Non-existence of universals for classes like reduced torsion free abelian groups under embeddings which are not necessarily pure

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Abstract

We consider a class K of structures, e.g. trees with $\omega + 1$ levels, metric spaces and mainly, classes of Abelian groups like the one mentioned in the title and the class of reduced separable (Abelian) p-groups. We say $M \in K$ is universal for K if any member N of K of cardinality not bigger than the cardinality of M can be embedded into M. This is a natural, often raised, problem. We try to draw consequences of cardinal arithmetic to non–existence of universal members for such natural classes.

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0 Introduction

Context. In this paper, model theoretic notions (like superstable, elementary classes) appear in the introduction but not in the paper itself (so the reader does not need to know them). Only naive set theory and basic facts on Abelian groups (all in [Fu]) are necessary for understanding the paper. The basic definitions are reviewed at the end of the introduction. On the history of the problem of the existence of universal members, see Kojman, Shelah [KjSh 409]; for more direct predecessors see Kojman, Shelah [KjSh 447], [KjSh 455] and [Sh 456], but we do not rely on them. For other advances see [Sh 457], [Sh 500] and Džamonja, Shelah [DjSh 614]. Lately [Sh 622] continue this paper.

A class $\mathfrak{K}$ is a class of structures with an embeddability notion. If not said otherwise, an embedding, is a one to one function preserving atomic relations and their negations. If $\mathfrak{K}$ is a class and λ is a cardinal, then $\mathfrak{K}_\lambda$ stands for the collection of all members of $\mathfrak{K}$ of cardinality λ.
We similarly define $\mathfrak{K}_{\leq\lambda}$.

A member M of $\mathfrak{K}_\lambda$ is universal, if every $N \in \mathfrak{K}_{\leq\lambda}$, embeds into M. An example is $M =: \bigoplus\limits_{\lambda}\mathbb{Q}$, which is universal in $\mathfrak{K}_\lambda$ if $\mathfrak{K}$ is the class of all torsion-free Abelian groups, under usual embeddings.

We give some motivation to the present paper by a short review of the above references. The general thesis in these papers, as well as the present one is:

Thesis 0.1 *General Abelian groups and trees with $\omega + 1$ levels behave in universality theorems like stable non-superstable theories.*

The simplest example of such a class is the class $\mathfrak{K}^{tr} =$: trees T with $(\omega+1)$-levels, i.e. $T \subseteq {}^{\omega\geq}\alpha$ for some α, with the relations $\eta E^0_n \nu =: \eta \restriction n = \nu \restriction n \;\&\; \lg(\eta) \geq n$. For $\mathfrak{K}^{tr}$ we know that $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ implies there is no universal for $\mathfrak{K}^{tr}_\lambda$ (by [KjSh 447]). Classes as $\mathfrak{K}^{rtf}$ (defined in the title), or $\mathfrak{K}^{rs(p)}$ (reduced separable Abelian p-groups) are similar (though they are not elementary classes) when we consider pure embeddings (by [KjSh 455]). But it is not less natural to consider usual embeddings (remembering they, the (Abelian) groups under consideration, are reduced). The problem is that the invariant has been defined using divisibility, and so under non-pure embedding those seemed to be erased.

Then in [Sh 456] the non-existence of universals is proved restricting ourselves to $\lambda > 2^{\aleph_0}$ and $(< \lambda)$-stable groups (see there). These restrictions hurt the generality of the theorem; because of the first requirement we lose some cardinals. The second requirement changes the class to one which is not established among Abelian group theorists (though to me it looks natural).

Our aim was to eliminate those requirements, or show that they are necessary. Note that the present paper is mainly concerned essentially with results in ZFC, but they have roots in "difficulties" in extending independence results thus providing a case for the

Thesis 0.2 *Even if you do not like independence results you better look at them, as you will not even consider your desirable ZFC results when they are camouflaged by the litany of many independence results you can prove things.*

Of course, independence has interest *per se*; still for a given problem in general a solution in ZFC is for me preferable on an independence result. But if it gives a method of forcing (so relevant to a series of problems) the independence result is preferable (of course, I assume there are no other major differences; the depth of the proof would be of first importance to me).

As occurs often in my papers lately, quotations of **pcf** theory appear.

This paper is also a case of

Thesis 0.3 *Assumption of cases of not GCH at singular (more generally $pp\lambda > \lambda^+$) are "good", "helpful" assumptions; i.e. traditionally uses of GCH proliferate mainly not from conviction but as you can prove many theorems assuming $2^{\aleph_0} = \aleph_1$ but very few from $2^{\aleph_0} > \aleph_1$, but assuming $2^{\beth_\omega} > \beth_\omega^+$ is helpful in proving.*

Unfortunately, most results are only almost in ZFC as they use extremely weak assumptions from **pcf**, assumptions whose independence is not known. So practically it is not tempting to try to remove them as they may be true, and it is unreasonable to try to prove independence results before independence results on **pcf** will advance.

In §1 we give an explanation of the earlier difficulties: the problem (of the existence of universals for $\mathfrak{K}^{rs(p)}$) is not like looking for $\mathfrak{K}^{tr}$ (trees with $\omega+1$ levels) but for $\mathfrak{K}^{tr}_{\langle\lambda_\alpha:\alpha<\omega\rangle}$ where

($\oplus$) $\lambda_n^{\aleph_0} < \lambda_{n+1} < \mu$, λ_n are regular and $\mu^+ < \lambda = \lambda_\omega = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ and $\mathfrak{K}^{tr}_{\langle\lambda_n:n<\omega\rangle}$ is

$$\{T : T \text{ a tree with } \omega+1 \text{ levels, in level } n<\omega \text{ there are } \lambda_n \text{ elements}\}.$$

We also consider $\mathfrak{K}^{tr}_{\langle\lambda_\alpha:\alpha\leq\omega\rangle}$, which is defined similarly but the level ω of T is required to have λ_ω elements.

For $\mathfrak{K}^{rs(p)}$ this is proved fully, for $\mathfrak{K}^{rtf}$ this is proved for the natural examples (but see [Sh 622]).

In §2 we define two such basic examples: one is $\mathfrak{K}^{tr}_{\langle\lambda_\alpha:\alpha\leq\omega\rangle}$, and the second is $\mathfrak{K}^{fc}_{\langle\lambda_\alpha:\alpha\leq\omega\rangle}$. The first is a tree with $\omega+1$ levels; in the second we have slightly less restrictions. We have ω kinds of elements and a function from the ω-th-kind to the nth kind. We can interpret a tree T as a member of the second example: $P^T_\alpha = \{x : x \text{ is of level } \alpha\}$ and

$$F_n(x) = y \quad \Leftrightarrow \quad x \in P^T_\omega \ \&\ y \in P^T_n \ \&\ y <_T x.$$

For the second we recapture the non-existence theorems.

But this is not one of the classes we considered originally.

In §3 we return to $\mathfrak{K}^{rtf}$ (reduced torsion free Abelian groups) and prove the non-existence of universal ones in λ if $2^{\aleph_0} < \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ and an additional very weak set theoretic assumption (the consistency of its failure is not known).

Note that (it will be proved in [Sh 622]):

($\otimes$) if $\lambda < 2^{\aleph_0}$ then $\mathfrak{K}^{rtf}_\lambda$ has no universal members.

Note: if $\lambda = \lambda^{\aleph_0}$ then $\mathfrak{K}^{tr}_\lambda$ has universal member also $\mathfrak{K}^{rs(p)}_\lambda$ (see [Fu]) and $\mathfrak{K}^{rtf}_\lambda$ (see [Sh 622]).

We have noted above that for $\mathfrak{K}^{rtf}_\lambda$ requiring $\lambda \geq 2^{\aleph_0}$ is reasonable: we can prove (i.e. in ZFC) that there is no universal member. What about $\mathfrak{K}^{rs(p)}_\lambda$? By §1 we should look at $\mathfrak{K}^{tr}_{\langle\lambda_i:i\leq\omega\rangle}$, $\lambda_\omega = \lambda < 2^{\aleph_0}$, $\lambda_n < \aleph_0$.

In §4 we prove the consistency of the existence of universals for $\mathfrak{K}^{tr}_{\langle\lambda_i:i\leq\omega\rangle}$ when $\lambda_n \leq \omega$, $\lambda_\omega = \lambda < 2^{\aleph_0}$ but of cardinality λ^+; this is not the original problem but it seems to be a reasonable variant, and more seriously, it shoots down the hope to use the present methods of proving non-existence of universals. Anyhow this is $\mathfrak{K}^{tr}_{\langle\lambda_i:i\leq\omega\rangle}$ not $\mathfrak{K}^{rs(p)}_{\lambda_\omega}$, so we proceed to reduce this problem to the previous one under a mild variant of MA. The intentions are to deal with "there is universal of cardinality λ" in Džamonja Shelah [DjSh 614].

The reader should remember that the consistency of e.g.

> $2^{\aleph_0} > \lambda > \aleph_0$ *and there is no* M *such that* $M \in \mathfrak{K}^{rs(p)}$ *is of cardinality* $< 2^{\aleph_0}$ *and universal for* $\mathfrak{K}^{rs(p)}_\lambda$

is much easier to obtain, even in a wider context (just add many Cohen reals).

As in §4 the problem for $\mathfrak{K}^{rs(p)}_\lambda$ was reasonably resolved for $\lambda < 2^{\aleph_0}$ (and for $\lambda = \lambda^{\aleph_0}$, see [KjSh 455]), we now, in §5 turn to $\lambda > 2^{\aleph_0}$ (and μ, λ_n) as in $(\oplus)$ above. As in an earlier proof we use $\langle C_\delta : \delta \in S\rangle$ guessing clubs for λ (see references or later here), so C_δ is a subset of δ (so the invariant depends on the representation of G but this disappears when we divide by suitable ideal on λ). What we do is: rather than trying to code a subset of C_δ (for $\bar{G} = \langle G_i : i < \lambda\rangle$ a representation or filtration of the structure G as the union of an increasing continuous sequence of structures of smaller cardinality) by an element of G, we do it, say, by some set $\bar{x} = \langle x_t : t \in \mathrm{Dom}(I)\rangle$, I an ideal on $\mathrm{Dom}(I)$ (really by $\bar{x}/I$). At first glance if $\mathrm{Dom}(I)$ is infinite we cannot list *a priori* all possible such sequences for a candidate H for being a universal member, as their number is $\geq \lambda^{\aleph_0} = \mu^{\aleph_0}$. But we can find a family

$$\mathcal{F} \subseteq \{\langle x_t : t \in A\rangle :\ A \subseteq \mathrm{Dom}(I),\ A \notin I,\ x_t \in \lambda\}$$

of cardinality $< \mu^{\aleph_0}$ such that for any $\bar{x} = \langle x_t : t \in \mathrm{Dom}(I)\rangle$, for some $\bar{y} \in \mathcal{F}$ we have $\bar{y} = \bar{x} \restriction \mathrm{Dom}(\bar{y})$.

As in §3 there is such $\mathcal{F}$ except when some set theoretic statement related to **pcf** holds. This statement is extremely strong, also in the sense that we do not know to prove its consistency at present. But again, it seems unreasonable to try to prove its consistency before the **pcf** problem was dealt with. Of course, we may try to improve the combinatorics to avoid the use of this statement, but are naturally discouraged by the possibility that the **pcf** statement can be proved in ZFC; thus we would retroactively get the non-existence of universals in ZFC.

In §6, under weak **pcf** assumptions, we prove: if there is a universal member in $\mathfrak{K}^{fc}_\lambda$ then there is one in $\mathfrak{K}^{rs(p)}_\lambda$; so making the connection between the combinatorial structures and the algebraic ones closer.

In §7 we give other weak **pcf** assumptions which suffice to prove non-existence of universals in $\mathfrak{K}^x_{\langle\lambda_\alpha:\alpha\leq\omega\rangle}$ (with x one of the "legal" values): $\max\mathrm{pcf}\{\lambda_n : n < \omega\} = \lambda$ and $\mathcal{P}(\{\lambda_n : n < \omega\})/J_{<\lambda}\{\lambda_n : n < \omega\}$ is an infinite Boolean Algebra (and $(\oplus)$ holds, of course).

In [KjSh 409], for singular λ results on non-existence of universals (there on orders) can be gotten from these weak **pcf** assumptions.

In §8 we get parallel results from, in general, more complicated assumptions.

In §9 we turn to a closely related class: the class of metric spaces with (one to one) continuous embeddings, similar results hold for it. We also phrase a natural criterion for deducing the non-existence of universals from one class to another.

In §10 we deal with modules and in §11 we discuss the open problems of various degrees of seriousness.

The sections are written in the order the research was done.

Notation 0.4 Note that we deal with trees with $\omega + 1$ levels rather than, say, with $\kappa + 1$, and related situations, as those cases are quite popular. But inherently the proofs of §1-§3, §5-§9 work for $\kappa + 1$ as well (in fact, **pcf** theory is stronger).
For a structure M, $\|M\|$ is its cardinality.
For a model, i.e. a structure, M of cardinality λ, where λ is regular uncountable, we say that $\bar{M}$ is a representation (or filtration) of M if $\bar{M} = \langle M_i : i < \lambda \rangle$ is an increasing continuous sequence of submodels of cardinality $< \lambda$ with union M.
For a set A, we let $[A]^\kappa = \{B : B \subseteq A \text{ and } |B| = \kappa\}$.
For a set C of ordinals,

$$\mathrm{acc}(C) = \{\alpha \in C : \alpha = \sup(\alpha \cap C)\}, \text{(set of accumulation points)}$$

$$\mathrm{nacc}(C) = C \setminus \mathrm{acc}(C) \ (= \text{ the set of non-accumulation points}).$$

We usually use η, ν, ρ for sequences of ordinals; let $\eta \triangleleft \nu$ means η is an initial segment of ν.

Let $\mathrm{cov}(\lambda, \mu, \theta, \sigma) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{<\mu}$, and for every $A \in [\lambda]^{<\theta}$ for some $\alpha < \sigma$ and $B_i \in \mathcal{P}$ for $i < \alpha$ we have $A \subseteq \bigcup_{i<\alpha} B_i\}$. Remember that for an ordinal α, e.g. a natural number, $\alpha = \{\beta : \beta < \alpha\}$.

Notation 0.5 $\mathfrak{K}^{rs(p)}$ is the class of (Abelian) groups which are p-groups (i.e. $(\forall x \in G)(\exists n)[p^n x = 0]$) reduced (i.e. have no divisible non-zero subgroups) and separable (i.e. every cyclic pure subgroup is a direct summand). See [Fu].
For $G \in \mathfrak{K}^{rs(p)}$ define a norm $\|x\| = \inf\{2^{-n} : p^n \text{ divides } x\}$. Now every $G \in \mathfrak{K}^{rs(p)}$ has a basic subgroup $B = \bigoplus_{\substack{n<\omega \\ i<\lambda_n}} \mathbb{Z} x_i^n$, where x_i^n has order p^{n+1}, and every $x \in G$ can be represented as $\sum_{\substack{n<\omega \\ i<\lambda_n}} a_i^n x_i^n$, where for each n, $w_n(x) = \{i < \lambda_n : a_i^n x_i^n \neq 0\}$ is finite and for some n, $p^n x = 0$.
$\mathfrak{K}^{rtf}$ is the class of Abelian groups which are reduced and torsion free (i.e. $G \models nx = 0, n > 0 \quad \Rightarrow \quad x = 0$).
For a group G and $A \subseteq G$ let $\langle A \rangle_G$ be the subgroup of G generated by A, we may omit the subscript G if clear from the context.
Group will mean an Abelian group, even if not stated explicitly.
Let $H \subseteq_{pr} G$ means H is a pure subgroup of G.
Let $nG = \{nx : x \in G\}$ and let $G[n] = \{x \in G : nx = 0\}$.

Notation 0.6 $\mathfrak{K}$ will denote a class of structures with the same vocabulary, with a notion of embeddability, equivalently a notion $\leq_{\mathfrak{K}}$ of submodel.

1 Their prototype is $\mathfrak{K}^{tr}_{\langle\lambda_n:n<\omega\rangle}$ not $\mathfrak{K}^{tr}$!

If we look for universal member in $\mathfrak{K}^{rs(p)}_\lambda$, thesis 0.1 suggests to us to think it is basically $\mathfrak{K}^{tr}_\lambda$ (trees with $\omega+1$ levels, i.e. $\mathfrak{K}^{tr}_\lambda$ is our prototype), a way followed in [KjSh 455], [Sh 456]. But, as explained in the introduction, this does not give answer for the case of usual embedding for the family of all such groups. Here we show that for this case the thesis should be corrected. More concretely, the choice of the prototype means the choice of what we expect is the division of the possible classes. That is for a family of classes a choice of a prototype assert that we believe that they all behave in the same way.

We show that looking for a universal member G in $\mathfrak{K}^{rs(p)}_\lambda$ is like looking for it among the G's with density $\leq\mu$ (λ,μ, as usual, as in $(\oplus)$ from §0). For $\mathfrak{K}^{rtf}_\lambda$ we get weaker results which still cover the examples usually constructed, so showing that the restrictions in [KjSh 455] (to pure embeddings) and [Sh 456] (to $(<\lambda)$-stable groups) were natural.

Proposition 1.1 *Assume that* $\mu=\sum\limits_{n<\omega}\lambda_n=\limsup\limits_n\lambda_n$, $\mu\leq\lambda\leq\mu^{\aleph_0}$, *and* G *is a reduced separable p-group such that*

$$|G|=\lambda \quad \textit{and} \quad \lambda_n(G)=:\dim((p^nG)[p]/(p^{n+1}G)[p])\leq\mu$$

(this is a vector space over $\mathbb{Z}/p\mathbb{Z}$, hence the dimension is well defined).
Then *there is a reduced separable p-group H such that $|H|=\lambda$, H extends G and $(p^nH)[p]/(p^{n+1}H)[p]$ is a group of dimension λ_n (so if $\lambda_n\geq\aleph_0$, this means cardinality λ_n).*

Remark 1.2 So for H the invariants from [KjSh 455] are trivial.

Proof (See Fuchs [Fu]). We can find z^n_i (for $n<\omega$, $i<\lambda_n(G)\leq\mu$) such that:

(a) z^n_i has order p^n,

(b) $B=\sum\limits_{n,i}\langle z^n_i\rangle_G$ is a direct sum,

(c) B is dense in G in the topology induced by the norm

$$||x||=\min\{2^{-n}:p^n \text{ divides } x \text{ in } G\}.$$

For each $n<\omega$ and $i<\lambda_n(G)$ $(\leq\mu)$ choose $\eta^n_i\in\prod\limits_{m<\omega}\lambda_m$, pairwise distinct such that for $(n^1,i^1)\neq(n^2,i^2)$ for some $n(*)$ we have:

$$\lambda_n\geq\lambda_{n(*)} \qquad\Rightarrow\qquad \eta^{n^1}_{i^1}(n)\neq\eta^{n^2}_{i^2}(n).$$

Let H be generated by G, x_i^m $(i < \lambda_m,\ m < \omega)$, $y_i^{n,k}$ $(i < \lambda_n,\ n < \omega,\ n \leq k < \omega)$ freely except for:

(α) the equations of G,

(β) $y_i^{n,n} = z_i^n$,

(γ) $py_i^{n,k+1} - y_i^{n,k} = x^k_{\eta_i^n(k)}$,

(δ) $p^{n+1}x_i^n = 0$,

(ε) $p^{k+1}y_i^{n,k} = 0$.

Now check. $\square_{1.1}$

Definition 1.3 1. $\mathbf{t}$ denotes a sequence $\langle t_i : i < \omega \rangle$, t_i a natural number > 1.

2. For a group G we define

$$G^{[\mathbf{t}]} = \{x \in G : \bigwedge_{j<\omega} [x \in (\prod_{i<j} t_i)G]\}.$$

3. We can define a semi-norm $|| - ||_{\mathbf{t}}$ on G

$$||x||_{\mathbf{t}} = \min\{2^{-i} : x \in (\prod_{j<i} t_j)G\}$$

and so the semi-metric

$$d_{\mathbf{t}}(x, y) = ||x - y||_{\mathbf{t}}.$$

Remark 1.4 So, if $|| - ||_{\mathbf{t}}$ is a norm, G has a completion under $|| - ||_{\mathbf{t}}$, which we call $|| - ||_{\mathbf{t}}$-completion; if $\mathbf{t} = \langle i! : i < \omega \rangle$ we refer to $|| - ||_{\mathbf{t}}$ as $\mathbb{Z}$-adic norm, and this induces $\mathbb{Z}$-adic topology, so we can speak of $\mathbb{Z}$-adic completion.

Proposition 1.5 *Suppose that*

$(\otimes_0)$ $\mu = \sum_n \lambda_n$ *and* $\mu \leq \lambda \leq \mu^{\aleph_0}$ *for simplicity,* $2 < 2 \cdot \lambda_n \leq \lambda_{n+1}$ *(maybe* λ_n *is finite!),*

$(\otimes_1)$ G *is a torsion free group,* $|G| = \lambda$, *and* $G^{[\mathbf{t}]} = \{0\}$,

$(\otimes_2)$ $G_0 \subseteq G$, G_0 *is free and* G_0 *is* $\mathbf{t}$*-dense in* G *(i.e. in the topology induced by the metric* $d_{\mathbf{t}}$*), where* $\mathbf{t}$ *is a sequence of primes.*

Then *there is a torsion free group* H, $G \subseteq H$, $H^{[\mathbf{t}]} = \{0\}$, $|H| = \lambda$ *and, under* $d_{\mathbf{t}}$, H *has density* μ.

Proof Let $\{x_i : i < \lambda\}$ be a basis of G_0. Let $\eta_i \in \prod_{n<\omega} \lambda_n$ for $i < \mu$ be distinct such that $\eta_i(n+1) \geq \lambda_n$ and

$$i \neq j \quad \Rightarrow \quad (\exists m)(\forall n)[m \leq n \quad \Rightarrow \quad \eta_i(n) \neq \eta_j(n)].$$

Let H be generated by

$$G, \ \ x_i^m \text{ (for } i < \lambda_m,\, m < \omega), \ \ y_i^n \text{ (for } i < \mu,\, n < \omega)$$

freely except for

(a) the equations of G,

(b) $y_i^0 = x_i$,

(c) $t_n\, y_i^{n+1} + y_i^n = x^n_{\eta_i(n)}$.

Fact A H extends G and is torsion free.
[Why? As H can be embedded into the divisible hull of G.]

Fact B $H^{[\mathbf{t}]} = \{0\}$.
Proof Let K be a countable pure subgroup of H such that $K^{[\mathbf{t}]} \neq \{0\}$. Now without loss of generality K is generated by

(i) $K_1 \subseteq G \cap$ [the $d_{\mathbf{t}}$–closure of $\langle x_i : i \in I\rangle_G$]], where I is a countable infinite subset of λ and $K_1 \supseteq \langle x_i : i \in I\rangle_G$,

(ii) y_i^m, x_j^n for $i \in I$, $m < \omega$ and $(n, j) \in J$, where $J \subseteq \omega \times \lambda$ is countable and

$$i \in I,\ n < \omega \quad \Rightarrow \quad (n, \eta_i(n)) \in J.$$

Moreover, the equations holding among those elements are deducible from the equations of the form

(a)$^-$ equations of K_1,

(b)$^-$ $y_i^0 = x_i$ for $i \in I$,

(c)$^-$ $t_n\, y_i^{n+1} + y_i^n = x^n_{\eta_i(n)}$ for $i \in I, n < \omega$.

We can find $\langle k_i : i < \omega\rangle$ such that

$$[n \geq k_i \ \&\ n \geq k_j \ \&\ i \neq j \quad \Rightarrow \quad \eta_i(n) \neq \eta_j(n)].$$

Let $y \in K \setminus \{0\}$. Then for some j, $y \notin (\prod_{i<j} t_i)G$, so for some finite $I_0 \subseteq I$ and finite $J_0 \subseteq J$ and

$$y^* \in \langle \{x_i : i \in I_0\} \cup \{x^n_\alpha : (n,\alpha) \in J_0\} \rangle_K$$

we have $y - y^* \in (\prod_{i<j} t_i)G$. Without loss of generality $J_0 \cap \{(n, \eta_i(n)) : i \in I,\ n \geq k_i\} = \emptyset$. Now there is a homomorphism φ from K into the divisible hull K^{**} of

$$K^* = \langle \{x_i : i \in I_0\} \cup \{x^n_j : (n,j) \in J_0\} \rangle_G$$

such that $\mathrm{Rang}(\varphi)/K^*$ is finite. This is enough.

