i)$,
- (ε) $h_{i+1} = h_i \cup \{(y_i, \mathcal{Y}_i)\}$,
- (i) if $x_1, x_2 \in X_i$ and $u_{x_1}^i \cap u_{x_2}^i \neq \emptyset$ then $u_{x_1}^i \cap u_{x_2}^i = \{0_{n_i}\}$,
- (j) if $x \in X_i$, $\beta \in w_i \cap \alpha_x^i$, $\varrho^* \in u_x^i$ and $h(x) \triangleleft \mathcal{Y} \in W$ then for some $j \in (i, \omega)$ we have $x_j = x$, $\beta_j = \beta$, $\varrho^* \leq \varrho_j^*$ and $\mathcal{Y}_j = \mathcal{Y}$,
- (k) the numbers $f_{y_i}^{i+1}(\varrho_1^{y_1}, \varrho_2^{y_1})$, $f_{y_i}^{i+1}(\varrho_2^{y_2}, \varrho_1^{y_2})$ are distinct and do not belong to $\bigcup \{\text{Rang}(f_x^i) : x \in X_i\}$.

There is no problem to carry out the definition. We then let

$$T_n = \{(\eta, \nu) : f_x^i(\eta', \nu') = n \text{ for some } i < \omega, x \in X_i \text{ and } \eta', \nu' \in u_x^i, \\ \text{and } (\eta, \nu) = (\eta' \upharpoonright l, \nu' \upharpoonright l) \text{ for some } l \leq n_i\}$$

and $\bar{T} = \langle T_n : n < \omega \rangle$. Now it is straightforward to compute the rank.

(5) By the completeness theorem for $L_{\omega_1, \omega}(Q)$ (see Keisler [Ke71]).

(6) By the proof of 1.13. $\blacksquare_{3.2}$

Now we turn to κ -Suslin sets.

DEFINITION 3.3. Let T be a $(2, 2, \kappa)$ -tree. Let $\text{set}(T)$ be the set of all pairs (u, f) such that

$$(\exists n = n(u, f))[u \subseteq {}^n 2 \ \& \ f : u \times u \rightarrow {}^n \kappa \ \& \ \eta, \nu \in u \Rightarrow (\eta, \nu, f(\eta, \nu)) \in T].$$

We want to define $\text{degsq}_T(x)$ for $x \in \text{set}(T)$. By induction on the ordinal α we define when $\text{degsq}_T(x) \geq \alpha$:

CASE 1. $\alpha = -1$:

$$\text{degsq}_T(u, f) \geq \alpha \quad \text{iff} \quad (u, f) \in \text{set}(T).$$

CASE 2. α limit:

$$\text{degsq}_T(u, f) \geq \alpha \quad \text{iff} \quad \text{degsq}_T(u, f) \geq \beta \text{ for every } \beta < \alpha.$$

CASE 3. $\alpha = \beta + 1$: $\text{degsq}_T(u, f) \geq \alpha$ iff for every $\eta^* \in u$, for some $m > n(u, f)$ there are $(u^*, f^*) \in \text{set}(T)$ and functions h_0, h_1 such that $\text{degsq}_T(u^*, f^*) \geq \beta$ and:

- (i) $n(u^*, f^*) = m$,
- (ii) h_i is a function from u to ${}^m 2$,
- (iii) $\eta \triangleleft h_i(\eta)$ for $i < 2$,
- (iv) for $\eta \in u$ we have $h_0(\eta) \neq h_1(\eta) \Leftrightarrow \eta = \eta^*$,
- (v) for $\eta_1 \neq \eta_2 \in u, i < 2$ we have $f(\eta_1, \eta_2) \triangleleft f^*(h_i(\eta_1), h_i(\eta_2))$,
- (vi) for $\eta \in u^*$ we have $f(\eta \upharpoonright n, \eta \upharpoonright n) \triangleleft f^*(\eta, \eta)$.

Lastly, $\text{degsq}_T(u, f) = \alpha$ iff $[\beta \leq \alpha \Leftrightarrow \text{degsq}_T(u, f) \geq \beta]$ for every ordinal β .

Also let $\text{degsq}(T) = \text{degsq}_T(\{\langle \rangle\}, \{\langle \rangle, \langle \rangle >\})$.

CLAIM 3.4. (1) For a $(2, 2, \kappa)$ -tree T and $(u, f) \in \text{set}(T)$, $\text{degsq}_T(u, f)$ is an ordinal or infinity or $= -1$, and similarly for $\text{degsq}(T)$. All are absolute. Also $\text{degsq}(T) \geq \kappa^+$ implies $\text{degsq}(T) = \infty$ and similarly for $\text{degsq}_T(u, f)$.

(2) $\text{degsq}(T) = \infty$ iff $\Vdash_{\mathbf{P}}$ “ $\text{prj lim } T (= \{(\eta, \nu) \in {}^\omega 2 \times {}^\omega 2 : \text{for some } \varrho \in {}^\omega \kappa, \bigwedge_{n < \omega} (\eta \upharpoonright n, \nu \upharpoonright n, \varrho \upharpoonright n) \in T\})$ contains a perfect square” for every forcing notion \mathbf{P} including a trivial one, i.e. $\mathbf{V}^{\mathbf{P}} = \mathbf{V}$ iff $\Vdash_{\mathbf{P}}$ “ $\text{prj lim}(T)$ contains a 2^{\aleph_0} -square” for the forcing notion \mathbf{P} which is adding λ Cohen reals for $\lambda = \lambda_{\kappa^+}^3(\kappa)$ iff for some \mathbf{P} , $\Vdash_{\mathbf{P}}$ “ $\text{prj lim}(T)$ contains a $\lambda_{\kappa^+}(\aleph_0)$ -square”.

(3) If $\alpha(*) = \text{deg}\text{sq}(T) < \kappa^+$, then $\text{prj}\lim(T)$ contains no $\lambda_{\alpha(*)+1}(\kappa)$ -square.

Proof. (1) Easy.

(2) Assume that $\text{deg}\text{sq}(T) = \infty$, and note that $\alpha^* = \{\text{deg}\text{sq}_T(u, f) : (u, f) \in \text{set}(T) \text{ and } \text{deg}\text{sq}_T(u, f) < \infty\} \setminus \{\infty\}$ is an ordinal so $(u, f) \in \text{set}(T) \ \& \ \text{deg}\text{sq}_T(u, f) \geq \alpha^* \Rightarrow \text{deg}\text{sq}_T(u, f) = \infty$ (in fact, any ordinal $\alpha \geq \sup\{\text{deg}\text{sq}_T(u, f) + 1 : (u, f) \in \text{set}(T)\}$ will do). Let $\text{set}^\infty(T) = \{(u, f) \in \text{set}(T) : \text{deg}\text{sq}_T(u, f) = \infty\}$. Now

(*)₁ There is $(u, f) \in \text{set}^\infty(T)$.

(*)₂ For every $(u, f) \in \text{set}^\infty(T)$ and $\varrho \in u$ we can find $(u^+, f^+) \in \text{set}^\infty(T)$ and for $e=1, 2$, $h_e : u \rightarrow u^+$ such that $(\forall \eta \in u)(\eta \triangleleft h_e(\eta))$, $(\forall \eta, \nu \in u)(f(\eta, \nu) \triangleleft f^+(h_e(\eta), h_e(\nu)))$, $(\forall \eta \in u)[h_1(\eta) = h_2(\eta) \Leftrightarrow \eta = \varrho]$.

[Why? As $\text{deg}\text{sq}_T(u, f) = \infty$ it is $\geq \alpha^* + 1$ so by the definition we can find (u^+, f^+) and h_1, h_2 as above but only with $\text{deg}\text{sq}_T(u^+, f^+) \geq \alpha^*$, and this implies $\text{deg}\text{sq}_T(u^+, f^+) = \infty$.]

(*)₃ For every $(u, f) \in \text{set}^\infty(T)$ with $u = \{\eta_\varrho : \varrho \in {}^n 2\} \subseteq ({}^{n_1} 2)$ (no repetition) we can find $n_2 > n_1$ and $(u^+, f^+) \in \text{set}^\infty(T)$ with $u^+ = \{\eta_\varrho : \varrho \in {}^{n+1} 2\} \subseteq ({}^{n_2} 2)$ (no repetitions) such that:

- (i) $\varrho \in {}^{n+1} 2 \Rightarrow \eta_{\varrho \upharpoonright n} \triangleleft \eta_\varrho$,
- (ii) for $\varrho_1, \varrho_2 \in {}^{n+1} 2$, $\varrho_1 \upharpoonright n \neq \varrho_2 \upharpoonright n \Rightarrow f(\eta_{\varrho_1 \upharpoonright n}, \eta_{\varrho_2 \upharpoonright n}) \triangleleft f^+(\eta_{\varrho_1}, \eta_{\varrho_2})$,
- (iii) for $\varrho \in {}^{n+1} 2$, $f(\eta_{\varrho \upharpoonright n}, \eta_{\varrho \upharpoonright n}) \triangleleft f(\eta_\varrho, \eta_\varrho)$.

