Endomorphism Rings of Modules Whose Cardinality Is Cofinal to Omega

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1. Introduction

We want to consider torsion-free R-modules over a ring R. In Section 3 the ring R will be a principal ideal domain and in Section 4 we allow more general commutative rings R. However generally we assume that R has a distinguished countable, multiplicatively closed subset S of non-zero divisors. We also may assume that $1 \in S$ and say that an R-module G is torsion-free if $gs = 0$ ($g \in G$, $s \in S$) only holds if $g = 0$. Moreover, G is reduced (for S) if $\bigcap_{s \in S} Gs = 0$. Throughout we suppose that R is reduced and torsion-free (for S). The reader will observe that under these restrictions two kinds of realization theorems for R-algebras A as endomorphism algebras of suitable modules G are known. If we are lucky, then we find an R-module G with

$$\operatorname{End}_R G = A. \qquad \text{(STRONG)}$$

This first case we shall call a strong realization theorem. The constructed module is an A-module and multiplication by $a \in A$ is an R–endomorphism of G because R is commutative, hence $A \subseteq \operatorname{End}_R G$ where $\operatorname{End}_R G$ is the endomorphism ring of G and the construction of G shows how to get rid of the endomorphisms not in A.

The first deep result for a strong realization theorem is Corner's theorem [**1**] mentioned at several places in this volume. Note that Corner at this time was interested in $R = \mathbb{Z}$ and A torsion-free, reduced of cardinality $\aleph_0$ with special

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emphasis to rings A of finite rank. Extensions and interesting applications of this result are due to Adalberto Orsatti [**21, 22, 23**] whom we want to honour by including this paper into a volume of articles on module theory for his 60th birthday.

Corner's result was extended in a number of papers which we do not want to discuss in detail. The reader is asked to consult the 'unified treatment' in Corner, Göbel [**4**] which extends known results and also summarizes the new developments in the early eighties. Paper [**4**] is based on new combinatorial techniques first used for p-groups in Shelah [**28**], then refined in [**29, 30, 31**] to what is called after [**4**] 'Shelah's Black Box'. We only mention some of the main contributions obtained [**1, 2, 3, 4, 6, 7, 8, 10, 18, 19, 17, 28, 29, 30, 31**] and surveys in [**11, 14**].

Besides the case of strong realizations it will happen quite often that mathematical interest leads to less lucky cases. We cannot expect a strong realization theorem due to unavoidable endomorphisms. Classical examples for this second kind of realization theorems are those for are abelian p-groups G where it is known from early results that many small endomorphisms automatically belong to $\operatorname{End} G$, see Fuchs [**15**]. In order to derive a realization theorem for a 'decent' ring (like the p-adic integers $A = J_p$) we must replace (STRONG) by a weaker demand $\operatorname{End} G/\operatorname{Small} G \cong A$, which was investigated in Corner [**2**] and for cardinals $\geq 2^{\aleph_0}$ in Shelah [**28**], see Dugas, Göbel [**6**] for an extension.

If we change the category from p-groups to some other class of modules, the ideal $\operatorname{Small} G$ must be replaced by some suitable ideal depending on that category. A useful definition of such an ideal should also reduce to well-known ideals like $\operatorname{Small} G$ or $\operatorname{Fin} G$ for well-studied categories. This idea was followed up in joint work with Dugas [**8**], in [**12**] and [**4**], and lead to the ideal $\operatorname{Ines} G$ which (up to small adjustments like purity) is the collection of all those $\sigma \in \operatorname{End} G$ which do extend to any

$$\widehat{\sigma} : G' \longrightarrow G \quad (G' \supseteq G)$$

where G' belongs to the category under consideration. The desired weak realization theorem is of the form

$$\operatorname{End}_R G = A \ltimes \operatorname{Ines} G \tag{WEAK}$$

where (WEAK) is a little stronger than $\operatorname{End}_R G/\operatorname{Ines} G \cong A$ and denotes a ring split extension.

The strong realization theorem is still a special case ($\operatorname{Ines} G = 0$) and can be obtained if A is cotorsion-free, i.e. if $\operatorname{Hom}_R(\widehat{R}, A) = 0$ where $\widehat{R}$ denotes the completion of R in the S-topology (generated by Rs, $s \in S$). Cotorsion-free modules G of cardinality λ with (STRONG) have been constructed earlier, see [**4**, p. 456]. They all satisfy

$$|G| = \lambda > |A| \text{ with } \lambda^{\aleph_0} = \lambda. \tag{CARD}$$

We note that such cardinals λ are not cofinal to ω by König's lemma, hence cardinals like $\lambda = \aleph_\omega$ are excluded. The proof of the Black Box uses $\lambda^{\aleph_0} = \lambda$, so the restriction seems to be due to the Black Box. The same holds for weak realization theorems.

It is the aim of this paper to study this drawback more closely. We want to deal with this in two more definite classical cases where either $\operatorname{Ines} G = 0$ or $\operatorname{Ines} G = \operatorname{Fin} G$ where $\operatorname{Fin} G$ denotes the ideal of all $\sigma \in \operatorname{End} G$ with $\operatorname{Im} \sigma$ of finite rank. The latter case comes up naturally for two classical categories, separable abelian groups and $\aleph_0$-cotorsion-free modules. Separable modules are pure submodules of products $\prod R$ of the ring R and $\aleph_0$-cotorsion-free G are defined by the requirement that they are reduced and torsion-free such that every homomorphism from a complete module into G has finite p-adic rank, see [**4**].

If A is $\aleph_0$-cotorsion-free, then we can find (e.g. in [**4**, p. 470]) $\aleph_0$-cotorsion-free R-modules G with

$$\text{(WEAK)}, \operatorname{Ines} G = \operatorname{Fin} G \text{ and (CARD)}. \tag{FIN}$$

Similarly, if A is R-free and countable, then we can find separable R-modules G with (FIN); see [**10**] and Corner, Göbel [**5**]. If A is uncountable, then we must add a technical condition discussed in [**5, 10**].