Fact C $H_0 =: \langle x^n_i : n < \omega, i < \lambda_n \rangle_H$ is dense in H by $d_{\mathbf{t}}$.
Proof Straight as each x_i is in the $d_{\mathbf{t}}$-closure of H_0 inside H.

Noting then that we can increase the dimension easily, we are done. $\square_{1.5}$

2 On structures like $(\prod_n \lambda_n, E_m)_{m<\omega}$, $\eta E_m \nu =: \eta(m) = \nu(m)$

Discussion 2.1 We discuss the existence of universal members in cardinality λ, $\mu^+ < \lambda < \mu^{\aleph_0}$, for certain classes of groups. The claims in §1 indicate that the problem is similar not to the problem of the existence of a universal member in $\mathfrak{K}^{tr}_\lambda$ (the class of trees with λ nodes, $\omega + 1$ levels) but to the one where the first ω levels, are each with $< \mu$ elements. We look more carefully and see that some variants are quite different.

The major concepts and Lemma (2.4) are similar to those of §3, but easier. Since detailed proofs are given in §3, here we give somewhat shorter proofs.

Definition 2.2 For a sequence $\bar\lambda = \langle \lambda_i : i \leq \delta \rangle$ of cardinals we define:

(A) $\mathfrak{K}^{tr}_{\bar\lambda} = \{T : T$ is a tree with $\delta + 1$ levels (i.e. a partial order such that for $x \in T$, $\mathrm{lev}_T(x) =: \mathrm{otp}(\{y : y < x\})$ is an ordinal $\leq \delta$) such that: $\mathrm{lev}_i(T) =: \{x \in T : \mathrm{lev}_T(x) = i\}$ has cardinality $\leq \lambda_i\}$,

(B) $\mathfrak{K}^{fc}_{\bar\lambda} = \{M : M = (|M|, P_i, F_i)_{i \leq \delta}$, $|M|$ is the disjoint union of $\langle P_i : i \leq \delta \rangle$, F_i is a function from P_δ to P_i, $\|P_i\| \leq \lambda_i$, F_δ is the identity (so can be omitted)$\}$,

(C) If $[i \leq \delta \quad \Rightarrow \quad \lambda_i = \lambda]$ then we write λ, $\delta + 1$ instead of $\langle \lambda_i : i \leq \delta \rangle$.

Definition 2.3 Embeddings for $\mathfrak{K}^{tr}_{\bar\lambda}$, $\mathfrak{K}^{fc}_{\bar\lambda}$ are defined naturally: for $\mathfrak{K}^{tr}_{\bar\lambda}$ embeddings preserve $x < y$, $\neg x < y$, $\mathrm{lev}_T(x) = \alpha$; for $\mathfrak{K}^{fc}_{\bar\lambda}$ embeddings are defined just as for models.

If $\delta^1 = \delta^2 = \delta$ and $[i < \delta \quad \Rightarrow \quad \lambda^1_i \le \lambda^2_i]$ and $M^\ell \in \mathfrak{K}^{fc}_{\bar\lambda^\ell}$, (or $T^\ell \in \mathfrak{K}^{tr}_{\bar\lambda^\ell}$) for $\ell = 1, 2$, then an embedding of M^1 into M^2 (T^1 into T^2) is defined naturally.

Lemma 2.4 *Assume $\bar\lambda = \langle \lambda_i : i \le \delta\rangle$ and θ, χ satisfy (for some $\bar C$):*

(a) *λ_δ, θ are regular, $\bar C = \langle C_\alpha : \alpha \in S\rangle$, $S \subseteq \lambda =: \lambda_\delta$, $C_\alpha \subseteq \alpha$, for every club E of λ for some α we have $C_\alpha \subseteq E$, $\lambda_\delta < \chi \le |C_\alpha|^\theta$ and* $\mathrm{otp}(C_\alpha) \ge \theta$,

(b) $\lambda_i \le \lambda_\delta$,

(c) *there are θ pairwise disjoint sets $A \subseteq \delta$ such that $\prod\limits_{i \in A} \lambda_i \ge \lambda_\delta$.*

Then

(α) *there is no universal member in $\mathfrak{K}^{fc}_{\bar\lambda}$; moreover*

(β) *if $M_\alpha \in \mathfrak{K}^{fc}_{\bar\lambda}$ or even $M_\alpha \in \mathfrak{K}^{fc}_{\lambda_\delta}$ for $\alpha < \alpha^* < \chi$* then *some $M \in \mathfrak{K}^{fc}_{\bar\lambda}$ cannot be embedded into any M_α.*

Remark 2.5 Note that clause (β) is relevant to our discussion in §1: the non-universality is preserved even if we increase the density and, also, it is witnessed even by non-embeddability in many models.

Proof Let $\langle A_\varepsilon : \varepsilon < \theta\rangle$ be as in clause (c) and let $\eta^\varepsilon_\alpha \in \prod\limits_{i\in A_\varepsilon} \lambda_i$ for $\alpha < \lambda_\delta$ be pairwise distinct. We fix $M_\alpha \in \mathfrak{K}^{fc}_{\lambda_\delta}$ for $\alpha < \alpha^* < \chi$.
For $M \in \mathfrak{K}^{fc}_{\bar\lambda}$, let $\bar M = (|M|, P^M_i, F^M_i)_{i \le \delta}$ and let $\langle M_\alpha : \alpha < \lambda_\delta\rangle$ be a representation (=filtration) of M; for $\alpha \in S$, $x \in P^M_\delta$, let

$$\mathrm{inv}(x, C_\alpha; \bar M) = \Big\{\beta \in C_\alpha : \begin{array}{l}\text{for some } \varepsilon < \theta \text{ and } y \in M_{\min(C_\alpha \setminus (\beta+1))}\\ \text{we have } \bigwedge\limits_{i \in A_\varepsilon} F^M_i(x) = F^M_i(y)\\ \underline{\text{but}} \text{ there is no such } y \in M_\beta\end{array}\Big\}.$$

$$\mathrm{Inv}(C_\alpha, \bar M) = \{\mathrm{inv}(x, C_\alpha, \bar M) : x \in P^M_\delta\}.$$
$$\mathrm{INv}(\bar M, \bar C) = \langle \mathrm{Inv}(C_\alpha, \bar M) : \alpha \in S\rangle.$$
$$\mathrm{INV}(\bar M, \bar C) = \mathrm{INv}(\bar M, \bar C)/\mathrm{id}^a(\bar C).$$

Recall that

$$\mathrm{id}^a(\bar C) = \{T \subseteq \lambda : \text{ for some club } E \text{ of } \lambda \text{ for no } \alpha \in T \text{ is } C_\alpha \subseteq E\}.$$

The rest should be clear (for more details see proofs in §3), noticing

Fact 2.6 *1.* $\mathrm{INV}(\bar{M}, \bar{C})$ *is well defined, i.e. if* $\bar{M}^1$, $\bar{M}^2$ *are representations (=filtrations) of* M *then* $\mathrm{INV}(\bar{M}^1, \bar{C}) = \mathrm{INV}(\bar{M}^2, \bar{C})$.

2. $\mathrm{Inv}(C_\alpha, \bar{M})$ *has cardinality* $\leq \lambda$.

3. $\mathrm{inv}(x, C_\alpha; \bar{M})$ *is a subset of* C_α *of cardinality* $\leq \theta$.

$\square_{2.4}$

Conclusion 2.7 *If* $\mu = \sum_{n<\omega} \lambda_n$ *and* $\lambda_n^{\aleph_0} < \lambda_{n+1}$ *and* $\mu^+ < \lambda_\omega = \mathrm{cf}(\lambda_\omega) < \mu^{\aleph_0}$, then *in* $\mathfrak{K}^{fc}_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}$ *there is no universal member and even in* $\mathfrak{K}^{fc}_{\langle \lambda_\omega : \alpha \leq \omega \rangle}$ *we cannot find a member universal for it.*

Proof Should be clear or see the proof in §3. $\square_{2.7}$

3 Reduced torsion free groups: Non-existence of universals

We try to choose torsion free reduced groups and define invariants so that in an extension to another such group H something survives. To this end it is natural to stretch "reduced" near to its limit.

Definition 3.1 1. $\mathfrak{K}^{tf}$ is the class of torsion free (abelian) groups.

2. $\mathfrak{K}^{rtf} = \{G \in \mathfrak{K}^{tf} : \mathbb{Q}$ is not embeddable into G (i.e. G is reduced)$\}$.

3. $\mathbf{P}^*$ denotes the set of primes.

4. For $x \in G$, $\mathbf{P}(x, G) =: \{p \in \mathbf{P}^* : \bigwedge_n x \in p^n G\}$.

5. $\mathfrak{K}^x_\lambda = \{G \in \mathfrak{K}^x : \|G\| = \lambda\}$.

6. If $H \in \mathfrak{K}^{rtf}_\lambda$, we say $\bar{H}$ is a representation or filtration of H if $\bar{H} = \langle H_\alpha : \alpha < \lambda \rangle$ is increasing continuous and $H = \bigcup_{\alpha<\lambda} H_\alpha$, $H \in \mathfrak{K}^{rtf}$ and each H_α has cardinality $< \lambda$.

Proposition 3.2 *1. If* $G \in \mathfrak{K}^{rtf}$, $x \in G \setminus \{0\}$, $Q \cup \mathbf{P}(x, G) \subsetneqq \mathbf{P}^*$, G^+ *is the group generated by* $G, y, y_{p,\ell}$ $(\ell < \omega$, $p \in Q)$ *freely, except for the equations of* G *and*

$$y_{p,0} = y, \quad p y_{p,\ell+1} = y_{p,\ell} \quad \textit{and} \quad y_{p,\ell} = z \textit{ when } z \in G, p^\ell z = x$$

then $G^+ \in \mathfrak{K}^{rtf}$, $G \subseteq G^+$.

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2. *If* $G_i \in \mathfrak{K}^{rtf}$ $(i < \alpha)$ *is* $\subseteq_{pr}$*-increasing* then $G_i \subseteq_{pr} \bigcup_{j<\alpha} G_j \in \mathfrak{K}^{rtf}$ *for every* $i < \alpha$.

The proof of the following lemma introduces a method quite central to this paper.

Lemma 3.3 *Assume that*

$(*)^1_\lambda$ $2^{\aleph_0} + \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$,

$(*)^2_\lambda$ *for every* $\chi < \lambda$, *there is* $S \subseteq [\chi]^{\leq \aleph_0}$, *such that:*

- **(i)** $|S| < \lambda$,
- **(ii)** *if* D *is a non-principal ultrafilter on* ω *and* $f : D \longrightarrow \chi$ then *for some* $a \in S$ *we have*

$$\bigcap \{X \in D : f(X) \in a\} \notin D.$$

Then

(α) *in* $\mathfrak{K}^{rtf}_\lambda$ *there is no universal member (under usual embeddings (i.e. not necessarily pure)),*

(β) *moreover,* <u>*for any*</u> $G_i \in \mathfrak{K}^{rtf}_\lambda$, *for* $i < i^* < \mu^{\aleph_0}$ <u>*there is*</u> $G \in \mathfrak{K}^{rtf}_\lambda$ *not embeddable into any one of* G_i.

Before we prove 3.3 we consider the assumptions of 3.3 in 3.4, 3.5.

Claim 3.4 *1. In 3.3 we can replace* $(*)^1_\lambda$ *by*

$(**)^1_\lambda$ **(i)** $2^{\aleph_0} < \mu < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$,

(ii) *there is* $\bar{C} = \langle C_\delta : \delta \in S^* \rangle$ *such that* S^* *is a stationary subset of* λ, *each* C_δ *is a subset of* δ *with* $\mathrm{otp}(C_\delta)$ *divisible by* μ, C_δ *closed in* $\sup(C_\delta)$ *(which normally* δ, *but not necessarily so) and*

$$(\forall \alpha)[\alpha \in \mathrm{nacc}(C_\delta) \quad \Rightarrow \quad \mathrm{cf}(\alpha) > 2^{\aleph_0}]$$

(where nacc *stands for "non-accumulation points"), and such that* $\bar{C}$ *guesses clubs of* λ *(i.e. for every club* E *of* λ, *for some* $\delta \in S^*$ *we have* $C_\delta \subseteq E$*) and* $[\delta \in S^* \quad \Rightarrow \quad \mathrm{cf}(\delta) = \aleph_0]$.

2. *In* $(*)^1_\lambda$ *and in* $(*)^2_\lambda$, *without loss of generality* $(\forall \theta < \mu)[\theta^{\aleph_0} < \mu]$ *and* $\mathrm{cf}(\mu) = \aleph_0$.

Proof 1) This is what we actually use in the proof (see below).
2) Replace μ by $\mu' = \min\{\mu_1 : \mu_1^{\aleph_0} \geq \mu$ (equivalently $\mu_1^{\aleph_0} = \mu^{\aleph_0}$)$\}$. $\square_{3.4}$

Compare to, say, [KjSh 447], [KjSh 455]; the new assumption is $(*)^2_\lambda$, note that it is a very weak assumption, in fact it might be that it is always true.

Claim 3.5 *Assume that $2^{\aleph_0} < \mu < \lambda < \mu^{\aleph_0}$ and $(\forall \theta < \mu)[\theta^{\aleph_0} < \mu]$ (see 3.4(2)). Then each of the following is a sufficient condition to $(*)^2_\lambda$:*

(α) $\lambda < \mu^{+\omega_1}$,

(β) *if $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \setminus \mu$ and $|\mathfrak{a}| \leq 2^{\aleph_0}$ then we can find $h : \mathfrak{a} \longrightarrow \omega$ such that:*

$$\lambda > \sup\{\max \mathrm{pcf}(\mathfrak{b}) : \mathfrak{b} \subseteq \mathfrak{a} \textit{ countable, and } h \restriction \mathfrak{b} \textit{ constant}\}.$$

Proof Clause (α) implies Clause (β): just use any one-to-one function $h : \mathrm{Reg} \cap \lambda \setminus \mu \longrightarrow \omega$.

Clause (β) implies (by [Sh 410, §6] + [Sh 430, §2]) that for $\chi < \lambda$ there is $S \subseteq [\chi]^{\aleph_0}$, $|S| < \lambda$ such that for every $Y \subseteq \chi$, $|Y| = 2^{\aleph_0}$, we can find Y_n such that $Y = \bigcup\limits_{n<\omega} Y_n$ and $[Y_n]^{\aleph_0} \subseteq S$. (Remember: $\mu > 2^{\aleph_0}$.) Without loss of generality (as $2^{\aleph_0} < \mu < \lambda$):

$(*)$ S is downward closed.

So if D is a non-principal ultrafilter on ω and $f : D \longrightarrow \chi$ then letting $Y = \mathrm{Rang}(f)$ we can find $\langle Y_n : n < \omega \rangle$ as above. Let $h : D \longrightarrow \omega$ be defined by $h(A) = \min\{n : f(A) \in Y_n\}$. So

$$X \subseteq D \;\;\&\;\; |X| \leq \aleph_0 \;\;\&\;\; h \restriction X \text{ constant } \Rightarrow f''(X) \in S \quad (\text{remember } (*)).$$

Now for each n, for some countable $X_n \subseteq D$ (possibly finite or even empty) we have:

$$h \restriction X_n \;\text{ is constantly } n,$$

$$\ell < \omega \;\&\; (\exists A \in D)(h(A) = n \;\&\; \ell \notin A) \Rightarrow (\exists B \in X_n)(\ell \notin B).$$

Let $A_n =: \bigcap\{A : A \in X_n\} = \bigcap\{A : A \in D \text{ and } h(X) = n\}$. If the desired conclusion fails, then $\bigwedge\limits_{n<\omega} A_n \in D$. So

$$(\forall A)[A \in D \quad \Leftrightarrow \quad \bigvee_{n<\omega} A \supseteq A_n].$$

So D is generated by $\{A_n : n < \omega\}$ but then D cannot be a non-principal ultrafilter. $\square_{3.5}$

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Remark 3.6 The case when D is a principal ultrafilter is trivial.

Proof of Lemma 3.3 Let $\bar{C} = \langle C_\delta : \delta \in S^* \rangle$ be as in $(**)^1_\lambda$ (ii) from 3.4 (for 3.4(1) its existence is obvious, for 3.3 - use [Sh:e, VI, old III 7.8]). Let us suppose that $\bar{A} = \langle A_\delta : \delta \in S^* \rangle$, $A_\delta \subseteq \mathrm{nacc}(C_\delta)$ has order type ω (A_δ like this will be chosen later) and let η_δ enumerate A_δ increasingly. Let G_0 be freely generated by $\{x_i : i < \lambda\}$.

Let R be

$\{\bar{a}:$ $\bar{a} = \langle a_n : n < \omega \rangle$ is a sequence of pairwise disjoint subsets of $\mathbf{P}^*$, with union $\mathbf{P}^*$ for simplicity, such that for infinitely many n, $a_n \neq \emptyset\}$.

Let G be a group generated by

$$G_0 \cup \{y^{\alpha,n}_{\bar{a}}, z^{\alpha,n}_{\bar{a},p} : \ \alpha < \lambda,\ \bar{a} \in R,\ n < \omega,\ p \text{ prime}\}$$

freely except for:

(a) the equations of G_0,

(b) $pz^{\alpha,n+1}_{\bar{a},p} = z^{\alpha,n}_{\bar{a},p}$ when $p \in a_n$, $\alpha < \lambda$,

(c) $z^{\delta,0}_{\bar{a},p} = y^{\delta,n}_{\bar{a}} - x_{\eta_\delta(n)}$ when $p \in a_n$ and $\delta \in S^*$.

Now $G \in \mathfrak{K}^{rtf}_\lambda$ by inspection.

Before continuing the proof of 3.3 we present a definition and some facts.

Definition 3.7 For a representation $\bar{H}$ of $H \in \mathfrak{K}^{rtf}_\lambda$, and $x \in H$, $\delta \in S^*$ let

1. $\mathrm{inv}(x, C_\delta; \bar{H}) =: \{\alpha \in C_\delta :$ for some $Q \subseteq \mathbf{P}^*$, there is $y \in H_{\min[C_\delta \setminus (\alpha+1)]}$ such that $Q \subseteq \mathbf{P}(x - y, H)$ but for no $y \in H_\alpha$ is $Q \subseteq \mathbf{P}(x - y, H)\}$ (so $\mathrm{inv}(x, C_\delta; \bar{H})$ is a subset of C_δ of cardinality $\leq 2^{\aleph_0}$).
2. $\mathrm{Inv}^0(C_\delta, \bar{H}) =: \{\mathrm{inv}(x, C_\delta; \bar{H}) : x \in \bigcup_i H_i\}$.
3. $\mathrm{Inv}^1(C_\delta, \bar{H}) =: \{a : a \subseteq C_\delta$ countable and for some $x \in H$, $a \subseteq \mathrm{inv}(x, C_\delta; \bar{H})\}$.
4. $\mathrm{INv}^\ell(\bar{H}, \bar{C}) =: \mathrm{Inv}^\ell(H, \bar{H}, \bar{C}) =: \langle \mathrm{Inv}^\ell(C_\delta; \bar{H}) : \delta \in S^* \rangle$ for $\ell \in \{0, 1\}$.
5. $\mathrm{INV}^\ell(H, \bar{C}) =: \mathrm{INv}^\ell(H, \bar{H}, \bar{C}) / \mathrm{id}^a(\bar{C})$, where

   $\mathrm{id}^a(\bar{C}) =: \{T \subseteq \lambda :$ for some club E of λ for no $\delta \in T$ is $C_\delta \subseteq E\}$.

6. If ℓ is omitted, $\ell = 0$ is understood.

Fact 3.8 *1.* $\mathrm{INV}^{\ell}(H, \bar{C})$ *is well defined.*

2. *The δ-th component of* $\mathrm{INv}^{\ell}(\bar{H}, \bar{C})$ *is a family of $\leq \lambda$ subsets of C_δ each of cardinality $\leq 2^{\aleph_0}$ and if $\ell = 1$ each member is countable and the family is closed under subsets.*

3. If $G_i \in \mathfrak{K}^{rtf}_\lambda$ *for $i < i^*$, $i^* < \mu^{\aleph_0}$, $\bar{G}^i = \langle \bar{G}_{i,\alpha} : \alpha < \lambda \rangle$ is a representation of G_i,*

   then *we can find $A_\delta \subseteq \mathrm{nacc}(C_\delta)$ of order type ω such that: $i < i^*$, $\delta \in S^*$ $\Rightarrow$ for no a in the δ-th component of* $\mathrm{INv}^{\ell}(G_i, \bar{G}^i, \bar{C})$ *do we have $|a \cap A_\delta| \geq \aleph_0$.*

Proof Straightforward. (For (3) note $\mathrm{otp}(C_\delta) \geq \mu$, so there are $\mu^{\aleph_0} > \lambda$ pairwise almost disjoint subsets of C_δ each of cardinality $\aleph_0$ and every $A \in \mathrm{Inv}(C_\delta, \bar{G}^i)$ disqualifies at most $2^{\aleph_0}$ of them.) $\square_{3.8}$

Fact 3.9 *Let G be as constructed above for $\langle A_\delta : \delta \in S^* \rangle, A_\delta \subseteq \mathrm{nacc}(C_\delta)$, $\mathrm{otp}(A_\delta) = \omega$ (where $\langle A_\delta : \delta \in S^* \rangle$ are chosen as in 3.8(3) for the sequence $\langle G_i : i < i^* \rangle$ given for proving 3.3, see (β) there).*
Assume $G \subseteq H \in \mathfrak{K}^{rtf}_\lambda$ and $\bar{H}$ is a filtration of H. Then

$$\begin{aligned} B &=: \{\delta : A_\delta \textit{ has infinite intersection with some} a \in \mathrm{Inv}(C_\delta, \bar{H})\} \\ &= \lambda \mod \mathrm{id}^a(\bar{C}). \end{aligned}$$

Proof We assume otherwise and derive a contradiction. Let for $\alpha < \lambda$, $S_\alpha \subseteq [\alpha]^{\leq \aleph_0}$, $|S_\alpha| < \lambda$ be as guaranteed by $(*)^2_\lambda$.

Let $\chi > 2^\lambda$, $\mathfrak{A}_\alpha \prec (H(\chi), \in, <^*_\chi)$ for $\alpha < \lambda$ increasing continuous, $\|\mathfrak{A}_\alpha\| < \lambda$, $\langle \mathfrak{A}_\beta : \beta \leq \alpha \rangle \in \mathfrak{A}_{\alpha+1}$, $\mathfrak{A}_\alpha \cap \lambda$ an ordinal and:

$$\langle S_\alpha : \alpha < \lambda \rangle,\ G,\ H,\ \bar{C},\ \langle A_\delta : \delta \in S^* \rangle,\ \bar{H},\ \langle x_i, y^\delta_{\bar{a}}, z^{\delta,n}_{\bar{a},p} \ :\ i, \delta, \bar{a}, n, p \rangle$$

all belong to $\mathfrak{A}_0$ and $2^{\aleph_0} + 1 \subseteq \mathfrak{A}_0$. Then $E = \{\delta < \lambda : \mathfrak{A}_\delta \cap \lambda = \delta\}$ is a club of λ. Choose $\delta \in S^* \cap E \setminus B$ such that $C_\delta \subseteq E$. (Why can we? As to $\mathrm{id}^a(\bar{C})$ belong all non stationary subsets of λ, in particular $\lambda \setminus E$, and $\lambda \setminus S^*$ and B, but $\lambda \notin \mathrm{id}^a(\bar{C})$.) Remember that η_δ enumerates A_δ (in the increasing order). For each $\alpha \in A_\delta$ (so $\alpha \in E$ hence $\mathfrak{A}_\alpha \cap \lambda = \alpha$ but $\bar{H} \in \mathfrak{A}_\alpha$ hence $H \cap \mathfrak{A}_\alpha = H_\alpha$) and $Q \subseteq \mathbf{P}^*$ choose, if possible, $y_{\alpha,Q} \in H_\alpha$ such that:

$$Q \subseteq \mathbf{P}(x_\alpha - y_{\alpha,Q}, H).$$

Let $I_\alpha =: \{Q \subseteq \mathbf{P}^* : y_{\alpha,Q} \text{ well defined}\}$. Note (see 3.4 $(**)^1_\lambda$ and remember $\eta_\delta(n) \in A_\delta \subseteq \mathrm{nacc}(C_\delta)$) that $\mathrm{cf}(\alpha) > 2^{\aleph_0}$ (by (ii) of 3.4 $(**)^1_\lambda$) and hence for some $\beta_\alpha < \alpha$,

$$\{y_{\alpha,Q} : Q \in I_\alpha\} \subseteq H_{\beta_\alpha}.$$

Now:

$\otimes_1$ I_α is downward closed family of subsets of $\mathbf{P}^*$, $\mathbf{P}^* \notin I_\alpha$ for $\alpha \in A_\delta$.