[Why? Repeat (*)₂ 2^n times.]

So we can find $\langle \eta_\varrho : \varrho \in {}^n 2 \rangle$ and f_n by induction on n such that $\langle \eta_\varrho : \varrho \in {}^n 2 \rangle$ is with no repetition, $\text{deg}\text{sq}_T(\{\eta_\varrho : \varrho \in {}^n 2\}, f_n) = \infty$, and for each n clauses (i)–(iii) of (*)₃ hold, i.e. $\varrho_1, \varrho_2 \in {}^{n+1} 2$, $\varrho_1 \upharpoonright n \neq \varrho_2 \upharpoonright n \Rightarrow f_n(\eta_{\varrho_1 \upharpoonright n}, \eta_{\varrho_2 \upharpoonright n}) \triangleleft f_{n+1}(\eta_{\varrho_1}, \eta_{\varrho_2})$ and for $\varrho \in {}^{n+1} 2$ we have $f_n(\eta_{\varrho \upharpoonright n}, \eta_{\varrho \upharpoonright n}) \triangleleft f_{n+1}(\eta_\varrho, \eta_\varrho)$ and of course $\{\eta_\varrho : \varrho \in {}^m 2\} \subseteq ({}^{k_n} 2)$ with $k_n < k_{n+1} < \omega$.

So we have proved that the first clause implies the second (about the forcing: the $\text{deg}\text{sq}(T) = \infty$ is absolute so holds also in $\mathbf{V}^{\mathbf{P}}$ for any forcing notion \mathbf{P}). Trivially the second clause implies the third and fourth. So assume the third clause and we shall prove the first. By 1.8, $\lambda_{\kappa^+}^2(\kappa)$ is well defined (e.g. $\leq \beth_{\kappa^+}$), but $\lambda_{\kappa^+}^3(\kappa) = \lambda_{\kappa^+}^2(\kappa)$ by 1.5(3). Let \mathbf{P} be the forcing notion adding $\lambda_{\kappa^+}^2(\kappa)$ Cohen reals. By 1.11(2) in $\mathbf{V}^{\mathbf{P}}$, $\lambda_{\kappa^+}(\kappa) \leq \lambda_{\kappa^+}^2(\kappa) \leq 2^{\aleph_0}$, and so there are pairwise disjoint $\eta_i \in {}^\omega 2$ for $i < \lambda_{\kappa^+}(\kappa)$ such that $(\eta_i, \eta_j) \in \text{prj}\lim(T)$ for $i, j < \lambda_{\kappa^+}(\kappa)$. Lastly, we prove that the fourth clause implies the first in $\mathbf{V}^{\mathbf{P}}$. By part (3) of the claim proved below, for every $\alpha < \kappa^+$ we get $\neg[\alpha = \text{deg}\text{sq}(T)]$ as $\lambda_\alpha(\kappa) \leq \lambda_{\kappa^+}(\kappa)$. Hence $\text{deg}\text{sq}(T) \geq \kappa^+$, but by part (1) this implies $\text{deg}\text{sq}(T) = \infty$.

(3) Just like the proof of 3.2(3). ■_{3.4}

We shall prove in [Sh 532]

CLAIM 3.5. Assume $\alpha(*) < \kappa^+$ and $\lambda < \lambda_{\alpha(*)}(\kappa)$.

(1) For some c.c.c. forcing notion \mathbf{P} , in $\mathbf{V}^{\mathbf{P}}$ there is a κ -Suslin subset $A = \text{prj lim}(T)$ (where T is a $(2, 2, \kappa)$ -tree) such that

(*) A contains a λ -square but $\text{degsq}(T) \leq \alpha(*)$.

(2) For given $B \subseteq (\omega 2 \times \omega 2)^{\mathbf{V}}$ of cardinality $\leq \lambda$ we can replace (*) by

(*)' $A \cap (\omega 2 \times \omega 2)^{\mathbf{V}} = B$ but $\text{degsq}(T) \leq \alpha(*)$.

REMARK 3.6. The following says in fact that “colouring of pairs is enough”, say for the Hanf number of $L_{\omega_1, \omega}$ below the continuum; for clarification see 3.8.

CLAIM 3.7 (MA). Assume $\lambda < 2^{\aleph_0}$ and $\alpha(*) < \omega_1$ is a limit ordinal, $\lambda < \mu := \lambda_{\alpha(*)}(\aleph_0)$. Then for some symmetric 2-place function F from λ to ω we have

(*) $_{\lambda, \mu, F}$ for no 2-place (symmetric) function F' from μ to ω do we have:

(**) for every $n < \omega$ and pairwise distinct $\beta_0, \dots, \beta_{n-1} < \mu$ there are pairwise distinct $\alpha_0, \dots, \alpha_{n-1} < \lambda$ such that

$$\bigwedge_{k < l < n} F'(\beta_k, \beta_l) = F(\alpha_k, \alpha_l).$$

REMARK 3.8. (1) This is close to Gilchrist–Shelah [GcSh 491].

(2) The proof of 3.7 says that letting $R_n = \{(\alpha, \beta) : F(\alpha, \beta) = n\}$ and $N = (\lambda, R_n)_{n < \omega}$, we have $\text{rk}(N) < \alpha(*)$.

(3) So 3.7 improves §2 by saying that the examples for the Hanf number of $L_{\omega_1, \omega}$ below the continuum being large can be very simple, speaking only about “finite patterns” of colouring pairs by countably many colours.

PROOF (of Claim 3.7). Let M be a model of cardinality λ with a countable vocabulary and $\text{rk}^1(M) < \alpha(*)$. We can assume that it has universe λ , relation $<$ and individual constants c_α for $\alpha \leq \omega$. Let k^M, φ^M be as in the proof of 1.13.

Let \mathbf{P} be the set of triples (u, f, \mathbf{w}) such that:

(a) u is a finite subset of λ ,

(b) f is a symmetric 2-place function from u to ω ,

(c) \mathbf{w} is a family of nonempty subsets of u such that

(d) if $\alpha \in u$ then $\{\alpha\} \in \mathbf{w}$,

(e) if $w = \{\alpha_0, \dots, \alpha_{n-1}\} \in \mathbf{w}$ (increasing enumeration), $k = k^M(w)$, $\alpha \in u \setminus w$ and $(\forall l)[l < n \ \& \ l \neq k \Rightarrow f(\alpha_l, \alpha) = f(\alpha_l, \alpha_k)]$ then $w \cup \{\alpha\}$ belongs to \mathbf{w} , $(\forall m \neq k)(\alpha < \alpha_m \Leftrightarrow \alpha_k < \alpha_m)$ and $k = k^M(w \cup \{\alpha\} \setminus \{\alpha_k\})$ and $M \models \varphi^M(w)[\alpha_0, \dots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \dots, \alpha_{n-1}]$,

(f) if $w^i = \{\alpha_0^i, \dots, \alpha_{n-1}^i\} \subseteq u$ (increasing enumeration, so with no repetition), $i = 0, 1$, and $(\forall l < k < n)[f(\alpha_l^0, \alpha_k^0) = f(\alpha_l^1, \alpha_k^1)]$ then $w^0 \in \mathbf{w} \Leftrightarrow w^1 \in \mathbf{w}$ and if $w^i \in \mathbf{w}$ then $\varphi^M(w^0) = \varphi^M(w^1)$, $k^M(w^0) = k^M(w^1)$ and $\text{rk}^1(w^0, M) = \text{rk}^1(w^1, M)$.

The order is the natural one.

It is easy to check that:

\oplus_1 \mathbf{P} satisfies the c.c.c.,

\oplus_2 for every $\alpha < \lambda$, $\mathcal{L}_\alpha = \{(u, f, \mathbf{w}) \in \mathbf{P} : \alpha \in u\}$ is dense.

Hence there is a directed $G \subseteq \mathbf{P}$ not disjoint from \mathcal{L}_α , for every $\alpha < \lambda$. Let $F = \bigcup \{f : (u, f, \mathbf{w}) \in G \text{ for some } u, \mathbf{w}\}$. We shall show that it is as required. Clearly F is a symmetric 2-place function from λ to ω ; so the only thing that can fail is if there is a symmetric 2-place function F' from μ to ω such that $(**)$ of 3.7 holds. By the compactness theorem for propositional logic, there is a linear order $<^*$ of μ such that

$(*)'$ for every $n < \omega$ and $\beta_0 <^* \dots <^* \beta_{n-1}$ from μ there are $\alpha_0 < \dots < \alpha_{n-1} < \lambda$ such that $\bigwedge_{k < l < n} F'(\beta_k, \beta_l) = F(\alpha_k, \alpha_l)$.

