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Rigid $\aleph_\varepsilon$-saturated models of superstable theories

by

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Abstract. In a countable superstable NDOP theory, the existence of a rigid $\aleph_\varepsilon$-saturated model implies the existence of 2^λ rigid $\aleph_\varepsilon$-saturated models of power λ for every $\lambda > 2^{\aleph_0}$.

1. Introduction. Ehrenfeucht conjectured that given a theory T, the class of cardinals for which T has a rigid model is quite well behaved. Shelah refuted Ehrenfeucht's conjecture showing that this class can be quite complicated.

In this paper we deal with problems related to this question in the context of stability. More specifically, we will study the existence of stable rigid models which satisfy an additional saturation property (note that if no saturation property is required, then a very simple example of a stable rigid model can be found, namely the model whose language is $\{P_n \mid n < \omega\}$ and consists of the disjoint union of the P_n's, each of which has exactly one element.)

We will give a partial solution to the following questions:

1. What classes of superstable theories have a rigid $\aleph_\varepsilon$-saturated model?
2. Assuming that there exists a rigid $\aleph_\varepsilon$-saturated model, what can be said about the number of $\aleph_\varepsilon$-saturated models, or perhaps even about the number of rigid $\aleph_\varepsilon$-saturated models, in large enough cardinality?

In Section 3 we consider two properties of a superstable theory T:

(1) T is strongly deep.

(2) T does not admit a nontrivial nonorthogonal automorphism of some saturated model.

We prove that (1) is a necessary condition for the existence of a rigid $\aleph_\varepsilon$-saturated model, and that (2) is a sufficient condition.

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In Section 4, we assume that T is a superstable NDOP theory. We prove that (2) is actually equivalent to the existence of a rigid $\aleph_\varepsilon$-saturated model. We can then conclude that the existence of a single rigid $\aleph_\varepsilon$-saturated model implies that T has 2^λ such models for every sufficiently large λ.

In this paper, the notations will be very similar to Shelah's notations in [Sh-C]: T will denote any complete stable theory with no finite models in some language L, and κ, λ, μ will denote cardinals.

We work in some huge saturated model $\mathcal{M}$. Sets $A, B, C, \ldots$ will be subsets of $\mathcal{M}$, with cardinality strictly less than the cardinality of $\mathcal{M}$; $\overline{a}, \overline{b}, \overline{c}, \ldots$ will denote finite tuples in $\mathcal{M}$; M, N will always be elementary submodels of $\mathcal{M}$; p, q, r will denote types, usually complete over some set A; $S(A)$ will denote the set of complete types over A. Also, for a tuple $\overline{b}$ and a set A, $\overline{b}/A$ denotes the type of $\overline{b}$ over A.

2. Building a dimensionally diverse $\aleph_\varepsilon$-saturated model. In this section, T denotes any stable theory. We give a brief outline of some standard constructions following [Sh-C]. Here $\kappa(T)$ will denote the smallest infinite cardinal κ such that there is no chain $\{p_\alpha \in S(A_\alpha) \mid \alpha < \kappa\}$ such that for all $\alpha < \beta < \kappa$, p_β is a forking extension of p_α. Recall that M is F^a_μ-*saturated* if for every $A \subseteq M$ such that $|A| < \mu$, every strong type over A is realized in M. M is F^a_μ-*prime over* A if M is F^a_μ-saturated and for every F^a_μ-saturated model N such that $A \subseteq N$ there is an elementary embedding of M into N over A. We say that M is $\aleph_\varepsilon$-*saturated* (resp. $\aleph_\varepsilon$-*prime over* A) if M is F^a_ω-saturated (resp. F^a_ω-prime over A).

Definition 1. We say that an $\aleph_\varepsilon$-saturated model M (of a superstable T) is *dimensionally diverse* if for any two stationary regular types p, q over finite subsets of M, $p \perp q$ if and only if $\dim(p, M) \neq \dim(q, M)$.

We also recall the following standard definition:

Definition 2. For superstable T, we say that T is *multidimensional* if for every cardinal α, there are nonalgebraic p_i, $i < \alpha$, which are pairwise orthogonal.

Our aim in this section is to prove the following standard theorem.

Theorem 1. *Let T be superstable, and let μ, δ be cardinals such that $\aleph_\delta = \delta$, $\mu < \delta$ and $2^{|T|} < \delta$. Then*:

(1) *There exists a μ-saturated model M of cardinality δ which is dimensionally diverse* (*in particular if $\mu \geq \aleph_1$, then M is $\aleph_\varepsilon$-saturated*).

(2) *If T is multidimensional*, *then for every increasing sequence of cardinals $\overline{\mu} = \langle \mu_\alpha \mid \alpha < \delta \rangle$ with $\mu_\alpha \in (\mu, \delta)$ there exists a μ-saturated and $\aleph_\varepsilon$-saturated model M of cardinality δ which is dimensionally diverse*, *and such that for every $\alpha < \delta$ there is a stationary and regular* (= *s.r.*) *type p_α*

(over a finite subset of M) such that for every s.r. type $\overline{p}$, $\dim(\overline{p}, M) = \mu_\alpha$ if and only if $\overline{p} \not\perp p_\alpha$. Moreover, for every s.r. type $\overline{p}$ (over a finite subset of M) $\dim(\overline{p}, M) = \mu_\beta$ for some $\beta < \delta$.

