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The distributivity numbers of finite products of $\mathcal{P}(\omega)/\mathrm{fin}$

by

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Abstract. Generalizing [ShSp], for every $n < \omega$ we construct a ZFC-model where $\mathfrak{h}(n)$, the distributivity number of r.o.$(\mathcal{P}(\omega)/\mathrm{fin})^n$, is greater than $\mathfrak{h}(n+1)$. This answers an old problem of Balcar, Pelant and Simon (see [BaPeSi]). We also show that both Laver and Miller forcings collapse the continuum to $\mathfrak{h}(n)$ for every $n < \omega$, hence by the first result, consistently they collapse it below $\mathfrak{h}(n)$.

Introduction. For λ a cardinal let $\mathfrak{h}(\lambda)$ be the least cardinal κ for which r.o.$(\mathcal{P}(\omega)/\mathrm{fin})^\lambda$ is not κ-distributive, where by $(\mathcal{P}(\omega)/\mathrm{fin})^\lambda$ we mean the (full) λ-product of $\mathcal{P}(\omega)/\mathrm{fin}$ in the forcing sense; so $f \in (\mathcal{P}(\omega)/\mathrm{fin})^\lambda$ if and only if $f : \lambda \to \mathcal{P}(\omega)/\mathrm{fin} \setminus \{0\}$, and the ordering is coordinatewise.

In [ShSp] the consistency of $\mathfrak{h}(2) < \mathfrak{h}$ (where $\mathfrak{h} = \mathfrak{h}(1)$) with ZFC has been proved, which provided a (partial) answer to a question of Balcar, Pelant and Simon in [BaPeSi]. This inequality holds in a model obtained by forcing with a countable support iteration of length ω_2 of Mathias forcing over a model of GCH. That $\mathfrak{h} = \omega_2$ in this model is folklore, but the proof of $\mathfrak{h}(2) = \omega_1$ is long and difficult.

The two main theorems which imply this are the following:

(a) Whenever some $r \in V^{P_{\omega_2}} \cap [\omega]^\omega$ (where P_{ω_2} is the above iteration) induces a Ramsey ultrafilter on $V \cap [\omega]^\omega$ which is a P-filter in $V^{P_{\omega_2}}$ then this filter is induced by some $r_1 \in V^{Q_0} \cap [\omega]^\omega$ (where Q_0 is the first iterand of P_{ω_2}) and hence belongs to V^{Q_0}.

(b) Whenever some $r \in V^{Q_0} \cap [\omega]^\omega$ induces a Ramsey ultrafilter on $V \cap [\omega]^\omega$ then this filter is Rudin–Keisler equivalent to the canonical Ramsey filter induced by the first Mathias real, and this equivalence is witnessed by some element of $V \cap \omega^\omega$.

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The following are the key properties of Mathias forcing (M.f.) which are essential to the proofs of these (see [ShSp] or below for precise definitions):

(1) M.f. factors into a σ-closed and a σ-centered forcing.

(2) M.f. is Suslin-proper, which means that, firstly, it is simply definable, and, secondly, it permits generic conditions over every countable model of ZF^-.

(3) Every infinite subset of a Mathias real is also a Mathias real.

(4) M.f. does not change the cofinality of any cardinal from above $\mathfrak{h}$ to below $\mathfrak{h}$.

(5) M.f. has the pure decision property and it has the Laver property.

In this paper we present a forcing Q^n, where $0 < n < \omega$, which is an n-dimensional version of M.f. which satisfies all the analogues of the five key properties of M.f. The following list indicates where the analogues of these properties will be proved:

(1) $\leftrightarrow$ Lemma 1.5,
(2) $\leftrightarrow$ Corollary 1.12,
(3) $\leftrightarrow$ Corollary 1.11,
(4) $\leftrightarrow$ Corollary 1.14,
(5) $\leftrightarrow$ Lemma 1.16 and Lemma 1.18.

In this paper we only prove these. Once this has been done the proof of [ShSp] can be generalized in a straightforward way to prove (a′) and (b′), analogues of (a) and (b) above, where (a′) is like (a) except that M.f. is replaced by Q^n, and (b′) is as follows:

(b′) Whenever some $r \in V^{Q^n} \cap [\omega]^\omega$ induces a Ramsey ultrafilter on $V \cap [\omega]^\omega$ then this filter is Rudin–Keisler equivalent to one of the n (pairwise non-RK-equivalent) canonical Ramsey ultrafilters induced by the length-n-sequence of Q^n-generic reals, and the equivalence is witnessed by some function from V.

Then as in [ShSp] we obtain the following:

Theorem. *Suppose* $V \models \mathrm{ZFC} + \mathrm{GCH}$. *If* P *is a countable support iteration of* Q^n *of length* ω_2 *and* G *is* P*-generic over* V, *then* $V[G] \models \mathfrak{h}(n+1) = \omega_1 \wedge \mathfrak{h}(n) = \omega_2$.

Besides the fact that the consistency of $\mathfrak{h}(n+1) < \mathfrak{h}(n)$ was an open problem in [BaPeSi], our motivation for working on it was that in [GoReShSp] it was shown that both Laver and Miller forcings collapse the continuum to $\mathfrak{h}$. Moreover, using ideas from [GoJoSp] and [GoReShSp] it can be proved that these forcings do not collapse $\mathfrak{c}$ below $\mathfrak{h}(\omega)$. We do not know whether they do collapse it to $\mathfrak{h}(\omega)$. But in §2 we show that they collapse it to $\mathfrak{h}(n)$, for every $n < \omega$. Combining this with the first result we conclude that, for every $n < \omega$, consistently Laver and Miller forcings collapse $\mathfrak{c}$ strictly below $\mathfrak{h}(n)$.

The reader should have a copy of [ShSp] at hand. We do not repeat all the definitions from [ShSp] here. Notions as Ramsey ultrafilter, Rudin–Keisler ordering, Suslin-proper are explained there and references are given.