Let

$$\mathbf{W} = \bigcup \{\mathbf{w} : (u, f, \mathbf{w}) \in G \text{ for some } u, f\}.$$

Let $N = (\lambda, R_n)_{n < \omega}$ where $R_n = \{(\alpha, \beta) : F(\alpha, \beta) = n\}$, so $\text{rk}^1(w, N)$ for $w \in [\lambda]^*$ is well defined; in fact, we can restrict ourselves to formulas of the form $\varphi(x_0, \dots, x_{n-1}) = \bigwedge_{l < k < n} F(x_l, x_k) = c_{m(l,k)}$. Let $N' = (\mu, R'_n)_{n < \omega}$ where $R'_n = \{(\alpha, \beta) : F'(\alpha, \beta) = n\}$. Now first note that

\otimes_1 if $w \in \mathbf{W}$ then $\text{rk}^1(w, N) \leq \text{rk}^1(w, M)$

(this is the role of clause (e) in the definition of \mathbf{P} ; we prove it by induction on $\text{rk}^1(w, M)$ using the same “witness” k^M). Secondly,

\otimes_2 $\{\alpha\} \in \mathbf{W}$ for $\alpha < \lambda$

(this is the role of clause (d) in the definition of \mathbf{P}). Hence we conclude (by 1.1(2), \otimes_1 , \otimes_2) that

\otimes_3 $\text{rk}^1(N) < \alpha(*)$.

Lastly,

\otimes_4 if $\alpha_0 <^* \dots <^* \alpha_{m-1}$ are $< \mu$, $\beta_0 < \dots < \beta_{m-1} < \lambda$, $\bigwedge_{k < l < m} F'(\alpha_k, \alpha_l) = F(\beta_k, \beta_l)$ and $\{\beta_0, \dots, \beta_{m-1}\} \in \mathbf{W}$ then $\text{rk}^1(\{\alpha_0, \dots, \alpha_{m-1}\}, N') \leq \text{rk}^1(\{\beta_0, \dots, \beta_{m-1}\}, N)$

(again it can be proved by induction on $\text{rk}^1(\{\beta_l, \dots, \beta_{m-1}\}, N)$, the choice of N' and our assumption toward a contradiction that $(**)$ of the claim holds). Now by \otimes_3 , \otimes_4 and 1.1(2) (and \otimes_2) we have $\text{rk}^1(N') \leq \text{rk}^1(N) < \alpha(*)$, but this contradicts $\|N'\| = \mu = \lambda_{\alpha(*)}(\aleph_0)$. $\blacksquare_{3.7}$

CLAIM 3.9 (MA). *If λ, μ, F are as in 3.7 (i.e. $(*)_{\lambda, \mu, F}$ holds) then some Borel set $B \subseteq \omega^2 \times \omega^2$ (actually of the form $\bigcup_{n < \omega} \text{lim}(T_n)$) has a λ -square but no μ -square.*

REMARK 3.10. (1) The converse holds too, of course.

(2) We can use “all $\lambda < \mu$ ”.

PROOF (of Claim 3.9). We can assume $\text{cf}(\lambda) > \aleph_0$ (otherwise combine ω examples). Let F be a symmetric 2-place function from λ to ω such that $(*)_{\lambda, \mu, F}$. For simplicity let $f^* : \omega \rightarrow \omega$ be such that $(\forall n)(\exists^\infty m)(f^*(m) = n)$. We define a forcing notion \mathbf{P} as in 1.13 except that we require in addition for $p \in \mathbf{P}$:

- ⊗₁ $f^*(g^p(\alpha, \beta)) = F(\alpha, \beta)$,
- ⊗₂ if $\alpha' \neq \beta', \alpha'' \neq \beta''$ are from u^p , $k < \omega$, $\eta_{\alpha'}^p \upharpoonright k = \eta_{\alpha''}^p \upharpoonright k \neq \eta_{\beta'}^p \upharpoonright k = \eta_{\beta''}^p \upharpoonright k$ both not constantly 1 then $F(\alpha', \beta') = F(\alpha'', \beta'')$,
- ⊗₃ if $\eta, \nu \in {}^{n[p]}2$ then for at most one $m < m[p]$ we have $(\eta, \nu) \in t_m^p$,
- ⊗₄ if $n < \omega$, $n > 1$, $\eta_0 <_{1x} \dots <_{1x} \eta_{n-1}$ are pairwise distinct members of ${}^{n(p)}2$ and $k < l \Rightarrow (\eta_k, \eta_l) \in t_{g(k,l)}^p$ then for some pairwise distinct $\alpha_0, \dots, \alpha_{n-1}$ from $u[p^n]$, we have

$$\bigwedge_{k < l < n} f^*(h(k, l)) = F(\alpha_k, \alpha_l).$$

We have

- ⊕₁ if $\alpha < \beta < \lambda$, then there is a unique $n < \omega$ such that $(\eta_\alpha, \eta_\beta) \in \text{lim } \underline{T}_n$.

Thus $\bigcup_{n \in \omega} T_n$ contains a λ -square. In proving that it does not contain a μ -square we apply $(*)_{\lambda, \mu, F}$. For this the crucial fact is

- ⊕₂ if $n < \omega$, $\eta_0, \dots, \eta_{n-1} \in \omega^2$ are distinct, $(\eta_k, \eta_l) \in \text{lim}(T_{h(k,l)})$ then for some pairwise distinct $\alpha_0, \dots, \alpha_{n-1} < \lambda$,

$$\bigwedge_{k < l < n} f^*(h(k, l)) = F(\alpha_k, \alpha_l).$$

Instead of “ \mathcal{T}_α^3 is dense” it is enough to show

- ⊕₃ for some $p \in \mathbf{P}$, $p \Vdash$ “the number of $\alpha < \lambda$ such that $\alpha \in u^q$ for some $q \in \mathcal{G}_\mathbf{P}$ is λ ”. ■_{3.9}

Similarly we can show

CLAIM 3.11 (MA). *Assume $\aleph_0 < \lambda < 2^{\aleph_0}$. Then the following are equivalent:*

(a) *For some symmetric 2-place functions F_μ from μ to ω (for $\mu < \lambda$) we have*

$(*)_{\langle F_\mu : \mu < \lambda \rangle}$ *for no 2-place function F' from λ to ω do we have:*

(**) for every $n < \omega$ and pairwise distinct $\beta_0, \dots, \beta_{n-1} < \lambda$ there are $\mu < \lambda$ and pairwise distinct $\alpha_0, \dots, \alpha_{n-1} < \mu$ such that

$$\bigwedge_{k < l < n} F'(\beta_k, \beta_l) = F_\mu(\alpha_k, \alpha_l).$$

(b) Some Borel subset of ${}^\omega 2 \times {}^\omega 2$ contains a μ -square iff $\mu < \lambda$ (in fact, B is an F_σ set).

4. Rectangles. Simpler than squares are rectangles: subsets of ${}^\omega 2 \times {}^\omega 2$ of the form $X_0 \times X_1$, characterized by pairs of cardinals $\langle \lambda_0, \lambda_1 \rangle$ where $\lambda_l = |X_l|$. So we would like to define ranks and cardinals which characterize their existence just as $\text{rk}^l(w, M; \kappa)$, $\lambda_\alpha^l(\kappa)$, $\text{degsq}_{\bar{T}}(-)$, $\text{degsq}_T(-)$ have done for squares. Though the demands are weaker, the formulation is more cumbersome: we need two “kinds” of variables, one corresponding to λ_0 and one to λ_1 . So the models have two distinguished predicates, R_0, R_1 (corresponding to X_0, X_1 respectively), and in the definition of rank we connect only elements from distinct sides (in fact, in §3 we already concentrate on 2-place relations explaining not much is lost). This is very natural as except for inequality nothing connects two members of X_0 or two members of X_1 .

DEFINITION 4.1. We define $\text{Prrc}_\alpha^l(\lambda_1, \lambda_2; < \kappa, \theta_0, \theta_1)$, $\text{rkrc}^l((w_1, w_2), M, \lambda_1, \lambda_2; \kappa, \theta_0, \theta_1)$ as in 1.1 (but $w_l \in [R_l^M]^*$ and $|R_l^M| = \lambda_l$), replacing rk by rkrc etc. Let $\bar{\lambda} = (\lambda_1, \lambda_2)$ and $\bar{w} = (w_1, w_2)$.

(1) For $l < 6$, cardinals $\bar{\lambda} = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 \geq \kappa$ and $\bar{\theta} = (\theta_0, \theta_1)$ with $\theta_0 \leq \theta_1 < \lambda_1, \lambda_2$, and an ordinal α , let $\text{Prrc}_\alpha^l(\bar{\lambda}; < \kappa, \bar{\theta})$ mean that for every model M with vocabulary of cardinality $\leq \theta_0$ such that $\bigwedge_{i=1}^2 |R_i^M| = \lambda_i$, $R_1^M \cap R_2^M = \emptyset$, F^M is a 2-place function with range included in $\theta_1 = Q^M$, we have $\text{rkrc}^l(M; < \kappa) \geq \alpha$ (defined below).