[Why? See the definition for the first phrase and note also that H is reduced for the second phrase.]

$\otimes_2$ I_α is closed under unions of two members (hence is an ideal on $\mathbf{P}^*$).

[Why? If $Q_1, Q_2 \in I_\alpha$ then (as $x_\alpha \in G \subseteq H$ witnesses this):

$$(\mathcal{H}(\chi), \in, <^*_\chi) \models \quad (\exists x)(x \in H \ \&\ Q_1 \subseteq \mathbf{P}(x - y_{\alpha,Q_1}, H) \ \& \\ Q_2 \subseteq \mathbf{P}(x - y_{\alpha,Q_2}, H)).$$

All the parameters are in $\mathfrak{A}_\alpha$ so there is $y \in \mathfrak{A}_\alpha \cap H$ such that

$$Q_1 \subseteq \mathbf{P}(y - y_{\alpha,Q_1}, H) \quad \text{and} \quad Q_2 \subseteq \mathbf{P}(y - y_{\alpha,Q_2}, H).$$

By algebraic manipulations,

$$Q_1 \subseteq \mathbf{P}(x_\alpha - y_{\alpha,Q_1}, H),\ Q_1 \subseteq \mathbf{P}(y - y_{\alpha,Q_1}, H) \quad \Rightarrow \quad Q_1 \subseteq \mathbf{P}(x_\alpha - y, H);$$

similarly for Q_2. So $Q_1 \cup Q_2 \subseteq \mathbf{P}(x_\alpha - y, H)$ and hence $Q_1 \cup Q_2 \in I_\alpha$.]

$\otimes_3$ If $\bar{Q} = \langle Q_n : n \in \Gamma\rangle$ are pairwise disjoint subsets of $\mathbf{P}^*$, for some infinite $\Gamma \subseteq \omega$, then for some $n \in \Gamma$ we have $Q_n \in I_{\eta_\delta(n)}$.

[Why? Otherwise let a_n be Q_n if $n \in \Gamma$, and $\emptyset$ if $n \in \omega \setminus \Gamma$, and let $\bar{a} = \langle a_n : n < \omega\rangle$. Now $n \in \Gamma \quad \Rightarrow \quad \eta_\delta(n) \in \mathrm{inv}(y^{\delta,0}_{\bar{a}}, C_\delta; \bar{H})$ and hence

$$A_\delta \cap \mathrm{inv}(y^{\delta,0}_{\bar{a}}, C_\delta; \bar{H}) \supseteq \{\eta_\delta(n) : n \in \Gamma\},$$

which is infinite, contradicting the choice of A_δ.]

$\otimes_4$ for all but finitely many n the Boolean algebra $\mathcal{P}(\mathbf{P}^*)/I_{\eta_\delta(n)}$ is finite.

[Why? If not, then by $\otimes_1$ second phrase, for each n there are infinitely many non-principal ultrafilters D on $\mathbf{P}^*$ disjoint to $I_{\eta_\delta(n)}$, so for $n < \omega$ we can find an ultrafilter D_n on $\mathbf{P}^*$ disjoint to $I_{\eta_\delta(n)}$, distinct from D_m for $m < n$. Thus we can find $\Gamma \in [\omega]^{\aleph_0}$ and $Q_n \in D_n$ for $n \in \Gamma$ such that $\langle Q_n : n \in \Gamma\rangle$ are pairwise disjoint (as $Q_n \in D_n$ clearly $|Q_n| = \aleph_0$). Why? Look: if $B_n \in D_0 \setminus D_1$ for $n \in \omega$ then

$$(\exists^\infty n)(B_n \in D_n) \quad \text{or} \quad (\exists^\infty n)(\mathbf{P}^* \setminus B_n \in D_n),$$

etc. Let $Q_n = \emptyset$ for $n \in \omega \setminus \Gamma$, now $\bar{Q} = \langle Q_n : n < \omega\rangle$ contradicts $\otimes_3$.]

$\otimes_5$ If the conclusion (of 3.9) fails, then for no $\alpha \in A_\delta$ is $\mathcal{P}(\mathbf{P}^*)/I_\alpha$ finite.

[Why? If not, choose such an α and $Q^* \subseteq \mathbf{P}^*$, $Q^* \notin I_\alpha$ such that $I = I_\alpha \restriction Q^*$ is a maximal ideal on Q^*. So $D =: \mathcal{P}(Q^*) \setminus I$ is a non-principal ultrafilter. Remember $\beta = \beta_\alpha < \alpha$ is such that $\{y_{\alpha,Q} : Q \in I_\alpha\} \subseteq H_\beta$. Now, $H_\beta \in \mathfrak{A}_{\beta+1}$, $|H_\beta| < \lambda$. Hence $(*)^2_\lambda$ from 3.3 (note that it does not matter whether we consider an ordinal $\chi < \lambda$ or a cardinal $\chi < \lambda$, or any other set of cardinality $< \lambda$) implies that there is $S_{H_\beta} \in \mathfrak{A}_{\beta+1}$, $S_{H_\beta} \subseteq [H_\beta]^{\leq\aleph_0}$, $|S_{H_\beta}| < \lambda$ as there. Now it does not matter if we deal with functions from an ultrafilter on ω <u>or</u> an ultrafilter on Q^*. We define $f : D \longrightarrow H_\beta$ as follows: for $U \in D$ we let $f(U) = y_{\alpha,Q^*\setminus U}$. (Note: $Q^* \setminus U \in I_\alpha$, hence $y_{\alpha,Q^*\setminus U}$ is well defined.) So, by the choice of S_{H_β} (see (ii) of $(*)^2_\lambda$), for some countable $f' \subseteq f$, $f' \in \mathfrak{A}_{\beta+1}$ and $\bigcap\{U : U \in \mathrm{Dom}(f')\} \notin D$ (reflect for a minute). Let $\mathrm{Dom}(f') = \{U_0, U_1, \dots\}$. Then $\bigcup\limits_{n<\omega} (Q^* \setminus U_n) \notin I_\alpha$. But as in the proof of $\otimes_2$, as

$$\langle y_\alpha, (Q^* \setminus U_n) : n < \omega\rangle \in \mathfrak{A}_{\beta+1} \subseteq \mathfrak{A}_\alpha,$$

we have $\bigcup\limits_{n<\omega} (Q^* \setminus U_n) \in I_\alpha$, an easy contradiction.]

Now $\otimes_4$, $\otimes_5$ give a contradiction. $\square_{3.3}$

Remark 3.10 We can deal similarly with R-modules, $|R| < \mu$ <u>if</u> R has infinitely many prime ideals I. Also the treatment of $\mathfrak{K}^{rs(p)}_\lambda$ is similar to the one for modules over rings with one prime.
Note: if we replace "reduced" by

$$x \in G \setminus \{0\} \quad \Rightarrow \quad (\exists p \in \mathbf{P}^*)(x \notin pG)$$

then here we could have defined

$$\mathbf{P}(x, H) =: \{p \in \mathbf{P}^* : x \in pH\}$$

and the proof would go through with no difference (e.g. choose a fixed partition $\langle \mathbf{P}^*_n : n < \omega\rangle$ of $\mathbf{P}^*$ to infinite sets, and let $\mathbf{P}'(x, H) = \{n : x \in pH$ for every $p \in \mathbf{P}^*_n\}$). Now the groups are less divisible.

Remark 3.11 We can get that the groups are slender, in fact, the construction gives it.

4 Below the continuum there may be universal structures

Both in [Sh 456] (where we deal with universality for $(<\lambda)$-stable (Abelian) groups, like $\mathfrak{K}^{rs(p)}_\lambda$) and in §3, we restrict ourselves to $\lambda > 2^{\aleph_0}$, a restriction which does not appear in [KjSh 447], [KjSh 455]. Is this restriction necessary? In this section we shall show that at least to some extent, it is.

We first show under MA that for $\lambda < 2^{\aleph_0}$, any $G \in \mathfrak{K}_\lambda^{rs(p)}$ can be embedded into a "nice" one; our aim is to reduce the consistency of "there is a universal in $\mathfrak{K}_\lambda^{rs(p)}$" to "there is a universal in $\mathfrak{K}^{tr}_{\langle \aleph_0 : n<\omega\rangle \hat{} \langle \lambda \rangle}$". Then we proceed to prove the consistency of the latter. Actually a weak form of MA suffices.

Definition 4.1 1. $G \in \mathfrak{K}_\lambda^{rs(p)}$ is *tree-like* if:

(a) we can find a basic subgroup $B = \bigoplus_{\substack{i<\lambda_n \\ n<\omega}} \mathbb{Z}x_i^n$, where

$$\lambda_n = \lambda_n(G) =: \dim\left((p^n G)[p]/p^{n+1}(G)[p]\right)$$

(see Fuchs [Fu]) such that: $\mathbb{Z}x_i^n \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$ and

$\otimes_0$ every $x \in G$ has the form

$$\sum\{a_i^n x_i^n : n < k, i < \lambda_n\} + \sum_{n,i}\{a_i^n p^{n-k} x_i^n : n \in [k,\omega) \text{ and } i < \lambda\}$$

where $a_i^n \in \mathbb{Z}$ and

$$\begin{aligned} k \leq n < \omega \quad &\Rightarrow \quad w_n[x] =: \{i : a_i^n\, p^{n-k} x_i^n \neq 0\} \text{ is finite and} \\ n < k \quad &\Rightarrow \quad w_n[x] = \{i : a_i^n x_i^n \neq 0\} \text{ is finite} \end{aligned}$$

(this applies to any $G \in \mathfrak{K}_\lambda^{rs(p)}$ we considered so far; we write $w_n[x] = w_n[x, \bar{Y}]$ when $\bar{Y} = \langle x_i^n : n, i\rangle$). Moreover

(b) $\bar{Y} = \langle x_i^n : n, i\rangle$ is tree-like inside G, which means that we can find $F_n : \lambda_{n+1} \longrightarrow \lambda_n$ such that letting $\bar{F} = \langle F_n : n < \omega\rangle$, G is generated by some subset of $\Gamma(G, \bar{Y}, \bar{F})$ where:

$$\Gamma(G,\bar{Y},\bar{F}) = \Big\{x : \text{ for some } \eta \in \prod_{n<\omega}\lambda_n, \text{ for each } n<\omega \text{ we have } F_n(\eta(n+1)) = \eta(n) \text{ and } x = \sum_{n\geq k} p^{n-k} x^n_{\eta(n)}\Big\}.$$

2. $G \in \mathfrak{K}_\lambda^{rs(p)}$ is *semi-tree-like* if above we replace (b) by

(b)′ we can find a set $\Gamma \subseteq \{\eta : \eta$ is a partial function from ω to $\sup_{n<\omega} \lambda_n$ with $\eta(n) < \lambda_n\}$ such that:

(α) $\eta_1 \in \Gamma,\ \eta_2 \in \Gamma,\ \eta_1(n) = \eta_2(n) \quad \Rightarrow \quad \eta_1 \restriction n = \eta_2 \restriction n$,

(β) for $\eta \in \Gamma$ and $n \in \mathrm{Dom}(\eta)$, there is

$$y_{\eta,n} = \sum\{p^{m-n} x^m_{\eta(m)} : m \in \mathrm{Dom}(\eta) \text{ and } m \geq n\} \in G,$$

(γ) G is generated by

$$\{x^n_i : n < \omega, i < \lambda_n\} \cup \{y_{\eta,n} : \eta \in \Gamma, n \in \mathrm{Dom}(\eta)\}.$$

3. $G \in \mathfrak{K}^{rs(p)}_\lambda$ is *almost tree-like* if in (b)′ we add

(δ) for some $A \subseteq \omega$ for every $\eta \in \Gamma$, $\mathrm{Dom}(\eta) = A$.

Proposition 4.2 *1. Suppose $G \in \mathfrak{K}^{rs(p)}_\lambda$ is almost tree-like, as witnessed by $A \subseteq \omega$, λ_n (for $n < \omega$), x^n_i (for $n \in A$, $i < \lambda_n$), and if $n_0 < n_2$ are successive members of A, $n_0 < n < n_2$ then $\lambda_n \geq \lambda_{n_0}$ or just*

$$\lambda_n \geq |\{\eta(n_0) : \eta \in \Gamma\}|.$$

Then G is tree-like (possibly with other witnesses).

2. If in 4.1(3) we just demand $\eta \in \Gamma \quad \Rightarrow \quad \bigvee_{n<\omega} \mathrm{Dom}(\eta)\backslash n = A\backslash n$; then changing the η's and the $y_{\eta,n}$'s we can regain the "almost tree-like".

Proof 1) For every successive members $n_0 < n_2$ of A for

$$\alpha \in S_{n_0} =: \{\alpha : (\exists \eta)[\eta \in \Gamma \ \& \ \eta(n_0) = \alpha]\},$$

choose ordinals $\gamma(n_0, \alpha, \ell)$ for $\ell \in (n_0, n_2)$ such that

$$\gamma(n_0, \alpha_1, \ell) = \gamma(n_0, \alpha_2, \ell) \quad \Rightarrow \quad \alpha_1 = \alpha_2.$$

We change the basis by replacing for $\alpha \in S_{n_0}$, $\{x^n_\alpha\} \cup \{x^\ell_{\gamma(n_0,\alpha,\ell)} : \ell \in (n_0, n_2)\}$ (note: $n_0 < n_2$ but possibly $n_0 + 1 = n_2$), by:

$$\Big\{x^{n_0}_\alpha + p x^{n_0+1}_{\gamma(n_0,\alpha,n_0+1)}, \quad x^{n_0+1}_{\gamma(n_0,\alpha,n_0+1)} + p x^{n_0+2}_{\gamma(n_0,\alpha,n_0+2)}, \ldots, \\ x^{n_2-2}_{\gamma(n_0,\alpha,n_2-2)} + p x^{n_2-1}_{\gamma(n_0,\alpha,n_2-1)}, x^{n_2-1}_{\gamma(n_0,\gamma,n_2-1)}\Big\}.$$

2) For $\eta \in \Gamma$ let $n(\eta) = \min\{n : n \in A \cap \mathrm{Dom}(\eta) \text{ and } \mathrm{Dom}(\eta)\backslash n = A\backslash n\}$, and let $\Gamma_n = \{\eta \in \Gamma : n(\eta) = n\}$ for $n \in A$. We choose by induction on $n < \omega$ the objects ν_η for $\eta \in \Gamma_n$ and ρ^n_α for $\alpha < \lambda_n$ such that: ν_η is a function with domain A, $\nu_\eta \restriction (A \setminus n(\eta)) = \eta \restriction (A \setminus n(\eta))$ and $\nu_\eta \restriction (A \cap n(\eta)) = \rho^n_{\eta(n)}$, $\nu_\eta(n) < \lambda_n$ and ρ^n_α is a function with domain $A \cap n$, $\rho^n_\alpha(\ell) < \lambda_\ell$ and $\rho^n_\alpha \restriction (A \cap \ell) = \rho^\ell_{\rho^n_\alpha(\ell)}$ for $\ell \in A \cap n$. There are no problems and $\{\nu_\eta : \eta \in \Gamma_n\}$ is as required. $\square_{4.2}$

Theorem 4.3 (MA0) *Let $\lambda < 2^{\aleph_0}$. Any $G \in \mathfrak{K}^{rs(p)}_\lambda$ can be embedded into some $G' \in \mathfrak{K}^{rs(p)}_\lambda$ with countable density which is tree-like.*

Proof By 4.2 it suffices to get G' "almost tree-like" and $A \subseteq \omega$ which satisfies 4.2(1). The ability to make A thin helps in proving Fact E below. By 1.1 without loss of generality G has a base (i.e. a dense subgroup of the form) $B = \bigoplus\limits_{\substack{n<\omega \\ i<\lambda_n}} \mathbb{Z}x^n_i$, where $\mathbb{Z}x^n_i \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$ and $\lambda_n = \aleph_0$ (in fact λ_n can be $g(n)$ if $g \in {}^\omega\omega$ is not bounded (by algebraic manipulations), this will be useful if we consider the forcing from [Sh 326, §2]).

Let B^+ be the extension of B by $y^{n,k}_i$ ($k < \omega$, $n < \omega$, $i < \lambda_n$) generated freely except for $py^{n,k+1}_i = y^{n,k}_i$ (for $k < \omega$), $y^{n,\ell}_i = p^{n-\ell}x^n_i$ for $\ell \leq n$, $n < \omega$, $i < \lambda_n$. So B^+ is a divisible p-group, let $G^+ =: B^+ \bigoplus\limits_B G$. Let $\{z^0_\alpha : \alpha < \lambda\} \subseteq G[p]$ be a basis of $G[p]$ over $\{p^n x^n_i : n, i < \omega\}$ (as a vector space over $\mathbb{Z}/p\mathbb{Z}$ i.e. the two sets are disjoint, their union is a basis); remember $G[p] = \{x \in G : px = 0\}$. So we can find $z^k_\alpha \in G$ (for $\alpha < \lambda$, $k < \omega$ and $k \neq 0$) such that

$$pz^{k+1}_\alpha - z^k_\alpha = \sum_{i \in w(\alpha,k)} a^{k,\alpha}_i x^k_i,$$

where $w(\alpha, k) \subseteq \omega$ is finite (reflect on the Abelian group theory).

We define a forcing notion P as follows: a condition $p \in P$ consists of (in brackets are explanations of intentions):

(a) $m < \omega$, $M \subseteq m$,

[M is intended as $A \cap \{0, \ldots, m-1\}$]

(b) a finite $u \subseteq m \times \omega$ and $h : u \longrightarrow \omega$ such that $h(n, i) \geq n$,

[our extensions will not be pure, but still we want that the group produced will be reduced, now we add some $y^{n,k}_i$'s and h tells us how many]

(c) a subgroup K of B^+:

$$K = \langle y^{n,k}_i : (n, i) \in u, k < h(n, i)\rangle_{B^+},$$

(d) a finite $w \subseteq \lambda$,

[w is the set of $\alpha < \lambda$ on which we give information]

(e) $g : w \to m + 1$,

[$g(\alpha)$ is in what level $m' \leq m$ we "start to think" about α]

(f) $\bar{\eta} = \langle \eta_\alpha : \alpha \in w \rangle$ (see (i)),

[of course, η_α is the intended η_α restricted to m and the set of all η_α forms the intended Γ]

(g) a finite $v \subseteq m \times \omega$,

[this approximates the set of indices of the new basis]

(h) $\bar{t} = \{t_{n,i} : (n,i) \in v\}$ (see (j)),

[approximates the new basis]

(i) $\eta_\alpha \in {}^M\omega$, $\bigwedge_{\alpha \in w} \bigwedge_{n \in M} (n, \eta_\alpha(n)) \in v$,

[toward guaranteeing clause (δ) of 4.1(3) (see 4.2(2))]

(j) $t_{n,i} \in K$ and $\mathbb{Z}t_{n,i} \cong \mathbb{Z}/p^n\mathbb{Z}$,

(k) $K = \bigoplus_{(n,i) \in v} (\mathbb{Z}t_{n,i})$,

[so K is an approximation to the new basic subgroup]

(l) if $\alpha \in w$, $g(\alpha) \leq \ell \leq m$ and $\ell \in M$ then

$$z_\alpha^\ell - \sum\{t^{n-\ell}_{n,\eta_\alpha(n)} : \ell \leq n \in \mathrm{Dom}(\eta_\alpha)\} \in p^{m-\ell}(K + G),$$

[this is a step toward guaranteeing that the full difference (when $\mathrm{Dom}(\eta_\alpha)$ is possibly infinite) will be in the closure of $\bigoplus_{\substack{n \in [i,\omega) \\ i < \omega}} \mathbb{Z}x_i^n$].

We define the order by:
$p \leq q$ if and only if

(α) $m^p \leq m^q$, $M^q \cap m^p = M^p$,

(β) $u^p \subseteq u^q$, $h^p \subseteq h^q$,

(γ) $K^p \subseteq_{pr} K^q$,

(δ) $w^p \subseteq w^q$,

(ε) $g^p \subseteq g^q$,

(ζ) $\eta_\alpha^p \trianglelefteq \eta_\alpha^q$, (i.e. η_α^p is an initial segment of η_α^q)

(η) $v^p \subseteq v^q$,

(θ) $t^p_{n,i} = t^q_{n,i}$ for $(n,i) \in v^p$.

A Fact $(P, \leq)$ is a partial order.

Proof of the Fact: Trivial.

B Fact P satisfies the c.c.c. (even is σ-centered).

Proof of the Fact: It suffices to observe the following.
Suppose that

$(*)$**(i)** $p, q \in P$,

(ii) $M^p = M^q$, $m^p = m^q$, $h^p = h^q$, $u^p = u^q$, $K^p = K^q$, $v^p = v^q$, $t^p_{n,i} = t^q_{n,i}$,

(iii) $\langle \eta^p_\alpha : \alpha \in w^p \cap w^q \rangle = \langle \eta^q_\alpha : \alpha \in w^p \cap w^q \rangle$,

(iv) $g^p \restriction (w^p \cap w^q) = g^q \restriction (w^p \cap w^q)$.

Then the conditions p, q are compatible (in fact have an upper bound with the same common parts): take the common values (in (ii)) or the union (for (iii)).

C Fact For each $\alpha < \lambda$ the set $\mathcal{I}_\alpha =: \{p \in P : \alpha \in w^p\}$ is dense (and open).

Proof of the Fact: For $p \in P$ let q be like p except that:

$$w^q = w^p \cup \{\alpha\} \quad \text{and} \quad g^q(\beta) = \begin{cases} g^p(\beta) & \text{if } \beta \in w^p \\ m^p & \text{if } \beta = \alpha,\ \beta \notin w^p. \end{cases}$$

D Fact For $n < \omega$, $i < \omega$ the following set is a dense subset of P:

$$\mathcal{J}^*_{(n,i)} = \{p \in P : \quad \begin{array}{l} \text{if } m^p > n \text{ then } x^n_i \in K^p \ \& \\ (\forall n < m^p)(\{n\} \times \omega) \cap u^p \text{ has } > m^p \text{ elements}\}. \end{array}$$

Proof of the Fact: Should be clear.

E Fact For each $m < \omega$ the set $\mathcal{J}_m =: \{p \in P : m^p \geq m\}$ is dense in P.

Proof of the Fact: Let $p \in P$ be given such that $m^p < m$. Let $w^p = \{\alpha_0, \dots, \alpha_{r-1}\}$ be without repetitions; we know that in G, $pz^0_{\alpha_\ell} = 0$ and $\{z^0_{\alpha_\ell} : \ell < r\}$ is independent mod B, hence also in $K + G$ the set $\{z^0_{\alpha_\ell} : \ell < r\}$ is independent mod K. Clearly

(A) $pz^{k+1}_{\alpha_\ell} = z^k_{\alpha_\ell} \mod K$ for $k \in [g(\alpha_\ell), m^p)$, hence

(B) $p^{m^p} z^{m^p}_{\alpha_\ell} = z^{g(\alpha_\ell)}_{\alpha_\ell} \mod K$.