1. The forcing

DEFINITION 1.1. Suppose that $D_0, \ldots, D_{n-1}$ are ultrafilters on ω. The game $G(D_0, \ldots, D_{n-1})$ is defined as follows: In his mth move player I chooses $\langle A_0, \ldots, A_{n-1}\rangle \in D_0 \times \ldots \times D_{n-1}$ and player II responds playing $k_m \in A_{m \bmod n}$. Finally, player II wins if and only if for every $i < n$, $\{k_j : j = i \bmod n\} \in D_i$ holds.

LEMMA 1.2. *Suppose $D_0, \ldots, D_{n-1}$ are Ramsey ultrafilters which are pairwise not RK-equivalent. Let $\langle m(l) : l < \omega\rangle$ be an increasing sequence of integers. There exists a subsequence $\langle m(l_j) : j < \omega\rangle$ and sets $Z_i \in D_i$, $i < n$, such that*:

(1) $l_{j+1} - l_j \geq 2$ *for all* $j < \omega$,
(2) $Z_i \subseteq \bigcup_{j=i \bmod n}[m(l_j), m(l_{j+1}))$ *for all* $i < n$,
(3) $Z_i \cap [m(l_j), m(l_{j+1}))$ *has precisely one member for every* $i < n$ *and* $j = i \bmod n$.

P r o o f. For $j < 3$, $k < \omega$ define

$$I_{j,k} = \bigcup_{s=(2n-1)(3k+j)}^{(2n-1)(3k+j+1)-1} [m_s, m_{s+1}), \qquad J_j = \bigcup_{k<\omega} I_{j,k}.$$

As the D_i are Ramsey ultrafilters, there exist $X_i \in D_i$ such that for every $i < n$:

(a) $X_i \subseteq J_j$ for some $j < 3$,
(b) if $X_i \subseteq J_j$, then $X_i \cap I_{j,k}$ contains precisely one member, for every $k < \omega$.

Next we want to find $Y_i \in D_i$, $Y_i \subseteq X_i$, such that for any distinct $i, i' < n$, Z_i and $Z_{i'}$ do not meet any adjacent intervals $I_{j,k}$.

Define $h : X_0 \to X_1$ as follows. Suppose $X_0 \subseteq J_j$. For every $k < \omega$, h maps the unique element of $X_0 \cap I_{j,k}$ to the unique element of X_1 which belongs either to $I_{j,k}$ or to one of the two intervals of the form $I_{j',k'}$ which are adjacent to $I_{j,k}$ (note that these are $I_{2,k-1}, I_{1,k}$ if $j = 0$, or $I_{0,k}, I_{2,k}$ if $j = 1$, or $I_{1,k}, I_{0,k+1}$ if $j = 2$). As h does not witness that D_0, D_1 are RK-equivalent, there exist $X'_i \in D_i$, $X'_i \subseteq X_i$ $(i < 2)$ such that $h[X'_0] \cap X'_1 = \emptyset$. Note that if $n = 2$, we can let $Y_i = X'_i$. Otherwise we repeat this procedure, starting from X_0' and X_2, and get X_0'' and X_2'. We repeat it again, starting from X_1' and X_2', and get X_1'' and X_2''. If $n = 3$ we are done. Otherwise we continue similarly. After finitely many steps we obtain Y_i as desired.

By the definition of $I_{j,k}$ it is now easy to add more elements to each Y_i in order to get Z_i as in the lemma. The "worst" case is when some Y_i contains integers $s < t$ such that $(s,t) \cap Y_u = \emptyset$ for all $u < n$. By construction there is some $I_{j,k} \subseteq (s,t)$. For every $u < n-1$ pick

$$x_u \in [m((2n-1)(3k+j)+2u+1), m((2n-1)(3k+j)+2u+2))$$

and add x_u to $Y_{i+u+1 \bmod n}$. The other cases are similar. ∎

Corollary 1.3. *Suppose $D_0, \ldots, D_{n-1}$ are Ramsey ultrafilters which are pairwise not RK-equivalent. Then in the game $G(D_0, \ldots, D_{n-1})$ player I does not have a winning strategy.*

Proof. Suppose σ is a strategy for player I. For every $m < \omega$, $i < n$ let $\mathcal{A}_i^m \subseteq D_i$ be the set of all ith coordinates of moves of player I in an initial segment of length at most $2m+1$ of a play in which player I follows σ and player II plays only members of m.

As the D_i are p-points and each $\mathcal{A}_i^m$ is finite, there exist $X_i \in D_i$ such that $\forall m \forall i < n \forall A \in \mathcal{A}_i^m (X_i \subseteq^* A)$. Moreover, we may clearly find a strictly increasing sequence $\langle m(l) : l < \omega \rangle$ such that $m(0) = 0$ and, for all $l < \omega$,

$$\forall i < n \forall A \in \mathcal{A}_i^{m(l)} (X_i \subseteq A \cup m(l+1) \wedge X_i \cap [m(l), m(l+1)) \neq \emptyset).$$

Applying Lemma 1.2, we obtain a subsequence $\langle m(l_j) : j < \omega \rangle$ and sets $Z_i \in D_i$.

Now let player II in his jth move play k_j, where k_j is the unique member of $[m(l_j), m(l_{j+1})) \cap X_{j \bmod n} \cap Z_{j \bmod n}$ if it exists, or otherwise is any member of $[m(l_j), m(l_{j+1})) \cap X_{j \bmod n}$ (note that this intersection is nonempty by the definition of $m(l_{j+1})$). Then this play is consistent with σ, moreover $X_i \cap Z_i \subseteq \{k_j : j = i \bmod n\}$ for every $i < n$, and hence it is won by player II. Consequently, σ could not have been a winning strategy for player I. ∎

Remark. It is easy to see that in 1.2 and 1.3 the assumption that the D_i are pairwise not RK-equivalent is necessary.