Let $\text{NPrrc}_\alpha^l(\bar{\lambda}; < \kappa, \bar{\theta})$ be the negation. Instead of $< \kappa^+$ we may write κ ; if $\kappa = \theta_0^+$ we may omit θ_0 ; if $\theta_0 = \aleph_0$, $\kappa = \aleph_1$, we may omit them. We may write θ_0, θ_1 instead of $\bar{\theta} = (\theta_0, \theta_1)$ and similarly for $\bar{\lambda}$. Lastly, we let $\lambda\text{rc}_\alpha^l(\kappa, \bar{\theta}) = \min\{\lambda : \text{Prrc}_\alpha^l(\lambda, \lambda; < \kappa, \bar{\theta})\}$.

(2) For a model M ,

$$\begin{aligned} \text{rkrc}^l(M; < \kappa) = \sup\{ & \text{rkrc}^l(\bar{w}, M; < \kappa) + 1 : \bar{w} = \langle w_1, w_2 \rangle \text{ where} \\ & w_i \subseteq R_i^M \text{ are finite nonempty for } i = 1, 2 \text{ and} \\ & (\exists c \in Q^M)(\forall a, b)[a \in w_1 \ \& \ b \in w_2 \Rightarrow F(a, b) = c] \} \end{aligned}$$

where rkrc is as defined in part (3) below.

(3) For a model M and

$$\begin{aligned} \bar{w} \in [M]^\otimes := \{ & \bar{u} = (u_1, u_2), u_i \subseteq R_i^M \text{ are finite nonempty and} \\ & (\exists c)(\forall a, b)[a \in u_1 \ \& \ b \in u_2 \Rightarrow F(a, b) = c] \} \end{aligned}$$

we shall define the truth value of $\text{rkrc}^l(\bar{w}, M; < \kappa) \geq \alpha$ by induction on the ordinal α (for $l = 0, 1$, κ can be omitted). If we write w instead of w_1 , w_2 we mean $w_1 = w \cap R_1^M$, $w_2 = w \cap R_2^M$ (here $R_1^M \cap R_2^M = \emptyset$ helps). Then we can note

$$(*)_0 \quad \alpha \leq \beta \ \& \ \text{rkrc}^l(\bar{w}, M; < \kappa) \geq \beta \Rightarrow \text{rkrc}^l(\bar{w}, M; < \kappa) \geq \alpha,$$

$$(*)_1 \quad \text{rkrc}^l(\bar{w}, M; < \kappa) \geq \delta \ (\delta \text{ limit}) \quad \text{iff} \quad \bigwedge_{\alpha < \delta} \text{rkrc}^l(\bar{w}, M; < \kappa) \geq \alpha,$$

$$(*)_2 \quad \text{rkrc}^l(\bar{w}, M; < \kappa) \geq 0 \quad \text{iff} \quad \bar{w} \in [M]^{\otimes}.$$

So we can define $\text{rkrc}^l(\bar{w}, M; < \kappa)$ to be the maximal α such that $\text{rkrc}^l(\bar{w}, M; < \kappa) \geq \alpha$, and ∞ if this holds for every α (and -1 if $\text{rkrc}^l(\bar{w}, M; < \kappa) \not\geq 0$).

Now the inductive definition of $\text{rkrc}^l(\bar{w}, M; < \kappa) \geq \alpha$ was already done above for $\alpha = 0$ and α limit, so for $\alpha = \beta + 1$ we let

$$(*)_3 \quad \text{rkrc}^l(\bar{w}, M; < \kappa) \geq \beta + 1 \quad \text{iff} \quad (\text{letting } w = w_1 \cup w_2, n = |w|, w = \{a_0, \dots, a_{n-1}\}), \text{ for every } k < n \text{ and quantifier free formula}$$

$$\varphi(x_0, \dots, x_{n-1}) =$$

$$\bigwedge_{i < j} x_i \neq x_j \ \& \ \bigwedge \{R_1(x_i) \wedge R_2(x_j) \wedge \varphi_{i,j}(x_i, x_j) : R_1(a_i) \ \& \ R_2(a_j)\}$$

(in the vocabulary of M) for which $M \models \varphi[a_0, \dots, a_{n-1}]$ we have:

CASE 1: $l = 1$. There are $a_m^i \in M$ for $m < n$, $i < 2$ such that:

$$(a) \quad \text{rkrc}^l(\{a_m^i : i < 2, m < n\}, M; < \kappa) \geq \beta,$$

$$(b) \quad M \models \varphi[a_0^i, \dots, a_{n-1}^i] \text{ (for } i = 1, 2), \text{ so there is no repetition in } a_0^i, \dots, a_{n-1}^i \text{ and } [a_m^i \in R_j^M \Leftrightarrow a_m \in R_j^M] \text{ for } j = 1, 2,$$

$$(c) \quad a_k^0 \neq a_k^1 \text{ but if } m < n \text{ and } (a_m \in R_1^M \Leftrightarrow a_k \notin R_1^M) \text{ then } a_m^0 = a_m^1,$$

$$(d) \quad \text{if } a_{m_1} \in R_1^M \text{ and } a_{m_2} \in R_2^M \text{ then for any } i, j \ (\in \{1, 2\}) \text{ we have } F^M(a_{m_1}^i, a_{m_2}^j) = F^M(a_{m_1}, a_{m_2}).$$

CASE 2: $l = 0$. As for $l = 1$ but in addition

$$(e) \quad \bigwedge_m a_m = a_m^0.$$

CASE 3: $l = 3$. The definition is like case 1 but $i < \kappa$; i.e. there are $a_m^i \in M$ for $m < n$, $i < \kappa$ such that:

$$(a) \quad \text{for } i < j < \kappa \text{ we have } \text{rkrc}^l(\{a_m^i, a_m^j : m < n\}, M; < \kappa) \geq \beta,$$

$$(b) \quad M \models \varphi[a_0^i, \dots, a_{n-1}^i] \text{ (for } i < \kappa; \text{ so there are no repetitions in } a_0^i, \dots, a_{n-1}^i),$$

$$(c) \quad \text{for } i < j < \kappa, a_k^i \neq a_k^j \text{ but if } m < n \text{ and } (a_m \in R_1^M \Leftrightarrow a_k \notin R_1^M) \text{ then } a_m^i = a_m^j,$$

$$(d) \quad \text{if } a_{m_1} \in R_1^M \text{ and } a_{m_2} \in R_2^M \text{ then for any } i, j, F^M(a_{m_1}^i, a_{m_2}^j) = F^M(a_{m_1}, a_{m_2}).$$

CASE 4: $l = 2$. Like case 3 but in addition

(e) $a_m = a_m^0$ for $m < n$.

CASE 5: $l = 5$. Like case 3 except that we replace clause (a) by

(a)⁻ for every function H with $\text{Dom}(H) = \kappa$ and $|\text{Rang}(H)| < \kappa$, for some $i < j < \kappa$ we have $H(i) = H(j)$ and

$$\text{rkrc}^l(\{a_m^i, a_m^j : m < n\}, M; < \kappa) \geq \beta.$$

CASE 6: $l = 4$. Like case 4 with (a)⁻ instead (a).

(4) For M as above and $c \in Q^M$ we define $\text{rkrc}^l(M, c; < \kappa)$ as

$$\sup\{\text{rkrc}^l(\bar{w}, M; < \kappa) + 1 : \bar{w} \in [M]^\otimes \text{ and } (\forall a \in w_1)(\forall b \in w_2)[F(a, b) = c]\}.$$

(5) Let $\text{Prd}_\alpha^l(\bar{\lambda}, \kappa, \bar{\theta})$ mean $\text{rkrc}^l(M, c; < \kappa, \bar{\theta}) \geq \alpha$ for every M for some $c \in M$ when M is such that $|R_1^M| = \lambda_1$, $|R_2^M| = \lambda_2$, $|\tau(M)| \leq \theta_0$, $F^M : R_1^M \times R_2^M \rightarrow Q^M$ and $|Q^M| = \theta_1$. Let $\text{NPrd}_\alpha^l(\bar{\lambda}, \kappa, \bar{\theta})$ mean its negation and $\lambda\text{rd}_\alpha^l(\kappa, \bar{\theta})$ be the minimal λ such that $\text{Prd}_\alpha^l(\lambda, \lambda, \kappa, \bar{\theta})$.