FACT 1. *Let $M = \bigcup_{i<\alpha} M_i$, where $\langle M_i \mid i < \alpha \rangle$ is an increasing and continuous sequence of κ-saturated models, let $A \subseteq M_0$ be such that $|A| < \kappa$, and let $p \in S(A)$ be a stationary regular type. Then* $\dim(p, M) = \dim(p, M_0) + \sum_{i<\alpha} \dim(p_i, M_{i+1})$, *where p_i is the stationarization of p to M_i.*

FACT 2. *If $\mathrm{cf}(\lambda) \geq \kappa(T)$, M is F^a_λ-prime over A, and $I \subseteq M$ is an indiscernible sequence over A, then $|I| \leq \lambda$.*

Following are two claims which we will use in our proof of Theorem 1 (although weaker versions thereof would suffice).

CLAIM 1. *Suppose $\mathrm{cf}(\mu) \geq \kappa = \kappa(T)$ and $M \models T$ is F^a_κ-saturated. Let $A \subseteq \mathcal{M}$ be such that $|A| \leq \mu$, and suppose M^+ is F^a_μ-prime over $M \cup A$. In addition, let $B \subseteq M^+$ be such that $|B| < \kappa$, and suppose $p \in S(B)$ is stationary and $\lambda = \dim(p, M^+) > \mu$. Then $p \not\perp M$. Moreover, if p is regular then there is a stationary regular type $q \in S(B^*)$, where $B^* \subseteq M$ and $|B^*| < \kappa$, such that $\dim(q, M) = \lambda$ and $q \not\perp p$.*

Proof. Left to the reader.

CLAIM 2. *Suppose $\mathrm{cf}(\mu) \geq \kappa = \kappa(T)$ and $M \models T$ is F^a_κ-saturated. Let $p_i \in S(M)$, $i < \alpha$, be pairwise orthogonal and let $E = \bigcup_{i<\alpha} E_i$ where E_i is a Morley sequence of p_i. In addition, suppose N is F^a_μ-prime over $M \cup E$, and let $B \subseteq N$ be such that $|B| < \kappa$. If $q \in S(B)$ is stationary and regular and $\dim(q, N) > \mu$, then $q \not\perp M$.*

Proof. Assume, for contradiction, that $q \perp M$. Let M^+ be F^a_μ-prime over $M \cup B$ with $M^+ \prec N$, and let $\widetilde{q} \in S(M^+)$ be the stationarization of q. Then either $\dim(q, M^+) > \mu$ or $\dim(\widetilde{q}, N) > \mu$, by Fact 1. Now, there exists $S \subseteq \alpha$ with $|S| < \kappa$ such that $p_i \perp \mathrm{tp}(M^+/M)$ for all $i \in \alpha \setminus S$. Thus $\dim(q, M^+) > \mu$ contradicts Claim 1, and $\dim(\widetilde{q}, N) > \mu$ contradicts the above and Fact 2.

FACT 3. *If $\mathrm{cf}(\delta) \geq \kappa(T)$, and $\langle M_i \mid i < \delta \rangle$ is an elementary chain of λ-saturated models, then $M^* = \bigcup_{i<\delta} M_i$ is λ-saturated. In particular, if T is superstable, then the union of any elementary chain of λ-saturated models is λ-saturated.*

Theorem 1 easily follows from the above:

Proof of Theorem 1 (*sketch*). Define by induction an increasing elementary chain $\langle M_\alpha \mid \alpha < \delta \rangle$ of μ-saturated models. At the αth step, choose a nonalgebraic stationary type p_α (over a finite set) which is orthogonal to p_β for all $\beta < \alpha$, and define M_α to be μ-prime over $\bigcup_{\beta<\alpha} M_\beta \cup \mathrm{Dom}(p_\alpha) \cup I_\alpha$, where I_α is a Morley sequence of p_α with cardinality μ_α. Now let $M^* = \bigcup_{\alpha<\delta} M_\alpha$. From Fact 1, Claim 1 and Claim 2 it follows that M^* realizes the desired dimensions (μ_α), and by Fact 3, M^* is μ-saturated.

3. The existence of a rigid $\aleph_\varepsilon$-saturated model. In this section T is assumed to be superstable. We say that a superstable theory is *strongly deep* if the depth of every type is positive (if and only if the depth of every type is infinity). We prove that whenever T has a rigid $\aleph_\varepsilon$-saturated model, T is strongly deep. We also introduce the notion of a *nontrivial nonorthogonal automorphism* and prove that if some saturated model does not have such an automorphism, then in arbitrarily large cardinality, T has a maximal number of rigid $\aleph_\varepsilon$-saturated models (i.e. 2^λ such models in cardinality λ).