Definition 1.4. Let $n < \omega$ be fixed. The forcing Q (really Q^n) is defined as follows: Its members are $(w, \overline{A}) \in [\omega]^{<\omega} \times [\omega]^\omega$. If $\langle k_j : j < \omega \rangle$ is the increasing enumeration of $\overline{A}$ we let $\overline{A}_i = \{k_j : j = i \bmod n\}$ for $i < n$, and if $\langle l_j : j < m \rangle$ is the increasing enumeration of w then let $w_i = \{l_j : j = i \bmod n\}$, for $i < n$.

Let $(w, \overline{A}) \leq (v, \overline{B})$ if and only if $w \cap (\max(v)+1) = v$, $w_i \setminus v_i \subseteq \overline{B}_i$ and $\overline{A}_i \subseteq \overline{B}_i$, for every $i < n$.

If $p \in Q$, then $w^p, w_i^p, \bar{A}^p, \bar{A}_i^p$ have the obvious meaning. We write $p \leq^0 q$ and say "p is a pure extension of q" if $p \leq q$ and $w^p = w^q$.

If $D_0, \ldots, D_{n-1}$ are ultrafilters on ω, let $Q(D_0, \ldots, D_{n-1})$ denote the subordering of Q containing only those $(w, \bar{A}) \in Q$ with the property $\bar{A}_i \in D_i$, for every $i < n$.

LEMMA 1.5. *The forcing Q is equivalent to $(\mathcal{P}(\omega)/\mathrm{fin})^n * Q(\dot{G}_0, \ldots \dot{G}_{n-1})$, where $(\dot{G}_0, \ldots, \dot{G}_{n-1})$ is the canonical name for the generic object added by $(\mathcal{P}(\omega)/\mathrm{fin})^n$, which consists of n pairwise not RK-equivalent Ramsey ultrafilters.*

Proof. Clearly, $(\mathcal{P}(\omega)/\mathrm{fin})^n$ is σ-closed and hence does not add reals. Moreover, members $\langle x_0, \ldots, x_{n-1}\rangle \in (\mathcal{P}(\omega)/\mathrm{fin})^n$ with the property that if $\bar{A} = \bigcup\{x_i : i < n\}$, then $x_i = \bar{A}_i$ for every $i < n$, are dense. Hence the map $(w, \bar{A}) \mapsto (\langle \bar{A}_0, \ldots, \bar{A}_{n-1}\rangle, (w, \bar{A}))$ is a dense embedding of the respective forcings.

That $\dot{G}_0, \ldots, \dot{G}_{n-1}$ are ($(\mathcal{P}(\omega)/\mathrm{fin})^n$-forced to be) pairwise not RK-equivalent Ramsey ultrafilters follows by an easy genericity argument and again the fact that no new reals are added. ∎

NOTATION. We will usually abbreviate the decomposition of Q from Lemma 1.5 by writing $Q = Q' * Q''$. So members of Q' are $\bar{A}, \bar{B} \in [\omega]^\omega$ ordered by $\bar{A}_i \subseteq \bar{B}_i$ for all $i < n$; Q'' is $Q(\dot{G}_0, \ldots, \dot{G}_{n-1})$. It is easy to see that Q'' is σ-centered. If G is a Q-generic filter, we denote by $G' * \dot{G}''$ its decomposition according to $Q = Q' * \dot{Q}''$, and we write $G' = (G'_0, \ldots, G'_{n-1})$.

DEFINITION 1.6. Let $I \subseteq Q(D_0, \ldots, D_{n-1})$ be open dense. We define a rank function rk_I on $[\omega]^{<\omega}$ as follows. Let $\mathrm{rk}_I(w) = 0$ if and only if $(w, \bar{A}) \in I$ for some $\bar{A}$. Let $\mathrm{rk}_I(w) = \alpha$ if and only if α is minimal such that there exists $A \in D_{|w| \bmod n}$ with the property that for every $k \in A$, $\mathrm{rk}_I(w \cup \{k\}) = \beta$ for some $\beta < \alpha$. Let $\mathrm{rk}_I(w) = \infty$ if for no ordinal α, $\mathrm{rk}_I(w) = \alpha$.

LEMMA 1.7. *If $D_0, \ldots, D_{n-1}$ are Ramsey ultrafilters which are pairwise not RK-equivalent and $I \subseteq Q(D_0, \ldots, D_{n-1})$ is open dense, then for every $w \in [\omega]^{<\omega}$, $\mathrm{rk}_I(w) \neq \infty$.*

Proof. Suppose we had $\mathrm{rk}_I(w) = \infty$ for some w. We define a strategy σ for player I in $G(D_0, \ldots, D_{n-1})$ as follows: $\sigma(\emptyset) = \langle A_0, \ldots, A_{n-1}\rangle \in D_0 \times \ldots \times D_{n-1}$ such that for every $k \in A_{|w| \bmod n}$, $\mathrm{rk}_I(w \cup \{k\}) = \infty$. This choice is possible by assumption and by the fact that the D_i are ultrafilters. In general, suppose that σ has been defined for plays of length $2m$ such that whenever $k_0, \ldots, k_{m-1}$ are moves of player II which are consistent with σ, then $k_0 < k_1 < \ldots < k_{m-1}$ and for every $\{k_{i_0} < \ldots < k_{i_{l-1}}\} \subseteq \{k_0, \ldots, k_{m-1}\}$ with $i_j = j \bmod n$, $j < l$, we have $\mathrm{rk}_I(w \cup \{k_{i_0}, \ldots, k_{i_{l-1}}\}) = \infty$. Let S be the set of all $\{k_{i_0} < \ldots < k_{i_{l-1}}\} \subseteq \{k_0, \ldots, k_{m-1}\}$ with $i_j = j \bmod n$,

$j < l$, and $l = m \bmod n$. As $D_{|w|+m \bmod n}$ is an ultrafilter, by induction hypothesis, if we let

$$A_{|w|+m \bmod n} = \{k > k_{m-1} : \forall s \in S(\mathrm{rk}_I(w \cup s \cup \{k\}) = \infty)\},$$

we have $A_{|w|+m \bmod n} \in D_{|w|+m \bmod n}$. For $i \neq |w| + m \bmod n$, choose $A_i \in D_i$ arbitrarily, and define

$$\sigma\langle k_0, \dots k_{m-1}\rangle = \langle A_0, \dots, A_{n-1}\rangle.$$