REMARK 4.2. The reader may wonder why in addition to Prrc we use the variant Prd . The point is that for the existence of the rectangle $X_1 \times X_2$ with $F \upharpoonright (X_1 \times X_2)$ constantly c^* , this constant plays a special role. So in our main claim 4.6, to get a model as there, we need to choose it, one out of θ_1 , but the other choices are out of κ . So though the difference between the two variants is small (see 4.5 below) we actually prefer the Prd version.

CLAIM 4.3. *The parallels of 1.2 (+statements in 1.1), also 1.3, 1.5(2), 1.6, 1.10 hold. ■*

CLAIM 4.4. (1) *If $w_i \in [R_i^M]^*$ for $i = 1, 2$ then*

$$\omega \times \text{rkrc}^l(\langle w_1, w_2 \rangle, M; \kappa) \geq \text{rk}^l(w_1 \cup w_2, M; \kappa).$$

(2) *If $R_1^M = R_2^M$ (abuse of notation) then $\text{rkrc}^l(M; < \kappa) \geq \text{rk}^l(M; \kappa)$.*

(3) *If $\lambda_1 = \lambda_2 = \lambda$ then $\text{Pr}_\alpha(\lambda, \kappa) \Rightarrow \text{Prrc}_\alpha(\lambda_1, \lambda_2; \kappa, \kappa)$. ■*

CLAIM 4.5. $\lambda\text{rd}_\alpha^l(\kappa, \theta) = \lambda\text{rc}_\alpha^l(\kappa, \theta)$ *if α is a successor ordinal or $\text{cf}(\alpha) > \theta$.*

CLAIM 4.6. *Assume $\kappa \leq \theta < \lambda_1, \lambda_2$. Then the following are equivalent:*

(A) $\text{Prd}_{\kappa^+}^1(\lambda_1, \lambda_2; \kappa, \theta)$.

(B) *Assume M is a model with a countable vocabulary, $|R_l^M| = \lambda_l$ for $l = 1, 2$, $P^M = \kappa$, $Q^M = \theta$, F^M is a 2-place function (really just $F \upharpoonright (R_1^M \times R_2^M)$ interests us), the range of $F \upharpoonright (R_1^M \times R_2^M)$ is included in Q^M and G is a function from $[R_1^M]^* \times [R_2^M]^*$ to P^M . Then we can find $\tau(M)$ -models M_0, N and elements c^*, a_η, b_η (for $\eta \in {}^\omega 2$) such that:*

- (i) N is a model with the vocabulary of M (but functions may be interpreted as partial ones, i.e. as relations),
- (ii) $a_\eta \in R_1^N$ and $b_\eta \in R_2^N$ are pairwise distinct and $F^N(a_\eta, b_\eta) = c^* \in N$,
- (iii) M_0 is countable, $M_0 \subseteq M$, $c^* \in Q^M$, M_0 is the closure of $(M_0 \cap P^M) \cup \{c^*\}$ in M , in fact for some $M'_0 \prec M$ we have $M_0 = \text{closure of } P^{M'_0} \cup \{c^*\}$, $c^* \in M'_0$,
- (iv) $M_0 \subseteq N$, $P^{M_0} = P^N$,
- (v) $|N| = \{\sigma(a_\eta, b_\nu, \bar{d}) : \sigma \text{ is a } \tau(M)\text{-term, } \eta \in {}^\omega 2, \nu \in {}^\omega 2 \text{ and } \bar{d} \subseteq M_0\}$,
- (vi) for $\{\eta_l : l < l(*)\}, \{\nu_m : m < m(*)\} \subseteq {}^\omega 2$ (both without repetitions nonempty) there is $d^1 \in P^{M_0}$ such that if $\bar{d} \subseteq P^{M_0}$ and quantifier free formulas $\varphi_{l,m}$ are such that

$$N \models \bigwedge_{l < l(*), m < m(*)} \varphi_{l,m}[a_{\eta_l}, b_{\nu_m}, \bar{d}]$$

then for some $\{a_l : l < l(*)\} \subseteq R_1^M$ and $\{b_m : m < m(*)\} \subseteq R_2^M$ (both with no repetition) we have

$$M \models \bigwedge_{l < l(*), m < m(*)} \varphi_{l,m}[a_l, b_m, \bar{d}]$$

and $G(\{a_l : l < l(*)\}, \{b_m : m < m(*)\}) = d^1$,

- (vii) for every quantifier free first order $\varphi = \varphi(x, y, z_0, \dots) \in L(\tau(M))$ and $d_0, \dots \in M_0$ there is $k < \omega$ such that for every $\eta_1, \eta_2, \nu_1, \nu_2 \in {}^\omega 2$ such that $\eta_1 \upharpoonright k = \eta_2 \upharpoonright k$ and $\nu_1 \upharpoonright k = \nu_2 \upharpoonright k$ we have

$$N \models \varphi[a_{\eta_1}, b_{\nu_1}, d_1, \dots] = \varphi[a_{\eta_2}, b_{\nu_2}, d_1, \dots],$$

- (vii)⁺ for every $n < \omega$, first order $\varphi = \varphi(x_0, \dots, x_{n-1}) \in L(\tau(M))$ quantifier free and $d_2, d_3, \dots \in M_0$ there is $n^* < \omega$ such that for every $k \in (n^*, \omega)$, $\eta_0, \eta_1 \in {}^\omega 2$, $\nu_0, \nu_1 \in {}^\omega 2$ satisfying

$$\eta_0 \upharpoonright k = \eta_1 \upharpoonright k \quad \text{and} \quad \nu_0 \upharpoonright k = \nu_1 \upharpoonright k$$

we have

$$N \models \varphi[a_{\eta_0}, b_{\nu_0}, d_2, \dots] \equiv \varphi[a_{\eta_1}, b_{\nu_1}, d_2, \dots],$$

- (viii) if φ is an existential sentence in $\tau(M)$ satisfied by N then φ is satisfied by M .

(B)⁻ Like (B) without (vii)⁺, (viii).

Proof. (B)⁻ \Rightarrow (A). Toward a contradiction assume $\text{NPrd}_{\kappa^+}(\lambda_1, \lambda_2; \kappa)$. Hence there is a model M' witnessing it, so $|\tau(M')| \leq \kappa$. So $c \in Q^M \Rightarrow \text{rkrc}^1(M', c; \kappa) < \kappa^+$ (note that Prd was defined by cases of $\text{rkrc}(M, c, \kappa)$).

Let $\{\varphi_i(x, y) : i < \kappa\}$ list the quantifier free formulas in $L_{\omega, \omega}(\tau(M'))$ with free variables x, y . Let $\{u_i : i < \kappa\}$ list the finite subsets of κ . For $c \in Q^{M'}$ and $a_0, \dots, a_{l(*)-1} \in R_1^{M'}, b_0, \dots, b_{m(*)-1} \in R_2^{M'}$ ($\bar{a} = \langle a_0, \dots, a_{l(*)-1} \rangle$, $\bar{b} = \langle b_0, \dots, b_{m(*)-1} \rangle$) and we set $a_{n+1+l} = b_l$ let

$$\alpha_{c, \bar{a}, \bar{b}} = \text{rkrc}^1(\langle \{a_0, \dots, a_{l(*)-1}\}, \{b_0, \dots, b_{m(*)-1}\} \rangle, M', c; \kappa),$$

and let $k_{c, \bar{a}, \bar{b}}, \varphi_{c, \bar{a}, \bar{b}}$ be witnesses for $\text{rkrc}^1(\langle \bar{a}, \bar{b} \rangle, M', c; \kappa, \theta) \not\geq \alpha_{c, \bar{a}, \bar{b}} + 1$. Let $i(c, \bar{a}, \bar{b}) < \kappa$ be such that $\varphi_{c, \bar{a}, \bar{b}}$ is a conjunction of formulas of the form $\varphi_j(x_l, y_m)$ for $j \in u_{i(c, \bar{a}, \bar{b})}$. We define M :

- the universe is $|M'|$,
- the function $F^{M'}$, relations $R_1^{M'}, R_2^{M'}, Q^{M'}, P^{M'}$, the pairing function on ordinals,
- $R_n = \{(i, a, b) : a \in R_1^M, b \in R_2^M \text{ and if } |u_i| > n \text{ then } M \models \varphi_j[a, b] \text{ where } j \text{ is the } n\text{th member of } u_i\}$,
- let H_c be one-to-one from $\omega \times \text{rkrc}(M', c; \kappa, \theta) \times \kappa$ into κ ,
- we define the function G :

$$G_c(\bar{a}, \bar{b}) = H(G_{c,0}(\bar{a}, \bar{b}), G_{c,1}(\bar{a}, \bar{b}), G_{c,2}(\bar{a}, \bar{b})) = H(k_{c, \bar{a}, \bar{b}}, \alpha_{c, \bar{a}, \bar{b}}, i_{c, \bar{a}, \bar{b}}).$$

Now we can apply statement (B)⁻ of 4.6 which we are assuming and get $M_0, N, c^*, a_\eta, b_\eta$ (for $\eta \in \omega^2$) satisfying clauses (i)–(vii) there. So $c^* \in M_0 \subseteq M' \cap N$, hence $\beta^* = \text{rkrc}(M', c; \kappa)$ satisfies $\beta^* < \infty$, even $< \kappa^+$. Clearly $\text{rkrc}^1(N, c^*; \kappa) \leq \beta^*$.