DEFINITION 3. We say that T is *strongly deep* if for every $\aleph_\varepsilon$-saturated model M, and for every (without loss of generality regular) type $p \in S(M)$, if N is $\aleph_\varepsilon$-prime over $M \cup \{\overline{a}\}$ where $\overline{a} \models p$, then $q \perp M$ for some $q \in S(N)$.

LEMMA 1. *Let N_0 be $\aleph_\varepsilon$-saturated, let $p \in S(N_0)$ be regular, and let $\langle \overline{e}_i \mid i < \alpha \rangle$ be a Morley sequence of p. Suppose N^+ is $\aleph_\varepsilon$-prime over $N_0 \cup \bigcup_{i<\alpha} \overline{e}_i$. Then the following are equivalent*:

(i) *There is $p_1 \in S(N_1)$ such that $p_1 \perp N_0$, where N_1 is $\aleph_\varepsilon$-prime over $N_0 \cup \overline{e}_0$.*

(ii) *There is $p^+ \in S(N^+)$ such that $p^+ \perp N_0$.*

Proof. (i)$\Rightarrow$(ii). We may assume that $N_1 \prec N^+$. If there is such a p_1, choose p^+ to be the nonforking extension of p_1 to N^+.

(ii)$\Rightarrow$(i). Assume that such a p^+ is given; then p^+ is strongly based on some finite subset A of N^+. Therefore $\mathrm{tp}(A/N_0 \cup \bigcup_{i<\alpha} \overline{e}_i)$ is $F^a_{\aleph_0}$-isolated, and we may assume $\alpha = n < \omega$. Now, if every $p \in S(N_1)$ is nonorthogonal to N_0, then by induction on n we have $p^+ \not\perp N_0$ (recall that the depth of parallel types is the same), which is a contradiction.

THEOREM 2. *If T has an $\aleph_\varepsilon$-saturated model N such that $|\mathrm{Aut}(N)| < 2^{\aleph_0}$, then T is strongly deep.*

Proof. Suppose not; then there is some depth 0 type p. Let $N_0 \prec N$ be $\aleph_\varepsilon$-prime over $\emptyset$ (without loss of generality $p \in S(N_0)$). Let $I \subseteq N$ be a maximal Morley sequence of p (without loss of generality I is infinite) and let M be a maximal model such that $N_0 \prec M \prec N$ and $M \underset{N_0}{\cup} I$ (so M is $\aleph_\varepsilon$-saturated). We claim that N is $\aleph_\varepsilon$-minimal over $M \cup I$. Otherwise, let $P \prec N$ be $\aleph_\varepsilon$-prime over $M \cup I$, so for some $\overline{b} \in N$, $\overline{b}/P$ is a nonalgebraic regular type.

CLAIM 1. *$\overline{b}/P \perp N_0$.*

Indeed, if not then (by [Sh-C]) there is $\bar{c} \in N$ such that $\bar{c}/P$ is regular, $\bar{c}/P \not\perp \bar{b}/P$ and $\bar{c} \underset{N_0}{\cup} P$. Thus $\{\bar{c}, M, I\}$ is independent over N_0, contrary to the definition of M.

Claim 2. *For all* $\bar{d} \in N$ *if* $\bar{d}/M \perp N_0$ *then* $\bar{d} \in M$.

Indeed, let $\bar{d}$ satisfy the above, so $\bar{d} \underset{M}{\cup} I$ and $M \underset{N_0}{\cup} I$. Therefore $M \cup \bar{d} \underset{N_0}{\cup} I$, so $\bar{d} \in M$ by the maximality of M.

Claim 3. $\bar{b}/P \perp M$.

Otherwise $\bar{b}/P \not\perp M$, so there is $\bar{c} \in N$ with $\bar{c}/P$ regular such that $\bar{c} \underset{M}{\cup} P$ and $\bar{c}/P \not\perp \bar{b}/P$. Hence by Claim 1, $\bar{c}/P \perp N_0$, so $\bar{c}/M \perp N_0$. But then applying Claim 2, we get $\bar{c} \in M$, which is a contradiction.

Now, continuing the proof of the theorem, we recall that $\mathrm{tp}(I/M)$ does not fork over N_0, so Claim 3 and Lemma 1 imply that for all $\bar{e} \in I$, $\mathrm{Depth}(\bar{e}/M) > 0$. But $\mathrm{Depth}(\bar{e}/M) = \mathrm{Depth}(\bar{e}/N_0) = 0$, a contradiction. So, we have shown that N is $\aleph_\varepsilon$-minimal over $M \cup I$, and therefore $\aleph_\varepsilon$-prime over $M \cup I$. By the uniqueness of $\aleph_\varepsilon$-prime models and the fact that $M \underset{N_0}{\cup} I$, we conclude that every permutation of I induces an automorphism of N, thus $|\mathrm{Aut}(N)| \geq 2^{\aleph_0}$, which is a contradiction.