Since by Lemma 1.2, σ is not a winning strategy for player I, there exist $k_0 < \dots < k_m < \dots$ which are moves of player II consistent with σ, such that, letting $\overline{A} = \{k_m : m < \omega\}$, we have $(w, \overline{A}) \in Q(D_0, \dots, D_{n-1})$. By construction we see that for every $(v, \overline{B}) \leq (w, \overline{A})$, $\mathrm{rk}_I(v) = \infty$. This contradicts the assumption that I is dense. ■

Definition 1.8. Let $p \in Q$. A set of the form $w^p \cup \{k_{|w|} < k_{|w|+1} < \dots\} \in [\omega]^\omega$ is called a *branch* of p if and only if $\max(w^p) < k_{|w|}$ and $\{k_j : j = i \bmod n\} \subseteq \overline{A}_i^p$ for every $i < n$. A set $F \subseteq [\omega]^{<\omega}$ is called a *front* in p if for every $w \in F$, $(w, \overline{A}^p) \leq p$ and for every branch B of p, $B \cap m \in F$ for some $m < \omega$.

Lemma 1.9. *Suppose $D_0, \dots, D_{n-1}$ are pairwise not RK-equivalent Ramsey ultrafilters. Suppose $p \in Q(D_0, \dots, D_{n-1})$ and $\langle I_m : m < \omega\rangle$ is a family of open dense sets in $Q(D_0, \dots, D_{n-1})$. There exists $q \in Q(D_0, \dots, D_{n-1})$, $q \leq^0 p$, such that for every m, $\{w \in [\omega]^{<\omega} : (w, \overline{A}^q) \in I_m \wedge (w, \overline{A}^q) \leq q\}$ is a front in q.*

Proof. First we prove this in the case $I_m = I$ for all $m < \omega$, by induction on $\mathrm{rk}_I(w^p)$. We define a strategy σ for player I in $G(D_0, \dots, D_{n-1})$ as follows. Generally we require that

$$\sigma\langle k_0, \dots, k_r\rangle_i \subseteq \sigma\langle k_0, \dots, k_s\rangle_i$$

for every $s < r$ and $i < n$, where $\sigma\langle k_0, \dots, k_r\rangle_i$ is the ith coordinate of $\sigma\langle k_0, \dots, k_r\rangle$. We also require that σ ensures that the moves of II are increasing. Define $\sigma(\emptyset) = \langle A_0, \dots, A_{n-1}\rangle$ such that for every $k \in A_{|w^p| \bmod n}$, $\mathrm{rk}_I(w^p \cup \{k\}) < \mathrm{rk}_I(w^p)$.

Suppose now that σ has been defined for plays of length $2m$, and let $\langle k_0, \dots, k_{m-1}\rangle$ be moves of II, consistent with σ. The interesting case is that of $m - 1 = 0 \bmod n$. Let us assume this first. By the definition of $\sigma(\emptyset)$ and the general requirement on σ we conclude $\mathrm{rk}_I(w^p \cup \{k_{m-1}\}) < \mathrm{rk}_I(w^p)$. By induction hypothesis there exists $\langle A_0, \dots, A_{n-1}\rangle \in D_0 \times \dots \times D_{n-1}$ such that, letting $\overline{A} = \bigcup_{i<n} A_i$, we have $(w^p, \overline{A}) \leq p$ and

$$\{v \in [\omega]^{<\omega} : (v, \overline{A}) \in I \wedge (v, \overline{A}) \leq (w^p \cup \{k_{m-1}\}, \overline{A})\}$$

is a front in $(w^p \cup \{k_{m-1}\}, \bar{A})$. We shrink $\bar{A}$ so that, letting

$$\sigma\langle k_0, \ldots, k_{m-1}\rangle = \langle A_0, \ldots, A_{n-1}\rangle,$$

the general requirements on σ above are satisfied.

In the case of $m - 1 \neq 0 \bmod n$, define $\sigma\langle k_0, \ldots, k_{m-1}\rangle$ arbitrarily, but consistently with the rules and the general requirements above.

Let $\bar{A} = \{k_i : i < \omega\}$ be moves of player II witnessing that σ is not a winning strategy. Let $q = (w^p, \bar{A})$. Let $B = w^p \cup \{l_{|w^p|} < l_{|w^p|+1} < \ldots\}$ be a branch of q. Hence $l_{|w^p|} = k_j$ for some $j = 0 \bmod n$. Then $w^p \cup \{k_j\} \cup \{l_{|w^p|+1}, l_{|w^p|+2}, \ldots\}$ is a branch of $(w^p \cup \{k_j\}, \sigma\langle k_0, \ldots, k_j\rangle)$. By the definition of σ there exists m such that $(B \cap m, \sigma\langle k_0, \ldots, k_j\rangle) \in I$. As $(B \cap m, \bar{A}) \leq (B \cap m, \sigma\langle k_0, \ldots, k_j\rangle)$ and I is open we are done.

For the general case where we have infinitely many I_m, we make a diagonalization, using the first part of the present proof. Define a strategy σ for player I satisfying the same general requirements as in the first part as follows. Let $\sigma(\emptyset) = \langle A_0, \ldots, A_{n-1}\rangle$ be such that, letting $\bar{A} = \bigcup\{A_i : i < n\}$, $(w^p, \bar{A}) \leq^0 p$ and it satisfies the conclusion of the lemma for I_0. In general, let $\sigma\langle k_0, \ldots, k_{m-1}\rangle = \langle A_0, \ldots, A_{n-1}\rangle$ be such that, letting $\bar{A} = \bigcup\{A_i : i < n\}$, for every $v \subseteq \{k_i : i < m\}$ and $j \leq m$, $(w^p \cup v, \bar{A}) \leq^0 (w^p \cup v, \bar{A}^p)$ and it satisfies the conclusion of the lemma for I_j (in fact we do not have to consider all such v here, but it does not hurt doing it). Then if $\bar{A} = \{k_i : i < \omega\}$ are moves of player II witnessing that σ is not a winning strategy for I, similarly to the first part it can be verified that $q = (w^p, \bar{A})$ is as desired. ∎