Remember

(C) $z^{m^p}_{\alpha_\ell} = \sum\{a^{k,\alpha_\ell}_i p^{k-m^p} x^k_i : k \geq m^p, i \in w(\alpha_\ell, k)\}$,

and so, in particular, (from the choice of $z^0_{\alpha_\ell}$)

$$p^{m^p+1} z^{m^p}_{\alpha_\ell} = 0 \quad \text{and} \quad p^{m^p} z^{m^p}_{\alpha_\ell} \neq 0.$$

For $\ell < r$ and $n \in [m^p, \omega)$ let

$$s^n_\ell =: \sum \left\{a^{k,\alpha_\ell}_i p^{k-m^p} x^k_i : k \geq m^p \text{ but } k < n \text{ and } i \in w(\alpha_\ell, k)\right\}.$$

But $p^{k-m^p} x^k_i = y^{k,m^p}_i$, so

$$s^n_\ell = \sum \left\{a^{k,\alpha_\ell}_i y^{k,m^p}_i : k \in [m^p, n) \text{ and } i \in (\alpha_\ell, k)\right\}.$$

Hence, for some $m^* > m, m^p$ we have: $\{p^m s^{m^*}_\ell : \ell < r\}$ is independent in $G[p]$ over $K[p]$ and also n in $\langle x^k_i : k \in [m^p, m^*], i < \omega\rangle$. Let

$$s^*_\ell = \sum \left\{a^{k,\alpha_\ell}_i y^{k,mp}_i : k \in [m^p, m^*) \text{ and } i \in w(\alpha_\ell, k)\right\}.$$

Then $\{s^*_\ell : \ell < r\}$ is independent in

$$B^+_{[m,m^*)} = \langle y^{l,m^*-1}_i : k \in [m^p, m^*) \text{ and } i < \omega\rangle.$$

Let $i^* < \omega$ be such that: $w(\alpha_\ell, k) \subseteq \{0, \ldots, i^* - 1\}$ for $k \in [m^p, m^*)$, $\ell = 1, \ldots, r$. Let us start to define q:

$$\begin{gathered} m^q = m^*, \quad M^q = M^p \cup \{m^* - 1\}, \quad w^q = w^p, \quad g^q = g^p, \\ u^q = u^p \cup ([m^p, m^*) \times \{0, \ldots, i^* - 1\}), \\ h^q \text{ is } h^p \text{ on } u^p \text{ and } h^q(k, i) = m^* - 1 \text{ otherwise}, \\ K^q \text{ is defined appropriately, let } K' = \langle x^n_i : n \in [m^p, m^*), i < i^*\rangle. \end{gathered}$$

Complete $\{s^*_\ell : \ell < r\}$ to $\{s^*_\ell : \ell < r^*\}$, a basis of $K'[p]$, and choose $\{t_{n,i} : (n, i) \in v^*\}$ such that: $[p^m t_{n,i} = 0 \Leftrightarrow m > n]$, and for $\ell < r$

$$p^{m^*-1-\ell} t_{m^*-1,\ell} = s^*_\ell.$$

The rest should be clear.

The generic gives a variant of the desired result: almost tree-like basis; the restriction to M and g but by 4.2 we can finish. $\square_{4.1}$

Conclusion 4.4 ($\mathbf{MA}_\lambda$(σ-centered)) *For $(*)_0$ to hold it suffices that $(*)_1$ holds where*

$(*)_0$ *in $\mathfrak{K}_\lambda^{rs(p)}$, there is a universal member,*

$(*)_1$ *in $\mathfrak{K}_{\bar\lambda}^{tr}$ there is a universal member, where:*

> **(a)** $\lambda_n = \aleph_0$, $\lambda_\omega = \lambda$, $\ell g(\bar\lambda) = \omega + 1$ <u>or</u>
>
> **(b)** $\lambda_\omega = \lambda$, $\lambda_n \in [n, \omega)$, $\ell g(\bar\lambda) = \omega + 1$.

Remark 4.5 Any $\langle \lambda_n : n < \omega \rangle$, $\lambda_n < \omega$ which is not bounded suffices.

Proof For case (a) - by 4.3.
For case (b) - the same proof. $\square_{4.4}$

Theorem 4.6 *Assume $\lambda < 2^{\aleph_0}$ and*

(a) *there are $A_i \subseteq \lambda$, $|A_i| = \lambda$ for $i < 2^\lambda$ such that $i \neq j \Rightarrow |A_i \cap A_j| \leq \aleph_0$.*

Let $\bar\lambda = \langle \lambda_\alpha : \alpha \leq \omega \rangle$, $\lambda_n = \aleph_0$, $\lambda_\omega = \lambda$.
Then *there is P such that:*

(α) *P is a c.c.c. forcing notion,*

(β) *$|P| = 2^\lambda$,*

(γ) *in V^P, there is $T \in \mathfrak{K}_{\bar\lambda}^{tr}$ into which every $T' \in (\mathfrak{K}_{\bar\lambda}^{tr})^V$ can be embedded.*

Proof Let $\bar T = \langle T_i : i < 2^\lambda \rangle$ list the trees T of cardinality $\leq \lambda$ satisfying

$${}^{\omega>}\omega \subseteq T \subseteq {}^{\omega\geq}\omega \quad \text{and} \quad T \cap {}^\omega\omega \text{ has cardinality } \lambda \text{, for simplicity.}$$

Let $T_i \cap {}^\omega\omega = \{\eta^i_\alpha : \alpha \in A_i\}$.

We shall force $\rho_{\alpha,\ell} \in {}^\omega\omega$ for $\alpha < \lambda$, $\ell < \omega$, and for each $i < 2^\lambda$ a function $g_i : A_i \longrightarrow \omega$ such that: there is an automorphism f_i of $({}^{\omega>}\omega, \triangleleft)$ which induces an embedding of T_i into $\big(({}^{\omega>}\omega) \cup \{\rho_{\alpha, g_i(\alpha)} : \alpha < \lambda\}, \triangleleft\big)$. We shall define $p \in P$ as an approximation.
A condition $p \in P$ consists of:

(a) $m < \omega$ and a finite subset u of ${}^{m\geq}\omega$, closed under initial segments such that $\langle\rangle \in u$,

(b) a finite $w \subseteq 2^\lambda$,

(c) for each $i \in w$, a finite function g_i from A_i to ω,

(d) for each $i \in w$, an automorphism f_i of $(u, \triangleleft)$,

(e) a finite $v \subseteq \lambda \times \omega$,

(f) for $(\alpha, n) \in v$, $\rho_{\alpha,n} \in u \cap ({}^m\omega)$,

such that

(g) if $i \in w$ and $\alpha \in \mathrm{Dom}(g_i)$ then:

- (α) $(\alpha, g_i(\alpha)) \in v$,
- (β) $\eta^i_\alpha \restriction m \in u$,
- (γ) $f_i(\eta^i_\alpha \restriction m) = \rho_{\alpha, g_i(\alpha)}$,

(h) $\langle \rho_{\alpha,n} : (\alpha, n) \in v \rangle$ is with no repetition (all of length m),

(i) for $i \in w$, $\langle \eta^i_\alpha \restriction m : \alpha \in \mathrm{Dom}(g_i) \rangle$ is with no repetition.

The order on P is: $p \leq q$ if and only if:

(α) $u^p \subseteq u^q$, $m^p \leq m^q$,

(β) $w^p \subseteq w^q$,

(γ) $f^p_i \subseteq f^q_i$ for $i \in w^p$,

(δ) $g^p_i \subseteq g^q_i$ for $i \in w^p$,

(ε) $v^p \subseteq v^q$,

(ζ) $\rho^p_{\alpha,n} \trianglelefteq \rho^q_{\alpha,n}$, when $(\alpha, n) \in v^p$,

(η) if $i \neq j \in w^p$ then for every $\alpha \in A_i \cap A_j \setminus (\mathrm{Dom}(g^p_i) \cap \mathrm{Dom}(g^p_j))$ we have $g^q_i(\alpha) \neq g^q_j(\alpha)$ (possibly $\alpha \notin \mathrm{Dom}(g^q_i)$ and/or $\alpha \notin \mathrm{Dom}(g^q_j)$).

A Fact $(P, \leq)$ is a partial order.

Proof of the Fact: Trivial.

B Fact For $i < 2^\lambda$ the set $\{p : i \in w^p\}$ is dense in P.

Proof of the Fact: If $p \in P$, $i \in 2^\lambda \setminus w^p$, define q like p except $w^q = w^p \cup \{i\}$, $\mathrm{Dom}(g^q_i) = \emptyset$.

C Fact If $p \in P$, $m_1 \in (m^p, \omega)$, $\eta^* \in u^p$, $m^* < \omega$, $i \in w^p$, $\alpha \in \lambda \setminus \mathrm{Dom}(g^p_i)$ *then* we can find q such that $p \leq q \in P$, $m^q > m_1$, $\eta^{*\frown}\langle m^* \rangle \in u^q$, $i \in w^q$, $\alpha \in \mathrm{Dom}(g^q_i)$ and $\langle \eta^j_\beta \restriction m^q : j \in w^q \text{ and } \beta \in \mathrm{Dom}(g^q_j) \rangle$ is with no repetition, more exactly $\eta^{j(1)}_{\beta_1} \restriction m^q = \eta^{j(2)}_{\beta_2} \restriction m^q \Rightarrow \eta^{j(1)}_{\beta_1} = \eta^{j(2)}_{\beta_2}$.

Proof of the Fact: Let $n_0 \leq m^p$ be maximal such that $\eta^i_\alpha \restriction n_0 \in u^p$. Let $n_1 < \omega$ be minimal such that $\eta^i_\alpha \restriction n_1 \notin \{\eta^i_\beta \restriction n_1 : \beta \in \mathrm{Dom}(g^p_i)\}$ and moreover the sequence

$$\langle \eta^j_\beta \restriction n_1 : j \in w^p \ \&\ \beta \in \mathrm{Dom}(g^p_j) \quad \text{or} \quad j = i \ \&\ \beta = \alpha \rangle$$

is with no repetition. Choose a natural number $m^q > m^p+1, n_0+1, n_1+2, m$ and let $k^* =: 3 + \sum_{i \in w^p} |\mathrm{Dom}(g^p_i)|$. Choose $u^q \subseteq {}^{m^q \geq}\omega$ such that:

(i) $u^p \subseteq u^q \subseteq {}^{m^q \geq}\omega$, u^q is downward closed,

(ii) for every $\eta \in u^q$ such that $\ell g(\eta) < m^q$, for exactly k^* numbers k, $\eta\hat{\ }\langle k \rangle \in u^q \setminus u^p$,

(iii) $\eta^j_\beta \restriction \ell \in u^q$ when $\ell \leq m^q$ and $j \in w^p$, $\beta \in \mathrm{Dom}(g^p_j)$,

(iv) $\eta^i_\alpha \restriction \ell \in u^q$ for $\ell \leq m^q$,

(v) $\eta^{*}\hat{\ }\langle m^* \rangle \in u^q$.

Next choose $\rho^q_{\beta,n}$ (for pairs $(\beta, n) \in v^p$) such that:

$$\rho^p_{\beta,n} \trianglelefteq \rho^q_{\beta,n} \in u^q \cap {}^{m^q}\omega.$$

For each $j \in w^p$ separately extend f^p_j to an automorphism f^q_j of $(u^q, \triangleleft)$ such that for each $\beta \in \mathrm{Dom}(g^p_j)$ we have:

$$f^q_j(\eta^j_\beta \restriction m^q) = \rho^q_{\beta, g_j(\beta)}.$$

This is possible, as for each $\nu \in u^p$, and $j \in w^p$, we can separately define

$$f^q_j \restriction \{\nu' : \nu \triangleleft \nu' \in u^q \ \text{ and } \ \nu' \restriction (\ell g(\nu) + 1) \notin u^p\}$$

–its range is

$$\{\nu' : f^p_j(\nu) \triangleleft \nu' \in u^q \ \text{ and } \ \nu' \restriction (\ell g(\nu) + 1) \notin u^p\}.$$

The point is: by Clause (ii) above those two sets are isomorphic and for each ν at most one $\rho^p_{\beta,n}$ is involved (see Clause (h) in the definition of $p \in P$) and we can take care of clause (h). Next let $w^q = w^p$, $g^q_j = g^p_j$ for $j \in w \setminus \{i\}$, $g^q_i \restriction \mathrm{Dom}(g^p_i) = g^p_i$, $g^q_i(\alpha) = \min(\{n : (\alpha, n) \notin v^p\})$, $\mathrm{Dom}(g^q_i) = \mathrm{Dom}(g^p_i) \cup \{\alpha\}$, and $\rho^q_{\alpha, g^q_i(\alpha)} = f^q_i(\eta^i_\alpha \restriction m^q)$ and $v^q = v^p \cup \{(\alpha, g^q_i(\alpha))\}$.

D Fact P satisfies the c.c.c.

Proof of the Fact: Assume $p_\varepsilon \in P$ for $\varepsilon < \omega_1$. By Fact C, without loss of generality each

$$\langle \eta^j_\beta \restriction m^{p_\varepsilon} : j \in w^{p_\varepsilon} \text{ and } \beta \in \mathrm{Dom}(g^{p_\varepsilon}_j)\rangle$$

is with no repetition. Without loss of generality, for all $\varepsilon < \omega_1$

$$U_\varepsilon =: \{\alpha < 2^\lambda : \alpha \in w^{p_\varepsilon} \text{ or } \bigvee_{i \in w^p} [\alpha \in \mathrm{Dom}(g_i)] \text{ or } \bigvee_k (k, \alpha) \in v^{p_\varepsilon}\}$$

has the same number of elements and for $\varepsilon \neq \zeta < \omega_1$, there is a unique one-to-one order preserving function from U_ε onto U_ζ which we call $\mathrm{OP}_{\zeta,\varepsilon}$, which also maps p_ε to p_ζ (so $m^{p_\zeta} = m^{p_\varepsilon}$; $u^{p_\zeta} = u^{p_\varepsilon}$; $\mathrm{OP}_{\zeta,\varepsilon}(w^{p_\varepsilon}) = w^{p_\zeta}$; if $i \in w^{p_\varepsilon}$, $j = \mathrm{OP}_{\zeta,\varepsilon}(i)$, then $f_i \circ \mathrm{OP}_{\varepsilon,\zeta} \equiv f_j$; and *if* $\beta = \mathrm{OP}_{\zeta,\varepsilon}(\alpha)$ and $\ell < \omega$ *then*

$$(\alpha, \ell) \in v^{p_\varepsilon} \quad \Leftrightarrow \quad (\beta, \ell) \in v^{p_\zeta} \quad \Rightarrow \quad \rho^{p_\varepsilon}_{\alpha,\ell} = \rho^{p_\zeta}_{\beta,\ell}.$$

Also this mapping is the identity on $U_\zeta \cap U_\varepsilon$ and $\langle U_\zeta : \zeta < \omega_1\rangle$ is a Δ-system.

Let $w =: w^{p_0} \cap w^{p_1}$. As $i \neq j \;\Rightarrow\; |A_i \cap A_j| \leq \aleph_0$, without loss of generality

$(*)$ if $i \neq j \in w$ then

$$U_\varepsilon \cap (A_i \cap A_j) \subseteq w.$$

We now start to define $q \geq p_0, p_1$. Choose m^q such that $m^q \in (m^{p_\varepsilon}, \omega)$ and

$$m^q > \max\{\ell g(\eta^{i_0}_{\alpha_0} \cap \eta^{i_1}_{\alpha_1}) + 1 : \; i_0 \in w^{p_0},\ i_1 \in w^{p_1},\ \mathrm{OP}_{1,0}(i_0) = i_1,\ \alpha_0 \in \mathrm{Dom}(g^{p_0}_{i_0}),\ \alpha_1 \in \mathrm{Dom}(g^{p_1}_{i_1}),\ \mathrm{OP}_{1,0}(\alpha_0) = \alpha_1\}.$$

Let $u^q \subseteq {}^{m^q \geq}\omega$ be such that:

(A) $u^q \cap ({}^{m^{p_0} \geq}\omega) = u^q \cap ({}^{m^{p_1} \geq}\omega) = u^{p_0} = u^{p_1}$,

(B) for each $\nu \in u^q$, $m^{p_0} \leq \ell g(\nu) < m^q$, for exactly two numbers $k < \omega$, $\nu\hat{\ }\langle k\rangle \in u^q$,

(C) $\eta^i_\alpha \restriction \ell \in u^q$ for $\ell \leq m^q$ <u>when</u>: $i \in w^{p_0}$, $\alpha \in \mathrm{Dom}(g^{p_0}_i)$ <u>or</u> $i \in w^{p_1}$, $\alpha \in \mathrm{Dom}(g^{p_1}_i)$.

[Possible as $\{\eta^i_\alpha \restriction m^{p_\varepsilon} : i \in w^{p_\varepsilon}, \alpha \in \mathrm{Dom}(g^{p_\varepsilon}_i)\}$ is with no repetitions (the first line of the proof).]

Let $w^q =: w^{p_0} \cup w^{p_1}$ and $v^q =: v^{p_0} \cup v^{p_1}$ and for $i \in w^q$

$$g^q_i = \begin{cases} g^{p_0}_i & \underline{\text{if}} \quad i \in w^{p_0} \setminus w^{p_1}, \\ g^{p_1}_i & \underline{\text{if}} \quad i \in w^{p_1} \setminus w^{p_0}, \\ g^{p_0}_i \cup g^{p_1}_i & \underline{\text{if}} \quad i \in w^{p_0} \cap w^{p_1}. \end{cases}$$

Next choose $\rho^q_{\alpha,\ell}$ for $(\alpha, \ell) \in v^q$ as follows. Let $\nu_{\alpha,\ell}$ be $\rho^{p_0}_{\alpha,\ell}$ if defined, $\rho^{p_1}_{\alpha,\ell}$ if defined (no contradiction). If $(\alpha, \ell) \in v^q$ choose $\rho^q_{\alpha,\ell}$ as any ρ such that:

$\otimes_0$ $\nu_{\alpha,\ell} \triangleleft \rho \in u^q \cap {}^{(m^q)}\omega$.

But not all choices are O.K., as we need to be able to define f_i^q for $i \in w^q$. A possible problem will arise only when $i \in w^{p_0} \cap w^{p_1}$. Specifically we need just (remember that $\langle \rho^{p_e}_{\alpha,\ell} : (\alpha,\ell) \in v^{p_e} \rangle$ are pairwise distinct by clause (b) of the Definition of $p \in P$):

$\otimes_1$ if $i_0 \in w^{p_0}, (\alpha_0, \ell) = (\alpha_0, g_{i_0}(\alpha_0)), \alpha_0 \in \mathrm{Dom}(g^{p_0}_{i_0})$, $i_1 = \mathrm{OP}_{1,0}(i_0)$ and $\alpha_1 = \mathrm{OP}_{1,0}(\alpha_0)$ and $i_0 = i_1$

then $\ell g(\eta^{i_0}_{\alpha_0} \cap \eta^{i_1}_{\alpha_1}) = \ell g(\rho^q_{\alpha_0,\ell} \cap \rho^q_{\alpha_1,\ell})$.

We can, of course, demand $\alpha_0 \neq \alpha_1$ (otherwise the conclusion of $\otimes_1$ is trivial). Our problem is expressible for each pair $(\alpha_0,\ell),(\alpha_1,\ell)$ separately as: first the problem is in defining the $\rho^q_{(\alpha,\ell)}$'s and second, if $(\alpha'_1,\ell'),(\alpha'_2,\ell)$ is another such pair then $\{(\alpha_1,\ell),(\alpha_2,\ell)\}$, $\{(\alpha'_1,\ell'),(\alpha'_2,\ell')\}$ are either disjoint or equal. Now for a given pair $(\alpha_0,\ell),(\alpha_1,\ell)$ how many $i_0 = i_1$ do we have? Necessarily $i_0 \in w^{p_0} \cap w^{p_1} = w$. But if $i'_0 \neq i''_0$ are like that then $\alpha_0 \in A_{i'_0} \cap A_{i''_0}$, contradicting $(*)$ above because $\alpha_0 \neq \alpha_1 = \mathrm{OP}_{1,0}(\alpha_0)$. So there is at most one candidate $i_0 = i_1$, so there is no problem to satisfy $\otimes_1$. Now we can define f_i^q (i$\in w^q$) as in the proof of Fact C.

The rest should be clear. $\square_{4.4}$

Conclusion 4.7 *Suppose $V \models GCH$, $\aleph_0 < \lambda < \chi$ and $\chi^\lambda = \chi$. Then for some c.c.c. forcing notion P of cardinality χ, not collapsing cardinals nor changing cofinalities, in V^P:*

(i) $2^{\aleph_0} = 2^\lambda = \chi$,

(ii) *$\mathfrak{K}^{tr}_\lambda$ has a universal family of cardinality λ^+,*

(iii) *$\mathfrak{K}^{rs(p)}_\lambda$ has a universal family of cardinality λ^+.*

Proof First use a preliminary forcing Q^0 of Baumgartner [B], adding $\langle A_\alpha : \alpha < \chi \rangle$, $A_\alpha \in [\lambda]^\lambda$, $\alpha \neq \beta \Rightarrow |A_\alpha \cap A_\beta| \leq \aleph_0$ (we can have $2^{\aleph_0} = \aleph_1$ here, or $[\alpha \neq \beta \Rightarrow A_\alpha \cap A_\beta$ finite], but not both). Next use an FS iteration $\langle P_i, \dot{Q}_i : i < \chi \times \lambda^+ \rangle$ such that each forcing from 4.4 appears and each forcing as in 4.6 appears. $\square_{4.7}$

Remark 4.8 We would like to have that there is a universal member in $\mathfrak{K}^{rs(p)}_\lambda$; this sounds very reasonable but we did not try.

In our framework, the present result shows limitations to ZFC results which the methods applied in the previous sections can give.

5 Back to $\mathfrak{K}^{rs(p)}$, real non-existence results

By §1 we know that if G is an Abelian group with set of elements λ, $C \subseteq \lambda$, then for an element $x \in G$ the distance from $\{y : y < \alpha\}$ for $\alpha \in C$ does not code an appropriate invariant. If we have infinitely many such distance functions, e.g. have infinitely many primes, we can use more complicated invariants related to x as in §3. But if we have one prime, this approach does not help.

If one element fails, can we use infinitely many? A countable subset X of G can code a countable subset of C:

$$\{\alpha \in C : \text{ closure}(\langle X \rangle_G) \cap \alpha \not\subseteq \sup(C \cap \alpha)\},$$

but this seems silly - we use heavily the fact that C has many countable subsets (in particular $> \lambda$) and λ has at least as many. However, what if C has a small family (say of cardinality $\leq \lambda$ or $< \mu^{\aleph_0}$) of countable subsets such that every subset of cardinality, say continuum, contains one? Well, we need more: we catch a countable subset for which the invariant defined above is infinite (necessarily it is at most of cardinality $2^{\aleph_0}$, and because of §4 we are not trying any more to deal with $\lambda \leq 2^{\aleph_0}$). The set theory needed is expressed by $\mathbf{U}_J$ below, and various ideals also defined below, and the result itself is 5.9.

Of course, we can deal with other classes like torsion free reduced groups, as they have the characteristic non-structure property of unsuperstable first order theories; but the relevant ideals will vary: the parallel to $I^0_{\bar\mu}$ for $\bigwedge_n \mu_n = \mu$, $J^2_{\bar\mu}$ seems to be always O.K.

Definition 5.1 1. For $\bar\mu = \langle \mu_n : n < \omega \rangle$ let $B_{\bar\mu}$ be

$$\bigoplus\{K^n_\alpha : n < \omega, \alpha < \mu_n\}, \qquad K^n_\alpha = \langle {}^* t^n_\alpha \rangle_{K^n_\alpha} \cong \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

Let $B_{\bar\mu \restriction n} = \bigoplus\{K^m_\alpha : \alpha < \mu_m, m < n\} \subseteq B_{\bar\mu}$ (they are in $\mathfrak{K}^{rs(p)}_{\leq \sum_n \mu_n}$).

Let $\hat{B}$ be the p-torsion completion of B (i.e. completion under the norm $\|x\| = \min\{2^{-n} : p^n \text{ divides } x\}$ restrited to the set of all x such that $p^n x = 0$ for some n).

2. Let $I^1_{\bar\mu}$ be the ideal on $\hat{B}_{\bar\mu}$ generated by $I^0_{\bar\mu}$, where

$$I^0_{\bar\mu} = \Big\{A \subseteq \hat{B}_{\bar\mu} : \begin{array}{l} \text{for every large enough } n, \\ \text{for no } y \in \bigoplus\{K^m_\alpha : m \leq n \text{ and } \alpha < \mu_m\} \\ \text{but } y \notin \bigoplus\{K^m_\alpha : m < n \text{ and } \alpha < \mu_m\} \text{ we have :} \\ \text{for every } m \text{ for some } z \in \langle A \rangle \text{ we have:} \\ p^m \text{ divides } z - y\Big\}. \end{array}$$

(We may write $I^0_{\hat{B}_{\bar{\mu}}}$, but the ideal depends also on $\langle \bigoplus_{\alpha<\mu_n} K^n_\alpha : n < \omega\rangle$ not just on $\hat{B}_{\bar{\mu}}$ itself).

