A TREE-ARROWING GRAPH

S. SHELAH[1]
Hebrew University,
Jerusalem, Israel.

AND

E. C. MILNER[2]
University of Calgary,
Calgary, Canada.

Dedicated to the memory of Eric Milner

Abstract. We answer a variant of a question of Rödl and Voigt by showing that, for a given infinite cardinal λ, there is a graph G of cardinality $\kappa = (2^\lambda)^+$ such that for any colouring of the edges of G with λ colours, there is an induced copy of the κ-tree in G in the set theoretic sense with all edges having the same colour.

Keywords: Partition relation, graph, tree, cardinal number, stationary set, normal filter.
AMS Subject Classification (1991): 03, 04

1. Introduction

$\mathcal{G} = (V, E)$ is a graph with *vertex set* V and *edge set* E, where $E \subseteq [V]^2$. The graph $\mathcal{H} = (W, F)$ is a *subgraph* of $\mathcal{G}$ if $W \subseteq V$ and $F \subseteq E$, it is an *induced subgraph* if $F = E \cap [W]^2$. If λ is a cardinal, the partition relation

$$\mathcal{G} \to (\mathcal{H})^2_\lambda, \tag{1}$$

means that if $c : E \to \lambda$ is any colouring of the edges of $\mathcal{G}$ with λ colours, then there is an induced copy of $\mathcal{H}$ in $\mathcal{G}$ in which all the edges have the same colour. There is a related notion $\mathcal{G} \to (\mathcal{H})^1_\lambda$, for vertex colourings of graphs. However, there is an essential difference since, for any given graph $\mathcal{H}$ and any λ, there is

[1]Paper Sh 578 in Shelah's publication list. Research supported by "The Israel Science Foundation" administered by The Israel Academy of Sciences and Humanities.
[2]Research supported by NSERC grant #69-0982.

C.A. Di Prisco et al. (ed.), Set Theory, 175-182.

some $\mathcal{G}$ such that $\mathcal{G} \to (\mathcal{H})^1_\lambda$ holds. This is not true for edge-colourings; Hajnal and Komjath [2] proved the consistency of a negative answer, and Shelah [5] proved that a positive answer is also consistent. It is therefore of some interest to have instances of graphs $\mathcal{H}$ such that (1) holds for some $\mathcal{G}$, and then, of course, one can ask for the smallest such $\mathcal{G}$.

Rödl and Voigt [4] (see also [3]) proved a result of this kind by showing that for any infinite cardinal λ and a suitably large κ, there is a graph $\mathcal{G}_\kappa$ of cardinality κ such that

$$\mathcal{G}_\kappa \to (\mathcal{T}_\kappa)^2_\lambda \tag{2}$$

holds, where $\mathcal{T}_\kappa$ is the tree in which every vertex has degree κ (see below). More precisely, 'suitably large' means that the ordinary partition relation

$$\mathrm{cf}(\kappa) \to (\omega)^3_\lambda$$

holds so that, by [1], $\kappa \geq (2^{2^\lambda})^+$; in fact, they showed in this case that the ubiquitous *shift-graph* on κ works. Rödl and Voigt [4] then asked, what is the smallest cardinal κ such that (2) holds? It is easily seen that (2) is false if $\kappa \leq 2^\lambda$, and they conjectured that it holds (for some suitable graph $\mathcal{G}_\kappa$) if $\kappa = (2^\lambda)^+$. In this paper we prove that (2) holds with $\mathcal{T}_\kappa$ replaced by $\mathcal{T}(\kappa)$, a related graph which we call *the transitive κ-tree* defined in the next section.