Consider all sequences

$$\langle \langle \eta_l : l < l(*) \rangle, \langle \nu_m : m < m(*) \rangle, d_1, \bar{d}, \langle \varphi_{l,m} : l < l(*), m < m(*) \rangle, \langle a_l : l < l(*) \rangle, \langle b_m : m < m(*) \rangle \rangle$$

which are as in clause (vi) of (B).

Among those tuples choose one with $\alpha^* = \text{rkrc}^1(\langle \{a_l : l < l(*)\}, \{b_m : m < m(*)\} \rangle, N, c^*; \kappa)$ minimal. Let this rank not being $\geq \alpha^* + 1$ be exemplified by φ and $k < l(*) + m(*)$, so by symmetry we can assume $k < l(*)$.

Choose $k^* < \omega$ large enough for clause (vi) of (B) to hold for all formulas $\varphi(x, y)$ appearing in $\{\varphi_j(x, y) : j \in u_{i_{c^*, \bar{a}, \bar{b}}}\}$ where $i_{c^*, \bar{a}, \bar{b}} = G_{c^*, 2}(\bar{a}, \bar{b})$ and $\langle \eta_l \upharpoonright k^* : l < l(*) \rangle, \langle \nu_m \upharpoonright k^* : m < m(*) \rangle$ are with no repetition. Choose $\eta_{l(*)} \in 2^\omega \setminus \{\eta_k\}$ such that $\eta_{l(*)} \upharpoonright k^* = \eta_k \upharpoonright k^*$. Now apply clause (vi) of (B)⁻ to $\bar{\eta}' = \langle \eta_l : l \leq l(*) \rangle, \bar{\nu}' = \langle \nu_m : m < m(*) \rangle, \bigwedge_m \varphi_{l(*), m}(x_{l(*)}, x_m), d'_1, \bar{d}'$. By the choice of φ , these clearly satisfy $\text{rkrc}^1(\langle \{a'_l : l < l(*)\}, \{b'_m : m < m(*)\} \rangle, N, c^*; \kappa, \theta) < \text{rkrc}^1(\langle \{a_l : l < l(*)\}, \{b_m : m < m(*)\} \rangle, N, c^*; \kappa, \theta) = \alpha^*$, but this easily contradicts the minimality of α^* .

(A) \Rightarrow (B). As in the proof of 2.5, 2.1 (with a fixed c).

(B) \Rightarrow (B)⁻. Trivial. $\blacksquare_{4.6}$

DISCUSSION 4.7. (1) When applying 4.6(A) \Rightarrow (B), or 2.5 we can use M which is an expansion of $(\mathcal{H}(\chi), \in, <^*)$ by Skolem functions, $P^M = \kappa$, χ large enough, so for $\eta, \nu \in {}^\omega 2$, $N_{\eta, \nu} := \text{cl}_N(M_0 \cup \{a_\eta, a_\nu\})$ is a model of ZFC, not well founded but with standard ω and more: its $\{i : i < \kappa\}$ is a part of the true κ . In [Sh 532] we will have (as in 2.1) $M_0 \subseteq N$, $M_0 \prec M$, $M_0 \prec N_{\eta, \nu}$, and if $N_{\eta, \nu} \models \text{“}\varphi(\eta, \nu)$, ν, η are (essentially) in ${}^\omega 2$, $\varphi \in M_0$ is a κ -Suslin relation”, then $\mathbf{V} \models \varphi(\eta, \nu)$.

(2) We can give a rank to subsets of $\lambda_1 \times \lambda_2$ and have parallel theorems.

CLAIM 4.8. *If $\varphi \subseteq {}^\omega 2 \times {}^\omega 2$ is $\bigvee_{i < \theta} \varphi_i$, each φ_i is κ -Suslin, φ contains a (λ_1, λ_2) -rectangle, and $\text{Prd}_{\kappa^+}^1(\lambda_1, \lambda_2; \theta)$ then φ contains a perfect rectangle.*

PROOF. Let $\varphi_i(\eta, \nu) = (\exists \varrho)((\eta, \nu, \varrho) \in \text{lim}(T_i))$ where T_i is a $(2, 2, \kappa)$ -tree. Let M be $(\mathcal{H}(\chi), \in, <_\chi^*, \langle T_i : i < \theta \rangle, h, \mathcal{Y}, \lambda_1, \lambda_2, R_1, R_2, Q, n)_{n < \omega}$ expanded by Skolem functions, where $Q^M = \theta$ and, for some $\eta_i \in {}^\omega 2$ for $i < \lambda_1$ pairwise distinct and $\nu_j \in {}^\omega 2$ for $j < \lambda_2$ pairwise distinct, $R_1^M = \{\eta_i : i < \lambda_1\}$ and $R_2^M = \{\nu_j : j < \lambda_2\}$; finally, let \mathcal{Y}, h be functions such that $(\eta_i, \nu_j, \mathcal{Y}(\eta_i, \nu_j)) \in \text{lim}(T_{h(\eta_i, \nu_j)})$. So let N, M_0, c^*, a_η (for $\eta \in {}^\omega 2$), b_η (for $\eta \in {}^\omega 2$) be as in clause (B) of 4.6. Now M has elimination of quantifiers, so there are quantifier free formulas $\varphi_n^1(x)$ saying (in M) that $x \in R_1$ & $x(n) = 1$, and let $H_n(x, y)$ be such that $x \in R_1^M$ & $y \in R_2^M \Rightarrow (\mathcal{Y}(x, y))(n) = H_n(x, y) \in \kappa = P^M$.

So for $\eta \in {}^\omega 2$ we can define $\sigma_\eta^1 \in {}^\omega 2$ by $\sigma_\eta^1(n) = 1 \Leftrightarrow N \models \varphi_n^1(a_\eta)$, and $\sigma_\eta^2 \in {}^\omega 2$ by $\sigma_\eta^2(n) = 1 \Leftrightarrow N \models \varphi_n^2(b_\eta)$, and we define, for $\eta, \nu \in {}^\omega 2$, a sequence $\sigma_{\eta, \nu} \in {}^{\omega >} (P^{M_0}) \subseteq {}^{\omega >} \kappa$ by $\sigma_{\eta, \nu}(n) = H_n(a_\eta, b_\nu)$.

Now $A := \{\sigma_\eta^1 : \eta \in {}^\omega 2\}$ and $B := \{\sigma_\eta^2 : \eta \in {}^\omega 2\}$ are perfect and for $\eta, \nu \in {}^\omega 2$, $(\sigma_\eta^1, \sigma_\nu^2, \sigma_{\eta, \nu}) \in \text{lim}(T)$, hence $A \times B$ is a perfect rectangle inside $\text{prj lim}(T)$. $\blacksquare_{4.8}$

FACT 4.9. (1) *Assume that $\varphi \subseteq {}^\omega 2 \times {}^\omega 2$ is θ_1 -Suslin, $\kappa < \theta_1$ and $\theta = \text{cf}(\mathcal{S}_{\leq \kappa}(\theta_1), \subseteq)$. Then φ can be represented as $\bigvee_{i < \theta} \varphi_i$ with each φ_i κ -Suslin.*

(2) *If φ is co- κ -Suslin, then it can be represented as $\bigvee_{i < \kappa^+} \varphi_i$ with each φ_i κ -Borel (i.e. can be obtained from clopen sets by unions and intersections of size $\leq \kappa$).*

PROOF. (1) Easy.

(2) Let $\neg\varphi$ be represented as $\text{prj lim}(T)$ with T a $(2, 2, \kappa)$ -tree.