COROLLARY 1.10. *Let $D_0, \ldots, D_{n-1}$ be pairwise not RK-equivalent Ramsey ultrafilters. Suppose $\bar{A} \in [\omega]^\omega$ is such that for every $i < n$ and $X \in D_i$, $\bar{A}_i \subseteq^* X$. Then $\bar{A}$ is $Q(D_0, \ldots, D_{n-1})$-generic over V.*

Proof. Let $I \subseteq Q(D_0, \ldots, D_{n-1})$ be open dense. Let $w \in [\omega]^{<\omega}$. It is easy to see that the set

$$I_w = \{(v, \bar{B}) \in Q(D_0, \ldots, D_{n-1}) : \\ (w \cup [v \setminus \min\{k \in v_{|w| \bmod n} : \max(w)\}], \bar{B}) \in I\}$$

is open dense. If we apply Lemma 1.9 to $p = (\emptyset, \omega, \ldots, \omega)$ and the countably many open dense sets I_w where $w \in [\omega]^{<\omega}$, we obtain $q = (\emptyset, \bar{B})$. Let $\langle a_i : i < \omega\rangle$ be the increasing enumeration of $\bar{A}$. Choose m large enough so that for each $i < n$, $\bar{A}_i \setminus \{a_j : j < mn\} \subseteq \bar{B}_i$. Let $w = \{a_j : j < mn\}$. By construction, there exists $v \subseteq \bar{A} \cap \bar{B} \setminus (a_{mn-1} + 1)$ such that $(v, \bar{B}) \in I_w$ and $w \cup v = \bar{A} \cap k$ for some $k < \omega$. Hence $(w \cup v, \bar{B}) \in I$, and so the filter on $Q(D_0, \ldots, D_{n-1})$ determined by $\bar{A}$ intersects I. As I was arbitrary, we are done. ∎

An immediate consequence of Lemma 1.5 and Corollary 1.10 is the following.

COROLLARY 1.11. *Suppose* $\overline{A} \in [\omega]^\omega$ *is* Q*-generic over* V, *and* $\overline{B} \in [\omega]^\omega$ *is such that* $\overline{B}_i \subseteq \overline{A}_i$ *for every* $i < n$. *Then* $\overline{B}$ *is* Q*-generic over* V *as well.*

Recall that a forcing is called *Suslin* if its underlying set is an analytic set of reals and its order and incompatibility relations are analytic subsets of the plane. A forcing P is called *Suslin-proper* if it is Suslin and for every countable transitive model $(N, \in)$ of ZF^- which contains the real coding P and for every $p \in P \cap N$, there exists an (N, P)-generic condition extending p. See [JuSh] for the theory of Suslin-proper forcing and [ShSp] for its properties which are relevant here.

COROLLARY 1.12. *The forcing* Q *is Suslin-proper.*

Proof. It is trivial to note that Q is Suslin, without parameter in its definition. Let $(N, \in)$ be a countable model of ZFC^-, and let $p \in Q \cap N$. Without loss of generality, $|w^p| = 0 \bmod n$. Let $\overline{A} \in [\omega]^\omega \cap V$ be Q-generic over N such that p belongs to its generic filter. Hence $w_i^p \subseteq \overline{A}_i \subseteq w_i^p \cup (\overline{A}_i^p \setminus (\max(w^p) + 1))$ for all $i < n$. But if $q = (w^p, \overline{A})$, then clearly $q \leq^0 p$ and q is (N, Q)-generic, as every $\overline{B} \in [\omega]^\omega$ which is Q-generic over V and contains q in its generic filter is a subset of $\overline{A}$ and hence $Q \cap N$-generic over N by Corollary 1.11 applied in N. ∎

The following is an immediate consequence of Corollary 1.12.

COROLLARY 1.13. *If* $p \in Q$ *and* $\langle \tau_n : n < \omega \rangle$ *are* Q*-names for members of* V, *there exist* $q \in Q$, $q \leq^0 p$ *and* $\langle X_n : n < \omega \rangle$ *such that* $X_n \in V \cap [V]^\omega$ *and* $q \Vdash_Q \forall n (\tau_n \in X_n)$.

COROLLARY 1.14. *Forcing with* Q *does not change the cofinality of any cardinal* λ *with* $\mathrm{cf}(\lambda) \geq \mathfrak{h}(n)$ *to a cardinal below* $\mathfrak{h}(n)$.

Proof. Suppose there were a cardinal $\kappa < \mathfrak{h}(n)$ and a Q-name $\dot{f}$ for a cofinal function from κ to λ. Working in V and using Corollary 1.13, for every $\alpha < \kappa$ we may construct a maximal antichain $\langle p_\beta^\alpha : \beta < \mathfrak{c} \rangle$ in Q and $\langle X_\beta^\alpha : \beta < \mathfrak{c} \rangle$ such that for all $\beta < \mathfrak{c}$, $w^{p_\beta^\alpha} = \emptyset$, $X_\beta^\alpha \in [V]^\omega \cap V$ and $p_\beta^\alpha \Vdash_Q \dot{f}(\alpha) \in X_\beta^\alpha$.