2. Preliminaries

For an infinite cardinal κ we denote by ${}^{<\omega}\kappa$ the set of all increasing finite sequences of ordinals in κ. The *length* of an element $s = \langle s_0, \dots, s_{n-1}\rangle \in {}^{<\omega}\kappa$ is denoted by $\ell n(s) = n$. Also, we define

$$\max(s) = \begin{cases} -1 & \text{if } s = \langle\rangle, \text{ the empty sequence,} \\ s_{\ell n(s)-1} & \text{if } \ell n(s) > 0. \end{cases}$$

If $s = \langle s_0, \dots, s_{n-1}\rangle$ and $t = \langle t_0, \dots, t_{m-1}\rangle$ are two elements of ${}^{<\omega}\kappa$, we write $s \lhd t$ to denote the fact that s is a proper initial segment of t, that is $n < m$ and $s_i = t_i$ for $i < n$, and in this case we write $s = t|n$. We also write $s = t_*$ if $m = n+1$ and $s \lhd t$. If s, t are distinct and $\lhd$-incomparable we write $s \perp t$. The *κ-tree of height ω* is the graph $\mathcal{T}_\kappa$ on ${}^{<\omega}\kappa$ with edge set

$$E_\kappa = \{\{s,t\} : s, t \in {}^{<\omega}\kappa \wedge s = t_*\}.$$

We shall also consider a related graph, *the transitive κ-tree of height ω*, which is the graph $\mathcal{T}(\kappa)$ on ${}^{<\omega}\kappa$ with edge set

$$F_\kappa = \{\{s,t\} : s, t \in {}^{<\omega}\kappa \wedge s \lhd t\}.$$

We shall prove the following theorem.

Theorem 2.1. *Let λ be an infinite cardinal, and let $\kappa = (2^\lambda)^+$. Then there is a graph G_κ of cardinality κ such that*

$$G_\kappa \to (\mathcal{T})^2_\lambda,$$

where $\mathcal{T}$ is $\mathcal{T}(\kappa)$.

Remark. *Instead of $\kappa = (2^\lambda)^+$, it is enough that κ be any regular cardinal such that $|\alpha|^\lambda < \kappa$ holds for all $\alpha < \kappa$. The same proof works.*

The construction of a suitable $\mathcal{G}_\kappa$ depends upon the following (slightly weaker version of a) theorem of Shelah [7] (or more [8, 3.5]):

($\bullet$) *Let λ be an infinite cardinal, $\kappa = (2^\lambda)^+$, $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda^+\}$. Then there are a sequence $\overline{C} = \langle C_\delta : \delta \in S\rangle$ and a sequence $\overline{h^*} = \langle h^*_\delta : \delta \in S\rangle$ such that C_δ is a club in δ having order type λ^+, $h^*_\delta : C_\delta \to 2$ and such that, for any club K in κ, there is a stationary subset B_K of $S \cap K$ such that for each $\delta \in B_K$ and each $i < 2$, $\min(C_\delta) \in K$ and the set*

$$D_K(\delta, i) = \{\alpha \in C_\delta \cap K : h^*_\delta(\alpha) = i \wedge \min(C_\delta \setminus (\alpha+1)) \in K\}$$

is cofinal in δ.

Remarks. *1. The result is also true if 2, the range of each h^*_δ, is replaced by λ; also, if $\kappa = \lambda^{++}$, we can also require that $D_K(\delta, i)$ be a stationary subset of δ for each $\delta \in B_K$ and $i < \lambda$ (see [8]).*

2. If $2^\lambda > \lambda^+$, then the following stronger assertion is true (see Shelah [6]): ($\bullet\bullet$) *There is a sequence $\bar{C} = \langle C_\delta : \delta \in S\rangle$ such that C_δ is a club in δ having order type λ^+ and, for any club K in κ and any stationary subset $S' \subseteq S$, there is a stationary subset $B_K \subseteq S' \cap K$ such that $C_\delta \subseteq K$ for each $\delta \in B_K$. Using this result instead of ($\bullet$), the proof of Theorem 2.1 for the case when $2^\lambda > \lambda^+$ may be slightly simplified.*

We will prove that Theorem 2.1 holds with the graph $G_\kappa = (\kappa, \mathcal{E})$, where

$$\mathcal{E} = \{\{\alpha, \beta\} : \beta \in S \wedge \min(C_\beta) < \alpha < \beta \wedge h^*_\beta(\sup(\alpha \cap C_\beta)) = 0\},$$

and the C_β and h^*_β are as described in ($\bullet$).