3. For $X, A \subseteq \hat{B}_{\bar{\mu}}$,

$$\text{recall} \quad \langle A\rangle_{\hat{B}_{\bar{\mu}}} = \Big\{ \sum_{n<n^*} a_n y_n : y_n \in A,\ a_n \in \mathbb{Z} \text{ and } n^* \in \mathbb{N}\Big\},$$

$$\text{and let} \quad c\ell_{\hat{B}_{\bar{\mu}}}(X) = \{x : (\forall n)(\exists y \in X)(x - y \in p^n \hat{B}_{\bar{\mu}})\}.$$

4. Let $J^1_{\bar{\mu}}$ be the ideal which $J^{0.5}_{\bar{\mu}}$ generates, where

$$J^{0.5}_{\bar{\mu}} = \Big\{A \subseteq \prod_{n<\omega} \mu_n : \begin{array}{l}\text{for some } n < \omega \text{ for no } m \in [n,\omega)\\ \text{and } \beta < \gamma < \mu_m \text{ do we have :}\\ \text{for every } k \in [m,\omega) \text{ there are } \eta, \nu \in A \text{ such}\\ \text{that: } \eta(m) = \beta,\ \nu(m) = \gamma,\ \eta \restriction m = \nu \restriction m\\ \text{and } \eta \restriction (m,k) = \nu \restriction (m,k)\Big\}.\end{array}$$

5.

$$J^0_{\bar{\mu}} = \{A \subseteq \prod_{n<\omega} \mu_n : \text{ for some } n < \omega \text{ and } k, \text{ the mapping } \eta \mapsto \eta \restriction n \text{ is } (\leq k)\text{-to-one }\}.$$

6. $J^2_{\bar{\mu}}$ is the ideal of nowhere dense subsets of $\prod_n \mu_n$ (under the following natural topology: a neighbourhood of η is $U_{\eta,n} = \{\nu : \nu \restriction n = \eta \restriction n\}$ for some n).

7. $J^3_{\bar{\mu}}$ is the ideal of meagre subsets of $\prod_n \mu_n$, i.e. subsets which are included in countable union of members of $J^2_{\bar{\mu}}$.

Observation 5.2 *1. $I^0_{\bar{\mu}}$, $J^0_{\bar{\mu}}$, $J^{0.5}_{\bar{\mu}}$ are $(< \aleph_1)$-based, i.e. for $I^0_{\bar{\mu}}$: if $A \subseteq \hat{B}_{\bar{\mu}}$, $A \notin I^0_{\bar{\mu}}$ then there is a countable $A_0 \subseteq A$ such that $A_0 \notin I^0_{\bar{\mu}}$.*

2. $I^1_{\bar{\mu}}$, $J^0_{\bar{\mu}}$, $J^1_{\bar{\mu}}$, $J^2_{\bar{\mu}}$, $J^3_{\bar{\mu}}$ are ideals, $J^3_{\bar{\mu}}$ is $\aleph_1$-complete.

3. $J^0_{\bar{\mu}} \subseteq J^1_{\bar{\mu}} \subseteq J^2_{\bar{\mu}} \subseteq J^3_{\bar{\mu}}$.

4. There is a function g from $\prod_{n<\omega} \mu_n$ into $\hat{B}_{\bar{\mu}}$ such that for every $X \subseteq \prod_{n<\omega} \mu_n$:

$$X \notin J^1_{\bar{\mu}} \quad \Rightarrow \quad g''(X) \notin I^1_{\bar{\mu}}.$$

Proof E.g. 4) Let $g(\eta) = \sum\limits_{n<\omega} p^n({}^*t^n_{\eta(n)})$.

Let $X \subseteq \prod\limits_{n<\omega} \mu_n$, $X \notin J^1_{\bar\mu}$. Assume $g''(X) \in \bar{I}^1_{\bar\mu}$, so for some ℓ^* and $A_\ell \subseteq \hat{B}_{\bar\mu}$, $(\ell < \ell^*)$ we have $A_\ell \in I^0_{\bar\mu}$, and $g''(X) \subseteq \bigcup\limits_{\ell<\ell^*} A_\ell$, so $X = \bigcup\limits_{\ell<\ell^*} X_\ell$, where

$$X_\ell =: \{\eta \in X : g(\eta) \in A_\ell\}.$$

As $J^1_{\bar\mu}$ is an ideal, for some $\ell < \ell^*$, $X_\ell \notin J^1_{\bar\mu}$. So by the definition of $J^1_{\bar\mu}$, for some infinite $\Gamma \subseteq \omega$ for each $m \in \Gamma$ we have $\beta_m < \gamma_m < \mu_m$ and for every $k \in [m,\omega)$ we have $\eta_{m,k}, \nu_{m,k}$, as required in the definition of $J^1_{\bar\mu}$. So $g(\eta_{m,k}), g(\nu_{m,k}) \in A_\ell$ (for $m \in \Gamma$, $k \in (m,\omega)$). Now

$${}^*t^m_{\gamma_m} - {}^*t^m_{\beta_m} = g(\eta_{m,k}) - g(\nu_{m,k}) \mod p^k \hat{B}_{\bar\mu},$$

but $g(\eta_{m,k}) - g(\nu_{m,k}) \in \langle A_\ell \rangle_{\hat{B}_{\bar\mu}}$. Hence

$$(\exists z \in \langle A_\ell\rangle_{\hat{B}_{\bar\mu}})[{}^*t^m_{\gamma_m} - {}^*t^m_{\beta_m} = z \mod p^k \hat{B}_{\bar\mu}],$$

as this holds for each k, ${}^*t^m_{\gamma_m} - {}^*t^m_{\beta_m} \in c\ell(\langle A_\ell\rangle_{\hat{B}_{\bar\mu}})$.

This contradicts $A_\ell \in I^0_{\bar\mu}$. $\square_{5.2}$

Definition 5.3 Let $I \subseteq \mathcal{P}(X)$ be downward closed (and for simplicity $\{\{x\} : x \in X\} \subseteq I$). Let $I^+ = \mathcal{P}(X) \setminus I$. Let

$$\mathbf{U}^{<\kappa}_I(\mu) =: \min\big\{|\mathcal{P}| : \begin{array}{l}\mathcal{P} \subseteq [\mu]^{<\kappa}, \text{ and for every } f : X \longrightarrow \mu \text{ for some} \\ Y \in \mathcal{P}, \text{ we have } \{x \in X : f(x) \in Y\} \in I^+\big\}.\end{array}$$

Instead of $< \kappa^+$ in the superscript of $\mathbf{U}$ we write κ. If $\kappa > |\mathrm{Dom}(I)|^+$, we may omit it (since then its value does not matter).

Remark 5.4 1. If $2^{<\kappa} + |\mathrm{Dom}(I)|^{<\kappa} \leq \mu$ we can find $F \subseteq$ partial functions from $\mathrm{Dom}(I)$ to μ such that:

(a) $|F| = \mathbf{U}^{<\kappa}_I(\mu)$,

(b) $(\forall f : X \longrightarrow \mu)(\exists Y \in I^+)[f \restriction Y \in F]$.

2. Such functions (as $\mathbf{U}^{<\kappa}_I(\mu)$) are investigated in **pcf** theory ([Sh:g], [Sh 410, §6], [Sh 430, §2], [Sh 513]).

3. If $I \subseteq J \subseteq \mathcal{P}(X)$, then $\mathbf{U}^{<\kappa}_I(\mu) \leq \mathbf{U}^{<\kappa}_J(\mu)$, hence by 5.2(3), and the above

$$\mathbf{U}^{<\kappa}_{J^0_{\bar\mu}}(\mu) \leq \mathbf{U}^{<\kappa}_{J^1_{\bar\mu}}(\mu) \leq \mathbf{U}^{<\kappa}_{J^2_{\bar\mu}}(\mu) \leq \mathbf{U}^{<\kappa}_{J^3_{\bar\mu}}(\mu)$$

and by 5.2(4) we have $\mathbf{U}^{<\kappa}_{I^1_{\bar\mu}} \leq \mathbf{U}^{<\kappa}_{J^1_{\bar\mu}}(\mu)$.

4. On $\mathrm{IND}_\theta(\bar\kappa)$ (see 5.5 below) see [Sh 513].

Definition 5.5 $\mathrm{IND}'_\theta(\langle\kappa_n : n<\omega\rangle)$ means that for every model M with universe $\bigcup_{n<\omega}\kappa_n$ and $\leq\theta$ functions, for some $\Gamma\in[\omega]^{\aleph_0}$ and $\eta\in\prod_{n<\omega}\kappa_n$ we have:

$$n\in\Gamma \quad\Rightarrow\quad \eta(n)\notin c\ell_M\{\eta(\ell):\ell\neq n\}.$$

Remark 5.6 Actually if $\theta\geq 2^{\aleph}_0$, this implies that we can fix Γ, hence replacing $\langle\kappa_n:n<\omega\rangle$ by an infinite subsequence we can have $\Gamma=\omega$.

Theorem 5.7 *1. If $\mu_n\to(\kappa_n)^2_{2^\theta}$ and* $\mathrm{IND}'_\theta(\langle\kappa_n:n<\omega\rangle)$ then $\prod_{n<\omega}\mu_n$ *is not the union of $\leq\theta$ sets from $J^1_{\bar\mu}$.*

2. If $\theta=\theta^{\aleph_0}$ and $\neg\mathrm{IND}'_\theta(\langle\mu_n:n<\omega\rangle)$ then $\prod_{n<\omega}\mu_n$ is the union of $\leq\theta$ members of $J^1_{\bar\mu}$.

3. If $\limsup_n \mu_n$ is ≥ 2, then $\prod_{n<\omega}\mu_n\notin J^3_{\bar\mu}$ (so also the other ideals defined above are not trivial by 5.2(3), (4)).

Proof 1) Suppose $\prod_{n<\omega}\mu_n$ is $\bigcup_{i<\theta}X_i$, and each $X_i\in J^1_{\bar\mu}$, as $\theta\geq\aleph_o$ wlog $x_i\in J^{0.5}_{\bar\mu}$. We define for each $i<\theta$ and $n<k<\omega$ a two-place relation $R^{n,k}_i$ on μ_n:

$\beta R^{n,k}_i\gamma$ if and only if
there are $\eta,\nu\in X_i\subseteq\prod_{\ell<k}\mu_\ell$ such that

$$\eta\restriction[0,n)=\nu\restriction[0,n)\quad\text{and}\quad \eta\restriction(n,k)=\nu\restriction(n,k)\quad\text{and}\quad \eta(n)=\beta,\ \nu(n)=\gamma.$$

Note that $R^{n,k}_i$ is symmetric and

$$n<k_1<k_2\ \&\ \beta R^{n,k_2}_i\gamma\quad\Rightarrow\quad\beta R^{n,k_1}_i\gamma.$$

As $\mu_n\to(\kappa_n)^2_{2^\theta}$, we can find $A_n\in[\mu_n]^{\kappa_n}$ and a truth value $\mathbf{t}^{n,k}_i$ such that for all $\beta<\gamma$ from A_n, the truth value of $\beta R^{n,k}_i\gamma$ is $\mathbf{t}^{n,k}_i$. If for some i the set

$$\Gamma_i =: \{n<\omega:\ \text{for every } k\in(n,\omega) \text{ we have } \mathbf{t}^{n,k}_i=\ \text{true}\}$$

is infinite, we get a contradiction to "$X_i\in J^1_{\bar\mu}$", so for some $n(i)<\omega$ we have $n(i)=\sup(\Gamma_i)$.

For each $n<k<\omega$ and $i<\theta$ we define a partial function $F^{n,k}_i$ from $\prod_{\ell<k,\,\ell\neq n}A_\ell$ into A_n:

$F(\alpha_0 \dots \alpha_{n-1}, \alpha_{n+1}, \dots, \alpha_k)$ is the first $\beta \in A_n$ such that for some $\eta \in X_i$ we have

$$\eta \restriction [0,n) = \langle \alpha_0, \dots, \alpha_{n-1} \rangle, \quad \eta(n) = \beta,$$
$$\eta \restriction (n,k) = \langle \alpha_{n+1}, \dots, \alpha_{k-1} \rangle.$$

So as $\mathrm{IND}'_\theta(\langle \kappa_n : n < \omega \rangle)$ there is $\eta = \langle \beta_n : n < \omega \rangle \in \prod_{n<\omega} A_n$ such that for infinitely many n, β_n is not in the closure of $\{\beta_\ell : \ell < \omega, \ell \neq n\}$ by the $F_i^{n,k}$'s. As $\eta \in \prod_{n<\omega} A_n \subseteq \prod_{n<\omega} \mu_n = \bigcup_{i<\theta} X_i$, necessarily for some $i < \theta$, $\eta \in X_i$. Let $n \in (n(i), \omega)$ be such that β_n is not in the closure of $\{\beta_\ell : \ell < \omega$ and $\ell \neq n\}$ and let $k > n$ be such that $\mathbf{t}_i^{n,k} =$ false. Now $\gamma =: F_i^{n,k}(\beta_0, \dots, \beta_{n-1}, \beta_{n+1}, \dots, \beta_{k-1})$ is well defined $\leq \beta_n$ (as β_n exemplifies that there is such β) and is $\neq \beta_n$ (by the choice of $\langle \beta_\ell : \ell < \omega \rangle$), so by the choice of $n(i)$ (so of n, k and earlier of $\mathbf{t}_i^{n,k}$ and of A_n) we get contradiction to "$\gamma < \beta_n$ are from A_n".

2) Let M be an algebra with universe $\sum_{n<\omega} \mu_n$ and $\leq \theta$ functions (say F_i^n for $i < \theta$, $n < \omega$, F_i^n is n-place) exemplifying $\neg \mathrm{IND}'_\theta(\langle \mu_n : n < \omega \rangle)$. Let

$$\Gamma =: \{ \langle (k_n, i_n) : n^* \leq n < \omega \rangle : n^* < \omega \text{ and } \bigwedge_n n < k_n < \omega \text{ and } i_n < \theta \}.$$

For $\rho = \langle (k_n, i_n) : n^* \leq n < \omega \rangle \in \Gamma$ let

$$A_\rho =: \Big\{ \eta \in \prod_{n<\omega} \mu_n : \text{for every } n \in [n^*, \omega) \text{ we have}$$
$$\eta(n) = F_{i_n}^{k_n - 1}(\eta(0), \dots, \eta(n-1), \eta(n+1), \dots, \eta(k_n)) \Big\}.$$

So, by the choice of M, $\prod_{n<\omega} \mu_n = \bigcup_{\rho \in \Gamma} A_\rho$. On the other hand, it is easy to check that $A_\rho \in J^1_{\bar\mu}$.

3) left for the reader. $\square_{5.7}$

Theorem 5.8 *If $\mu = \sum_{n<\omega} \lambda_n$, $\lambda_n^{\aleph_0} < \lambda_{n+1}$ and $\mu < \lambda = \mathrm{cf}(\lambda) < \mu^{+\omega}$ then $\mathbf{U}^{\aleph_0}_{I^0_{\langle \lambda_n : n<\omega \rangle}}(\lambda) = \lambda$ and even $\mathbf{U}^{\aleph_0}_{J^3_{\langle \lambda_n : n<\omega \rangle}}(\lambda) = \lambda$.*

Proof See [Sh 410, §6], [Sh 430, §2], and [Sh 513] for considerably more.

Lemma 5.9 *Assume $\lambda > 2^{\aleph_0}$ and*

(*)(a) $\prod_{n<\omega} \mu_n < \mu$ *and* $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$,

(b) $\hat{B}_{\bar\mu} \notin I^0_{\bar\mu}$ *and* $\lim_n \sup \mu_n$ *is infinite,*

(c) $\mathbf{U}^{\aleph_0}_{I^0_{\bar\mu}}(\lambda) = \lambda$ *(note $I^0_{\bar\mu}$ is not required to be an ideal).*

Then *there is no universal member in $\mathfrak{K}^{rs(p)}_\lambda$.*

Proof Let $S \subseteq \lambda$, $\bar C = \langle C_\delta : \delta \in S\rangle$ guesses clubs of λ, chosen as in the proof of 3.3 (so $\alpha \in \mathrm{nacc}(C_\delta) \Rightarrow \mathrm{cf}(\alpha) > 2^{\aleph_0}$). Instead of defining the relevant invariant we prove the theorem directly, but we could define them, somewhat cumbersomely (like [Sh:e, III,§3]).

Assume $H \in \mathfrak{K}^{rs(p)}_\lambda$ is a pretender to universality; without loss of generality with the set of elements of H equal to λ.

Let $\chi = \beth_7(\lambda)^+$, $\bar{\mathfrak{A}} = \langle \mathfrak{A}_\alpha : \alpha < \lambda\rangle$ be an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$, $\bar{\mathfrak{A}} \restriction (\alpha+1) \in \mathfrak{A}_{\alpha+1}$, $\|\mathfrak{A}_\alpha\| < \lambda$, $\mathfrak{A}_\alpha \cap \lambda$ an ordinal, $\mathfrak{A} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$ and $\{H, \langle \mu_n : n < \omega\rangle, \mu, \lambda\} \in \mathfrak{A}_0$, so $B_{\bar\mu}, \hat B_{\bar\mu} \in \mathfrak{A}_0$ (where $\bar\mu = \langle \mu_n : n < \omega\rangle$, of course).

For each $\delta \in S$, let $\mathcal{P}_\delta =: [C_\delta]^{\aleph_0} \cap \mathfrak{A}$. Choose $A_\delta \subseteq C_\delta$ of order type ω almost disjoint from each $a \in \mathcal{P}_\delta$, and from A_{δ_1} for $\delta_1 \in \delta \cap S$; its existence should be clear as $\lambda < \mu^{\aleph_0}$. So

$(*)_0$ every countable $A \in \mathfrak{A}$ is almost disjoint to A_δ.

By 5.2(2), $I^0_{\bar\mu}$ is $(<\aleph_1)$-based so by 5.4(1) and the assumption (c) we have

$(*)_1$ for every $f : \hat B_{\bar\mu} \longrightarrow \lambda$ for some countable $Y \subseteq \hat B_{\bar\mu}$, $Y \notin I^0_{\bar\mu}$, we have $f \restriction Y \in \mathfrak{A}$

(remember $(\prod_{n<\omega} \mu_n)^{\aleph_0} = \prod_{n<\omega} \mu_n$).

Let B be $\bigoplus\{G^n_{\alpha,i} : n < \omega, \alpha < \lambda,\ i < \sum_{k<\omega} \mu_k\}$, where

$$G^n_{\alpha,i} = \langle x^n_{\alpha,i}\rangle_{G^n_{\alpha,i}} \cong \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

So B, $\hat B$, $\langle (n, \alpha, i, x^n_{\alpha,i}) : n < \omega, \alpha < \lambda, i < \sum_{k<\omega} \mu_k\rangle$ are well defined. Let G be the subgroup of $\hat B$ generated by:

$$B \cup \Big\{x \in \hat B : \begin{array}{l}\text{for some } \delta \in S,\ x \text{ is in the closure of}\\ \bigoplus\{G^n_{\alpha,i} : n < \omega, i < \mu_n, \alpha \text{ is the } n\text{th element of } A_\delta\}\end{array}\Big\}.$$

As $\prod_{n<\omega} \mu_n < \mu < \lambda$, clearly $G \in \mathfrak{K}^{rs(p)}_\lambda$, without loss of generality the set of elements of G is λ and let $h : G \longrightarrow H$ be an embedding. Let

$$E_0 =: \{\delta < \lambda : (\mathfrak{A}_\delta, h \restriction \delta,\ G \restriction \delta) \prec (\mathfrak{A}, h, G)\},$$

$$E =: \{\delta < \lambda : \mathrm{otp}(E_0 \cap \delta) = \delta\}.$$

They are clubs of λ, so for some $\delta \in S$, $C_\delta \subseteq E$ (and $\delta \in E$ for simplicity). Let η_δ enumerate A_δ increasingly.

There is a natural embedding $g = g_\delta$ of $B_{\bar\mu}$ into G:

$$g({}^*t_i^n) = x^n_{\eta_\delta(n),i}.$$

Let $\hat g_\delta$ be the unique extension of g_δ to an embedding of $\hat B_{\bar\mu}$ into G; those embeddings are pure, (in fact $g_\delta''(\hat B_{\bar\mu}) \setminus g_\delta''(B_\mu) \subseteq G \setminus G \cap \mathfrak{A}_\delta$). So $h \circ \hat g_\delta$ is an embedding of $\hat B_{\bar\mu}$ into H, not necessarily pure but still an embedding, so the distance function can become smaller but not zero and

$$h \circ \hat g_\delta(\hat B_{\bar\mu}) \setminus h \circ g_\delta(B_\mu) \subseteq H \setminus \mathfrak{A}_\delta.$$

Remember $\hat B_{\bar\mu} \subseteq \mathfrak{A}_0$ (as it belongs to $\mathfrak{A}_0$ and has cardinality $\prod_{n<\omega} \mu_n < \lambda$ and $\lambda \cap \mathfrak{A}_0$ is an ordinal). By $(*)_1$ applied to $f = h \circ \hat g$ there is a countable $Y \subseteq \hat B_{\bar\mu}$ such that $Y \notin I^0_{\bar\mu}$ and $f \restriction Y \in \mathfrak{A}$. But, from $f \restriction Y$ we shall below reconstruct some countable set not almost disjoint to A_δ, reconstruct meaning in $\mathfrak{A}$, in contradiction to $(*)_0$ above.

As $Y \notin I^0_{\bar\mu}$ we can find an infinite $S^* \subseteq \omega \setminus m^*$ and for $n \in S^*$, $z_n \in \bigoplus_{\alpha<\mu_n} K^n_\alpha \setminus \{0\}$ and $y^\ell_n \in \hat B_{\bar\mu}$ (for $\ell < \omega$) such that:

$(*)_2$ $z_n + y_{n,\ell} \in \langle Y \rangle_{\hat B_{\bar\mu}}$, and

$(*)_3$ $y_{n,\ell} \in p^\ell \hat B_{\bar\mu}$.

Without loss of generality $pz_n = 0 \neq z_n$ hence $p y^\ell_n = 0$. Let

$$\nu_\delta(n) =: \min(C_\delta \setminus (\eta_\delta(n)+1)), \quad z^*_n = (h \circ \hat g_\delta)(z_n) \quad \text{and} \quad y^*_{n,\ell} = (h \circ \hat g_\delta)(y_{n,\ell}).$$

Now clearly $\hat g_\delta(z_n) = g_\delta(z_n) = x^n_{\eta_\delta(n),i} \in G \restriction \nu_\delta(n)$, hence $(h \circ \hat g_\delta)(z_n) \notin H \restriction \eta_\delta(n)$, that is $z^*_n \notin H \restriction \eta_\delta(n)$.

So $z^*_n \in H_{\nu_\delta(n)} \setminus H_{\eta_\delta(n)}$ belongs to the p-adic closure of $\mathrm{Rang}(f \restriction Y)$. As H, G, h and $f \restriction Y$ belongs to $\mathfrak{A}$, also K, the closure of $\mathrm{Rang}(f \restriction Y)$ in H by the p-adic topology belongs to $\mathfrak{A}$, and clearly $|K| \leq 2^{\aleph_0}$, hence

$$A^* = \{\alpha \in C_\delta : K \cap H_{\min(C_\delta \setminus (\alpha+1))} \setminus H_\alpha \text{ is not empty}\}$$

is a subset of C_δ of cardinality $\leq 2^{\aleph_0}$ which belongs to $\mathfrak{A}$, hence $[A^*]^{\aleph_0} \subseteq \mathfrak{A}$ but $A_\delta \subseteq A^*$ so $A_\delta \in \mathfrak{A}$, a contradiction. $\square_{5.9}$

6 Implications between the existence of universals

Theorem 6.1 *Let $\bar{n} = \langle n_i : i < \omega\rangle$, $n_i \in [1, \omega)$. Remember*

$$J^2_{\bar{n}} = \{A \subseteq \prod_{i<\omega} n_i : A \text{ is nowhere dense}\}.$$

Assume $\lambda \geq 2^{\aleph_0}$, $\mathbf{U}^{\aleph_0}_{J^3_{\bar{n}}}(\lambda) = \lambda$ or just $\mathbf{U}^{\aleph_0}_{J^2_{\bar{n}}}(\lambda) = \lambda$ for every such $\bar{n}$, and

$$n < \omega \quad \Rightarrow \quad \lambda_n \leq \lambda_{n+1} \leq \lambda_\omega = \lambda \quad \text{and}$$

$$\lambda \leq \prod_{n<\omega} \lambda_n \quad \text{and} \quad \bar{\lambda} = \langle \lambda_i : i \leq \omega\rangle.$$

1. *If in $\mathfrak{K}^{fc}_{\bar{\lambda}}$ there is a universal member*

   then *in $\mathfrak{K}^{rs(p)}_{\lambda}$ there is a universal member.*

2. *If in $\mathfrak{K}^{fc}_{\lambda}$ there is a universal member for $\mathfrak{K}^{fc}_{\bar{\lambda}}$*

   then *in $\mathfrak{K}^{rs(p)}_{\bar{\lambda}} =: \{G \in \mathfrak{K}^{rs(p)}_{\lambda} : \lambda_n(G) \leq \lambda\}$ there is a universal member (even for $\mathfrak{K}^{rs(p)}_{\lambda}$).*

($\lambda_n(G)$ were defined in 1.1).