Then clearly $\mathcal{A}_\alpha = \langle \langle \overline{A}_i^{p_\beta^\alpha} : i < n \rangle : \beta < \mathfrak{c} \rangle$ is a maximal antichain in $(\mathcal{P}(\omega)/\mathrm{fin})^n$. By $\kappa < \mathfrak{h}(n)$, $\langle \mathcal{A}_\alpha : \alpha < \kappa \rangle$ has a refinement, say $\mathcal{A}$. Choose $\langle \overline{A}_i : i < n \rangle \in \mathcal{A}$. Let $\overline{A} = \bigcup \{\overline{A}_i : i < n\}$. We may assume that the $\overline{A}_i$ also have the meaning from Definition 1.4 with respect to $\overline{A}$. For each $\alpha < \kappa$ there exists $\beta(\alpha)$ such that $\langle \overline{A}_i : i < n \rangle \leq_{(\mathcal{P}(\omega)/\mathrm{fin})^n} \langle \overline{A}_i^{p_{\beta(\alpha)}^\alpha} : i < n \rangle$. Then clearly

$$(\emptyset, \overline{A}) \Vdash_Q \text{ range}(\dot{f}) \subseteq \bigcup\{X^\alpha_{\beta(\alpha)} : \alpha < \kappa\}.$$

But as $\mathrm{cf}(\lambda) \geq \mathfrak{h}(n)$ and $\kappa < \mathfrak{h}(n)$, we have a contradiction. ■

LEMMA 1.15. *Suppose $D_0, \ldots, D_{n-1}$ are pairwise not RK-equivalent Ramsey ultrafilters. Then $Q(D_0, \ldots, D_{n-1})$ has the pure decision property (for finite disjunctions), i.e. given a $Q(D_0, \ldots, D_{n-1})$-name τ for a member of $\{0,1\}$ and $p \in Q(D_0, \ldots, D_{n-1})$, there exist $q \in Q(D_0, \ldots, D_{n-1})$ and $i \in \{0,1\}$ such that $q \leq^0 p$ and $q \Vdash_{Q(D_0,\ldots,D_{n-1})} \tau = i$.*

Proof. The set $I = \{r \in Q(D_0, \ldots, D_{n-1}) : r \text{ decides } \tau\}$ is open dense. By a similar induction on rk_I as in the proof of Lemma 1.9 we may find $q \in Q(D_0, \ldots, D_{n-1})$, $q \leq^0 p$, such that for every $q' \leq q$, if q' decides τ then $(w^{q'}, \overline{A}^q)$ decides τ. Now again by induction on rk_I we may assume that for every $k \in \overline{A}^q_{|w^q| \bmod n}$, $(w^q \cup \{k\}, \overline{A}^q)$ satisfies the conclusion of the lemma, and hence by the construction of q, $(w^q \cup \{k\}, \overline{A}^q)$ decides τ. But then clearly a pure extension of q decides τ, and hence q does. ■

LEMMA 1.16. *Lemma 1.15 holds if $Q(D_0, \ldots, D_{n-1})$ is replaced by Q.*

Proof. Suppose $p \in Q$, τ is a Q-name and $p \Vdash_Q \tau \in \{0,1\}$. As $\overline{A}^p \Vdash_{Q'}$ "$p \in Q(\dot{G}_0, \ldots, \dot{G}_{n-1})$", by Lemma 1.15 there exists a Q'-name $\dot{\overline{A}}$ such that

$$\overline{A}^p \Vdash_{Q'} \text{“}(w^p, \dot{\overline{A}}) \in Q'' \wedge (w^p, \dot{\overline{A}}) \leq p \wedge (w^p, \dot{\overline{A}}) \text{ decides } \tau\text{”}.$$

As Q' does not add reals there exist $\overline{A}_1, \overline{A}_2 \in [\omega]^\omega \cap V$ such that $\overline{A}_1 \subseteq \overline{A}^p$ and $\overline{A}_1 \Vdash_{Q'} \dot{\overline{A}} = \overline{A}_2$. Letting $\overline{B} = \overline{A}_1 \cap \overline{A}_2$ we conclude $(w^p, \overline{B}) \in Q$, $(w^p, \overline{B}) \leq^0 p$ and $(w^p, \overline{B})$ decides τ. ■

The rest of this section is devoted to the proof that if the forcing Q is iterated with countable supports, then in the resulting model $\mathrm{cov}(\mathcal{M}) = \omega_1$, where $\mathcal{M}$ is the ideal of meagre subsets of the real line, and $\mathrm{cov}(\mathcal{M})$ is the least number of meagre sets needed to cover the real line. Hence for every $n < \omega$, we obtain the consistency of $\mathrm{cov}(\mathcal{M}) < \mathfrak{h}(n)$.

DEFINITION 1.17. A forcing P is said to have the *Laver property* if for every P-name $\dot{f}$ for a member of ${}^\omega\omega$, $g \in {}^\omega\omega \cap V$ and $p \in P$, if

$$p \Vdash_P \forall n < \omega(\dot{f}(n) < g(n)),$$

then there exist $H : \omega \to [\omega]^{<\omega}$ and $q \in P$ such that $H \in V$, $\forall n < \omega$ $(|H(n)| \leq 2^n)$, $q \leq p$ and

$$q \Vdash_P \forall n < \omega(\dot{f}(n) \in H(n)).$$

It is not difficult to see that a forcing with the Laver property does not add Cohen reals. Moreover, by [Shb, 2.12, p. 207] the Laver property is preserved by a countable support iteration of proper forcings. See also [Go, 6.33, p. 349] for a more accessible proof.

LEMMA 1.18. *The forcing Q has the Laver property.*

Proof. Suppose $\dot{f}$ is a Q-name for a member of ${}^{\omega}\omega$ and $g \in {}^{\omega}\omega \cap V$ such that $p \Vdash_Q \forall n < \omega(\dot{f}(n) < g(n))$. We shall define $q \leq^0 p$ and $\langle H(i) : i < \omega \rangle$ such that $|H(i)| \leq 2^i$ and $q \Vdash_Q \forall i(\dot{f}(i) \in H(i))$. We may assume $|w^p| = 0 \bmod n$ and $\min(\bar{A}^p) > \max(w^p)$.

By Lemma 1.15 choose $q_0 \leq^0 p$ and K^0 such that $q_0 \Vdash_Q \dot{f}(0) = K^0$, and let $H(0) = \{K^0\}$.