3. The case $\mathcal{T} = \mathcal{T}(\kappa)$

We prove the result for the case of the transitive tree $T(\kappa)$.

Proof: Let $c : \mathcal{E} \to \lambda$ be any λ-colouring of the edges of G_κ. For each $\zeta \in \lambda$ consider the following two-person game $\mathcal{G}_\zeta$. The game has ω moves. At the n-th stage the first player P_1 chooses ordinals α_n, β_n, and then the second player P_2 chooses two ordinals γ_n, δ_n so that

$$\alpha_n < \beta_n < \gamma_n < \delta_n < \kappa, \tag{3}$$

$$\delta_m < \alpha_n \quad (m < n). \tag{4}$$

The player P_2 is declared the winner in a play of the game if he succeeds in choosing the γ_n so that

$$\{\gamma_m, \gamma_n\} \in \mathcal{E}, \qquad c(\{\gamma_m, \gamma_n\}) = \zeta \quad (m < n < \omega), \tag{5}$$

and

$$\{\xi, \gamma_n\} \notin \mathcal{E} \text{ for } \xi \in (\alpha_m, \beta_m) \text{ and } m \leq n < \omega. \tag{6}$$

(As usual, (α, β) denotes the open interval $\{\xi : \alpha < \xi < \beta\}$ and $[\alpha, \beta]$ is the corresponding closed interval.)

The proof of the theorem depends upon the following two facts:

Fact A: For some $\zeta < \lambda$, P_2 has a winning strategy for the game $\mathcal{G}_\zeta$.

Fact B: If P_2 can win $\mathcal{G}_\zeta$, then the graph G_κ contains an induced copy of $\mathcal{T}(\kappa)$ with all edges coloured ζ.

Proof of Fact B. We assume that $\zeta < \lambda$ and that the second player P_2 has a winning strategy σ_ζ for the game $\mathcal{G}_\zeta$. We shall define ordinals $\alpha_s, \beta_s, \gamma_s, \delta_s$ for s a vertex of $\mathcal{T}(\kappa)$ so that the following conditions are satisfied:

(a) For each s the sequence

$$\langle(\alpha_{s|i}, \beta_{s|i}, \gamma_{s|i}, \delta_{s|i}) : i < \ell n(s)\rangle$$

consists of the first $2\ell n(s)$ moves in a proper play of the game $\mathcal{G}_\zeta$ in which P_2 uses the winning strategy σ_ζ.

(b) $\gamma_s \neq \gamma_t$ if $s \neq t$.

(c) If $s \perp t$, then $\{\gamma_s, \gamma_t\} \notin \mathcal{E}$.

Since (5) holds, these conditions imply that the map $s \mapsto \gamma_s$ is an embedding of the tree $\mathcal{T}(\kappa)$ into the graph G_κ and all the edges of the image have colour ζ.

In fact, we shall choose the $\alpha_s, \beta_s, \gamma_s, \delta_s$ so that (a) holds and so that the following condition is satisfied:

(d) For any vertices s, t of $\mathcal{T}(\kappa)$, if $s \perp t$, then

EITHER *(i)* $[\gamma_s, \delta_s] \subset \bigcup_{i \leq \ell n(t)} (\alpha_{t|i}, \beta_{t|i})$,
OR *(ii)* $[\gamma_t, \delta_t] \subset \bigcup_{i \leq \ell n(s)} (\alpha_{s|i}, \beta_{s|i})$.

The conditions (a) and (d), and the fact that P_2 is using the winning strategy σ_ζ, ensure that (b) and (c) also hold.