Conclusion 1. *If* T *has an* $\aleph_\varepsilon$*-saturated model* N *such that* $|\mathrm{Aut}(N)| < 2^{\aleph_0}$, *then* T *is multidimensional* (*because even* $\mathrm{Depth}(T) > 0$ *implies that* T *is multidimensional*).

Definition 4. We say that $\sigma \in \mathrm{Aut}(M)$ (where M is $\aleph_\varepsilon$-saturated) is a *nontrivial nonorthogonal automorphism* (= n.n.a.) if for any nonalgebraic $p \in S(M)$, $p \not\perp \sigma(p)$, and $\sigma \neq \mathrm{id}$.

Remark 1. $\sigma \in \mathrm{Aut}(M)$ (with M as above) is a n.n.a. if and only if its unique extension σ^{eq} to M^{eq} is a n.n.a.

Theorem 3. *If* $\delta = \aleph_\delta > \beta \geq 2^{|T|}$ *and the saturated model of cardinality* β *does not have a nontrivial nonorthogonal automorphism*, *then* T *has* 2^δ *rigid* β*-saturated models of cardinality* δ.

Proof. It is enough to show that every dimensionally diverse β-saturated model of cardinality δ is rigid. This is indeed enough, since using Theorem 1 we may then take a dimensionally diverse β-saturated model N of cardinality δ. So N is rigid and by Conclusion 1, T is multidimensional. Then, by Theorem 1, for every subset D of (β, δ) which consists of cardinals, we may choose a β-saturated model which realizes exactly the dimensions in D (in the sense of Theorem 1(2)), from which the theorem follows.

So let M be a dimensionally diverse β-saturated model of cardinality δ, and let $\sigma \in \mathrm{Aut}(M)$. Assume by way of contradiction that $\sigma \neq \mathrm{id}$. We define by induction an increasing chain of elementary submodels of M: Let M_0 be a saturated model of cardinality β such that $\sigma|M_0 \neq \mathrm{id}$. For all $n < \omega$ let M_{i+1} be F^a_β-prime over $\bigcup_{i\in\mathbb{Z}} \sigma^i(M_n)$. Now, according to Fact 3, M_n is a saturated model of cardinality β for all $n < \omega$. Hence if $M_\omega = \bigcup_{i<\omega} M_i$, then M_ω is also a saturated model of cardinality β. Clearly, $\sigma|M_\omega$ is a nontrivial automorphism of M_ω, so by the assumption of the theorem, there is a regular type $p \in S(M_\omega)$ such that $p \perp \sigma(p)$. Hence for some finite $A \subset M_\omega$, p is strongly based on A, so $\dim(p|A, M) = \dim(\sigma(p|A), M)$, which contradicts the fact that M is dimensionally diverse.

4. A characterization for superstable NDOP countable theories. In this section T is assumed to be a superstable NDOP countable theory. We will use [Sh-401] to get the following characterization: such a theory has a rigid $\aleph_\varepsilon$-saturated model if and only if no $\aleph_\varepsilon$-saturated model has a nontrivial nonorthogonal automorphism. Then we will use this characterization to show that the existence of a single rigid $\aleph_\varepsilon$-saturated model implies the existence of a maximal number of such models in every sufficiently large cardinality.

We work in $\mathcal{M}^{\mathrm{eq}}$.

Definition 5. We say that A is *almost finite* if A is contained in the algebraic closure of some finite set.

4.1. *The $L_{\infty,\aleph_\varepsilon}(d.q)$-characterization theorem of* [Sh-401]. Let M_0, M_1 be $\aleph_\varepsilon$-saturated. We say that they are *$L_{\infty,\aleph_\varepsilon}(d.q)$-equivalent* if there is a family $\mathcal{F}$ which satisfies the following:

(1) Each $f \in \mathcal{F}$ is an elementary partial map from M_0 to M_1 such that $\mathrm{Dom}(f)$ is almost finite.

(2) $\mathcal{F}$ is closed under restriction.

(3) For every $f \in \mathcal{F}$, and every $\overline{a}_l \in M_l$ ($\overline{a}_l$ are finite sequences, $l = 0, 1$) there exists $g \in \mathcal{F}$ such that $f \subseteq g$, $\mathrm{acl}(\overline{a}_0) \subseteq \mathrm{Dom}(g)$, and $\mathrm{acl}(\overline{a}_1) \subseteq \mathrm{Rang}(g)$.

(4) Whenever $\mathrm{tp}(e/A)$ (where A is almost finite) is stationary and regular, then for some almost finite $A^* \supseteq A$, if $f \cup \{\langle e_0, e_1\rangle\} \in \mathcal{F}$ and $\mathrm{tp}(e_0/\mathrm{Dom}(f))$ is conjugate to the stationarization of $\mathrm{tp}(e/A)$ to A^*, then

$$\dim(p, \mathrm{Dom}(f), I_0) = \dim(f(p), \mathrm{Rang}(f), I_1),$$

where p denotes $e_0/\mathrm{Dom}(f)$, $I_0 = \{e \in M_0 \mid f \cup \{\langle e, e_1\rangle\} \in \mathcal{F}\}$, and $I_1 = \{e \in M_1 \mid f \cup \{\langle e_0, e\rangle\} \in \mathcal{F}\}$.