Suppose $q_i \leq^0 p$, $\langle H(j) : j \leq i \rangle$ have been constructed and let a^i be the set of the first $i+1$ members of $\bar{A}^{q_i}$. Let $\langle v^k : k < k^* \rangle$ list all subsets v of a^i such that $v_l \subseteq (a^i)_l$ for every $l < n$ (see Definition 1.4). Then clearly $k^* \leq 2^{i+1}$. By Lemma 1.15 we may shrink $\bar{A}^{q_i}$ k^* times so as to obtain $\bar{A}$ and $\langle K_k^{i+1} : k < k^* \rangle$ such that for every $k < k^*$, $(w^{q_i} \cup v^k, \bar{A}) \Vdash_Q \dot{f}(i+1) = K_k^{i+1}$. Without loss of generality, $\min(\bar{A}) > \max(a^i)$. Let q_{i+1} be defined by $w^{q_{i+1}} = w^p$ and $\bar{A}^{q_{i+1}} = a^i \cup \bar{A}'$, where $\bar{A}'$ is $\bar{A}$ without its first $(i+1) \bmod n$ members. Let $H(i+1) = \{K_k^{i+1} : k < k^*\}$. Then $q^{i+1} \Vdash_Q \dot{f}(i+1) \in H(i+1)$. Finally, let q be defined by $w^q = w^p$ and $\bar{A}^q = \bigcup\{a^i : i < \omega\}$. Then q and $\langle H(i) : i < \omega \rangle$ are as desired. ∎

As explained above, from Lemma 1.18 and Shelah's preservation theorem it follows that if P is a countable support iteration of Q and G is P-generic over V, then in $V[G]$ no real is Cohen over V; equivalently, the meagre sets in V cover all the reals of $V[G]$. Now starting with V satisfying CH we obtain the following theorem.

THEOREM 1.19. *For every $n < \omega$, the inequality $\operatorname{cov}(\mathcal{M}) < \mathfrak{h}(n)$ is consistent with ZFC.*

2. Both Laver and Miller forcings collapse the continuum below each $\mathfrak{h}(n)$

DEFINITION 2.1. Let $p \subseteq {}^{<\omega}\omega$ be a tree. For any $\eta \in p$ let $\operatorname{succ}_\eta(p) = \{n < \omega : \eta^\frown\langle n \rangle \in p\}$. We say that p has a *stem*, and denote it $\operatorname{stem}(p)$, if there is $\eta \in p$ such that $|\operatorname{succ}_\eta(p)| \geq 2$ and for every $\nu \subset \eta$, $|\operatorname{succ}_\nu(p)| = 1$. Clearly, $\operatorname{stem}(p)$ is uniquely determined, if it exists. If p has a stem, by p^- we denote the set $\{\eta \in p : \operatorname{stem}(p) \subseteq \eta\}$. We say that p is a *Laver tree* if p has a stem and for every $\eta \in p^-$, $\operatorname{succ}_\eta(p)$ is infinite. We say that p is *superperfect* if for every $\eta \in p$ there exists $\nu \in p$ with $\eta \subseteq \nu$ and $|\operatorname{succ}_\nu(p)| = \omega$. We denote by $\mathbb{L}$ the set of all Laver trees, ordered by reverse inclusion, and by $\mathbb{M}$ the set of all superperfect trees, ordered by reverse inclusion. $\mathbb{L}$, $\mathbb{M}$ is usually called *Laver*, *Miller forcing*, respectively.

THEOREM 2.2. *Suppose that G is $\mathbb{L}$-generic or $\mathbb{M}$-generic over V. Then in $V[G]$, $|\mathfrak{c}^V| = |\mathfrak{h}(n)|^V$.*

Proof. Completely similarly to [BaPeSi] for the case $n = 1$, a base tree T for $(\mathcal{P}(\omega)/\mathrm{fin})^n$ of height $\mathfrak{h}(n)$ can be constructed, i.e.

(1) $T \subseteq (\mathcal{P}(\omega)/\mathrm{fin})^n$ is dense;
(2) $(T, \supseteq^*)$ is a tree of height $\mathfrak{h}(n)$;
(3) each level T_α, $\alpha < \mathfrak{h}(n)$, is a maximal antichain in $(\mathcal{P}(\omega)/\mathrm{fin})^n$;
(4) every member of T has 2^ω immediate successors.

It follows easily that, firstly, every chain in T of length of countable cofinality has an upper bound, and secondly, every member of T has an extension in T_α for arbitrarily large $\alpha < \mathfrak{h}(n)$.

Using T, we will define an $\mathbb{L}$-name for a map from $\mathfrak{h}(n)$ onto $\mathfrak{c}$. For $p \in \mathbb{L}$ and $\{\eta_0, \ldots, \eta_{n-1}\} \in [p^-]^n$, let $\bar{A}^p_{\{\eta_i : i<n\}} = \langle \mathrm{succ}_{\eta_i}(p) : i < n \rangle$.

By induction on $\alpha < \mathfrak{c}$ we will construct $(p_\alpha, \delta_\alpha, \gamma_\alpha) \in \mathbb{L} \times \mathfrak{h}(n) \times \mathfrak{c}$ such that the following clauses hold:

(5) if $\{\eta_0, \ldots, \eta_{n-1}\} \in [p_\alpha]^n$, then $\bar{A}^{p_\alpha}_{\{\eta_i : i<\omega\}} \in T_{\delta_\alpha}$;

(6) if $\beta < \alpha$, $\delta_\beta = \delta_\alpha$, $\{\eta_0, \ldots, \eta_{n-1}\} \in [p_\alpha^-]^n \cap [p_\beta^-]^n$, then $\bar{A}^{p_\alpha}_{\{\eta_i : i<n\}}$, $\bar{A}^{p_\beta}_{\{\eta_i : i<n\}}$ are incompatible in $(\mathcal{P}(\omega)/\mathrm{fin})^n$;

(7) if $p \in \mathbb{L}$, $\gamma < \mathfrak{c}$, then for some $\alpha < \mathfrak{c}$, every extension of p_α is compatible with p and $\gamma_\alpha = \gamma$.