We now come to our main concern, the problem whether we are able to avoid the cardinality restriction on λ caused (virtually) by the use of the Black Box which is $\lambda = \lambda^{\aleph_0}$. The condition $\lambda^{\aleph_0} = \lambda$ is needed to complete an easy and transparent counting argument for predicting homomorphisms, see Appendix of [**4**]. Hence $\lambda^{\aleph_0} > \lambda$ requires at least changes of the Black Box. However, in Section 3 we will see that this obstacle is more basic and really not due to Shelah's Black Box but caused by the 'natural algebraic' setting which is prepared for its use. Recall that the desired R-modules in all cases are sandwiched between a base module B and its S-adic completion $\widehat{B}$, i.e.

$$B \subseteq G \subseteq \widehat{B}.$$

This initial step already removes the chance to work with λ such that $\operatorname{cf}(\lambda) = \omega$ in case (WEAK) as follows from one of our main result:

Corollary 3.7. *Let R be a principal ideal domain and G be a torsion-free, reduced R-module of cardinality λ such that* $\operatorname{cf}(\lambda) = \omega$. *Suppose $\mu^{|R|} < \lambda$ for all cardinals $\mu < \lambda$. If G has λ pairwise distinct pure injective submodules, then* $\operatorname{End}_R G/\operatorname{Fin} G$ *has rank $\lambda^{\aleph_0}$.*

If we want to construct $\aleph_0$-cotorsion-free R-modules realizing an $\aleph_0$-cotorsion-free (but not cotorsion-free) algebra A, then the base module B above is $\bigoplus_{\lambda} A_R$ and $|A| < \lambda$. Each copy of the R-module A_R has a non-trivial cotorsion submodule and B as well as G, if of size λ, satisfies the requirements of Corollary 3.7 above. If $A \cong \operatorname{End}_R G/\operatorname{Fin} G$ then $|A| < \lambda$ contradicts the conclusion of Corollary 3.7. Hence modules of cardinality λ do not have the desired endomorphism ring.

It is interesting to note that cofinality $\operatorname{cf} \lambda = \omega$ is used in the proof of (3.7) to conclude

$$|\operatorname{End}_R G/\operatorname{Fin} G| = \lambda^{\aleph_0} > \lambda \text{ from } |\operatorname{End}_R G| = \lambda^{\aleph_0} \text{ and } |\operatorname{Fin} G| \leq \lambda.$$

If A is cotorsion-free, we have seen that the construction by the Black Box must be improved. In Section 4 we distinguish two cases (A) and (B) depending on the algebra A. In case (A) we assume that A (as above) is cotorsion-free. A new combinatorial argument is introduced which is a mixture of the Black Box and an older combinatorial principle from [**25**], which was also used in Göbel, May [**19**] and named 'Shelah elevator'. The Shelah elevator was used originally in [**25**] for constructing arbitrary large indecomposable, torsion-free abelian groups. In [**19**] it was used to export modules from smaller to larger cardinality. Here we first construct a fully rigid system of R-modules G_X $(X \leq \mu)$ such that

$$\mathrm{Hom}_R(G_X, G_Y) = A \text{ and } G_X \subseteq G_Y \text{ if } X \subseteq Y \subseteq \mu$$

and

$$\mathrm{Hom}_R(G_X, G_Y) = 0 \text{ if } X \not\subseteq Y.$$

If $\mu = \mu^{\aleph_0} < \lambda$ is some cardinal $> |A|$, then the Black Box applies and we obtain a fully rigid system. In the second step we take a few members of this fully rigid system and put them into the Shelah elevator and lift them up to G with $|G| = \lambda$ as desired.

If A is not cotorsion-free, we have to work harder to circumvent the dead end by (3.7). We must avoid that G has too many pure injective submodules. This is done in case (B). Again, a basic idea is to carry information from a rigid system of R-modules of smaller cardinal $\mu < \lambda$ up to λ. However, this time the Black Box is used to obtain an even stronger fully rigid system. Here a family of R-modules $\{G_u : u \in \mu^{\leq\aleph_0}\}$ is called *essentially A-rigid over a directed subset $\mathfrak{U}$ of $\mu^{\leq\aleph_0}$*, if the G_u's are fully rigid as usually (see e.g. [**4**]):

$$\mathrm{Hom}_R(G_u, G_{u'}) = A\delta_{uu'}, \ltimes \mathrm{Fin}(G_u, G_{u'})$$

where $\delta_{uu'} = 1$ if $u \subseteq u'$ and $\delta_{uu'} = 0$ if $u \not\subseteq u'$. Moreover $G_u \subseteq G_{u'}$ for $u \subseteq u' \in \mu^{\leq\aleph_0}$.

Hence $G_{\mathfrak{U}} = \bigcup_{u\in\mathfrak{U}} G_u$ is a well defined R-module and 'rigidness' between any G_u and $G_{\mathfrak{U}}$ is required as well; see Definition 4.3.

Inspection of the proofs in [**4**, The torsion-free theory, pp. 464–465, Ines in other torsion-free theories (pp. 465–470)] shows that the existence of essentially rigid families can be replaced by these stronger essentially rigid families, see (4.4) and (4.5). The main burden in the rest of case (B) is to find a suitable directed system $\mathfrak{U}$ of size λ to ensure that $G_{\mathfrak{U}}$ is of size λ. Since we start from a family of modules of size μ given by the Black Box, λ must be close enough to that μ. If this is the case we derive a new realization theorem for algebras A with particular emphasis on cardinals λ with $\mathrm{cf}\,\lambda = \omega$.