We define $\alpha_s, \beta_s, \gamma_s, \delta_s$ by induction on $\max(s)$. Let $\alpha_{\langle\rangle} = 0$, $\beta_{\langle\rangle} = 1$, and then let $(\gamma_{\langle\rangle}, \delta_{\langle\rangle})$ be P_2's response in the game $\mathcal{G}_\zeta$ using his winning strategy σ_ζ. Now let $0 \leq \xi < \kappa$, and suppose that we have suitably defined $\alpha_s, \beta_s, \gamma_s, \delta_s$ for all vertices s of $\mathcal{T}(\kappa)$ such that $\max(s) < \xi$. We need to define these when $\max(s) = \xi$.

Let $\langle t_i : i < \theta(\xi)\rangle$ be an enumeration of all the nodes s of $\mathcal{T}(\kappa)$ with $\max(s) = \xi$. Then $1 \leq \theta(\xi) \leq 2^\lambda < \kappa$. Now inductively choose the α_{t_i}, β_{t_i}, γ_{t_i}, δ_{t_i} for $i < \theta(\xi)$ so that

$$\alpha_{t_i} = \delta_{(t_i)_*} + 1,$$

and if $i = 0$, $\beta_{t_i} = \alpha_{i_0} + 1$ and if $i > 0$

$$\beta_{t_i} = \sup\{\delta_s + 2 : \max(s) < \xi \text{ or } s = t_j \text{ for some } j < i\}.$$

The corresponding pairs $(\gamma_{t_i}, \delta_{t_i})$ are determined by the strategy σ_ζ. With these choices it is easily seen that (a) continues to hold; we have to check that (d) also holds when $s \perp t$ and $\max(s) = \xi$ or $\max(t) = \xi$.

If $\max(s) = \max(t) = \xi$, then $s = t_i$ and $t = t_j$, where say $i < j$. Then

$$\alpha_t = \delta_{t_*} + 1 < \beta_s < \gamma_s < \delta_s < \beta_t,$$

and so (d)(i) holds.

Suppose $\max(s) < \xi = \max(t)$. Then by the induction hypothesis, either (i) or (ii) of (d) holds when we replace t by t_*. Suppose first that (d)(i) holds. Then for some $m \leq \ell n(t_*)$ we have that

$$\alpha_{t_*|m} < \gamma_s < \delta_s < \beta_{t_*|m}.$$

It follows that (d)(i) also holds for s and t since $t|m = t_*|m$. Now suppose that (d)(ii) holds so that, for some $m \leq \ell n(s)$,

$$\alpha_{s|m} < \gamma_{t_*} < \delta_{t_*} < \beta_{s|m}.$$

Then, by the definitions of α_t and β_t, it follows that

$$\alpha_t = \delta_{t_*} + 1 \leq \beta_s < \gamma_s < \delta_s < \beta_t,$$

so that again (d)(i) holds for s and t. Similarly, if $\max(t) < \xi = \max(s)$. $\square$

Proof of Fact A. We have to show that P_2 wins the game $\mathcal{G}_\zeta$ for some $\zeta < \lambda$. Suppose for a contradiction that this is false. Since the games are open and hence determined, it follows that P_1 has a winning strategy, say τ_ζ, for the game $\mathcal{G}_\zeta$ for every $\zeta < \lambda$.

For convenience we write $c(\{\alpha,\beta\}) = -1$ if $\{\alpha,\beta\} \notin \mathcal{E}$, so that c is defined on all pairs $\{\alpha,\beta\} \in [\kappa]^2$. For each bounded subset $X \subseteq \kappa$ define an equivalence relation e_X on $S \setminus (\sup(X)+1)$ so that $\beta\, e_X\, \gamma$ holds if and only if

(i) $\beta, \gamma \in S$ and $\sup(X) < \beta, \gamma < \kappa$;

(ii) $c(\{\alpha,\beta\}) = c(\{\alpha,\gamma\})$ for all $\alpha \in X$;