At stage α, by a suitable bookkeeping we are given $\gamma < \mathfrak{c}$, $p \in \mathbb{L}$, and have to find δ_α, p_α such that (5)–(7) hold. For $\eta \in p^-$ let $B_\eta = \mathrm{succ}_\eta(p)$; for $\eta \in {}^{<\omega}\omega \setminus p^-$, $B_\eta = \omega$. Let $\langle \{\eta_0^i, \ldots, \eta_{n-1}^i\} : i < \omega \rangle$ list $[{}^{<\omega}\omega]^n$ so that every member is listed $\aleph_0$ times.

Inductively we define $\langle \xi_i : i < \omega \rangle$ and $\langle B_\eta^\varrho : \eta \in {}^{<\omega}\omega,\ \varrho \in {}^{<\omega}2 \rangle$ such that

(8) $B_\eta^\varrho \in [\omega]^\omega$ and $\langle \xi_i : i < \omega \rangle$ is a strictly increasing sequence of ordinals below $\mathfrak{h}(n)$;

(9) $B_\eta^\emptyset = B_\eta$;

(10) for every $i < \omega$, the map $\varrho \mapsto \langle B^\varrho_{\eta_0^i}, \ldots, B^\varrho_{\eta_{n-1}^i} \rangle$ is one-to-one from ${}^{i+1}2$ into T_{ξ_i};

(11) for every $i < k$ and $\varrho \in {}^{k+1}2$, $B_\eta^\varrho \subseteq^* B_\eta^{\varrho \restriction i+1} \subseteq^* B_\eta^\emptyset$.

Suppose that at stage i of the construction, $\langle \xi_j : j < i \rangle$ and $\langle B_\eta^\varrho : \eta \in \{\eta_0^j, \ldots, \eta_{n-1}^j : j < i\},\ \varrho \in {}^{\le i}2 \rangle$ have been constructed. For $\eta \in \{\eta_0^i, \ldots, \eta_{n-1}^i\}$ and $\varrho \in {}^{\le i}2$, if B_η^ϱ is not yet defined, there is no problem to choose it so that (8) and (11) hold. Next by the properties of T it is easy to find ξ_i and B_η^ϱ, for every $\varrho \in {}^{i+1}2$ and $\eta \in \{\eta_0^i, \ldots, \eta_{n-1}^i\}$, so that (8)–(11) hold up to i.

By the remark following the properties of T, letting $\delta_\alpha = \sup\{\xi_i : i < \omega\}$, for every $\eta \in {}^{<\omega}\omega$ and $\varrho \in {}^\omega 2$, there exists $B^\varrho_\eta \in [\omega]^\omega$ such that

(12) for all $i < \omega$, $B^\varrho_\eta \subseteq^* B^{\varrho \restriction i}_\eta$;
(13) for all $\{\eta_0, \ldots, \eta_{n-1}\} \in [{}^{<\omega}\omega]^n$, $\langle B^\varrho_{\eta_0}, \ldots, B^\varrho_{\eta_{n-1}} \rangle \in T_{\delta_\alpha}$.

For $\varrho \in {}^\omega 2$ let $p^\varrho \in \mathbb{L}$ be defined by

$$\operatorname{stem}(p^\varrho) = \operatorname{stem}(p_\alpha), \qquad \forall \eta \in (p^\varrho)^- (\operatorname{succ}_\eta(p^\varrho) = B^\varrho_\eta).$$

It is easy to see that every extension of p^ϱ is compatible with p_α. Moreover, if $\{\eta_0, \ldots, \eta_{n-1}\} \in [(p^\varrho)^-]$, then $\overline{A}^{p^\varrho}_{\{\eta_i : i<n\}} \in T_{\delta_\alpha}$ by construction. Hence we have to find $\varrho \in {}^\omega 2$ such that, letting $p_\alpha = p^\varrho$, (6) holds. Note that for every $\{\eta_0, \ldots, \eta_{n-1}\} \in [{}^{<\omega}\omega]^n$ and $\beta < \alpha$ with $\delta_\beta = \delta_\alpha$ and $\{\eta_0, \ldots, \eta_{n-1}\} \in [p^-_\beta]^n$ there exists at most one $\varrho \in {}^\omega 2$ such that $\{\eta_0, \ldots, \eta_{n-1}\} \in [(p^\varrho)^-]^n$ and $\overline{A}^{p^\varrho}_{\{\eta_i : i<n\}}$, $\overline{A}^{p_\beta}_{\{\eta_i : i<n\}}$ are compatible in $(\mathcal{P}(\omega)/\mathrm{fin})^n$. In fact, by construction and by the fact that T_{δ_α} is an antichain, either $\overline{A}^{p^\varrho}_{\{\eta_i : i<n\}} = \overline{A}^{p_\beta}_{\{\eta_i : i<n\}}$ or they are incompatible; and moreover, for $\varrho \neq \sigma$, $\overline{A}^{p^\varrho}_{\{\eta_i : i<n\}}$, $\overline{A}^{p^\sigma}_{\{\eta_i : i<n\}}$ are incompatible. Hence, as $\aleph_0 \cdot |\alpha| < \mathfrak{c}$ we may certainly find ϱ such that, letting $p_\alpha = p^\varrho$ and $\gamma_\alpha = \gamma$, (5)–(7) hold.

But now it is easy to define an $\mathbb{L}$-name $\dot{f}$ for a function from $\mathfrak{h}(n)$ to $\mathfrak{c}$ such that for every $\alpha < \mathfrak{c}$, $p_\alpha \Vdash_{\mathbb{L}} \dot{f}(\delta_\alpha) = \gamma_\alpha$. By (7) we conclude $\Vdash_{\mathbb{L}}$ "$\dot{f} : \mathfrak{h}(n)^V \to \mathfrak{c}^V$ is onto".

A similar argument works for Miller forcing. ∎

Combining Theorem 2.2 with $\mathrm{Con}(\mathfrak{h}(n+1) < \mathfrak{h}(n))$ from §1 we obtain the following:

COROLLARY 2.3. *For every $n < \omega$, it is consistent that both Laver and Miller forcings collapse the continuum* (*strictly*) *below* $\mathfrak{h}(n)$.

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