The main result is

Theorem 4.7. *Let A be an R-algebra, μ, λ be cardinals such that $|A| \leq \mu = \mu^{\aleph_0} < \lambda \leq 2^\mu$. If A is $\aleph_0$-cotorsion-free or A is countably free, respectively, then there exists an $\aleph_0$-cotorsion-free or a separable (reduced, torsion-free) R-module G respectively of cardinality $|G| = \lambda$ with* $\mathrm{End}_R\, G = A \oplus \mathrm{Fin}\, G$.

2. Basic definitions, examples and motivations

Let R be a principal ideal domain and $\chi = |R|^+$ be the successor of the cardinality $|R|$ of R, which is fixed throughout Section 2 and 3.

Definition 2.1. We will say that an R-module M of rank $\lambda \geq \chi$ has **many pure injectives** if there are λ pairwise distinct pure injective summands of M (purely) generated by $< \chi$ elements.

This definition may also be useful in the countable case as well, however we are mainly interested in application close to Black Box proofs, hence $\lambda \geq \aleph_1$.

Examples. Any module M of rank λ over a discrete valuation ring R possesses a basic submodule B, which is unique up to isomorphism; see Fuchs [**15**] or Eklof, Mekler [**14**, p.124]. Hence $B = \bigoplus_{i\in I} b_i R$ is a direct sum of pure cyclic submodules $b_i R$ $(i \in I)$. It is often the case that M has many pure injectives:

(a) If $R = J_p$ is the ring of p-adic integers, then M is a direct sum of a divisible module D and a reduced submodule M'. If D has rank λ, then M has enough pure injectives. Otherwise we may assume that $D = 0$ and M is a reduced J_p-module. If M is an abelian p-group, then a theorem of Kulikov applies, see Fuchs [**15**, p. 146]. It shows when $|I| \geq \lambda$, then M has many pure injectives. If M is torsion-free, then each summand of its basic module is pure injective. Hence M has many injectives if, again, $|I| \geq \lambda$. Also note that $B \subseteq M \subseteq \widehat{B}$ where $\widehat{B}$ is the p-adic completion of B; see Fuchs [**13**].

(b) The last remark relates to modules used in Black Box proofs for realizing rings as endomorphism rings; see Dugas, Göbel [**6, 7, 8**], Shelah [**30, 31**] or Corner, Göbel [**4**]. In any case (mixed, torsion-free or torsion)-constructions begin with an A-submodule $B = \bigoplus_{i\in\lambda} b_i A$ of the final module G with $\operatorname{End} G$ as required and

$$B \subseteq G \subseteq_* \widehat{B} \qquad (*)$$

where $\widehat{B}$ is the S-completion and $\subseteq_*$ denotes pure submodules. In all cases which are not cotorsion-free, the pure cyclic A-module $b_i A$ is not cotorsion-free and possesses a pure injective submodule $\neq 0$. Hence G has many pure injectives if G has rank $\lambda > |A|$. The final module has size $|G| = \lambda^{\aleph_0}$ which is λ only if $\operatorname{cf} \lambda > \omega$.

We want to investigate what happens if we require $|G| = \lambda$ and $\operatorname{cf} \lambda = \omega$. Surely, many pure injectives may prevent the existence of realization theorems. Hence we consider this possibility first.

In case $|G| = \lambda, \operatorname{cf} \lambda = \omega$ we note that (like in case $\operatorname{cf} \lambda > \omega$), the resulting module G has many pure injectives. If however $|G| = \lambda^{\aleph_0}$ this is no harm. (In fact it is not obvious from $(*)$ and surprisingly not true as we shall show that G (derived in the realization theorems) has many pure injectives.

If λ has cofinality ω, then we are bound to distinguish two cases. If the algebra A is cotorsion-free, then we will derive new realization theorems for modules of size λ cofinal to ω, which is similar to the known ones in [**4, 30, 31**]. If the module has many pure injectives, then we want to show that realization theorems (even modulo large ideals of inessential endomorphisms) do not exist. If the algebra A is not cotorsion-free. in Section 4 we also find a way around to construct modules of size λ with cf $\lambda = \omega$ for certain cardinals having a specified endomorphism ring as before.

3. Torsion-free R-modules — non existence of a realization theorem

Recall that R is a PID such that R is reduced (and torsion-free) for some fixed multiplicatively closed, countable subset S. Also recall that N_* denotes the pure closure of $N \subset M$ if the R-module M is torsion-free. Let J_p denote the p-adic integers for some prime p, this is the p-adic completion of R provided R is p-reduced.

We begin with a known result which appears in Dugas, Göbel [**7**], see the proof in [**7**] or in [**14**].

Proposition 3.1. *If M is a reduced, torsion-free R-module and $N \subset M$ with $N \cong J_p$, then $N_* \cong J_p$ and N_* is a summand of M.*

Corollary 3.2. *Let $\lambda > |R|$ be some cardinal. If M is torsion-free reduced with pairwise distinct submodules N_i $(i \in \lambda)$ which are pure-injective, then M has λ pure injectives which constitute a direct sum in M.*

Proof. Each N_i is a p-adic module. We replace the given family by an equipotent subfamily of J_p-modules for a fixed p. Similarly we may assume that $N_i \cong J_p$. If we replace the new family by $(N_i)_*$ $(i \leq \lambda)$, each $(N_i)_*$ may coincide with finitely many $(N_j)_*$ by (3.1). An equipotent subfamily $J_p \cong N_i' \sqsubset M$ satisfies $\bigoplus_{i\in\lambda} N_i' \subseteq M$. $\square$

The conclusion of the following Proposition 3.3 follows from the existence of a family of submodules similar to the one in 3.2. Under these conditions it will be possible to find many endomorphisms. These endomorphisms will destroy any hope for a realization theorem, even modulo some ideal of inessential endomorphisms. Moreover (3.3) illustrates that (3.2) must be strengthened in order to carry out (3.3) and its consequences. Notice that (3.3) is the main tool for proving the non-existence of a realization theorem.