Remark 6.2 1. Similarly for "there are $M_i \in \mathfrak{K}_{\lambda_1}$ $(i < \theta)$ with $\langle M_i : i < \theta\rangle$ being universal for $\mathfrak{K}_\lambda$".

2. The parallel of 1.1 holds for $\mathfrak{K}^{fc}_{\lambda}$.

3. By §5 only the case λ singular or $\lambda = \mu^+$ & $\mathrm{cf}(\mu) = \aleph_0$ & $(\forall \alpha < \mu)(|\alpha|^{\aleph_0} < \mu)$ is of interest for 6.1.

Proof 1) By 1.1, (2) $\Rightarrow$ (1).

More elaborately, by part (2) of 6.1 below there is $H \in \mathfrak{K}^{rs(p)}_{\bar{\lambda}}$ which is universal in $\mathfrak{K}^{rs(p)}_{\bar{\lambda}}$. Clearly $|G| = \lambda$ so $H \in \mathfrak{K}^{rs(p)}_{\lambda}$, hence for proving part (1) of 6.1 it suffices to prove that H is a universal member of $\mathfrak{K}^{rs(p)}_{\lambda}$. So let $G \in \mathfrak{K}^{rs(p)}_{\lambda}$, and we shall prove that it is embeddable into H. By 1.1 there is G' such that $G \subseteq G' \in \mathfrak{K}^{rs(p)}_{\bar{\lambda}}$. By the choice of H there is an embedding h of G' into H. So $h \restriction G$ is an embedding of G into H, as required.

2) Let T^* be a universal member of $\mathfrak{K}^{fc}_{\bar{\lambda}}$ (see §2) and let $P_\alpha = P^{T^*}_\alpha$.

Let $\chi > 2^\lambda$. Without loss of generality $P_n = \{n\} \times \lambda_n$, $P_\omega = \lambda$. Let

$$B_0 = \bigoplus \{G^n_t : n < \omega, t \in P_n\},$$

$$B_1 = \bigoplus\{G^n_t : n < \omega \text{ and } t \in P_n\},$$

where $G^n_t \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$, G^n_t is generated by x^n_t. Let $\mathfrak{B} \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\|\mathfrak{B}\| = \lambda$, $\lambda + 1 \subseteq \mathfrak{B}$, $T^* \in \mathfrak{B}$, hence B_0, $B_1 \in \mathfrak{B}$ and $\hat{B}_0, \hat{B}_1 \in \mathfrak{B}$ (the torsion completion of $B0, B_1$, resp.). Let $G^* = \hat{B}_1 \cap \mathfrak{B}$.

Let us prove that G^* is universal for $\mathfrak{K}^{rs(p)}_\lambda$ (by 1.1 this suffices). Let $G \in \mathfrak{K}^{rs(p)}_\lambda$, so by 1.1 without loss of generality $B_0 \subseteq G \subseteq \hat{B}_0$. We define R:

$$\begin{aligned} R = \{\eta :\ & \eta \in \prod_{n<\omega} \lambda_n \text{ and for some } x \in G \text{ letting} \\ & x = \sum\{a^n_i\, p^{n-k}\, x^n_i : n < \omega, i \in w_n(x)\} \text{ where} \\ & w_n(x) \in [\lambda_n]^{<\aleph_0}, a^n_i\, p^{n-k}\, x^n_i \neq 0 \text{ we have} \\ & \bigwedge_n \eta(n) \in w_n(x) \cup \{\ell : \ell + |w_n(x)| \leq n\}\}. \end{aligned}$$

Lastly let $M =: (R \cup \bigcup_{n<\omega} \{n\} \times \lambda_n, P_n, F_n)_{n<\omega}$ where $P_n = \{n\} \times \lambda_n$ and $F_n(\eta) = (n, \eta(n))$, so clearly $M \in \mathfrak{K}^{fc}_\lambda$. Consequently, there is an embedding $g : M \longrightarrow T^*$, so g maps $\{n\} \times \lambda_n$ into $P^{T^*}_n$ and g maps R into $P^{T^*}_\omega$. Let $g(n, \alpha) = (n, g_n(\alpha))$ (i.e. this defines g_n). Clearly $g \restriction (\cup P^M_n) = g \restriction (\bigcup_n \{n\} \times \lambda_n)$ induces an embedding g^* of B_0 to B_1 (by mapping the generators into the generators).

The problem is why:

(*) if $x = \sum\{a^n_i\, p^{n-k}\, x^n_i : n < \omega, i \in w_n(x)\} \in G$

then $g^*(x) = \sum\{a^n_i\, p^{n-k}\, g^*(x^n_i) : n < \omega, i \in w_n(x)\} \in G^*$.

As $G^* = \hat{B}_1 \cap \mathfrak{B}$, and $2^{\aleph_0} + 1 \subseteq \mathfrak{B}$, it is enough to prove $\langle g''(w_n(x)) : n < \omega\rangle \in \mathfrak{B}$. Now for notational simplicity $\bigwedge_n [|w_n(x)| \geq n + 1]$ (we can add an element of $G^* \cap \mathfrak{B}$ or just repeat the arguments). For each $\eta \in \prod_{n<\omega} w_n(x)$ we know that $g(\eta) = \langle g(\eta(n)) : n < \omega\rangle \in T^*$ hence is in $\mathfrak{B}$ (as $T^* \in \mathfrak{B}$, $|T^*| \leq \lambda$). Now by assumption there is $A \subseteq \prod_{n<\omega} w_n(x)$ which is not nowhere dense such that $g \restriction A \in \mathfrak{B}$, hence for some n^* and $\eta^* \in \prod_{\ell<n^*} w_\ell(x)$, A is dense above η^* (in $\prod_{n<\omega} w_n(x)$). Hence

$$\langle\{\eta(n) : \eta \in A\} : n^* \leq n < \omega\rangle = \langle w_n[x] : n^* \leq n < \omega\rangle,$$

but the former is in $\mathfrak{B}$ as $A \in \mathfrak{B}$, and from the latter the desired conclusion follows. $\square_{6.1}$

7 Non-existence of universals for trees with small density

For simplicity we deal below with the case $\delta = \omega$, but the proof works in general (as for $\mathfrak{K}^{fr}_{\bar\lambda}$ in §2). Section 1 hinted we should look at $\mathfrak{K}^{tr}_{\bar\lambda}$ not only for the case $\bar\lambda = \langle \lambda : \alpha \le \omega \rangle$ (i.e. $\mathfrak{K}^{tr}_{\lambda}$), but in particular for

$$\bar\lambda = \langle \lambda_n : n < \omega \rangle\hat{\ }\langle \lambda \rangle, \qquad \lambda_n^{\aleph_0} < \lambda_{n+1} < \mu < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}.$$

Here we get for this class (embeddings are required to preserve levels), results stronger than the ones we got for the classes of Abelian groups we have considered.

Theorem 7.1 *Assume that*

(a) $\bar\lambda = \langle \lambda_\alpha : \alpha \le \omega \rangle$, $\lambda_n < \lambda_{n+1} < \lambda_\omega$, $\lambda = \lambda_\omega$, *all are regular,*

(b) D *is a filter on* ω *containing cobounded sets,*

(c) $\mathrm{tcf}(\prod \lambda_n / D) = \lambda$ *(indeed, we mean* $=$*, we could just use* $\lambda \in \mathrm{pcf}_D(\{\lambda_n : n < \omega\})$*),*

(d) $(\sum\limits_{n<\omega} \lambda_n)^+ < \lambda < \prod\limits_{n<\omega} \lambda_n$.

Then *there is no universal member in* $\mathfrak{K}^{tr}_{\bar\lambda}$.

Proof We first notice that there is a sequence $\bar P = \langle P_\alpha : \sum\limits_{n<\omega} \lambda_n < \alpha < \lambda \rangle$ such that:

1. $|P_\alpha| < \lambda$,
2. $a \in P_\alpha \quad \Rightarrow \quad a$ is a closed subset of α of order type $\le \sum\limits_{n<\omega} \lambda_n$,
3. $a \in \bigcup\limits_{\alpha<\lambda} P_\alpha$ & $\beta \in \mathrm{nacc}(a) \quad \Rightarrow \quad a \cap \beta \in P_\beta$,
4. For all club subsets E of λ, there are stationarily many δ for which there is an $a \in \bigcup\limits_{\alpha<\lambda} P_\alpha$ such that

$$\mathrm{cf}(\delta) = \aleph_0 \ \&\ a \in P_\delta \ \&\ \mathrm{otp}(a) = \sum_{n<\omega} \lambda_n \ \&\ a \subseteq E.$$

[Why? If $\lambda = (\sum_{n<\omega} \lambda_n)^{++}$, then it is the successor of a regular, so we use [Sh 351, §4], i.e.

$$\{\alpha < \lambda : \mathrm{cf}(\alpha) \leq (\sum_{n<\omega} \lambda_n)\}$$

is the union of $(\sum_{n<\omega} \lambda_n)^+$ sets with squares and continue with [Sh:g, III 2.14(2)(c)].
If $\lambda > (\sum_{n<\omega} \lambda_n)^{++}$, then we can use [Sh 420, §1], which guarantees that there is a stationary $S \in I[\lambda]$ and then use [Sh:g, III 2.3(2)]'s proof.]

We can now find a sequence

$$\langle f_\alpha, g_{\alpha,a} : \alpha < \lambda, a \in P_\alpha \rangle$$

such that:

(a) $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is a $<_D$-increasing cofinal sequence in $\prod_{n<\omega} \lambda_n$,

(b) $g_{\alpha,a} \in \prod_{n<\omega} \lambda_n$,

(c) $\bigwedge_{\beta<\alpha} f_\beta <_D g_{\alpha,a} <_D f_{\alpha+1}$,

(d) $\lambda_n > |a| \ \& \ \beta \in \mathrm{nacc}(a) \quad \Rightarrow \quad g_{\beta,a\cap\beta}(n) < g_{\alpha,a}(n)$.

[How? Choose $\bar{f}$ by $\mathrm{tcf}(\prod_{n<\omega} \lambda_n / D) = \lambda$. Then choose g's by induction, possibly throwing out some of the f's; this is from [Sh:g, II, §1].]

Let $T \in \mathfrak{K}^{tr}_{\lambda}$.

We introduce for $x \in \mathrm{lev}_\omega(T)$ and $\ell < \omega$ the notation $F^T_\ell(x) = F_\ell(x)$ to denote the unique member of $\mathrm{lev}_\ell(T)$ which is below x in the tree order of T.

For $a \in \bigcup_{\alpha<\lambda} P_\alpha$, let $a = \{\alpha_{a,\xi} : \xi < \mathrm{otp}(a)\}$ be an increasing enumeration. We shall consider two cases. In the first one, we assume that the following statement $(*)$ holds. In this case, the proof is easier, and maybe $(*)$ always holds for some D, but we do not know this at present.

$(*)$ There is a partition $\langle A_n : n < \omega \rangle$ of ω into sets not disjoint to any member of D.

In this case, let for $n \in \omega$, D_n be the filter generated by D and A_n. Let for $a \in \bigcup_{\alpha<\lambda} P_\alpha$ with $\mathrm{otp}(a) = \sum_{n<\omega} \lambda_n$, and for $x \in \mathrm{lev}_\omega(T)$,

$$\mathrm{inv}(x, a, T) =: \langle \xi_n(x, a, T) : n < \omega \rangle,$$

where

$$\xi_n(x,a,T) =: \min\{\xi < \text{otp}(a): \text{ for some } m<\omega \text{ we have } \langle F^T_\ell(x):\ell<\omega\rangle <_{D_n} g_{\alpha',a'} \text{ where } \alpha' = \alpha_{a,\omega\xi+m} \text{ and } a' = a\cap\alpha'\}.$$

Let

$$\text{INv}(a,T) =: \{\text{inv}(x,a,T) : x\in T \ \&\ \text{lev}_T(x)=\omega\},$$

$$\text{INV}(T) =: \{c: \text{ for every club } E\subseteq\lambda, \text{ for some } \delta \text{ and } a \text{ we have otp}(a)=\textstyle\sum\lambda_n \ \&\ a\subseteq E \ \&\ a\in P_\delta \text{ and for some } x\in T \text{ of lev}_T(x)=\omega,\ c=\text{inv}(x,a,T)\}.$$

(Alternatively, we could have looked at the function giving each a the value $\text{INv}(a,T)$, and then divide by a suitable club guessing ideal as in the proof in §3, see Definition 3.7.)
Clearly

Fact: $\text{INV}(T)$ has cardinality $\le\lambda$.

The main point is the following

Main Fact: If $\mathbf{h}:T^1\longrightarrow T^2$ is an embedding, *then*

$$\text{INV}(T^1)\subseteq\text{INV}(T^2).$$

Proof of the **Main Fact** *under* $(*)$ We define for $n\in\omega$

$$E_n =: \{\delta<\lambda_n : \delta > \bigcup_{\ell<n}\lambda_\ell \text{ and } (\forall x\in\text{lev}_n(T^1))\,(\mathbf{h}(x)<\delta \Leftrightarrow x<\delta)\}.$$

We similarly define E_ω, so E_n ($n\in\omega$) and E_ω are clubs (of λ_n and λ respectively). Now suppose $c\in\text{INV}(T_1)\setminus\text{INV}(T_2)$. Without loss of generality E_ω is (also) a club of λ which exemplifies that $c\notin\text{INV}(T_2)$. For $h\in\prod_{n<\omega}\lambda_n$, let

$$h^+(n) =: \min(E_n\setminus h(n)), \quad \text{and} \quad \beta[h]=\min\{\beta<\lambda : h<f_\beta\}.$$

(Note that $h<f_{\beta[h]}$, not just $h<_D f_{\beta[h]}$.) For a sequence $\langle h_i : i<i^*\rangle$ of functions from $\prod_{n<\omega}\lambda_n$, we use $\langle h_i:i<i^*\rangle^+$ for $\langle h_i^+ : i<i^*\rangle$. Now let

$$E^* =: \{\delta<\lambda : \text{if } \alpha<\delta \text{ then } \beta[f^+_\alpha]<\delta \text{ and } \delta\in\text{acc}(E_\omega)\}.$$

Thus E^* is a club of λ. Since $c\in\text{INV}(T_1)$, there is $\delta<\lambda$ and $a\in P_\delta$ such that for some $x\in\text{lev}_\omega(T_1)$ we have

$$a\subseteq E^* \ \&\ \text{otp}(a)=\sum_{n<\omega}\lambda_n \ \&\ c=\text{inv}(x,a,T_1).$$

Let for $n \in \omega$, $\xi_n =: \xi_n(x, a, T_1)$, so $c = \langle \xi_n : n < \omega \rangle$. Also let for $\xi < \sum_{n<\omega} \lambda_n$, $\alpha_\xi =: \alpha_{a,\xi}$, so $a = \langle \alpha_\xi : \xi < \sum_{n<\omega} \lambda_n \rangle$ is an increasing enumeration. Now fix an $n < \omega$ and consider $\mathbf{h}(x)$. Then we know that for some m

(α) $\langle F_\ell^{T_1}(x) : \ell < \omega \rangle <_{D_n} g_{\alpha'}$ where $\alpha' = \alpha_{\omega\xi_n+m}$ and

(β) for no $\xi < \xi_n$ is there such an m.

Now let us look at $F_\ell^{T_1}(x)$ and $F_\ell^{T_2}(\mathbf{h}(x))$. They are not necessarily equal, but

(γ) $\min(E_\ell \setminus F_\ell^{T_1}(x)) = \min(E_\ell \setminus F_\ell^{T_2}(\mathbf{h}(x))$

(by the definition of E_ℓ). Hence

(δ) $\langle F_\ell^{T_1}(x) : \ell < \omega \rangle^+ = \langle F_\ell^{T_2}(\mathbf{h}(x)) : \ell < \omega \rangle^+$.

Now note that by the choice of g's

(ε) $(g_{\alpha_\varepsilon, a\cap\alpha_\varepsilon})^+ <_{D_n} g_{\alpha_{\varepsilon+1}, a\cap\alpha_{\varepsilon+1}}$.

From (δ) and (ε) it follows that $\xi_n(\mathbf{h}(x), a, T^2) = \xi_n(x, a, T^1)$. Hence $c \in \mathrm{INV}(T^2)$. $\square_{\text{Main Fact}}$

Now it clearly suffices to prove:

Fact A: For each $c = \langle \xi_n : n < \omega \rangle \in {}^\omega(\sum_{n<\omega} \lambda_n)$ we can find a $T \in \mathfrak{K}_\lambda^{tr}$ such that $c \in \mathrm{INV}(T)$.

Proof of the Fact A in case $(*)$ *holds* For each $a \in \bigcup_{\delta<\lambda} P_\delta$ with $\mathrm{otp}(a) = \sum_{n\in\omega} \lambda_n$ we define $x_{c,a} =: \langle x_{c,a}(\ell) : \ell < \omega \rangle$ by:

if $\ell \in A_n$, then $x_{c,a}(\ell) = \alpha_{a,\omega\xi_n+\delta}$.

Let

$$T = \bigcup_{n<\omega} \prod_{\ell<n} \lambda_\ell \cup \{x_{c,a} : a \in \bigcup_{\delta<\lambda} P_\delta \ \&\ \mathrm{otp}(a) = \sum_{n<\omega} \lambda_n\}.$$

We order T by $\triangleleft$.

It is easy to check that T is as required. $\square_A$

Now we are left to deal with the case that $(*)$ does not hold. Let

$$\mathrm{pcf}(\{\lambda_n : n < \omega\}) = \{\kappa_\alpha : \alpha \leq \alpha^*\}$$

be an enumeration in increasing order so in particular

$$\kappa_{\alpha^*} = \max \mathrm{pcf}(\{\lambda_n : n < \omega\}).$$

Without loss of generality $\kappa_{\alpha^*} = \lambda$ (by throwing out some elements if necessary) and $\lambda \cap \mathrm{pcf}(\{\lambda_n : n < \omega\})$ has no last element (this appears explicitly in [Sh:g], but is also straightforward from the pcf theorem). In particular, α^* is a limit ordinal. Hence, without loss of generality

$$D = \{A \subseteq \omega : \lambda > \max\mathrm{pcf}\{\lambda_n : n \in \omega \setminus A\}\}.$$

Let $\langle \mathfrak{a}_{\kappa_\alpha} : \alpha \leq \alpha^* \rangle$ be a generating sequence for $\mathrm{pcf}(\{\lambda_n : n < \omega\})$, i.e.

$$\max\mathrm{pcf}(\mathfrak{a}_{\kappa_\alpha}) = \kappa_\alpha \quad \text{and} \quad \kappa_\alpha \notin \mathrm{pcf}(\{\lambda_n : n < \omega\} \setminus \mathfrak{a}_{\kappa_\alpha}).$$

(The existence of such a sequence follows from the pcf theorem). Without loss of generality,

$$\mathfrak{a}_{\alpha^*} = \{\lambda_n : n < \omega\}.$$

Now note

Remark 7.2 If $\mathrm{cf}(\alpha^*) = \aleph_0$, then $(*)$ holds.

Why? Let $\langle \alpha(n) : n < \omega \rangle$ be a strictly increasing cofinal sequence in α^*. Let $\langle B_n : n < \omega \rangle$ partition ω into infinite pairwise disjoint sets and let

$$A_\ell =: \{k < \omega : \bigvee_{n \in B_\ell} [\lambda_k \in \mathfrak{a}_{\kappa_{\alpha(n)}} \setminus \bigcup_{m<n} \mathfrak{a}_{\kappa_{\alpha(m)}}]\}.$$

To check that this choice of $\langle A_\ell : \ell < \omega \rangle$ works, recall that for all α we know that α_{κ_α} does not belong to the ideal generated by $\{\mathfrak{a}_{\kappa_\beta} : \beta < \alpha\}$ and use the pcf calculus. $\square$7.2

Now let us go back to the general case, assuming $\mathrm{cf}(\alpha^*) > \aleph_0$. Our problem is the possibility that

$$\mathcal{P}(\{\lambda_n : n < \omega\})/J_{<\lambda}[\{\lambda_n : n < \omega\}].$$

is finite. Let now $A_\alpha =: \{n : \lambda_n \in \mathfrak{a}_\alpha\}$, and

$$\begin{array}{rcl} J_\alpha & =: & \{A \subseteq \omega : \max\mathrm{pcf}\{\lambda_\ell : \ell \in A\} < \kappa_\alpha\} \\ J'_\alpha & =: & \{A \subseteq \omega : \max\mathrm{pcf}(\{\lambda_\ell : \ell \in A\} \cap \mathfrak{a}_{\kappa_\alpha}) < \kappa_\alpha\}. \end{array}$$

We define for $T \in \mathfrak{K}^{tr}_\lambda$, $x \in \mathrm{lev}_\omega(T)$, $\alpha < \alpha^*$ and $a \in \bigcup_{\delta<\lambda} P_\delta$:

$$\xi^*_\alpha(x, a, T) =: \min\{\xi : \ \bigvee_m [\langle F^T_\ell(x) : \ell < \omega \rangle <_{J'_\alpha} g_{\alpha',a'} \text{ where } \alpha' = \alpha_{a,\omega\xi+m} \text{ and } a' = a \cap \alpha'\}.$$

Let

$$\mathrm{inv}_\alpha(x, a, T) =: \langle \xi^*_{\alpha+n}(x, a, T) : n < \omega \rangle,$$

$$\mathrm{INv}(a,T) =: \{\mathrm{inv}_\alpha(x,a,T) : x \in T \ \&\ \alpha < \alpha^* \ \&\ \mathrm{lev}_T(x) = \omega\},$$

and

$$\mathrm{INV}(T) = \{c: \begin{array}{l} \text{for every club } E^* \text{ of } \lambda \text{ for some } a \in \bigcup_{\delta<\lambda} P_\delta \\ \text{with } \mathrm{otp}(a) = \sum_{n<\omega} \lambda_n \text{ for arbitrarily large } \alpha < \alpha^*, \\ \text{there is } x \in \mathrm{lev}_\omega(T) \text{ such that } \mathrm{inv}_\alpha(x,a,T) = c\}. \end{array}$$

As before, the point is to prove the Main Fact.

Proof of the **Main Fact** *in general* Suppose $\mathbf{h} : T^1 \longrightarrow T^2$ and $c \in \mathrm{INV}(T^1)\setminus\mathrm{INV}(T^2)$. Let E' be a club of λ which witnesses that $c \notin \mathrm{INV}(T^2)$. We define E_n, E_ω as before, as well as E^* ($\subseteq E_\omega \cap E'$). Now let us choose $a \in \bigcup_{\delta<\lambda} P_\delta$ with $a \subseteq E^*$ and $\mathrm{otp}(a) = \sum_{n<\omega} \lambda_n$. So $a = \{\alpha_{a,\xi} : \xi < \sum_{n<\omega} \lambda_n\}$, which we shorten as $a = \{\alpha_\xi : \xi < \sum_{n<\omega} \lambda_n\}$. For each $\xi < \sum_{n<\omega} \lambda_n$, as before, we know that

$$(g_{\alpha_\xi, a\cap\alpha_\xi})^+ <_{J^*_\alpha} g_{\alpha_{\xi+1}, a\cap\alpha_{\xi+1}}.$$

Therefore, there are $\beta_{\xi,\ell} < \alpha^*$ ($\ell < \ell_\xi$) such that

$$\{\ell : g^+_{\alpha_\xi, a\cap\alpha_\xi}(\ell) \geq g_{\alpha_{\xi+1}, a\cap\alpha_{\xi+1}}(\ell)\} \supseteq \bigcup_{\ell<\ell_\xi} A_{\beta_{\xi,\ell}}.$$

Let $c = \langle \xi_n : n < \omega\rangle$ and let

$$\Upsilon = \{\beta_{\xi,\ell} : \text{for some } n \text{ and } m \text{ we have } \xi = \omega\xi_n + m \ \&\ \ell < \omega\}.$$

Thus $\Upsilon \subseteq \alpha^*$ is countable. Since $\mathrm{cf}(\alpha^*) > \aleph_0$, the set Υ is bounded in α^*. Now we know that c appears as an invariant for a and arbitrarily large $\delta < \alpha^*$, for some $x_{a,\delta} \in \mathrm{lev}_\omega(T_1)$. If $\delta > \sup(\Upsilon)$, $c \in \mathrm{INV}(T^2)$ is exemplified by $a, \delta, \mathbf{h}(x_{\alpha,\delta})$, just as before. □

We still have to prove that every $c = \langle \xi_n : n < \omega\rangle$ appears as an invariant; i.e. the parallel of Fact A.