Now $\varphi(\eta, \nu)$ iff $T_{(\eta, \nu)} = \{\varrho : \text{for some } n, (\eta \upharpoonright n, \nu \upharpoonright n, \varrho) \in T\}$ is well founded, which is equivalent to the existence of $\alpha < \kappa^+$ and $f : T_{(\eta, \nu)} \rightarrow \alpha$ such that $\varrho_1 < \varrho_2 \in T \Rightarrow f(\varrho_1) > f(\varrho_2)$. For each $\alpha < \kappa^+$, this property is κ -Borel. $\blacksquare_{4.9}$

CONCLUSION 4.10. *If φ is an \aleph_n -Suslin subset of ${}^\omega 2 \times {}^\omega 2$ containing a $(\lambda \text{rd}_{\omega_1}(\aleph_n), \lambda \text{rd}_{\omega_1}(\aleph_n))$ -rectangle then it contains a perfect rectangle. (Note: \aleph_n can be replaced by κ if $\text{cf}(\mathcal{S}_{\leq \aleph_0}(\kappa), \subseteq) = \kappa$, e.g. \aleph_{ω_4} , by [Sh g, IX, §4].)*

CONCLUSION 4.11. (1) *For $l < 6$, $\text{Prrc}_\infty^l(\kappa^+, (2^{\kappa^+})^+; \kappa)$.*

(2) *If $\mathbf{V} = \mathbf{V}_0^{\mathbf{P}}$, $\mathbf{P} \models \text{c.c.c.}$ and $\mathbf{V}_0 \models \text{GCH}$ then $\text{Prrc}_{\omega_1}(\aleph_1, \aleph_3)$.*

Proof. (1) For a model M , letting $(\lambda_1, \lambda_2) = (\kappa^+, (2^{\kappa^+})^+)$ choose (for $m = 1, 2$) a nonempty sequence \bar{a}_i^m from R_m^M for $i < \lambda_m$, with $\{\bar{a}_i^m : i < \lambda_m\}$ pairwise disjoint. For $(i, j) \in \lambda_1 \times \lambda_2$ let $\beta_{i,j} = \text{rkrc}(\bar{a}_i^1, \bar{a}_j^2, M)$ with witnesses $k^M(\bar{a}_i^1, \bar{a}_j^2)$, $\varphi^M(\bar{a}_i^1, \bar{a}_j^2)$ for $\neg \text{rkrc}^1(\bar{a}_i^1, \bar{a}_j^2, M) > \beta_{i,j}$. As $\lambda_2 = (2^{\lambda_1})^+$ and $|\tau(M)| \leq \kappa$, for some $B_2 \subseteq \lambda_2$ with $|B_2| = \lambda_2$ and for every $i < \lambda_1$ the following does not depend on $j \in B_2$:

$$k^M(\bar{a}_i^1, \bar{a}_j^2), \varphi^M(\bar{a}_i^1, \bar{a}_j^2).$$

Similarly, there is $B_1 \subseteq \lambda_1$ with $|B_1| = \lambda_1 (= \kappa^+)$ such that for $j = \min(B_2)$ the values $k^M(\bar{a}_i^1, \bar{a}_j^2), \varphi^M(\bar{a}_i^1, \bar{a}_j^2)$ are the same for all $i \in B_1$; but they do not depend on $j \in B_2$ either. So for $(i, j) \in B_1 \times B_2$ we have $k^M(\bar{a}_i^1, \bar{a}_j^2) = k^*$, $\varphi^M(\bar{a}_i^1, \bar{a}_j^2) = \varphi^*$. Let k^* “speak” about \bar{a}_i^1 , for definiteness only. Choose distinct i_ζ in B_1 (for $\zeta < \kappa^+$). We can assume $\text{rkrc}^1(\bar{a}_{i_0}^1, \bar{a}_j^2, M) \leq \text{rkrc}^1(\bar{a}_{i_\zeta}^1, \bar{a}_j^2, M)$.

Now $\bar{a}_{i_\zeta}^1$ give contradiction to $\text{rkrc}^l(\bar{a}_{i_0}^1, \bar{a}_j^2, M) \not\geq \beta_{i_0,j}$.

(2) This can be proved directly (or see [Sh 532]) through preservation by c.c.c. forcing notion of ranks which are relatives of rkrc similarly to 1.10.

REMARK 4.12. (1) If T is an (ω, ω) -tree and $A \times B \subseteq \lim(T)$ with $A, B \subseteq {}^\omega \omega$ uncountable (or just not scattered) then $\lim(T)$ contains a perfect rectangle. Instead of $\lim(T)$ (i.e. a closed set) we can use a countable intersection of open sets. The proof is just like 1.17.

(2) We can define a rank for $(2, 2, \kappa)$ -trees measuring whether $\text{prj } \lim(T) \subseteq {}^\omega 2 \times {}^\omega 2$ contains a perfect rectangle, and similarly for (ω, ω) -trees T measuring whether $\lim(T)$ contains a perfect rectangle. We then have theorems parallel to those of §1. See below and in [Sh 532].

The use of ${}^\omega \omega$ below is just notational change.

DEFINITION 4.13. For an (ω, ω) -tree T we define a function degrc_T (rectangle degree). Its domain is $\text{rcpr}(T) := \{(u_1, u_2) : \text{for some } l < \omega, u_1, u_2 \text{ are finite nonempty subsets of } {}^l \omega \text{ and } g \text{ a function from } u_1 \times u_2 \text{ to } \omega \text{ such that } (\eta_0, \eta_1) \in T \text{ for } \eta_i \in u_i\}$. Its value is an ordinal $\text{degrc}_T(u_1, u_2)$ (or -1 or ∞). For this we define the truth value of $\text{degrc}_T(u_1, u_2) \geq \alpha$ by induction on the ordinal α .

CASE 1, $\alpha = -1$:

$$\text{degrc}_T(u_1, u_2) \geq -1 \quad \text{iff} \quad (u_1, u_2) \text{ is in } \text{rcpr}(T).$$

CASE 2, α limit:

$$\text{degrc}_T(u_1, u_2) \geq \alpha \quad \text{iff} \quad \text{degrc}_T(u_1, u_2) \geq \beta \text{ for every } \beta < \alpha.$$

CASE 3, $\alpha = \beta + 1$: $\text{degrc}_T(u_1, u_2) \geq \alpha$ iff for $k \in \{1, 2\}$ and $\eta^* \in u_k$ we can find $l(*) < \omega$ and functions h_0, h_1 such that $\text{Dom}(h_i) = u_1 \cup u_2$, $[\eta \in u_1 \cup u_2 \Rightarrow \eta \triangleleft h_i(\eta) \in {}^{l(*)}\omega]$, $h_0(\eta^*) \neq h_1(\eta^*)$, $\eta \in u_{1-k} \Rightarrow h_0(\eta) = h_1(\eta)$ and letting $u_i^1 = \text{Rang}(h_0 \upharpoonright u_i) \cup \text{Rang}(h_1 \upharpoonright u_i)$ we have $\text{degrc}_T(u_0^1, u_1^1) \geq \beta$.

Lastly, define: $\text{degrc}_T(u_0, u_1) = \alpha$ iff $\bigwedge_{\beta} [\text{degrc}_T(u_0, u_1) \geq \beta \Leftrightarrow \alpha \geq \beta]$ (α an ordinal or ∞).

Also $\text{degrc}(T) = \text{degrc}_T(\{\langle \rangle\}, \{\langle \rangle\})$.

CLAIM 4.14. Assume T is an (ω, ω) -tree.

(1) For every $(u_0, u_1) \in \text{repr}(T)$, $\text{degrc}_T(u_0, u_1)$ is an ordinal or ∞ or -1 ; if f is an automorphism of $({}^\omega\omega, \triangleleft)$ then

$$\text{degrc}_T(u_0, u_1) = \text{degrc}_{f(T)}(f(u_0), f(u_1)).$$

(2) $\text{degrc}(T) = \infty$ iff there is a perfect rectangle in $\lim(T)$ iff $\omega_1 \leq \text{degrc}_T(u_0, u_1)$ for some (u_0, u_1) (so those statements are absolute).

(3) If $\text{degrc}(T) = \alpha(*) < \omega_1$ then $\lim(T)$ contains no $(\text{lr}_{\alpha(*)+1}(\aleph_0), \text{lr}_{\alpha(*)+1}(\aleph_0))$ -rectangle.

(4) If $\bar{T} = \langle T_n : n < \omega \rangle$ is a sequence of (ω, ω) -trees, $\text{degrc}(T_n) \leq \alpha(*)$, and $A = \bigcup_{n < \omega} \lim(T_n)$ then A contains no $(\text{lr}_{\alpha(*)+1}(\aleph_0), \text{lr}_{\alpha(*)+1}(\aleph_0))$ -rectangle.

(5) In part (4) we can replace ω by any infinite cardinal θ .

Proof. (1)–(3) Left to the reader.

(4) Follows from part (5).

(5) Let $\lambda = \text{lr}_{\alpha(*)+1}(\theta)$, $\bar{T} = \langle T_i : i < \theta \rangle$, $\text{degrc}(T_i) \leq \alpha(*)$ and $A = \bigcup_{i < \theta} \lim(T_i)$. Let $\{\eta_\alpha : \alpha < \lambda\} \times \{\nu_\beta : \beta < \lambda\} \subseteq A$ where $\alpha < \beta \Rightarrow \eta_\alpha \neq \eta_\beta$ & $\nu_\alpha \neq \nu_\beta$, and for simplicity $\{\eta_\alpha : \alpha < \lambda\} \cap \{\nu_\beta : \beta < \lambda\} = \emptyset$.