Condition (4) is the main assumption; roughly speaking, it implies that whenever $\mathcal{F}$ sends a stationary and regular type p to a type q, then the full structure of dimensions above p will be the same as the full structure of dimensions above q.

We can now state the Characterization Theorem for $\aleph_\varepsilon$-saturated models M_0, M_1:

M_0, M_1 are isomorphic if and only if they are $L_{\infty,\aleph_\varepsilon}(d.q)$-equivalent.

THEOREM 4. *If T has a nontrivial nonorthogonal automorphism of some $\aleph_\varepsilon$-saturated model, then no $\aleph_\varepsilon$-saturated model of T is rigid.*

Proof. By Remark 1, without loss of generality $T = T^{\rm eq}$. Suppose N_0 is an $\aleph_\varepsilon$-saturated model which has a n.n.a. σ_0, and let $a_0 \neq a_1$ be in N_0 such that $\sigma_0(a_0) = a_1$. Let N be some $\aleph_\varepsilon$-saturated model (of T); we will show that N is not rigid. Choose $b_0, b_1 \in N$ such that $\mathrm{tp}(b_0, b_1) = \mathrm{tp}(a_0, a_1)$.

Let $\mathcal{F}$ be the family of all partial elementary maps f from (N, b_0) to (N, b_1) with an almost finite domain, such that for some partial elementary map τ, with almost finite domain, $(\sigma_0 \mid A_0)\tau = \tau\sigma$, where $A_0 = \mathrm{Rang}(\tau)$. By [Sh-401] it is sufficient to show that $\mathcal{F}$ satisfies (1)–(4) above.

(1) and (2) are immediate. (3) is also immediate by the fact that N is $\aleph_\varepsilon$-saturated. To show (4), we will prove the following claim: if $\mathrm{tp}(e/A)$ is stationary and regular, where A is almost finite, then it can be replaced by a nonforking extension of it, denoted again by $\mathrm{tp}(e/A)$ (where A is almost finite and $A \cup \{e\} \subseteq N$), such that if $\sigma_i(A) = A'$ and $\sigma_i(e) = e_i$ for $\sigma_i \in \mathcal{F}$ $(i = 0, 1)$ then $e_0 \mathop{\not\perp}\limits_{A'} e_1$. (4) will follow from this, as the claim implies that for all $f \in \mathcal{F}$ with $\mathrm{Dom}(f) = A$, if $e_0^* = e$, $e_1^* = f(e)$, $M_0 = M_1 = N$ and I_0, I_1 are defined as in (4) of Subsection 4.1 (with e_i^* instead of e_i,) then $\dim(\mathrm{tp}(e_0^*/A), A, I_0) = \dim(\mathrm{tp}(e_1^*/f(A)), f(A), I_1) = 1$.

Proceeding to prove this claim, we note that by Theorem 2, T is without loss of generality strongly deep. Let M_0 be an $\aleph_\varepsilon$-prime model over $\emptyset$ such that $\mathrm{tp}(e/M_0)$ is a nonforking extension of $\mathrm{tp}(e/A)$, let M_0^+ be $\aleph_\varepsilon$-prime over $M_0 \cup \{e\}$ and let e^+ be such that $\mathrm{tp}(e^+/M_0^+) \perp M_0$, with $\mathrm{tp}(e^+/M_0^+)$ nonalgebraic and regular. Now let $B \subseteq M_0^+$ be finite such that $\mathrm{tp}(e^+/M_0^+)$ is strongly based on B, so there is some finite $C \subseteq M_0$ with $\mathrm{tp}(B/\mathrm{acl}(C \cup \{e\})) \vdash \mathrm{tp}(B/M_0 \cup \{e\})$. We may assume without loss of generality that $M_0^+ \subseteq N$ (because the $\aleph_\varepsilon$-prime model over the union of M_0 and a countable set is also $\aleph_\varepsilon$-prime over $\emptyset$; see [Sh-C]).

Now, from the way condition (4) was stated we may also assume that $A \supseteq C$. Assume, towards a contradiction, that $e_0 \mathop{\smile\!\!\!\!|}\limits_{A'} e_1$, and let σ_i^+ be in $\mathcal{F}$ with $\sigma_i \subseteq \sigma_i^+$ and $\mathrm{Dom}(\sigma_i^+) \supseteq B \cup \mathrm{acl}(A \cup \{e\})$, for $i = 0, 1$. Choose M_0' to be an $\aleph_\varepsilon$-prime model over $\emptyset$ such that $A' \subseteq M_0'$ and $\{e_0, e_1\} \mathop{\smile\!\!\!\!|}\limits_{A'} M_0'$, and set $B_i = \sigma_i(B)$. Since $\mathrm{tp}(e_0, e_1/M_0')$ determines the type $\mathrm{tp}(\mathrm{acl}(\{e_0\} \cup A'), \mathrm{acl}(\{e_1\} \cup A')/M_0')$, and $\mathrm{tp}(B/\mathrm{acl}(\{e\} \cup A)) \vdash \mathrm{tp}(B/\{e\} \cup M_0)$, we conclude that we may choose M_0^* and M_1^* which are $\aleph_\varepsilon$-prime over $M_0' \cup \{e_0\}$ and $M_1' \cup \{e_1\}$ respectively, and such that $B_i \subseteq M_i^*$ for $i = 0, 1$. So if we set