Proposition 3.3. *Suppose $G = \bigcup_{n\in\omega} G_n$ is the union of a chain of pure submodules G_n of cardinality λ_n $(n \in \omega)$ such that λ_n $(n \in \omega)$ is strictly increasing. Let $\{N_i^n \mid i \in \lambda_{n+1}, n \in \omega\}$ be a family of pure injective modules such that $\bigoplus_{i\in\lambda_{n+1}} N_i^n \subseteq G_{n+1}$ is direct and $G_n \oplus N_i^n \subseteq_* G_{n+1}$ is pure for any $n \in \omega$ and $i \in \lambda_{n+1}$. If $\eta \in \prod_{n\in\omega} \lambda_{n+1}$ then there exists an $h_\eta \in \operatorname{End}_R G$ with $\operatorname{Im} h_\eta = \bigoplus_{n\in\omega} N^n_{\eta(n)}$.*

Remark 3.4. Note that $\lambda = |G| = \sup_{n\in\omega} \lambda_n$ is cofinal to ω. The choice of $\operatorname{Im} h_\eta$ will ensure that h_η is not swallowed by $\operatorname{Ines} G$, the ideal of inessential endomorphisms of G. Obviously (3.3) will imply the existence of $\lambda^{\aleph_0} > \lambda$ such endomorphisms and $R \cong \operatorname{End} G / \operatorname{Ines} G$ would be impossible for any ring R with $|R| \leq \lambda$; see (3.6).

Proof of (3.3). Let $h_i^n : G_n \oplus N_i^n \longrightarrow N_i^n$ be the canonical projection. This projection extends to

$$h_i^n : G \longrightarrow N_i^n$$

because $G_n \oplus N_i^n$ is pure in G and N_i^n is pure injective. If $\eta \in \prod_{n \in \omega} \lambda_{n+1}$, then put

$$h_\eta = \sum_{n \in \omega} h_{\eta(n)}^n.$$

If $x \in G$, then there is $n \in \omega$ such that $x \in G_n$, hence $h_{\eta(m)}^m(x) = 0$ for all $m \geq n$ and the sum $h_\eta(x) = \sum_{n \in \omega} h_{\eta(n)}^n(x)$ is finite and hence well-defined in G. Clearly $h_\eta \in \operatorname{End} G$ and $\operatorname{Im} h_\eta = \bigoplus_{n \in \omega} N_{\eta(n)}^n$. □

In view of (3.3) we want to strengthen (3.2).

Lemma 3.5. *Let μ be a regular cardinal $> \chi$ and let G be a torsion-free, reduced R-module with the following properties.*

(a) *There is a family $N_i \subseteq G$ $(i < \mu)$ of pure injective pairwise distinct submodules purely generated by $< \chi$ elements.*

(b) *Let $K \subseteq G$ with $|K| \leq \kappa < \mu$ for some regular cardinal κ.*

Then we can find pure injective summands $0 \neq N_i'$ of G and $K' \subseteq G$ such that $K \subseteq K'$, $|K| \leq \kappa$ and $K' \oplus \bigoplus_{i<\mu} N_i' \subseteq_* G$.

Proof. By Corollary 3.2 we replace the given family by a new family of pure injective summands $N_i \neq 0$ $(i < \mu)$ such that $\bigoplus_{i<\mu} N_i$ is a direct sum. Inductively we enumerate a subfamily of the N_i's and choose $K = K_0, K_i \subseteq K_{i+1}$ with $N_i \subseteq K_{i+1}$ and $N_i \oplus K_i$ which is a strictly increasing, continuous chain K_i of submodules and elementary submodels of G with respect to a language L_χ of cardinality χ.

If K_i is given, we want to find N_i from the above family with $K_i \oplus N_i$. Then we let K_{i+1} be the elementary closure of $K_i \oplus N_i$ and proceed continuously. Recall that $J_p \cong N_j$ from (3.2) and let $g_j : J_p \hookrightarrow G$ be the given isomorphism. There is some $g = g_j$ with $g(1) \notin K_i$ by cardinality. The algebraic reason for taking the elementary closure is that G/K_i must be torsion-free, reduced. We want to show that $\operatorname{Im} g \cap K_i = 0$. Suppose $g(x) \in K_i$ for some $0 \neq x \in J_p$. If x is not pure in J_p, then $x = p^k x'$ for some pure $x' \in J_p$. Hence $g(x) = p^k g(x') \in K_i$ and K_i is pure in M and torsion-free. We also have $g(x') \in K_i$ and hence may assume that x is pure in J_p. There is a maximal p-power p^k such that

$p^k \mid g(1)$ modulo K_i because G/K_i is torsion-free reduced and $0 \not\equiv g(1) + K_i$.

We also find $n \in R$ such that $p^{k+1} \mid (n - x)$ in J_p, hence $p^{k+1} \mid (ng(1) - g(x))$ and $p^{k+1} \mid ng(1) \mod K_i$. We conclude $p \mid n$ and $p^{k+1} \mid (n - x)$ forces $p \mid x$ in J_p, contradicting purity. We have $N_i \oplus K_i$ for the above N_j renamed as N_i. The above family N_i, K_i $(i < \mu)$ is established.