We define a model M , with universe $\mathcal{H}((2^{\aleph_0})^+)$ and relations: all those definable in $(\mathcal{H}((2^{\aleph_1})^+), \in, <^*, R_1, R_2, g, \bar{T}, i)_{i \leq \theta}$ where $R_1^M = \{\eta_\alpha : \alpha < \lambda\}$, $R_2^M = \{\nu_\beta : \beta < \lambda\}$ and $g(\eta_\alpha, \nu_\beta) = \min\{i \leq \theta : (\eta_\alpha, \nu_\beta) \in \lim(T_i)\}$.

Next we prove

(*) if $w_l \in [R_l^M]^*$ for $l = 1, 2$, then

$$\begin{aligned} \text{rkrc}(\langle w_1, w_2 \rangle, M) &\leq \min\{\text{degrc}_{T_i}(\{\eta_\alpha \upharpoonright k : \alpha \in u_1\}, \{\nu_\beta \upharpoonright k : \beta \in u_2\}) : \\ &\quad u_1 \subseteq w_1, u_2 \subseteq w_2, u_1 \neq \emptyset, u_2 \neq \emptyset, k < \omega, \\ &\quad \langle \eta_\alpha \upharpoonright k : \alpha \in u_1 \rangle \text{ is with no repetitions,} \\ &\quad \langle \nu_\beta \upharpoonright k : \beta \in u_2 \rangle \text{ is with no repetitions}\}. \end{aligned}$$

We prove (*) by induction on the left side of the inequality. Now by the definitions we are done. ■_{4.14}

CLAIM 4.15. (1) For each $\alpha(*) < \omega_1$, there is an ω -sequence $\bar{T} = \langle T_n : n < \omega \rangle$ of (ω, ω) -trees such that:

- (α) for every $\mu < \text{lr}_{\alpha(*)}(\aleph_0)$, some c.c.c. forcing notion adds a (μ, μ) -rectangle to $\bigcup_{n < \omega} \text{lim}(T_n)$,
- (β) $\text{degrc}(T_n) = \alpha(*)$.

(2) If $\text{NPr}_{\alpha(*)}(\lambda_1, \lambda_2; \aleph_0)$ then for some (ω, ω) -tree T , some c.c.c. forcing notion adds a (λ_1, λ_2) -rectangle to $\text{lim}(T)$ such that $\alpha(*) = \text{degrc}(\bar{T})$ (consequently, if $\text{Pr}_{\alpha(*)}(\lambda'_1, \lambda'_1; \aleph_0)$ then there is no (λ'_1, λ'_1) -rectangle in $\text{lim}(T)$).

(3) Moreover, we can have for the tree T of (4): if $\mu < \text{lr}_{\alpha(*)}(\aleph_0)$ and A, B are disjoint subsets of ${}^\omega 2 \times {}^\omega 2$ of cardinality $\leq \mu$, then some c.c.c. forcing notion \mathbf{P} adds an automorphism f of $({}^{>\omega} \omega, \triangleleft)$ such that $A \subseteq \text{lim}^*[f(T)]$ and $B \cap \text{lim}^*[f(T)] = \emptyset$ (the lim^* means closure under finite changes).

PROOF. We define the forcing for part (2) and postpone the others to [Sh 532].

(2) It is enough to do it for successor $\alpha(*)$, say $\beta(*) + 1$. It is like 1.13; we will give the basic definition and the new points. Let M be a model as in Definition 4.1 with $|R_l^M| = \lambda_l$ and $\text{rkrc}^1(M) < \alpha(*)$, so $\text{rkrc}^1(M) \leq \beta(*)$. We assume that R_1^M are R_2^M are disjoint sets of ordinals. For nonempty $\bar{a}_l \subseteq R_l^M$ ($l < 2$; no repetition inside \bar{a}_l), let $\varphi^M(\bar{a}_1, \bar{a}_2), k^M(\bar{a}_1, \bar{a}_2) \in \text{Rang}(\bar{a}_1) \cup \text{Rang}(\bar{a}_2)$ be witnesses to the value of $\text{rkrc}^1(\bar{a}_1, \bar{a}_2, M)$ which is $< \alpha(*)$.

We define the forcing notion \mathbf{P} : a condition p consists of:

- (1) $\bar{u}^p = \langle u_0^p, u_1^p \rangle$ and $u_\varepsilon^p = u_\varepsilon[p]$ is a finite subset of R_ε^M for $\varepsilon < 2$.
- (2) $n^p = n[p] < \omega$ and $\eta_\alpha^p = \eta_\alpha[p] \in {}^{n[p]}\omega$ for $\alpha \in u_\varepsilon^p$ such that $\alpha \neq \beta \in u_\varepsilon^p \Rightarrow \eta_\alpha^p \neq \eta_\beta^p$.
- (3) $0 < m^p < \omega$ and $t_m^p \subseteq \bigcup \{ {}^l \omega \times {}^l \omega : l \leq n^p \}$ is closed under initial segments and such that the \triangleleft -maximal elements have length n^p and $\langle \rangle \in t^p$.
- (4) The domain of f^p is $\{ \bar{u} = (u_1, u_2) : \text{for some } l = l(u_1, u_2) \leq n^p \text{ and } m = m(u_1, u_2) < m^p \text{ we have } u_\varepsilon \subseteq t_\varepsilon^p \cap {}^l \omega \text{ and if } \alpha_1 \in u_1, \alpha_2 \in u_2, \eta_{\alpha_1}^p \upharpoonright l \in u_1 \text{ and } \eta_{\alpha_2}^p \upharpoonright l \in u_2 \text{ then } g^p(\alpha_1, \alpha_2) = m \}$ and $f^p(\bar{u}) = (f_0^p(\bar{u}), f_1^p(\bar{u}), f_2^p(\bar{u})) \in \alpha(*) \times (u_1 \cup u_2) \times L_{\omega_1, \omega}(\tau(M))$.
- (5) A function $g^p : u_1^p \times u_2^p \rightarrow \{0, \dots, m^p - 1\}$, $m^p < \omega$.
- (6) $t_m^p \cap ({}^{n^p} \omega) = \{ (\eta_\alpha^p, \eta_\beta^p) : \alpha \in u_0^p, \beta \in u_1^p \text{ and } m = g^p(\alpha, \beta) \}$.
- (7) If $\emptyset \neq u_\varepsilon \subseteq t_\varepsilon \cap {}^l \omega$, $f^p(u_1, u_2) = (\beta^*, \varrho^*, \varphi^*)$, $l < l(*) \leq n^p$, for $i = 0, 1$ a function $e_{i, \varepsilon}$ has domain u_ε , $[(\forall l) (\varrho \in u_\varepsilon \Rightarrow \varrho \triangleleft e_{i, \varepsilon}(\varrho) \in t_\varepsilon^p \cap {}^{l(*)} \omega)]$, $[\varrho^* \notin u_\varepsilon \ \& \ \varrho \in u_\varepsilon \Rightarrow e_{0, \varepsilon}(\varrho) = e_{1, \varepsilon}(\varrho)]$, $[\varrho^* \in u_\varepsilon \Rightarrow e_{0, \varepsilon}(\varrho^*) \neq e_{1, \varepsilon}(\varrho^*)]$ and $f^p(e_{0,1}(u_1) \cup e_{1,1}(u_1), e_{0,2}(u_2) \cup e_{1,2}(u_2)) = (\beta', \varrho', \varphi')$ (so well defined) then $\beta' < \beta^*$.

(8) If $l \leq n^p$, $u_\varepsilon \subseteq u_\varepsilon^p$ are nonempty for $\varepsilon = 1, 2$, the sequence $\langle \eta_\alpha^p \upharpoonright l : \alpha \in u_\varepsilon \rangle$ is with no repetition, and $u'_\varepsilon = \{\eta_\alpha^p \upharpoonright l : \alpha \in u_\varepsilon\}$ and $f^p(u'_1, u'_2)$ is well defined, then $f_2^p(u'_1, u'_2) = \varphi^M(u_1, u_2)$, $f_1^p(u'_1, u'_2) = \eta_\alpha^p \upharpoonright l$ where α is $k^M(u_1, u_2)$ and $f_0^p(u'_1, u'_2, h) = \text{rkrc}(u_1, u_2, M)$.

(9) If $(u'_1, u'_2) \in \text{Dom}(f^p)$ then there are l, u_1, u_2 as above.

(10) if $(\eta_1, \eta_2) \in t_m^p \cap ({}^n 2 \times {}^n 2)$ then for some $\alpha_1 \in u_1^p, \alpha_2 \in u_2^p$ we have $g^p(\alpha_1, \alpha_2) = m$ and $\eta_1 \trianglelefteq \eta_{\alpha_1}^p, \eta_2 \trianglelefteq \eta_{\alpha_2}^p$. ■_{4.14}

REMARK 4.16. We can generalize 4.13–4.15 to Suslin relations.

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