Proof of Fact A in the general case: Define for each $a \in \bigcup_{\delta<\lambda} P_\delta$ with $\mathrm{otp}(a) = \sum_{n<\omega} \lambda_n$ and $\beta < \alpha^*$

$$x_{c,a,\beta} = \langle x_{c,a,\beta}(\ell) : \ell < \omega\rangle,$$

where

$$x_{c,a,\beta}(\ell) = \begin{cases} \alpha_{a,\omega\xi_n+\delta} & \underline{\text{if}} \quad \lambda_\ell \in \mathfrak{a}_{\beta+k} \setminus \bigcup_{k'<k} \mathfrak{a}_{\beta+k'} \\ 0 & \underline{\text{if}} \quad \lambda_\ell \notin \mathfrak{a}_{\beta+k} \text{ for any } k < \omega. \end{cases}$$

Form the tree as before. Now for any club E of λ, we can find $a \in \bigcup_{\delta<\lambda} P_\delta$ with $\mathrm{otp}(a) = \sum_{n<\omega} \lambda_n$, $a \subseteq E$ such that $\langle x_{c,a,\beta} : \beta < \alpha^* \rangle$ shows that $c \in \mathrm{INV}(T)$. $\square_{7.1}$

Remark 7.3 1. Clearly, this proof shows not only that there is no one T which is universal for $\mathfrak{K}^{tr}_{\bar\lambda}$, but that any sequence of $< \prod_{n<\omega} \lambda_n$ trees will fail. This occurs generally in this paper, as we have tried to mention in each particular case.

2. The case "$\lambda < 2^{\aleph_0}$" is included in the theorem, though for the Abelian group application the $\bigwedge_{n<\omega} \lambda_n^{\aleph_0} < \lambda_{n+1}$ is necessary.

Remark 7.4 1. If $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \chi < \mu^{\aleph_0}$ and $\chi^{[\lambda]} < \mu^{\aleph_0}$ (or at least $T_{\mathrm{id}^a(\bar C)}(\chi) < \mu^{\aleph_0}$) we can get the results for "no $M \in \mathfrak{K}^x_\chi$ is universal for $\mathfrak{K}^x_\lambda$", see §8 (and [Sh 456]).

We can below (and subsequently in §8) use $J^3_{\bar m}$ as in §6.

Theorem 7.5 *Assume that* $2^{\aleph_0} < \lambda_0$, $\bar\lambda = \langle \lambda_n : n < \omega \rangle \hat{}\langle \lambda \rangle$, $\mu = \sum_{n<\omega} \lambda_n$, $\lambda_n < \lambda_{n+1}$, $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$.
__If__, for simplicity, $\bar m = \langle m_i : i < \omega \rangle = \langle \omega : i < \omega \rangle$ *(actually* $m_i \in [2,\omega]$ *or even* $m_i \in [2, \lambda_0)$, $\lambda_0 < \lambda$ *are O.K.) and* $\mathbf{U}^{<\mu}_{J^2_{\bar m}}(\lambda) = \lambda$ *(remember*

$$J^2_{\bar m} = \{A \subseteq \prod_{i<\omega} m_i : A \text{ is nowhere dense}\}$$

and definition 5.1),
__then__ in $\mathfrak{K}^{tr}_{\bar\lambda}$ *there is no universal member.*

Proof Let $S \subseteq \lambda$, $\bar C = \langle C_\delta : \delta \in S \rangle$ be a club guessing sequence on λ with $\mathrm{otp}(C_\delta) \geq \sup \lambda_n$. We assume that we have $\bar{\mathfrak{A}} = \langle \mathfrak{A}_\alpha : \alpha < \lambda \rangle$, $J^2_{\bar m}$, $T^* \in \mathfrak{A}_0$ (T^* is a candidate for the universal), $\bar C = \langle C_\delta : \delta \in S \rangle \in \mathfrak{A}_\alpha$, $\mathfrak{A}_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\chi = \beth_7(\lambda)^+$, $\|\mathfrak{A}_\alpha\| < \lambda$, $\mathfrak{A}_\alpha$ increasingly continuous, $\langle \mathfrak{A}_\beta : \beta \leq \alpha \rangle \in \mathfrak{A}_{\alpha+1}$, $\mathfrak{A}_\alpha \cap \lambda$ is an ordinal, $\mathfrak{A} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$ and

$$E =: \{\alpha : \mathfrak{A}_\alpha \cap \lambda = \alpha\}.$$

Note: $\prod \bar m \subseteq \mathfrak{A}$ (as $\prod \bar m \in \mathfrak{A}$ and $|\prod \bar m| = 2^{\aleph_0}$).
NOTE: By $\mathbf{U}^{<\mu}_{J^2_{\bar m}}(\lambda) = \lambda$,

$(*)$ <u>if</u> $x_\eta \in \mathrm{lev}_\omega(T^*)$ for $\eta \in \prod \bar{m}$

<u>then</u> for some $A \in (J^2_{\bar{m}})^+$ the set $\langle (\eta, x_\eta) : \eta \in A\rangle$ belongs to $\mathfrak{A}$. But then for some $\nu \in \bigcup\limits_k \prod\limits_{i<k} m_i$, the set A is dense above ν (by the definition of $J^2_{\bar{m}}$) and hence: if the mapping $\eta \mapsto x_\eta$ is continuous then $\langle x_\rho : \nu \triangleleft \rho \in \prod \bar{m}\rangle \in \mathfrak{A}$.

For $\delta \in S$ such that $C_\delta \subseteq E$ we let

$$P^0_\delta = P^0_\delta(\mathfrak{A}) = \left\{ \bar{x} : \begin{array}{l} \bar{x} = \langle x_\rho : \rho \in t\rangle \in \mathfrak{A} \text{ and } x_\rho \in \mathrm{lev}_{\ell g(\rho)} T^*, \\ \text{the mapping } \rho \mapsto x_\rho \text{ preserves all of the relations:} \\ \ell g(\rho) = n, \rho_1 \triangleleft \rho_2, \neg(\rho_1 \triangleleft \rho_2), \neg(\rho_1 = \rho_2), \\ \rho_1 \cap \rho_2 = \rho_3 \text{ (and so } \ell g(\rho_1 \cap \rho_2) = n \text{ is preserved);} \\ \text{and } t \subseteq \bigcup\limits_{\alpha \leq \omega} \prod\limits_{i<\alpha} m_i \end{array} \right\}.$$

Assume $\bar{x} = \langle x_\rho : \rho \in t\rangle \in P^0_\delta$. Let

$$\mathrm{inv}(\bar{x}, C_\delta, T^*, \bar{\mathfrak{A}}) =: \{\alpha \in C_\delta : (\exists \rho \in \mathrm{Dom}(\bar{x}))(x_\rho \in \mathfrak{A}_{\min(C_\delta \setminus (\alpha+1))} \setminus \mathfrak{A}_\alpha)\}.$$

Let $\mathrm{Inv}(C_\delta, T^*, \bar{\mathfrak{A}}) =:$

$$\{a : \text{for some } \bar{x} \in P^0_\delta, \ \ a \text{ is a countable subset of } \mathrm{inv}(\bar{x}, C_0, T^*, \bar{\mathfrak{A}})\}.$$

Note: $\mathrm{inv}(\bar{x}, C_0, T^*, \bar{\mathfrak{A}})$ has cardinality at most continuum, so $\mathrm{Inv}(C_0, T^*, \bar{\mathfrak{A}})$ is a family of $\leq 2^{\aleph_0} \times |\mathfrak{A}| = \lambda$ countable subsets of C_δ.

We continue as before. Let $\alpha_{\delta,\varepsilon}$ be the ε-th member of C_δ for $\varepsilon < \sum\limits_{n<\omega} \lambda_n$. So as $\lambda < \mu^{\aleph_0}, \mu > 2^{\aleph_0}$ clearly $\lambda < \mathrm{cf}([\lambda]^{\aleph_0}, \subseteq)$ (equivalently $\lambda < \mathrm{cov}(\mu, \mu, \aleph_1, 2)$) hence we can find $\gamma_n \in (\bigcup\limits_{\ell<n} \lambda_\ell, \lambda_n)$ limit such that for each $\delta \in S$, $a \in \mathrm{Inv}(C_\delta, T^*, \bar{\mathfrak{A}})$ we have $\{\gamma_n + \ell : n < \omega \text{ and } \ell < m_i\} \cap a$ is bounded in μ.

Now we can find T such that $\mathrm{lev}_n(T) = \prod\limits_{\ell<n} \lambda_\ell$ and

$$\mathrm{lev}_\omega(T) = \{\bar{\beta} : \begin{array}{l} \bar{\beta} = \langle \beta_\ell : \ell < \omega\rangle, \text{ and for some } \delta \in S, \text{ for every } \ell < \omega \\ \text{we have } \gamma'_\ell \in \{\alpha_{\delta, \gamma_\ell + m} : m < m_i\}\}. \end{array}$$

So, if T^* is universal there is an embedding $f : T \longrightarrow T^*$, and hence

$$E' = \{\alpha \in E : \mathfrak{A}_\alpha \text{ is closed under } f \text{ and } f^{-1}\}$$

is a club of λ. By the choice of $\bar{C}$ for some $\delta \in S$ we have $C_\delta \subseteq E'$. Now use $(*)$ with $x_\eta = f(\bar{\beta}^{\delta,\eta})$, where $\beta^{\delta,\eta}_\ell = \alpha_{\delta, \gamma_\ell + \eta(\ell)} \in \mathrm{lev}_\omega(T)$. Thus we get

$A \in (J^2_{\bar{m}})^+$ such that $\{(\eta, x_\eta) : \eta \in A\} \in \mathfrak{A}$, there is $\nu \in \bigcup_k \prod_{i<k} m_i$ such that A is dense above ν, hence as f is continuous, $\langle(\eta, x_\eta) : \nu \triangleleft \eta \in \prod \bar{m}\rangle \in \mathfrak{A}$. So $\langle x_\eta : \eta \in \prod \bar{m}, \nu \trianglelefteq \eta\rangle \in P^0_\delta(\mathfrak{A})$, and hence the set

$$\{\alpha_{\delta,\gamma_\ell+m} : \ell \in [\ell g(\nu), \omega) \text{ and } m < m_\ell\} \cup \{\alpha_{\delta,\gamma_i+\nu(i)} : \ell < \ell g(\nu)\}$$

is $\mathrm{inv}(\bar{x}, C_\delta, T^*, \mathfrak{A})$. Hence

$$a = \{\alpha_{\delta,\gamma_\ell} : \ell \in [\ell g(\nu), \omega)\} \in \mathrm{Inv}(C_\delta, T^*, \mathfrak{A}),$$

contradicting

"$\{\alpha_{\delta,\gamma_\ell} : \ell < \omega\}$ has finite intersection with any $a \in \mathrm{Inv}(C_\delta, T^*, \mathfrak{A})$". $\square$7.5

Remark 7.6 We can a priori fix a set of $\aleph_0$ candidates and say more on their order of appearance, so that $\mathrm{Inv}(\bar{x}, C_\delta, T^*, \bar{\mathfrak{A}})$ has order type ω. This makes it easier to phrase a true invariant, i.e. $\langle(\eta_n, t_n) : n < \omega\rangle$ is as above, $\langle \eta_n : n < \omega\rangle$ lists ${}^{\omega>}\omega$ with no repetition, $\langle t_n \cap {}^\omega\omega : n < \omega\rangle$ are pairwise disjoint. If $x_\rho \in \mathrm{lev}_\omega(T^*)$ for $\rho \in {}^\omega\omega$, $\bar{T}^* = \langle \bar{T}^*_\zeta : \zeta < \lambda\rangle$ representation we have

$$\mathrm{inv}(\langle x_\rho : \rho \in {}^\omega\omega\rangle, C_\delta, \bar{T}^*) =$$
$$\big\{\alpha \in C_\delta : \text{for some } n,\ (\forall\rho)[\rho \in t_n \cap {}^\omega\omega \quad \Rightarrow \quad x_\rho \in T^*_{\min(C_\delta\setminus(\alpha+1))} \setminus T^*_\alpha]\big\}.$$

Remark 7.7 If we have $\Gamma \in (J^2_{\bar{m}})^+$, Γ non-meagre, $J = J^2_m \restriction \Gamma$ and $\mathbf{U}^2_J(\lambda) < \lambda^{\aleph_0}$ then we can weaken the cardinal assumptions to:

$$\bar{\lambda} = \langle \lambda_n : n < \omega\rangle\hat{\ }\langle\lambda\rangle, \qquad \mu = \sum_n \lambda_n, \qquad \lambda_n < \lambda_{n+1},$$

$$\mu^+ < \lambda = \mathrm{cf}(\lambda) \qquad \text{and} \qquad \mathbf{U}^2_J(\lambda) < \mathrm{cov}(\mu, \mu, \aleph_1, 2) \text{(see 0.4)}.$$

The proof is similar.

8 Universals in singular cardinals

In §3, §5, 7.5, we can in fact deal with "many" singular cardinals λ. This is done by proving a stronger assertion on some regular λ. Here $\mathfrak{K}$ is a class of models.

Lemma 8.1 *1. There is no universal member in $\mathfrak{K}_{\mu^*}$ if for some $\lambda < \mu^*$, $\theta \geq 1$ we have:*

$\otimes_{\lambda,\mu^*,\theta}[\mathfrak{K}]$ *not only there is no universal member in $\mathfrak{K}_\lambda$ but if we assume:*

$$\langle M_i : i < \theta\rangle \text{ is given, } \|M_i\| \le \mu^* < \prod_n \lambda_n, \ M_i \in \mathfrak{K},$$

then there is a structure M from $\mathfrak{K}_\lambda$ (in some cases of a simple form) not embeddable in any M_i.

2. Assume

$\otimes_1^\sigma$ *$\langle \lambda_n : n < \omega\rangle$ is given, $\lambda_n^{\aleph_0} < \lambda_{n+1}$,*

$$\mu = \sum_{n<\omega} \lambda_n < \lambda = \mathrm{cf}(\lambda) \le \mu^* < \prod_{n<\omega} \lambda_n$$

and $\mu^+ < \lambda$ or at least there is a club guessing $\bar{C}$ as in $(**)^1_\lambda$ (ii) of 3.4 for (λ,μ).

Then there is no universal member in $\mathfrak{K}_{\mu^*}$ (and moreover $\otimes_{\lambda,\mu^*,\theta}[\mathfrak{K}]$ holds) in the following cases

$\otimes_2$**(a)** *for torsion free groups, i.e. $\mathfrak{K} = \mathfrak{K}^{rtf}_{\bar{\lambda}}$ if $\mathrm{cov}(\mu^*, \lambda^+, \lambda^+, \lambda) < \prod_{n<\omega} \lambda_n$, see notation 0.4 on* cov*)*

(b) *for $\mathfrak{K} = \mathfrak{K}^{tcf}_{\bar{\lambda}}$,*

(c) *for $\mathfrak{K} = \mathfrak{K}^{tr}_{\bar{\lambda}}$ as in 7.5 -* $\mathrm{cov}(\mathbf{U}_{J^3_{\bar{m}}}(\mu^*), \lambda^+, \lambda^+, \lambda) < \prod_{n<\omega} \lambda_n$*,*

(d) *for $\mathfrak{K}^{rs(p)}_{\bar{\lambda}}$: like case (c) (for appropriate ideals), replacing tr by $rs(p)$.*

Remark 8.2 1. For 7.5 as $\bar{m} = \langle \omega : i < \omega\rangle$ it is clear that the subtrees t_n are isomorphic. We can use $m_i \in [2,\omega)$, and use coding; anyhow it is immaterial since ${}^\omega\omega, {}^\omega 2$ are similar.

2. We can also vary $\bar{\lambda}$ in 8.1 $\otimes_2$, case (c).

3. We can replace cov in $\otimes_2$(a),(c) by

$$\sup \mathrm{pp}_{\Gamma(\lambda)}(\chi) : \mathrm{cf}(\chi) = \lambda, \lambda < \chi \le \mathbf{U}_{J^3_{\bar{m}}}(\mu^*)\}$$

(see [Sh 355, 5.4], 2.4).

Proof Should be clear, e.g.
Proof of Part 2), Case (c) Let $\langle T_i : i < i^*\rangle$ be given, $i^* < \prod_{n<\omega} \lambda_n$ such that

$$\|T_i\| \le \mu^* \quad \text{and} \quad \mu^\otimes =: \mathrm{cov}(\mathbf{U}_{J^3_{\bar{m}}}(\mu^*), \lambda^+, \lambda^+, \lambda) < \prod_{n<\omega} \lambda_n.$$

By [Sh 355, 5.4] and pp calculus ([Sh 355, 2.3]), $\mu^{\otimes} = \mathrm{cov}(\mu^{\otimes}, \lambda^+, \lambda^+, \lambda)$. Let $\chi = \beth_7(\lambda)^+$. For $i < i^*$ choose $\mathfrak{B}_i \prec (\mathcal{H}(\chi) \in <^*_\chi)$, $\|\mathfrak{B}_i\| = \mu^{\otimes}$, $T_i \in \mathfrak{B}_i$, $\mu^{\otimes} + 1 \subseteq \mathfrak{B}_i$. Let $\langle Y_\alpha : \alpha < \mu^{\otimes} \rangle$ be a family of subsets of T_i exemplifying the Definition of $\mu^{\otimes} = \mathrm{cov}(\mu^{\otimes}, \lambda^+, \lambda^+, \lambda)$.
Given $\bar{x} = \langle x_\eta : \eta \in {}^\omega\omega \rangle$, $x_\eta \in \mathrm{lev}_\omega(T_i)$, $\eta \mapsto x_\eta$ continuous (in our case this means $\ell g(\eta_1 \cap \eta_2) = \ell g(x_{\eta_1} \cap x_{\eta_2}) =: \ell g(\max\{\rho : \rho \triangleleft \eta_1 \ \&\ \rho \triangleleft \eta_2\})$. Then for some $\eta \in {}^{\omega>}\omega$,

$$\langle x_\rho : \eta \triangleleft \rho \in {}^\omega\omega \rangle \in \mathfrak{B}.$$

So given $\langle \langle x^\zeta_\eta : \eta \in {}^\omega\omega \rangle : \zeta < \lambda \rangle$, $x^\zeta_\eta \in \mathrm{lev}_\omega(T_i)$ we can find $\langle (\alpha_j, \eta_j) : j < j^* < \lambda \rangle$ such that:

$$\bigwedge_{\zeta<\lambda} \bigvee_j \langle x^\zeta_\eta : \eta_j \triangleleft \eta \in {}^\omega\omega \rangle \in Y_\alpha.$$

Closing Y_α enough we can continue as usual. $\square_{8.1}$

9 Metric spaces and implications

Definition 9.1 1. $\mathfrak{K}^{mt}$ is the class of metric spaces M (i.e. $M = (|M|, d)$, $|M|$ is the set of elements, d is the metric, i.e. a two-place function from $|M|$ to $\mathbb{R}^{\geq 0}$ such that $d(x, y) = 0 \quad \Leftrightarrow \quad x = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$ and $d(x, y) = d(y, x)$).

An embedding f of M into N is a one-to-one function from $|M|$ into $|N|$ which is continuous, i.e. such that:

if in M, $\langle x_n : n < \omega \rangle$ converges to x
then in N, $\langle f(x_n) : n < \omega \rangle$ converges to $f(x)$.

2. $\mathfrak{K}^{ms}$ is defined similarly but $\mathrm{Rang}(d) \subseteq \{2^{-n} : n < \omega\} \cup \{0\}$ and instead of the triangular inequality we require

$$d(x, y) = 2^{-i}, \qquad d(y, z) = 2^{-j} \quad \Rightarrow \quad d(x, z) \leq 2^{-\min\{i-1, j-1\}}.$$

3. $\mathfrak{K}^{tr[\omega]}$ is like $\mathfrak{K}^{tr}$ but $P^M_\omega = |M|$ and embeddings preserve $x\ E_n\ y$ (not necessarily its negation) are one-to-one, and remember $\bigwedge_n x\ E_n\ y \quad \Rightarrow$ $x \restriction n = y \restriction n$).

4. $\mathfrak{K}^{mt(c)}$ is the class of semi-metric spaces $M = (|M|, d)$, which means that for the constant $c \in \mathbb{R}^+$ the triangular inequality is weakened to $d(x, z) \leq cd(x, y) + cd(y, z)$ with embedding as in 9.1(1) (so for $c = 1$ we get $\mathfrak{K}^{mt}$).

5. $\mathfrak{K}^{mt[c]}$ is the class of pairs (A, d) such that A is a non-empty set, d a two-place symmetric function from A to $\mathbb{R}^{\geq 0}$ such that $[d(x, y) = 0 \Leftrightarrow x = y]$ and
$$d(x_0, x_n) \leq c \sum_{\ell < n} d(x_\ell, x_{\ell+1}) \quad \text{for any } n < \omega \text{ and } x_0, \ldots, x_n \in A.$$
6. $\mathfrak{K}^{ms(c)}$, $\mathfrak{K}^{ms[c]}$ are defined parallely.
7. $\mathfrak{K}^{rs(p),\mathrm{pure}}$ is defined like $\mathfrak{K}^{rs(p)}$ but the embeddings are pure.

Remark 9.2 There are, of course, other notions of embeddings; isometric embeddings if d is preserved, co-embeddings if the image of an open set is open, bi-continuous means an embedding which is a co-embedding. The isometric embedding is the weakest, its case is essentially equivalent to the $\mathfrak{K}^{tr}_\lambda$ case (as in 9.8(3)); for the open case there is a universal: discrete space. The universal for $\mathfrak{K}^{mt}_\lambda$ under bicontinuous case exist one in cardinality $\lambda^{\aleph_0}$.

Theorem 9.3 ([Ko57]) *For every infinite cardinal κ, the product of $\aleph_0$ copies of the hedgehod $J(\kappa)$ is a universal space for metrizable spaces of weight κ, where $J(\kappa)$ is the metric space obtained by taking κ copies of the unit interval and identifying their 0-points.*

Proof see [Ko57].

Definition 9.4 1. $\mathrm{Univ}^0(\mathfrak{K}^1, \mathfrak{K}^2) = \{(\lambda, \kappa, \theta)$: there are $M_i \in \mathfrak{K}^2_\kappa$ for $i < \theta$ such that any $M \in \mathfrak{K}^1_\lambda$ can be embedded into some $M_i\}$. We may omit θ if it is 1. We may omit the superscript 0.

2. $\mathrm{Univ}^1(\mathfrak{K}^1, \mathfrak{K}^2) = \{(\lambda, \kappa, \theta)$: there are $M_i \in \mathfrak{K}^2_\kappa$ for $i < \theta$ such that any $M \in \mathfrak{K}^1_\lambda$ can be represented as the union of $< \lambda$ sets A_ζ $(\zeta < \zeta^* < \lambda)$ such that each $M \restriction A_\zeta$ can be embedded into some $M_i\}$ and is a $\leq_{\mathfrak{K}^1}$-submodel of M.

3. If above $\mathfrak{K}^1 = \mathfrak{K}^2$ we write it just once; (naturally we usually assume $\mathfrak{K}^1 \subseteq \mathfrak{K}^2$).

Remark 9.5 1. We prove our theorems for $Univ^0$, we can get parallel things for Univ^1.

2. Many previous results of this paper can be rephrased using a pair of classes.

3. We can make 9.6 below deal with pairs and/or function H changing cardinality.

4. Univ^ℓ has the obvious monotonicity properties.

Proposition 9.6 *1. Assume $\mathfrak{K}^1, \mathfrak{K}^2$ has the same models as their members and every embedding for $\mathfrak{K}^2$ is an embedding for $\mathfrak{K}^1$.*
Then $\mathrm{Univ}(\mathfrak{K}^2) \subseteq \mathrm{Univ}(\mathfrak{K}^1)$.

2. *Assume there is for $\ell = 1, 2$ a function H_ℓ from $\mathfrak{K}^\ell$ into $\mathfrak{K}^{3-\ell}$ such that:*

   (a) $\|H_1(M_1)\| = \|M_1\|$ *for* $M_1 \in \mathfrak{K}^1$,

   (b) $\|H_2(M_2)\| = \|M_2\|$ *for* $M_2 \in \mathfrak{K}^2$,

   (c) *if* $M_1 \in \mathfrak{K}^1$, $M_2 \in \mathfrak{K}^2$, $H_1(M_1) \in \mathfrak{K}^2$ *is embeddable into* M_2 *then* M_1 *is embeddable into* $H_2(M_2) \in \mathfrak{K}^1$.

   Then $\mathrm{Univ}(\mathfrak{K}^2) \subseteq \mathrm{Univ}(\mathfrak{K}^1)$.