$q = \mathrm{tp}(e^+/B)$ and $q_i = \sigma_i^+(q)$ for $i = 0, 1$, then by the definition of $\mathcal{F}$, $q \not\perp q_i$ for $i = 0, 1$. Let $\overline{q}_i \in S(M_i^*)$ be a nonforking extension of q_i. Since $\overline{q}_i \perp M_0'$ (as M_i^* is $\aleph_\varepsilon$-prime over $M_i' \cup \{e_i\} \cup B_i$ and unique up to isomorphism over $M_i' \cup \{e_i\} \cup B_i$,) and since $M_0^* \underset{M_0'}{\cup} M_1^*$, we conclude that $q_0 \perp q_1$. Thus q, q_0 and q_1 contradict the fact that nonorthogonality is an equivalence relation on stationary regular types.

Theorem 5. *If T has a rigid $\aleph_\varepsilon$-saturated model then for every cardinal $\lambda \geq (2^{\aleph_0})^+$, T has 2^λ rigid $\aleph_\varepsilon$-saturated models of power λ.*

Before proving the theorem, we will need a combinatorial lemma. To that end, we first introduce the following notations.

Notations. 1) $\mathcal{T}$ denotes a subtree of ${}^{<\omega}\lambda$. Let $\eta, \nu \in \mathcal{T}$. Then:

(i) $\nu^- = \eta$ if and only if ν is a successor of η.
(ii) $\lg(\eta)$ denotes the length of η.
(iii) $\mathcal{T}_\nu = \{\eta \in \mathcal{T} \mid \nu \triangleleft \eta\}$, $\mathcal{T}_\nu^+ = \{\eta \in \mathcal{T} \mid \nu \triangleleft \eta \text{ or } \nu = \eta\}$.

2) $\mathcal{R} = \langle (N_\eta, a_\eta) \mid \eta \in \mathcal{T}\rangle$ denotes an $\aleph_\varepsilon$-representation (see Chapter X, Def. 5.2 in [Sh-C]).

3) For an $\aleph_\varepsilon$-representation $\mathcal{R} = \langle (N_\eta, a_\eta) \mid \eta \in \mathcal{T}\rangle$, $E_\mathcal{R}$ denotes the equivalence relation on $\mathcal{T}$ defined by: $E_\mathcal{R}(\eta, \eta')$ if and only if there exists $\nu \in \mathcal{T}$ such that $\eta = \nu^\wedge\langle\alpha\rangle$, $\eta' = \nu^\wedge\langle\alpha'\rangle$ for some α, α', and $\mathrm{tp}(a_\eta/N_\nu) = \mathrm{tp}(a_{\eta'}/N_\nu)$.

Definition 6. (1) We say that a subtree $\mathcal{T}$ of ${}^{<\omega}\lambda$ is *μ-wide* if for all $\eta \in \mathcal{T}$, $|\{\nu \in \mathcal{T} \mid \nu^- = \eta\}| \geq \mu$.

(2) Suppose $\mathcal{T}_0, \mathcal{T}_1 \subseteq {}^{<\omega}\lambda$ are subtrees. We say that $\mathcal{T}_0, \mathcal{T}_1$ are *μ-equivalent* if there exists $\overline{A}_i = \langle A_i^\varrho \mid \varrho \in \mathcal{T}_i\rangle$ $(i = 0, 1)$ such that

(a) for $i = 0, 1$ and all $\varrho \in \mathcal{T}_i$, $|A_i^\varrho| < \mu$,
(b) for $i = 0, 1$, $A_i^\varrho \subseteq \mathcal{T}_\varrho$, and
(c) $\mathcal{T}_0 \setminus \bigcup_{\tau \in U_0} (\mathcal{T}_0)_\tau$ and $\mathcal{T}_1 \setminus \bigcup_{\tau \in U_1} (\mathcal{T}_1)_\tau$ are isomorphic as partial orders, where $U_i = \bigcup_{\varrho \in \mathcal{T}_i} A_i^\varrho$.

(3) A tree $\mathcal{T} \subseteq {}^{<\omega}\lambda$ is called *μ-strongly rigid* if for every η and $\alpha_0 < \alpha_1 < \lambda$ (such that $\eta^\wedge\langle\alpha_i\rangle \in \mathcal{T}$), $\mathcal{T}^+_{\eta^\wedge\langle\alpha_0\rangle}$, $\mathcal{T}^+_{\eta^\wedge\langle\alpha_1\rangle}$ are not μ-equivalent, and $\mathcal{T}$ is μ-wide.