Let $S = \{\alpha < \mu : \mathrm{cf}(\alpha) \geq \chi\}$ which is stationary in μ. The following arguments do not use the specific structure of N_i. We only need that the N_i's are the pure closure of $< \chi$ elements. Also in the last paragraph we could have dropped the reference to (3.2). If $\delta \in S$, then

$$K_\delta \oplus N_\delta \subseteq G$$

by the above family. The elementary submodel K_δ over L_χ ensures that K_δ allows an elementary embedding $h_\delta : N_\delta \longrightarrow K_\delta$. Let

$$\Gamma_\delta \text{ be the set of equations } r \mid (x_i - c_r),\ (c_r \in K_\delta),\ (r \in R)$$

where $x_i \in L\chi$ corresponds to some generator a_i^δ of N_δ (say N_δ is purely generated by a set $\{a_i^\delta : i \in I_\delta\}$ of size $< \chi$) such that $r \mid (a_i^\delta - c_r)$ in M $(r \in R)$. Then Γ_δ has $< \chi$ variables and $|R| < |R|^+ = \chi$ equations. The elementary embedding ensures some strong purity.

$$\text{If } r \mid (x_i - c_r) \in \Gamma_\delta \text{ then } r \mid (h_\delta(a_i^\delta) - c_r) \text{ in } G. \tag{$*$}$$

Put $N'_\delta = \{x - h_\delta(x) : x \in N_\delta\} \subseteq N_\delta \oplus K_\delta \subseteq G$ and notice that

$$N_\delta \to N'_\delta(x \to x - h_\delta(x))$$

gives an isomorphism. Clearly $N_\delta \oplus K_\delta = N'_\delta \oplus K_\delta$ by definition of N'_δ. Let $N''_\delta = (N'_\delta)_*$ and note that N''_δ is pure injective as well by the above. Also note that $N''_\delta \cap K_\delta = 0$ and $N''_\delta \oplus K_\delta$ must be pure in G. The pure injective module N''_δ is the first of our μ candidates needed in (3.5). The others show up by an easy combinatorial trick based on Fodor's Lemma, see Jech [**20**]. Recall that $K_\delta \oplus N_\delta$ and $h_\delta : N_\delta \to K_\delta$ may be viewed as a regressive function and S is stationary. Copies of N_δ in K_δ can be enumerated by ordinals $< \delta$ as $\mathrm{cf}(\delta) \geq \chi$, hence $h_\delta : N_\gamma \to N_\delta$ for $\gamma < \delta$. By Fodor's Lemma there is a stationary subset $S_1 \subseteq S$ with $h_\delta(N_\delta) = N$ for some fixed N and all $\delta \in S_1$.

Choose $K' \subseteq_* K_{i_0}$ for some $i_0 \in S_1$ (minimal) with $N \cup K \subseteq K'$. Induction on δ with the last argument shows that

$$K' \oplus \bigoplus_{\delta \in I} N''_\delta \subseteq_* G \text{ for } I = \{\delta \in S_1, \delta \geq i_0\}.$$

Also note that $|I| = \mu$ and (3.5) follows. □

Theorem 3.6. *Let R be a PID with $|R|^+ = \chi$ and let $\lambda \geq \mu > \chi$ be cardinals with $\mathrm{cf}(\lambda) = \omega$ and $\mu^{|R|} < \lambda$ for all $\mu < \lambda$. If G is a torsion-free, reduced R-module of cardinality λ with a set of λ pairwise distinct pure injective submodules, then we find $\lambda^{\aleph_0}$ endomorphisms h_i $(i \in \lambda^{\aleph_0})$ with* $\mathrm{Im}\, h_i$ *pure and isomorphic to a direct sum of a countable infinite subset of some set of λ pure injective submodules.*

Proof. Let λ_n $(n \in \omega)$ be a strictly increasing sequence of cardinals with $\sup_{n\in\omega} \lambda_n = \lambda$. Replacing λ_n by its successor λ_n^+ if necessary, we may assume that each λ_n is a regular cardinal, moreover $\lambda_0 > \chi$. We apply Lemma 3.5 inductively to find a countable chain of pure submodules $G_n \subsetneq G$ such that $G = \bigcup_{n\in\omega} G_n$, $|G_n| = \lambda_n$ and such that there are pure injective modules N_i^n $(i \in \lambda_{n+1})$ with

$$G_n \oplus \bigoplus_{i\in\lambda_{n+1}} N_i^n \subseteq_* G_{n+1}.$$

Now we are in the position to apply Proposition 3.3 and find endomorphisms $h_\eta \in \operatorname{End} G$ $(\eta \in \prod_{n\in\omega} \lambda_{n+1})$ such that $\operatorname{Im} h_\eta = \bigoplus_{n\in\omega} N^n_{\eta(n)}$. □

Realization theorems for certain R-algebras A provide R-modules G with $A \cong \operatorname{End} G/J$ for some suitable ideal $J = \operatorname{Ines} G$ depending on the nature of A and modules G. If G is torsion-free, then either $J = 0$ (in case A is cotorsion-free) or $J = \operatorname{Fin} G$ is the ideal of those endomorphisms φ of G with $\operatorname{Im}\varphi$ of finite rank. More generally $J = \operatorname{Ines} G$ if $\operatorname{Im}\varphi$ is complete in the S-topology, see § 1 and 4.

Our first application is an easy counting argument.

Corollary 3.7. *Let (R, G) be as in Theorem 3.6, then* $\operatorname{End} G/\operatorname{Fin} G$ *has rank* $\lambda^{\aleph_0}$.

Remark 3.8. If the algebra A is not cotorsion-free, then A possesses a pure injective submodule $0 \neq N \subset A_R$ and any module G in construction by the Black Box has a pure submodule $\bigoplus_\lambda A$, hence the hypothesis of (3.7) holds, and $\operatorname{End} G/\operatorname{Fin} G \cong A$ is impossible. This is in contrast to cardinals λ with $\operatorname{cf}(\lambda) > \omega$, see [**4**] and [**17**].