Definition 9.7 We say $\mathfrak{K}^1 \leq \mathfrak{K}^2$ if the assumptions of 9.6(2) hold. We say $\mathfrak{K}^1 \equiv \mathfrak{K}^2$ if $\mathfrak{K}^1 \leq \mathfrak{K}^2 \leq \mathfrak{K}^1$ (so larger means with fewer cases of universality).

Theorem 9.8 *1. The relation "$\mathfrak{K}^1 \leq \mathfrak{K}^2$" is a quasi-order (i.e. transitive and reflexive).*

2. *If $(\mathfrak{K}^1, \mathfrak{K}^2)$ are as in 9.6(1) then $\mathfrak{K}^1 \leq \mathfrak{K}^2$ (use $H_1 = H_2 =$ the identity).*

3. *For $c_1 > 1$ we have $\mathfrak{K}^{mt(c_1)} \equiv \mathfrak{K}^{mt[c_1]} \equiv \mathfrak{K}^{ms[c_1]} \equiv \mathfrak{K}^{ms(c_1)]}$.*

4. $\mathfrak{K}^{tr[\omega]} \leq \mathfrak{K}^{rs(p)}$.

5. $\mathfrak{K}^{tr[\omega]} \leq \mathfrak{K}^{tr(\omega)}$.

6. $\mathfrak{K}^{tr(\omega)} \leq \mathfrak{K}^{rs(p),pure}$.

Proof 1) Check.
2) Check.
3) Choose $n(*) < \omega$ large enough and $\mathfrak{K}^1, \mathfrak{K}^2$ any two of the four. We define H_1, H_2 as follows. H_1 is the identity. For $(A, d) \in \mathfrak{K}^\ell$ let $H_\ell((A, d)) = (A, d^{[\ell]})$ where $d^{[\ell]}(x, y) = \inf\{1/(n+n(*)) : 2^{-n} \geq d(x, y)\}$ (the result is not necessarily a metric space, $n(*)$ is chosen so that the semi-metric inequality holds). The point is to check clause (c) of 9.6(2); so assume f is a function which $\mathfrak{K}^2$-embeds $H_1((A_1, d_1))$ into (A_2, d_2); but

$$H_1((A_1, d_1)) = (A_1, d_1), \quad H_2((A_2, d_2)) = (A_2, d_2^{[2]}),$$

so it is enough to check that f is a function which $\mathfrak{K}^1$-embeds $(A_1, d_1^{[1]})$ into $(A_2, d_2^{[2]})$ i.e. it is one-to-one (obvious) and preserves limit (check).
4) For $M = (A, E_n)_{n<\omega} \in \mathfrak{K}^{tr[\omega]}$, without loss of generality $A \subseteq {}^\omega\lambda$ and

$$\eta E_n \nu \qquad \Leftrightarrow \qquad \eta \in A \ \& \ \nu \in A \ \& \ \eta \restriction n = \nu \restriction n.$$

Let $B^+ = \{\eta \restriction n : \eta \in A \text{ and } n < \omega\}$. We define $H_1(M)$ as the (Abelian) group generated by

$$\{x_\eta : \eta \in A \cup B\} \cup \{y_{\eta,n} : \eta \in A, n < \omega\}$$

freely except

$$\begin{array}{rcl} p^{n+1}x_\eta = 0 & \underline{\text{if}} & \eta \in B, \ell g(\eta) = n \\ y_{\eta,0} = x_\eta & \underline{\text{if}} & \eta \in A \\ py_{\eta,n+1} - y_\eta = x_{\eta \restriction n} & \underline{\text{if}} & \eta \in A, n < \omega \\ p^{n+1}y_{\eta,n} = 0 & \underline{\text{if}} & \eta \in B, n < \omega. \end{array}$$

For $G \in \mathfrak{K}^{rs(p)}$ let $H_2(G)$ be $(A, E_n)_{n<\omega}$ with:

$$A = G, \quad xE_n y \quad \underline{\text{iff}} \quad G \models \text{“}p^n \text{ divides } (x-y)\text{”}.$$

$H_2(G) \in \mathfrak{K}^{tr[\omega]}$ as "G is separable" implies $(\forall x)(x \neq 0 \Rightarrow (\exists n)[x \notin p^n G])$. Clearly clauses (a), (b) of Definition 9.1(2) hold. As for clause (c), assume $(A, E_n)_{n<\omega} \in \mathfrak{K}^{tr[\omega]}$. As only the isomorphism type counts without loss of generality $A \subseteq {}^\omega\lambda$. Let $B = \{\eta \restriction n : n < \omega : \eta \in A\}$ and $G = H_1((A, E_n)_{n<\omega})$ be as above. Suppose that f embeds G into some $G^* \in \mathfrak{K}^{rs(p)}$, and let $(A^*, E_n^*)_{n<\omega}$ be $H_2(G^*)$. We should prove that $(A, E_n)_{n<\omega}$ is embeddable into (A^*, E_n^*).
Let $f^* : A \longrightarrow A^*$ be $f^*(\eta) = x_\eta \in A^*$. Clearly f^* is one to one from A to A^*; if $\eta E_n \nu$ then $\eta \restriction n = \nu \restriction n$ hence $G \models p^n \restriction (x_\eta - x_\nu)$ hence $(A^*, A_n^*)_{n<\omega} \models \eta E_n^* \nu$. $\square_{9.8}$

Remark 9.9 In 9.8(4) we can prove $\mathfrak{K}^{tr[\omega]}_{\bar\lambda} \leq \mathfrak{K}^{rs(p)}_{\bar\lambda}$.

Theorem 9.10 *1.* $\mathfrak{K}^{mt} \equiv \mathfrak{K}^{mt(c)}$ *for* $c \geq 1$.

2. $\mathfrak{K}^{mt} \equiv \mathfrak{K}^{ms[c]}$ *for* $c > 1$.

Proof 1) Let $H_1 : \mathfrak{K}^{mt} \longrightarrow \mathfrak{K}^{mt(c)}$ be the identity. Let $H_2 : \mathfrak{K}^{mt(c)} \longrightarrow \mathfrak{K}^{mt}$ be defined as follows:
$H_2((A, d)) = (A, d^{mt})$, where

$$d^{mt}(y, z) = \inf\Big\{\sum_{\ell=0}^{n} d(x_\ell, x_{\ell,n}) : n < \omega \ \& \ x_\ell \in A \text{ (for } \ell \leq n) \ \& \ x_0 = y \ \& \ x_n = z\Big\}.$$

Now

$(*)_1$ d^{mt} is a two-place function from A to $\mathbb{R}^{\geq 0}$, is symmetric, $d^{mt}(x,x)=0$ and it satisfies the triangular inequality.

This is true even on $\mathfrak{K}^{mt(c)}$, but here also

$(*)_2$ $d^{mt}(x,y)=0 \Leftrightarrow x=0$.

[Why? As by the Definition of $\mathfrak{K}^{mt[c]}$, $d^{mt}(x,y) \geq \frac{1}{c} d(x,y)$. Clearly clauses (a), (b) of 9.6(2) hold.]
Next,

$(*)_3$ If $M_1, N \in \mathfrak{K}^{mt}$, f is an embedding (for $\mathfrak{K}^{mt}$) of M_1 into N then f is an embedding (for $\mathfrak{K}^{mt[c]}$) of $H_1(M)$ into $H_1(N)$

[why? as $H_1(M)=M$ and $H_2(N)=N$],

$(*)_4$ If $M, N \in \mathfrak{K}^{mt[c]}$, f is an embedding (for $\mathfrak{K}^{mt[c]}$) of M into N then f is an embedding (for $\mathfrak{K}^{mt}$) of $H_2(M)$ into $H_1(M)$

[why? as H^*_ℓ preserves $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} x_n \neq x$].

So two applications of 9.6 give the equivalence.

2) We combine H_2 from the proof of (1) and the proof of 9.8(3). $\square_{9.10}$

Definition 9.11 1. If $\bigwedge\limits_n \mu_n = \aleph_0$ let

$$J^{mt} = J^{mt}_{\bar\mu} = \Big\{A \subseteq \prod_{n<\omega} \mu_n : \begin{array}{l}\text{for every } n \text{ large enough,}\\ \quad\text{for every } \eta \in \prod_{\ell<n} \mu_\ell\\ \text{the set } \{\eta'(n) : \eta \triangleleft \eta' \in A\} \text{ is finite}\Big\}.\end{array}$$

2. Let $T = \bigcup\limits_{\alpha\leq\omega} \prod\limits_{n<\alpha} \mu_n$, (T,d^*) be a metric space such that

$$\prod_{\ell<n} \mu_\ell \cap \text{closure}\left(\bigcup_{m<n} \prod_{\ell<m} \mu_\ell\right) = \emptyset;$$

now

$$I^{mt}_{(T,d^*)} =: \Big\{A \subseteq \prod_{n<\omega} \mu_n : \begin{array}{l}\text{for some } n, \text{ the closure of } A \text{ (in } (T,d^*))\\ \text{is disjoint to } \bigcup\limits_{m\in[n,\omega)} \prod\limits_{\ell<m} \mu_\ell\Big\}.\end{array}$$

3. Let $H \in \mathfrak{K}^{rs(p)}$, $\bar H = \langle H_n : n<\omega\rangle$, $H_n \subseteq H$ pure and closed, $n<m \Rightarrow H_n \subseteq H_m$ and $\bigcup\limits_{n<\omega} H_n$ is dense in H. Let

$$I^{rs(p)}_{H,\bar H} =: \Big\{A \subseteq H : \begin{array}{l}\text{for some } n \text{ the closure of } \langle A\rangle_H \text{ intersected with}\\ \bigcup\limits_{\ell<\omega} H_\ell \text{ is included in } H_n\Big\}.\end{array}$$

Proposition 9.12 *Suppose that $2^{\aleph_0} < \mu$ and $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ and*

$(*)_\lambda$ $\mathbf{U}_{J^{mt}_{\bar\mu}}(\lambda) = \lambda$ *or at least* $\mathbf{U}_{J^{mt}_{\bar\mu}}(\lambda) < \lambda^{\aleph_0}$ *for some* $\bar\mu = \langle \mu_n : n < \omega \rangle$ *such that* $\prod_{n<\omega} \mu_n < \lambda$.

Then $\mathfrak{K}^{mt}_\lambda$ has no universal member.

Proposition 9.13 *1. J^{mt} is $\aleph_1$-based.*

2. *The minimal cardinality of a set which is not in the σ-ideal generated by J^{mt} is $\mathfrak{b}$.*
3. *$I^{mt}_{(T,d^*)}$, $I^{rs(p)}_{H,\bar H}$ are $\aleph_1$-based.*
4. *J^{mt} is a particular case of $I^{mt}_{(T,d^*)}$ (i.e. for some choice of (T,d^*)).*
5. *$I^0_{\bar\mu}$ is a particular case of $I^{rs(p)}_{H,\bar H}$.*

Proof of 9.12. Let

$$T_\alpha = \{(\eta,\nu) \in {}^\alpha\lambda \times {}^\alpha(\omega+1) : \quad \text{for every } n \text{ such that } n+1 < \alpha \text{ we have } \nu(n) < \omega\}$$

and for $\alpha \le \omega$ let $T = \bigcup_{\alpha \le \omega} T_\alpha$. We define on T the relation $<_T$:

$$(\eta_1,\nu_1) \le (\eta_1,\nu_2) \quad \text{iff} \quad \eta_1 \trianglelefteq \eta_2 \ \&\ \nu_1 \triangleleft \nu_2.$$

We define a metric:
if $(\eta_1,\nu_1) \ne (\eta_2,\nu_2) \in T$ and (η,ν) is their maximal common initial segment and $(\eta,\nu) \in T$ then necessarily $\alpha = \ell g((\eta,\nu)) < \omega$ and we let:

if $\eta_1(\alpha) \ne \eta_2(\alpha)$ then

$$d((\eta_1,\nu_1),(\eta_2,\nu_2)) = 2^{-\sum\{\nu(\ell):\ell<\alpha\}},$$

if $\eta_1(\alpha) = \eta_2(\alpha)$ (so $\nu_1(\alpha) \ne \nu_2(\alpha)$ then

$$d((\eta_1,\nu_1),(\eta_2,\nu_2)) = 2^{-\sum\{\nu(\ell):\ell<\alpha\}} \times 2^{-\min\{\nu_1(\alpha),\nu_2(\alpha)\}}.$$

Now, for every $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$, and $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$, $\eta_\delta \in {}^\omega\delta$, η_δ increasing let M_η be $(T,d) \restriction A_{\bar\eta}$, where

$$A_{\bar\eta} = \bigcup_{n<\omega} T_n \cup \{(\eta_\delta,\nu) : \delta \in S,\ \nu \in {}^\omega\omega\}.$$

The rest is as in previous cases (note that $\langle(\eta\hat{}\langle\alpha\rangle, \nu\hat{}\langle n\rangle) : n < \omega\rangle$ converges to $(\eta\hat{}\langle\alpha\rangle, \nu\hat{}\langle\omega\rangle)$ and even if $(\eta\hat{}\langle\alpha\rangle, \nu\hat{}\langle n\rangle) \le (\eta_n,\nu_n) \in T_\omega$ then $\langle(\eta_n,\nu_n) : n < \omega\rangle$ converge to $(\eta\hat{}\langle\alpha\rangle, \nu\hat{}\langle\omega\rangle)$). $\square_{9.13}$

Proposition 9.14 *If* $\mathrm{IND}_{\chi'}(\langle \mu_n : n < \omega \rangle)$, *then* $\prod\limits_{n<\omega} \mu_n$ *is not the union of* $\leq \chi$ *members of* $I^0_{\bar{\mu}}$ *(see Definition 5.5 and Theorem 5.7).*

Proof Suppose that $A_\zeta = \{ \sum\limits_{n<\omega} p^n x^n_{\alpha_n} : \langle \alpha_n : n < \omega \rangle \in X_\zeta \}$ and $\alpha_n < \mu_n$ are such that if $\sum p^n x^n_{\alpha_n} \in A_\zeta$ then for infinitely many n for every $k < \omega$ there is $\langle \beta_n : n < \omega \rangle$,

$$(\forall \ell < k)[\alpha_\ell = \beta_\ell \quad \Leftrightarrow \quad \ell = n] \qquad \text{and} \qquad \sum_{n<\omega} p^n x^n_{\beta_n} \in A_\zeta \quad \text{(see §5)}.$$

This clearly follows. $\square_{9.14}$

10 On Modules

Here we present the straight generalization of the one prime case like Abelian reduced separable p-groups. This will be expanded in [Sh 622] (including the proof of 10.4).

Hypothesis 10.1 (A) R is a ring, $\bar{\mathfrak{e}} = \langle \mathfrak{e}_n : n < \omega \rangle$, $\mathfrak{e}_n$ is a definition of an additive subgroup of R-modules by an existential positive formula (finitary or infinitary) decreasing with n, we write $\mathfrak{e}_n(M)$ for this additive subgroup, $\mathfrak{e}_\omega(M) = \bigcap\limits_n \mathfrak{e}_n(M)$. Let $M_a \leq_{rp} M_2$ of $M_1 \subseteq M_2$ and $\mathfrak{e}_w(M_1) = \mathfrak{e}_w(M_2) \cap M_1$, let $M_1 \leq_{pr} M_2$ if $M_1 \subseteq M_2$ and $n < w \Rightarrow \mathfrak{e}_n(M_2) \cap M_1 = \mathfrak{e}_n(M_1)$.

(B) $\mathfrak{K}$ is the class of R-modules.

(C) $\mathfrak{K}^* \subseteq \mathfrak{K}$ is a class of R-modules, which is closed under direct summand, direct limit for $\leq_{pr}$-increasing chains and for which there is M^*, $x^* \in M^*$, $M^* = \bigoplus\limits_{\ell \leq n} M^*_\ell \oplus M^{**}_n$, $M^*_n \in \mathfrak{K}$, $x^*_n \in \mathfrak{e}_n(M^*_n) \setminus \mathfrak{e}_{n+1}(M^*)$, $x^* - \sum\limits_{\ell<n} x^*_\ell \in \mathfrak{e}_n(M^*)$.

Definition 10.2 For $M_1, M_2 \in \mathfrak{K}$, we say h is a $(\mathfrak{K}, \bar{\mathfrak{e}})$-homomorphism from M_1 to M_2 if it is a homomorphism and it maps $M_1 \setminus \mathfrak{e}_\omega(M_1)$ into $M_2 \setminus \mathfrak{e}_\omega(M_2)$; we say h is an $\bar{\mathfrak{e}}$-pure homomorphism if for each n it maps $M_1 \setminus \mathfrak{e}_n(M_1)$ into $M_2 \setminus \mathfrak{e}_n(M_2)$.

Definition 10.3 1. Let $H_n \subseteq H_{n+1} \subseteq H$, $\bar{H} = \langle H_n : n < \omega \rangle$, $c\ell$ is a closure operation on H, $c\ell$ is a function from $\mathcal{P}(H)$ to itself and

$$X \subseteq c\ell(X) = c\ell(c\ell(X)).$$

Define

$$I_{H,\bar{H},c\ell} = \{A \subseteq H : \text{for some } k < \omega \text{ we have } c\ell(A) \cap \bigcup_{n<\omega} H_n \subseteq H_k\}.$$

2. We can replace ω by any regular κ (so $H = \langle H_i : i < \kappa \rangle$).

Claim 10.4 *Assume $|R| + \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$, then for every $M \in \mathfrak{K}_\lambda$ there is $N \in \mathfrak{K}_\lambda$ with no $\bar{\mathfrak{e}}$-pure homomorphism from N into M.*

Remark 10.5 In the interesting cases $c\ell$ has infinitary character.
The applications here are for $\kappa = \omega$. For the theory, pcf is nicer for higher κ.

11 Open problems

Problem 11.1 1. If $\mu^{\aleph_0} \geq \lambda$ then any $(A, d) \in \mathfrak{K}^{mt}_\lambda$ can be embedded into some $M' \in \mathfrak{K}^{mt}_\lambda$ with density $\leq \mu$.

2. If $\mu^{\aleph_0} \geq \lambda$ then any $(A, d) \in \mathfrak{K}^{ms}_\lambda$ can be embedded into some $M' \in \mathfrak{K}^{ms}_\lambda$ with density $\leq \mu$.

Problem 11.2 1. Other inclusions on $\mathrm{Univ}(\mathfrak{K}^x)$ or show consistency of non inclusions (see §9).

2. Is $\mathfrak{K}^1 \leq \mathfrak{K}^2$ the right partial order? (see §9).

3. By forcing reduce consistency of $\mathbf{U}_{J_1}(\lambda) > \lambda + 2^{\aleph_0}$ to that of $\mathbf{U}_{J_2}(\lambda) > \lambda + 2^{\aleph_0}$.

Problem 11.3 1. The cases with the weak pcf assumptions, can they be resolved in ZFC? (the pcf problems are another matter).

2. Use [Sh 460], [Sh 513] to get ZFC results for large enough cardinals.

Problem 11.4 If $\lambda_n^{\aleph_0} < \lambda_{n+1}$, $\mu = \sum_{n<\omega} \lambda_n$, $\lambda = \mu^+ < \mu^{\aleph_0}$ can $(\lambda, \lambda, 1)$ belong to $\mathrm{Univ}(\mathfrak{K})$? For $\mathfrak{K} = \mathfrak{K}^{tr}, \mathfrak{K}^{rs(p)}, \mathfrak{K}^{trf}$?

Problem 11.5 1. If $\lambda = \mu^+$, $2^{<\mu} = \lambda < 2^\mu$ can $(\lambda, \lambda, 1) \in \mathrm{Univ}(\mathfrak{K}^{\mathrm{or}} =$ class of linear orders)?

2. Similarly for $\lambda = \mu^+$, μ singular, strong limit, $\mathrm{cf}(\mu) = \aleph_0$, $\lambda < \mu^{\aleph_0}$.

3. Similarly for $\lambda = \mu^+$, $\mu = 2^{<\mu} = \lambda^+ < 2^\mu$.

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Problem 11.6 1. Analyze the existence of universal member from $\mathfrak{K}_\lambda^{rs(p)}$, $\lambda < 2^{\aleph_0}$.

2. §4 for many cardinals, i.e. is it consistent that: $2^{\aleph_0} > \aleph_\omega$ and for every $\lambda < 2^{\aleph_0}$ there is a universal member of $\mathfrak{K}_\lambda^{rs(p)}$?

Problem 11.7 1. If there are $A_i \subseteq \mu$ for $i < 2^{\aleph_0}$, $|A_i \cap A_j| < \aleph_0$, $2^\mu = 2^{\aleph_0}$ find forcing adding $S \subseteq [{}^\omega\omega]^\mu$ universal for $\{(B, \triangleleft) : {}^{\omega>}\omega \subseteq B \subseteq {}^{\omega\geq}\omega, |B| \leq \lambda\}$ under (level preserving) natural embedding.

Problem 11.8 For simple countable T, $\kappa = \kappa^{<\kappa} < \lambda \subseteq \kappa$ force existence of universal for T in λ still $\kappa = \kappa^{<\kappa}$ but $2^\kappa = \chi$.

Problem 11.9 Make [Sh 457, §4], [Sh 500, §1] work for a larger class of theories more than simple.

See on some of these problems [DjSh 614], [Sh 622].

References

[B] J. Baumgartner. Decomposition of embedding of trees. *Notices Amer. Math. Soc.*, **17**:967, 1970.

[DjSh 614] Mirna Džamonja and Saharon Shelah. Universal Banach Spaces. *in preparation.*

[Fu] L. Fuchs. *Infinite Abelian Groups*, volume I, II. Academic Press, New York, 1970, 1973.

[KjSh 409] Menachem Kojman and Saharon Shelah. Non-existence of Universal Orders in Many Cardinals. *Journal of Symbolic Logic*, **57**:875–891, 1992.

[KjSh 447] Menachem Kojman and Saharon Shelah. The universality spectrum of stable unsuperstable theories. *Annals of Pure and Applied Logic*, **58**:57–72, 1992.

[KjSh 455] Menachem Kojman and Saharon Shelah. Universal Abelian Groups. *Israel Journal of Mathematics*, **92**:113–124, 1995.

[Ko57] H.J. Kowalsky. Einbettung metrischer Räume. *Arch. für Math.*, **8**:336–339, 1957.

[Sh:e] Saharon Shelah. *Non-structure theory*, accepted. Oxford University Press.

[Sh 622] Saharon Shelah. Non-existence of universal members in classes of Abelian groups. *in preparation.*

[Sh 513] Saharon Shelah. PCF and infinite free subsets. *Archive for Mathematical Logic*, submitted.

[Sh 460] Saharon Shelah. The Generalized Continuum Hypothesis revisited. *Israel Journal of Mathematics*, submitted.

[Sh 500] Saharon Shelah. Toward classifying unstable theories. *Annals of Pure and Applied Logic*, accepted.

[Sh 351] Saharon Shelah. Reflecting stationary sets and successors of singular cardinals. *Archive for Mathematical Logic*, **31**:25–53, 1991.

[Sh 326] Saharon Shelah. Viva la difference I: Nonisomorphism of ultrapowers of countable models. In *Set Theory of the Continuum*, volume 26 of *Mathematical Sciences Research Institute Publications*, pages 357–405. Springer Verlag, 1992.

[Sh 420] Saharon Shelah. Advances in Cardinal Arithmetic. In *Finite and Infinite Combinatorics in Sets and Logic*, pages 355–383. Kluwer Academic Publishers, 1993. N.W. Sauer et al (eds.).

[Sh 410] Saharon Shelah. More on Cardinal Arithmetic. *Archive for Mathematical Logic*, **32**:399–428, 1993.

[Sh 457] Saharon Shelah. The Universality Spectrum: Consistency for more classes. In *Combinatorics, Paul Erdős is Eighty*, volume 1, pages 403–420. Bolyai Society Mathematical Studies, 1993. Proceedings of the Meeting in honour of P.Erdős, Keszthely, Hungary 7.1993; A corrected version available as ftp://ftp.math.ufl.edu/pub/settheory/shelah/457.tex.

[Sh 355] Saharon Shelah. $\aleph_{\omega+1}$ has a Jonsson Algebra. In *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*, chapter II. Oxford University Press, 1994.

[Sh:g] Saharon Shelah. *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, 1994.

[Sh 430] Saharon Shelah. Further cardinal arithmetic. *Israel Journal of Mathematics*, **95**:61–114, 1996.

[Sh 456] Saharon Shelah. Universal in $(< \lambda)$-stable abelian group. *Mathematica Japonica*, **43**:to appear, 1996.