(iii) $X \cap C_\beta = X \cap C_\gamma$, (iv) for $\alpha \in X$, $\alpha \leq \min(C_\beta) \Leftrightarrow \alpha \leq \min(C_\gamma)$, $\mathrm{tp}(\alpha \cap C_\beta) = \mathrm{tp}(\alpha \cap C_\gamma)$ and $h^*_\beta(\sup(\alpha \cap C_\beta)) = h^*_\gamma(\sup(\alpha \cap C_\gamma))$ (for $\alpha > \min(C_\beta)$).

Note that the equivalence relation e_X has at most $(\lambda^+)^{|X|} \leq 2^{\lambda|X|}$ classes. Also, if $Y \subseteq X$, then $\beta \, e_X \, \gamma \Rightarrow \beta \, e_Y \, \gamma$.

Since $\kappa = (2^\lambda)^+$, there is a continuous increasing sequence of ordinals $\langle \rho_\eta : \eta < \kappa \rangle$ in κ such that the following two conditions hold:

(o) If $X \subseteq \rho_\eta$, $|X| \leq \lambda$ and $\rho_\eta < \beta < \kappa$, then there is some $\gamma \in (\rho_\eta, \rho_{\eta+1})$ such that $\beta e_X \gamma$

(oo) ρ_η is closed under τ_ζ for all $\zeta < \lambda$. In other words, if at the n-th stage of a play in the game $\mathcal{G}_\zeta$, player P_2 chooses $\gamma_n < \delta_n < \rho_\eta$, then P_1's response using τ_ζ is to choose $\alpha_{n+1}, \beta_{n+1}$ so that $\delta_n < \alpha_{n+1} < \beta_{n+1} < \rho_\eta$.

Since $K = \{\rho_\eta : \eta < \kappa\}$ is a club in κ, there is some $\delta \in S$ such that $\min(C_\delta) \in K$ and, for $\varepsilon \in \{0, 1\}$,

$$A_\varepsilon = \{\alpha \in C_\delta \cap K : h_\delta^*(\alpha) = \varepsilon \wedge \min(C_\delta \setminus (\alpha + 1)) \in K\}$$

is an unbounded subset of δ. Let $C_\delta = \{i_\sigma : \sigma < \lambda^+\}$, where $i_0 < i_1 < \cdots$.

We claim that the following assertion holds for some $\zeta < \lambda$.

$(*)_\zeta$: If $X \subseteq \delta$, $|X| \leq \lambda$, then there are $\sigma < \lambda^+$ and γ such that (a) $\sup(X) < i_\sigma < \gamma < i_{\sigma+1}$, (b) $i_\sigma \in A_0$, (c) $\gamma \, e_X \, \delta$, and (d) $c(\gamma, \delta) = \zeta$.

For suppose the claim is false. Then, for each $\zeta < \lambda$ there is a counter-example X_ζ. Let $X = \bigcup\{X_\zeta : \zeta < \lambda\}$. Then $X \subseteq \delta$ and $|X| \leq \lambda$ and so, for some $\alpha \in A_0$, $\sup(X) < \alpha < \delta$. There are $\eta < \kappa$ and $\sigma < \lambda^+$ such that $\alpha = \rho_\eta = i_\sigma$, and therefore, by the choice of $\rho_{\eta+1}$, there is γ such that $\rho_\eta < \gamma < \rho_{\eta+1}$ and $\gamma e_X \delta$. Since $\alpha = i_\sigma \in A_0$, $i_{\sigma+1} = \min(C_\delta \setminus (\alpha + 1)) \in K$. So $\rho_{\eta+1} \leq i_{\sigma+1}$. Therefore, $\sup(C_\delta \cap \gamma) = i_\sigma$, and since $\alpha = i_\sigma \in A_0$, we have that $h_\delta^*(\sup(C_\delta \cap \gamma)) = 0$. Therefore, $\{\gamma, \delta\}$ is an edge of G and there is some $\zeta \in \lambda$ such that $c(\gamma, \delta) = \zeta$. But this contradicts the choice of $X_\zeta \subseteq X$, and hence $(*)_\zeta$ holds for some $\zeta < \lambda$.