Proof of 3.7. Note that $|\operatorname{End} G| = \lambda^{\aleph_0}$ by (3.6) and $|\operatorname{Fin} G| = \lambda$, hence from $\lambda^{\aleph_0} > \lambda$ it follows $|\operatorname{End} G/\operatorname{Fin} G| = \lambda^{\aleph_0}$. □

The next application is based on the observation that endomorphisms h_i $(i \in \lambda^{\aleph_0})$ in Theorem 3.6 are not complete: Each $\operatorname{Im} h_i$ is pure and a countable direct sum of pure injectives. Hence $h_i \notin \operatorname{Ines} G$ and a suitable choice of $h_i's$ ensures that the following holds.

Corollary 3.9. *Let (R, G) be as in Theorem 3.6, then* $\operatorname{End} G/\operatorname{Ines} G$ *has rank* $\lambda^{\aleph_0}$ *as well.*

Sometimes the implication of Theorem 3.6 holds automatically, e.g. in case of certain classes of p-groups. In this case (3.9) follows by the given arguments. We leave it to the reader to check the details.

Remarks 3.8 applies *mutatis mutandis* for Corollary 3.9. This might lead to the impressions that realization theorems (which so far have only been established for cardinals λ with $\operatorname{cf}(\lambda) > \omega$ or if R has 'more than three primes') will always fail otherwise. Fortunately we will be able to extend the known results in Section 4.

4. Realizing algebras

Let R be any fixed commutative ring, with a distinguished countable multiplicatively closed subset S of non-zero-divisors as discussed in Section 1. We will consider torsion-free, reduced R-algebras A (for S).

In the first part (A) we concentrate on cotorsion-free R-algebras, so we require $\mathrm{Hom}_R(\widehat{R}, A) = 0$. Part (B) will be harder; we will deal with realization theorems of the (WEAK) form.

As in Section 3 we choose a cardinal χ with $|A|^+ = \chi$.

(A) In the cotorsion-free case we can follow an established road including only a little new work. However, we are mainly interested in cardinals λ cofinal to ω and modules of this size, where $\lambda > \chi$.

Theorem 4.1. *Let A, R and χ be as above and suppose μ is a cardinal with $\chi \le \mu = \mu^{\aleph_0} \le \lambda$. Then we can find an R-module G with* $\mathrm{End}\, G = A$ *and* $|G| = \lambda$.

We need a useful notion from Corner [**3**], used in [**4, 18**] and at many other places.

Definition 4.2. Let A be an R-algebra. A family $\{G_X : X \subseteq I\}$ of R-modules G_X is called fully A-rigid family over an indexing set I if for any subsets $X, Y, \subseteq I$ the following holds

$$\mathrm{Hom}_R(G_X, G_Y) = A \quad \text{and} \quad G_X \subseteq G_Y \quad \text{if} \quad X \subseteq Y$$
$$\mathrm{Hom}_R(G_X, G_Y) = 0 \quad \text{if} \quad X \not\subseteq Y.$$

Proof of (4.1). If $\lambda = \lambda^{\aleph_0}$, then the existence of a fully A-rigid family over λ follows from Corner, Göbel [**4**] by a proof based on Shelah's Black Box, see [**4**] and also [**27**]. In particular, if $\lambda = \mu$, let $\{G_X : X \subseteq \mu\}$ be such a family. Also note that $|G_X| = \mu$ follows from [**4**]. We choose a finite cotorsion-free rigid subfamily, taking a finite subset $I \subseteq \mu$ with $|I| \ge 6$ and

$$\mathfrak{F} = \{G_X : X \subseteq I\}.$$

Note that $|I| = 4$ would suffice, see [**19**]. This small family is the basic tool for applying a different combinatorial argument, the "Shelah's elevator", see [**19**] and also Shelah [**26**]. We will apply a version given in Corner [**3**] which can be used more directly to obtain a cotorsion-free R-module G of cardinality λ with $\mathrm{End}\, G = A$. □

(B) In order to find realization theorems for algebras as endomorphisms algebras A of R-modules G which have unavoidable inessential endomorphisms we have to work harder for $\mathrm{End}\, G / \mathrm{Ines}\, G \cong A$, where $\mathrm{Ines}\, G$ is the ideal of all inessential endomorphisms of G. Since we are primarily interested in G's of cardinality λ with $\mathrm{cf}(\lambda) = \omega$ we need different (new) combinatorial techniques because the second combinatorial principle used in (A) would break down. Nevertheless the new methods resembles ideas from this method which originates from [**25**]. While the proofs on this Shelah' elevator are based on a clever distribution of rigid pairs covering the forthcoming module, e.g. an indecomposable abelian group, the new method is no longer an elevator moving up from bottom to top (cardinals), see [**19**]. It only connects certain levels, needs more fuel and runs on a more powerful rigid system (even more powerful then a fully rigid system), which we explain first. For clarity we restrict to modules G with $\mathrm{Ines}\, G = \mathrm{Fin}\, G$, the unavoidable endomorphisms are those of finite rank. So we assume that the algebra A is $\aleph_0$-cotorsion-free (e.g.

$A = J_p$), which automatically leads to $\mathrm{Fin}\, G$, see Corner, Göbel [**4**]. Recall that an R-module G is $\aleph_0$-cotorsion-free if G is torsion-free, reduced and any cotorsion submodule has finite rank over $\widehat{R}$.

Definition 4.3.

(a) If I is an indexing set of cardinality μ, then $J = P(I)^{\leq \aleph_0}$ denotes all subsets of cardinality $\leq \aleph_0$. Obviously J is partially ordered by inclusion and we will abuse notation and write $\{i\} = i$ $(i \in I)$ for singletons.