(4) We say that an $\aleph_\varepsilon$-representation $\mathcal{R} = \langle (N_\eta, a_\eta) \mid \eta \in \mathcal{T}\rangle$ is

(i) *of maximal width* if for every $\nu \in \mathcal{T}$, $\{\mathrm{tp}(a_\eta/N_{\eta^-}) \mid \eta \in \mathcal{T}, \eta^- = \nu\}$ is a maximal set of pairwise orthogonal regular types (modulo equality) such that $\mathrm{tp}(a_\eta/N_{\eta^-}) \perp N_{\nu^-}$;
(ii) *equally divided* if $|\eta^\wedge\langle\alpha\rangle/E_\mathcal{R}| = |\eta^\wedge\langle\beta\rangle/E_\mathcal{R}|$ for all η, α, and β such that $\eta^\wedge\langle\alpha\rangle, \eta^\wedge\langle\beta\rangle \in \mathcal{T}$;
(iii) *μ-wide* if $|\eta/E_\mathcal{R}| \geq \mu$ for all $\eta \in \mathcal{T}$.

Before stating the following lemma, we would like to point out that much stronger versions of it have been proved by Shelah.

LEMMA 2. *For every $\lambda > \mu$ there exist 2^λ trees $\mathcal{T} \subseteq \lambda^{<\omega}$ which are μ-strongly rigid and μ-nonequivalent, of power λ.*

Proof. First, it is enough to show the existence of a single such tree in which the root has λ many successors. So, let us construct such a tree:

(a) Let $\langle A_n \mid n < \omega \rangle$ be a partition of the set of positive natural numbers, such that for all $n < \omega$, A_n is infinite and $\min(A_n) > n$. Also, suppose $A_n = \{k_l^n \mid l < \omega\}$, where $k_i^n < k_j^n$ for $i < j$.

(b) Let $h : \lambda \to \lambda$ be surjective such that for all $\alpha < \lambda$, $|h^{-1}(\alpha)| = \lambda$.

(c) Let $\mathcal{T}_0 = {}^{<\omega}\lambda = \{\eta_i \mid i < \lambda\}$ ($\eta_i \neq \eta_j$ for $i \neq j$).

(d) For every ordinal i, let $t_i = \{\nu \mid \nu$ is a strictly decreasing sequence of ordinals $< \omega + i\}$.

(e) Now, let us define our tree: $\mathcal{T}^* = \{\varrho \in {}^{<\omega}\lambda \mid$ for all $k < \lg(\varrho)$, if $(*)(k,\varrho)$ then $\varrho(k) < \mu\}$, where $(*)(k,\varrho)$ is the following statement: If $n < \omega$ is the unique natural number such that $k \in A_n$, and $l(*), i(*)$ are such that $k = k_{l(*)}^n$ and $\varrho \mid n = \eta_{i(*)}$, then $\langle h(\varrho(k_l^n)) \mid l < l(*)\rangle \notin t_{i(*)}$.

(f) The following ranks are useful. Let (n, S, η) be a triple with $n < \omega$, $S \subseteq {}^{<\omega}\lambda$ a subtree, and $\eta \in S$. We define an ordinal rank $\mathrm{rk}^n[\eta, S]$ by:

(i) $\mathrm{rk}^n[\eta, S] \geq 0$ for every $\eta \in S$.

(ii) $\mathrm{rk}^n[\eta, S] \geq \alpha + 1$ if there exist $\{\eta_i^* \mid i < \mu^+\}$, where η_i^* are distinct elements of S, each extending η and satisfying $\lg(\eta_i^*) \in A_n$ and $\mathrm{rk}^n[\eta_i^*, S] \geq \alpha$.

(g) It can be easily verified that:

(i) If $S \subseteq T \subseteq {}^{<\omega}\lambda$ are μ-equivalent subtrees, $\eta \in S$, and $n < \omega$, then $\mathrm{rk}^n[\eta, S] = \mathrm{rk}^n[\eta, T]$.

(ii) If $i < \lambda$, $n < \omega$ and $\lg(\eta_i) = n$ where $\eta_i \in \mathcal{T}^*$ then $\mathrm{rk}^n[\eta_i, \mathcal{T}^*] = \omega + i$.

Now, by (g) we conclude that $\mathcal{T}^*$ is μ-strongly rigid.