By induction on $n < \omega$ we now choose ordinals $\alpha_n, \beta_n, \gamma_n, \delta_n$ in δ and $\sigma(n) < \lambda^+$ so that the following conditions are satisfied:

A: $\langle (\alpha_m, \beta_m, \gamma_m, \delta_m) : m \leq n \rangle$ is an intial segment of a play in the game $\mathcal{G}_\zeta$ in which P_1 uses the winning strategy τ_ζ.

B: $\alpha_0, \beta_0 < \min(C_\delta)$.

C: $\gamma_n = \min\{\gamma : \gamma > i_{\sigma(2n)} \wedge \gamma \, e_{X_n} \, \delta \wedge c(\gamma, \delta) = \zeta\}$, where

$$X_n = \bigcup\{\{\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell\} : \ell < n\} \cup \{\alpha_n, \beta_n\} \cup \bigcup\{\{i_{\sigma(\ell)}, i_{\sigma(\ell)+1}\} : \ell < 2n\}.$$

D: $\delta_n = i_{\sigma(2n+1)}$.

E: For $n > 0$, $[\alpha_n, \beta_n] \subseteq (\delta_{n-1}, i_{\sigma(2n-1)+1})$.

F: $i_{\sigma(n)}$ belongs to A_0 or A_1 according as n is even or odd and $\sigma(n)+1 < \sigma(n+1)$.

We have to prove that it is possible to choose the α_n etc., so that these conditions are satisfied. Clearly (B) holds since, by (oo), the first moves by P_1 using the stategy τ_ζ are $\alpha_0 < \beta_0 < \rho_0$ and $\rho_0 \leq \min(C_\delta) \in K$. By $(*)_\zeta$, there are $\sigma(0) < \lambda^+$

and γ such that $i_{\sigma(0)} \in A_0$, $i_{\sigma(0)} < \gamma < i_{\sigma(0)+1}$, $\gamma\, e_{X_0}\, \delta$, where $X_0 = \{\alpha_0, \beta_0\}$ and $c(\gamma, \delta) = \zeta$; let γ_0 be the least such γ. Now let $\sigma(1) > \sigma(0) + 1$ be minimal so that $i_{\sigma(1)} \in A_1$, and put $\delta_0 = i_{\sigma(1)}$. Now suppose that $n > 0$ and that the $\alpha_m, \beta_m, \gamma_m, \delta_m$, $\sigma(2m)$ and $\sigma(2m+1)$ have been suitably defined for all $m < n$. Let $\rho \in K$ be minimal such that $\rho > \delta_{n-1}$. P_1 chooses α_n, β_n using the strategy τ_ζ so that $\delta_{n-1} < \alpha_n < \beta_n < \rho$. Since $\delta_{n-1} = i_{\sigma(2n-1)} \in A_1$, it follows that $i_{\sigma(2n-1)+1} \in K$ and hence $\rho \leq i_{\sigma(2n-1)+1}$. Now by $(*)_\zeta$, there are $\sigma(2n)$ and γ so that $i_{\sigma(2n)} \in A_0$, $i_{\sigma(2n)} < \gamma < i_{\sigma(2n)+1}$, $\gamma\, e_{X_n}\, \delta$ (where X_n is as described in (C)), and $c(\gamma, \delta) = \zeta$; let γ_n be the least such γ. Note that, since $i_{\sigma(2n)} \in A_0$, $i_{\sigma(2n)+1} = \min(C_\delta \setminus (i_{\sigma(2n)} + 1)) \in K$. Finally, choose a minimal ordinal $\sigma(2n+1) > \sigma(2n)+1$ so that $\delta_n = i_{\sigma(2n+1)} \in A_1$. This completes the definition of the α_n etc., so that (A)-(F) hold.