(b) Let A be an R-algebra and $\mathfrak{U}$ be a directed subset of J. A family of R-modules $\{G_u : u \in J\}$ will be called an essentially A-rigid family for $\mathfrak{U}$ (over $\mu^{\leq \aleph_0}$) if the following holds.

  (i) If $u = \{u_i : i \leq n\}$ and the u_i's in J are pairwise disjoint, then $G_u = \bigoplus_{i \leq n} G_{u_i}$

  (ii) $\{G_u : u \in J\}$ is directed, if $u \subseteq u'$ then $G_u \subseteq_* G_{u'}$. Let $G_{\mathfrak{U}} = \bigcup_{u \in \mathfrak{U}} G_u$.

  (iii) If $u \subseteq u'$, then $G_u \subseteq G_{u'}$ and $\mathrm{Hom}_R(G_u, G_{u'}) = A \oplus \mathrm{Fin}(G_u, G'_u)$.

  (iv) If $u \in J$, then $\mathrm{Hom}_R(G_u, G_{\mathfrak{U}}) = A \oplus \mathrm{Fin}(G_u, G_{\mathfrak{U}})$.

  (v) If $u \not\subseteq u'$, then $\mathrm{Hom}_R(G_u, G_{u'}) = \mathrm{Fin}(G_u, G_{u'})$.

  (vi) $|G_u| = \mu^{\aleph_0}$ for all $u \in J$.

An easy modification of the proof of the Main Theorem in [**4**] shows that we can strengthen this result to get

Proposition 4.4. *Let $|I| = \mu$ be a cardinal with $\mu^{\aleph_0} = \mu$ and $J, \mathfrak{U}$ be as in (4.3). Let A be an $\aleph_0$-cotorsion-free R-algebra with $|A| \leq \mu$.*

Then we can find an essentially A-rigid family $\{G_u : u \in J\}$ of $\aleph_0$-cotorsion-free R-modules for $\mathfrak{U}$.

Proof. By inspection of [**4**]. □

A similar result holds for separable modules. Separable R-modules are submodules of products $\prod R$.

Proposition 4.5. *Let $|I| = \mu$ be a cardinal with $\mu^{\aleph_0} = \mu$ and $J, \mathfrak{U}$ be as in (4.3). Let A be a free R-algebra which is countably generated or satisfies a technical ('nasty') condition discussed in* [**5, 8**] *with $|A| < \mu$.*

Then there exists an essentially rigid family $\{G_u : u \in J\}$ of separable R-modules for $\mathfrak{U}$.

Proof. See [**10**] and [**5**] for A countably generated. □

The modules in (4.4) and (4.5) give rise to the desired modules $G_{\mathfrak{U}}$ for a suitable directed system $\mathfrak{U}$. The relevant properties of $\mathfrak{U}$ are derived in our next

Proposition 4.6. *Let $W \subset [\lambda]^{\leq \aleph_0}$ with $|W| \leq \lambda$ and $\lambda^{\aleph_0} > \lambda > \kappa = \mathrm{cf}(\kappa) > \aleph_0$. Then we can find a directed subset $\mathcal{U} \subseteq [\lambda]^{\leq \aleph_0}$ and a coding function $\| \; \| : \lambda \to \mathcal{U}$ with the following properties*

(a) *$\| \; \|$ is a bijection*

(b) $W \cup [\lambda]^{\leq \aleph_0} \subseteq \mathcal{U}$

(c) *If* $\alpha : \kappa \to \lambda$, *then there exists* $u \in \mathcal{U}$ *with* $|\{i \in \kappa : \|\alpha(i)\| \subset u\}| \geq \aleph_0$.

Proof. Write $\lambda = \bigcup_{\zeta<\theta} A_\zeta$ for a decomposition of λ into θ subsets A_ζ of size λ for some regular cardinal θ with $\lambda > \theta > \kappa$.

Let $V_\zeta = \bigcup_{\beta<\zeta} A_\beta$ and $U_0 = W \cup [\lambda]^{<\aleph_0}, \| \|_0 = \emptyset$.

We want to define inductively $U_\zeta \subseteq [\lambda]^{\leq\aleph_0}, \| \|_\zeta : V_\zeta \to U_\zeta$ for each $\zeta \in \theta$ as ascending, continuous chains such that $\mathcal{U} = \bigcup_{\zeta\in\theta} U_\zeta$ and $\| \| = \bigcup_{\zeta\in\theta} \| \|_\zeta$. From $\operatorname{dom} \| \|_\zeta = V_\zeta$ follows $\operatorname{dom} \| \| = \lambda$ immediately. Note that $|U_0| = \lambda$, and a bijective map $\| \|_1 : A_0 \to U_0$ can be defined. Suppose $\xi \in \theta$ and $\| \|_\zeta : V_\zeta \to U_\zeta$ is defined for all $\zeta < \xi$. If ξ is a limit ordinal we take unions. Suppose $\xi = \zeta + 1$, then we must define $\| \|_\xi$ and U_ξ such that (a) and (c) hold for $U = U_\xi$ and those α with $\operatorname{Im}\alpha \subseteq V_\zeta$. Note that α is regular, so any α gives rise to some such ζ. Hence (a) and (c) will also hold for U. Condition (a) requires only that $\| \|_\zeta$ is extended to $\| \|_\xi$ as a bijection $\| \|_\xi : V_\xi \to U_\xi$. It remains to define any bijection $\| \| : A_\zeta \to U_\xi \setminus U_\zeta$ taking care of (c).

From $|A_\zeta| = \lambda$ we need $|U_\xi \setminus U_\zeta| = \lambda$, $U_\xi \supset U_\zeta$ and define U_ξ in three steps.