FACT 5 (Chapter X, [Sh-C]). *Suppose that T is a superstable NDOP theory. Let $\mathcal{R}_i = \langle (N_\eta^i, a_\eta^i) \mid \eta \in \mathcal{T}_i\rangle$ be $\aleph_\varepsilon$-representations which are $\aleph_1$-wide, let M_i be $\aleph_\varepsilon$-prime over $\bigcup_{\eta \in \mathcal{T}_i} N_\eta^i$ and let $\sigma : M_0 \to M_1$ be an isomorphism. Then there are $\mathcal{T}_i^* \subseteq \mathcal{T}_i$ such that $\mathcal{T}_i^*, \mathcal{T}_i$ are $\aleph_1$-equivalent for $i = 0, 1$ and an isomorphism $\widetilde{\sigma} : \mathcal{T}_0^* \to \mathcal{T}_1^*$ (of partial orders) such that for all $\eta_i \in \mathcal{T}_i^*$ ($i = 0, 1$) if $p_{\eta_i}^i = \mathrm{tp}(a_{\eta_i}^i / N_{\eta_i^-}^i)$, then*

$$\sigma(p_{\eta_0}^0) \not\perp p_{\eta_1}^1 \quad \textit{implies} \quad \widetilde{\sigma}((\eta_0/E_{\mathcal{R}_0}) \cap \mathcal{T}_0^*) = (\eta_1/E_{\mathcal{R}_1}) \cap \mathcal{T}_1^*.$$

Proof of Theorem 5. By Lemma 2, for every $\lambda > 2^{\aleph_0}$ there exist 2^λ trees $\mathcal{T} \subseteq {}^{<\omega}\lambda$ which are $2^{\aleph_0}$-strongly rigid and $2^{\aleph_0}$-nonequivalent, of power λ.

We will show that for every such $\mathcal{T}$, if $\mathcal{R} = \langle (M_\eta, e_\eta) \mid \eta \in \mathcal{T} \rangle$ is an equally divided $\aleph_\varepsilon$-representation of maximal width, and M is $\aleph_\varepsilon$-prime over it, then M is rigid. This is sufficient because by Fact 5, any two $\aleph_\varepsilon$-prime models over $2^{\aleph_0}$-nonequivalent trees are nonisomorphic.

So, assume $\sigma \in \mathrm{Aut}(M)$. We must show $\sigma = \mathrm{id}$: otherwise by Theorem 4, there exists a nonalgebraic regular type $p^* \in S(M)$ such that $p^* \perp \sigma p^*$. Since T has NDOP and M is $\aleph_\varepsilon$-minimal over $\bigcup_{\eta \in \mathcal{T}} M_\eta$, we have $p^* \not\perp M_{\eta_0}$ for some $\eta_0 \in \mathcal{T}$. As M is $\aleph_\varepsilon$-prime over an $\aleph_\varepsilon$-representation of maximal width, $p^* \not\perp p_{\eta_0{}^\wedge\langle\alpha^*\rangle}$ for some α^*. Therefore $\sigma p_{\eta_0{}^\wedge\langle\alpha^*\rangle} \not\perp \sigma p^* \perp p^* \not\perp p_{\eta_0{}^\wedge\langle\alpha^*\rangle}$. In particular,

$$(*) \qquad \sigma p_{\eta_0{}^\wedge\langle\alpha^*\rangle} \perp p_{\eta_0{}^\wedge\langle\alpha^*\rangle}.$$

But by Fact 5, there exist $\mathcal{T}_0^*, \mathcal{T}_1^* \subseteq \mathcal{T}$ (without loss of generality $\eta_0{}^\wedge\langle\alpha^*\rangle \in \mathcal{T}_0^*$), and an isomorphism of partial orders $\widetilde{\sigma} : \mathcal{T}_0^* \to \mathcal{T}_1^*$ such that $\mathcal{T}_i^*$ $(i = 0, 1)$ are $\aleph_1$-equivalent to $\mathcal{T}$, and η^* such that $\sigma p_{\eta_0{}^\wedge\langle\alpha^*\rangle} \not\perp p_{\eta^*}$. So by $(*)$, $p_{\eta^*} \neq p_{\eta_0{}^\wedge\langle\alpha^*\rangle}$ and by Fact 5, $\widetilde{\sigma}(\eta^\wedge\langle\alpha^*\rangle/E_{\mathcal{R}}) = \eta^*/E_{\mathcal{R}} \neq (\eta_0{}^\wedge\langle\alpha^*\rangle)/E_{\mathcal{R}}$, contradicting the fact that $\mathcal{T}$ is even $2^{\aleph_0}$-strongly rigid (here we used the fact that $\mathcal{T}$ is equally divided).

EXAMPLE 1. Let $L = \{E, f\}$ and let M be the following L-structure: $|M| = Z \times \omega$, E is the equivalence relation defined by $E[(m_0, k_0), (m_1, k_1)]$ if and only if $m_0 = m_1$, and $f : |M| \to |M|$ is any function with the properties:

(i) For all $(m, k) \in |M|$, $f(m, k) = (m + 1, k')$ for some k'.
(ii) For all $(m, k) \in |M|$, $f^{-1}(m, k)$ is infinite.

It is not hard to see that $T = \mathrm{Th}(M)$ is an ω-stable NDOP theory in which no saturated model has a n.n.a., and therefore T has 2^λ rigid $\aleph_\varepsilon$-saturated models of cardinality λ for every $\lambda > 2^{\aleph_0}$.

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