By (C) it follows that $c(\gamma_n, \delta) = \zeta$ for all $n < \omega$, and hence $c(\gamma_m, \gamma_n) = \zeta$ holds for all $m < n < \omega$ since $\gamma_m \in X_n$ and $\gamma_n\, e_{X_n}\, \delta$. There is no edge of G_κ from δ to (α_0, β_0) since $\beta_0 < \min(C_\delta)$. Since $\gamma_n\, e_{X_n}\, \delta$ and $\beta_0 \in X_n$, it follows that $\beta_0 < \min(C_{\gamma_n})$ also, and so there is no edge from γ_n to (α_0, β_0) either. By the construction, for $0 < m < \omega$, $i_{\sigma(2m-1)} < \alpha_m < \beta_m < i_{\sigma(2m-1)+1}$, and hence $C_\delta \cap (\alpha_m, \beta_m) = \emptyset$. Therefore, for any $\xi \in (\alpha_m, \beta_m)$, $h^*_\delta(\sup(\xi \cap C_\delta)) = h^*_\delta(i_{\sigma(2m-1)}) = 1$ by (F), and so there is no edge of G from δ to (α_m, β_m). If $0 < m < n < \omega$, then $\gamma_n\, e_{X_n}\, \delta$ and therefore,

$$\mathrm{tp}(\alpha_m \cap C_{\gamma_n}) = \mathrm{tp}(\alpha_m \cap C_\delta) = \mathrm{tp}(\beta_m \cap C_\delta) = \mathrm{tp}(\beta_m \cap C_{\gamma_n}).$$

Therefore, for any $\xi \in (\alpha_m, \beta_m)$, it follows that

$$h^*_{\gamma_n}(\sup(\xi \cap C_{\gamma_n})) = h^*_{\gamma_n}(\sup(\alpha_m \cap C_{\gamma_n})) = h^*_\delta(\sup(\alpha_m \cap C_\delta)) = 1$$

and so there are no edges of G from γ_n to (α_m, β_m) either.

Thus we have produced a play in the game $\mathcal{G}_\zeta$ in which P_1 uses the strategy τ_ζ but the second player P_2 wins! This contradicts the assumption that σ_ζ is a winning strategy for the first player, and completes the proof. □

References

1. P. Erdős and R. Rado, A partition calculus in set theory, *Bull. Amer. Math. Soc.* **62** (1956) 427-489.
2. A. Hajnal and P. Komjath, Embedding graphs into colored graphs, *Trans. Amer. Math. Soc.* **307** (1988), 395–409; Corrigendum: **332** (1992), 475.
3. P. Komjath and E.C. Milner, On a conjecture of Rödl and Voigt. J. Combin. Theory, Ser. B **61** (1994), 199-209.
4. V. Rödl and B. Voigt, Monochromatic trees with respect to edge partitions, *J. Combin. Theory Ser. B* **58** (1993), 291-298.
5. Saharon Shelah [Sh: 289], Consistency of positive partition theorems for graphs and models, in: *Set theory and its applications (Toronto, ON, 1987), Lecture Notes in Mathematics* **1401**, (J. Steprans and S. Watson, eds.), Springer, Berlin-New York, (1989) 167–193.

6. Saharon Shelah [Sh: 365], There are Jonsson algebras in many inaccessible cardinals, in: *Cardinal Arithmetic*, *Oxford Logic Guides* **29** chapter III, Oxford University Press, 1994.
7. Saharon Shelah [Sh: 413], More Jonsson Algebras and Colourings, *Archive for Mathematical Logic*, to appear.
8. Saharon Shelah [Sh: 572], Colouring and $\aleph_2$-cc not productive, *Annals of Pure and Applied Logic*, **84** (1997), 153-174..