First we take the ideal U'_ξ generated by U_ξ, that is

$$U'_\zeta = \{u \subset \bigcup E : E \subset U_\zeta, E \text{ finite}\}.$$

Then adjoin the $\| \|_\zeta$-closure set of U_ζ, the set $\bar{U}_\zeta$ of all elements

$$u' = \bigcup\{\|i\|_\zeta : i \in u \cap V_\zeta\}$$

where $u \in U_\zeta$. Note that $U''_\zeta = U'_\zeta \cup \bar{U}_\zeta$ has size λ while $\lambda^{\aleph_0} > \lambda$, hence $|[\lambda]^{\leq\aleph_0} \setminus U''_\zeta| = \lambda^{\aleph_0}$ and we also find a set $U'''_\zeta \subseteq [\lambda]^{\leq\aleph_0} \setminus U''_\zeta$ of cardinal λ. Now we define $U_\xi = U'''_\zeta \cup U''_\zeta$ and U is constructed.

The first step in the construction of U_ξ ensures that U is directed in

$$([\lambda]^{\leq\aleph_0}, \subseteq)$$

and the second step ensures (c). If $\alpha : \kappa \to \lambda$ then $\operatorname{Im}\alpha \subseteq V_\zeta$ for some $\zeta < \theta$. Next we consider $\|\operatorname{Im}\alpha\| = \{\|\alpha(i)\| : i \in \kappa\} \subseteq U$. The closure properties provide $u \in U$ with $\|\alpha(i)\| \subseteq U$ for infinitely many $i \in \kappa$. □

Theorem 4.7. *Let* A *be an* R*-algebra and* μ, λ *cardinals such that* $|A| \leq \mu = \mu^{\aleph_0} < \lambda \leq 2^\mu$.

(a) *If* A_R *is* $\aleph_0$*-cotorsion-free, then there exists an* $\aleph_0$*-cotorsion-free* R*-module* G *of cardinality* λ *with* $\operatorname{End}_R G = A \oplus \operatorname{Fin} G$.

(b) *If* A_R *is a free* R*-module which is either countably generated or satisfies the 'nasty' condition from* [**10**], *then there exists a separable (torsion-free)* R*-module* G *of cardinality* λ *with* $\operatorname{End}_R G = A \oplus \operatorname{Fin} G$.

Remark 4.8. Theorem 4.7 is new for cardinals λ cofinal to ω.

Proof. Let $\{X_i : i \in \lambda\}$ be an anti-chain of the power set $P(\mu)$ and choose U from Proposition 4.6. Then we build the new partially ordered set

$$\mathfrak{U} = \{\{X_i : i \in u\} : u \in \mathcal{U}\}.$$

By Proposition 4.4 or Proposition 4.3, respectively there is an essentially rigid family $\{G_u : u \in [\mu]^{\leq \aleph_0}\}$ for $\mathfrak{U}$ of $\aleph_0$-cotorsion-free R-modules or separable R-modules respectively. In particular $G = G_{\mathfrak{U}} = \bigcup_{u \in U} G_u$.

If $\sigma \in \operatorname{End}_R G$ and $u \in \mathfrak{U}$ then there exists $a_u \in A$ such that

$$(\sigma \restriction G_u) - a_u \in \operatorname{Fin}_u G$$

from Definition 4.3. Here we identify $a_u \in A$ with scalar multiplication a_u on G (or any submodule). Moreover $\operatorname{Fin}_u G$ denotes (partial) homomorphisms from G_u into G of finite rank. It follows immediately that a_u does not depend on u since G has infinite rank, is torsion-free and $\mathfrak{U}$ is directed: $(\sigma \restriction G_u) - a_u, (\sigma \restriction G_{u'}) - a_{u'} \subseteq (\sigma \restriction G_v) - a_v$ for some $v \in \mathfrak{U}$ implies $a = a_u = a_v = a_{u'}$, hence

$$(\sigma \restriction G_u) - a \in \operatorname{Fin}_u G \text{ for all } u \in \mathfrak{U}. \qquad (*)$$

We also claim that $\sigma - a \in \operatorname{Fin} G$. Otherwise $\operatorname{Im}(\sigma - a)$ has infinite rank, there are independent elements $y_i \in \operatorname{Im}(\sigma - a)$, $(i \in \omega)$. If $\sigma' = \sigma - a$, then we find $x_i \in G_{\mathfrak{U}}$ with $x_i\sigma' = y_i$ $(i \in \omega)$. We may assume $x_i, y_i \in G_{u_i}$ with $u_i \in \mathfrak{U}$ $(i \in \omega)$.

By Proposition 4.6 (a) there are $\alpha_i \in \lambda$ such that $\|\alpha_i\| = u_i$ $(i \in \omega)$. Moreover, by Proposition 4.6 (c) we can find $u \in U$ such that $\{i \in \omega : \|\alpha_i\| \subseteq u\}$ is infinite. Hence $x_i, y_i \in G_{u_i} \subseteq G_u$ for infinitely many $i \in \omega$ from (4.3) (ii). However these y_i belong to $\operatorname{Im}((\sigma - a) \restriction G_u)$ and are independent. The mapping $(\sigma - a) \restriction G_u$ has infinite rank and $(\sigma - a) \restriction G_u \notin \operatorname{Fin}_u G$ contradicts $(*)$. Hence (a) and (b) follows. □

Finally we want to state a result for the kind of cardinal λ not covered by (4.7).

Theorem 4.9. *Let A be an R-algebra and $\lambda > |A|$ be a strong limit cardinal of cofinality ω. Then (a) and (b) from (4.7) hold.*

Sketch of a proof. We can write $\lambda = \sup_{n \in \omega} \lambda_n$ such that $2^{\lambda}_n < \lambda_{n+1}$ and λ_n is a successor cardinal. Then we can use essentially rigid families for each cardinality λ_n using $\diamondsuit$-arguments, which give rise to the desired R-modules of size λ. □

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