

On a problem of Steve Kalikow

by

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Abstract. The Kalikow problem for a pair (λ, κ) of cardinal numbers, $\lambda > \kappa$ (in particular $\kappa = 2$) is whether we can map the family of ω -sequences from λ to the family of ω -sequences from κ in a very continuous manner. Namely, we demand that for $\eta, \nu \in {}^\omega \lambda$ we have: η, ν are almost equal if and only if their images are.

We show consistency of the negative answer, e.g., for \aleph_ω but we prove it for smaller cardinals. We indicate a close connection with the free subset property and its variants.

0. Introduction. In the present paper we are interested in the following property of pairs of cardinal numbers:

DEFINITION 0.1. Let λ, κ be cardinals. We say that the pair (λ, κ) has the *Kalikow property* (and then we write $\mathcal{KL}(\lambda, \kappa)$) if there is a sequence $\langle F_n : n < \omega \rangle$ of functions such that

$$F_n : {}^n \lambda \rightarrow \kappa \quad (\text{for } n < \omega)$$

and if $F : {}^\omega \lambda \rightarrow {}^\omega \kappa$ is given by

$$(\forall \eta \in {}^\omega \lambda)(\forall n \in \omega)(F(\eta)(n) = F_n(\eta \upharpoonright n))$$

then for every $\eta, \nu \in {}^\omega \lambda$,

$$(\forall^\infty n)(\eta(n) = \nu(n)) \quad \text{iff} \quad (\forall^\infty n)(F(\eta)(n) = F(\nu)(n)).$$

In particular we answer the following question of Kalikow:

KALIKOW PROBLEM 0.2. Is $\mathcal{KL}(2^{\aleph_0}, 2)$ provable in ZFC?

The Kalikow property of pairs of cardinals was studied in [Ka90]. Several results are known already. Let us mention some of them. First, one can easily notice that

$$\mathcal{KL}(\lambda, \kappa) \ \& \ \lambda' \leq \lambda \ \& \ \kappa' \geq \kappa \ \Rightarrow \ \mathcal{KL}(\lambda', \kappa').$$

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Also (“transitivity”)

$$\mathcal{KL}(\lambda_2, \lambda_1) \ \& \ \mathcal{KL}(\lambda_1, \lambda_0) \ \Rightarrow \ \mathcal{KL}(\lambda_2, \lambda_0)$$

and

$$\mathcal{KL}(\lambda, \kappa) \ \Rightarrow \ \lambda \leq \kappa^{\aleph_0}.$$

Kalikow proved that CH implies $\mathcal{KL}(2^{\aleph_0}, 2)$ (in fact that $\mathcal{KL}(\aleph_1, 2)$ holds true) and he conjectured that CH is equivalent to $\mathcal{KL}(2^{\aleph_0}, 2)$.

The question 0.2 is formulated in [Mi91, Problem 15.15, p. 653].

We shall prove that $\mathcal{KL}(\lambda, 2)$ is closely tied with some variants of the free subset property (both positively and negatively). First we present an answer to problem 0.2 proving the consistency of $\neg\mathcal{KL}(2^{\aleph_0}, 2)$ in 1.1 (see 2.8 too). Later we discuss variants of the proof (concerning the cardinal and the forcing). Then we deal with a positive answer, in particular $\mathcal{KL}(\aleph_n, 2)$, and we show that the negation of a relative of the free subset property for λ implies $\mathcal{KL}(\lambda, 2)$.

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NOTATION. We will use the Greek letters κ, λ, χ to denote (infinite) cardinals and the letters $\alpha, \beta, \gamma, \zeta, \xi$ to denote ordinals. Sequences of ordinals will be called $\bar{\alpha}, \bar{\beta}, \bar{\zeta}$ with the usual convention that $\bar{\alpha} = \langle \alpha_n : n < \text{lg}(\bar{\alpha}) \rangle$ etc. Sets of ordinals will be denoted by u, v, w (with possible indexes).

The quantifiers $(\forall^{\infty} n)$ and $(\exists^{\infty} n)$ are abbreviations for “for all but finitely many $n \in \omega$ ” and “for infinitely many $n \in \omega$ ”, respectively.

1. The negative result. For a cardinal χ , the forcing notion \mathbb{C}_χ for adding χ many Cohen reals consists of finite functions p such that for some $w \in [\chi]^{<\omega}$, $n < \omega$,

$$\text{dom}(p) = \{(\zeta, k) : \zeta \in w \ \& \ k < n\} \quad \text{and} \quad \text{rang}(p) \subseteq 2$$

ordered by inclusion.

THEOREM 1.1. *Assume $\lambda \rightarrow (\omega_1 \cdot \omega)_{2^\kappa}^{<\omega}$, $2^\kappa < \lambda \leq \chi$. Then*

$$\Vdash_{\mathbb{C}_\chi} \neg\mathcal{KL}(\lambda, \kappa) \quad \text{and hence} \quad \Vdash_{\mathbb{C}_\chi} \neg\mathcal{KL}(2^{\aleph_0}, 2).$$

Proof. Suppose that \mathbb{C}_χ -names \underline{F}_n (for $n \in \omega$) and a condition $p \in \mathbb{C}_\chi$ are such that

$$p \Vdash_{\mathbb{C}_\chi} \text{“}\langle \underline{F}_n : n < \omega \rangle \text{ exemplifies } \mathcal{KL}(\lambda, \kappa)\text{”}.$$

For $\bar{\alpha} \in {}^n\lambda$ choose a maximal antichain $\langle p_{\bar{\alpha}, l}^n : l < \omega \rangle$ of \mathbb{C}_χ deciding the values of $\underline{F}_n(\bar{\alpha})$. Thus we have a sequence $\langle \gamma_{\bar{\alpha}, l}^n : l < \omega \rangle \subseteq \kappa$ such that

$$p_{\bar{\alpha}, l}^n \Vdash_{\mathbb{C}_\chi} \underline{F}_n(\bar{\alpha}) = \gamma_{\bar{\alpha}, l}^n.$$

Let χ^* be a sufficiently large regular cardinal. Take an elementary submodel M of $(\mathcal{H}(\chi^*), \in, <_{\chi^*}^*)$ such that

- $\|M\| = \chi$, $\chi + 1 \subseteq M$,
- $\langle p_{\bar{\alpha}, l}^n : l < \omega, n \in \omega, \bar{\alpha} \in {}^n\lambda \rangle, \langle \gamma_{\bar{\alpha}, l}^n : l < \omega, n \in \omega, \bar{\alpha} \in {}^n\lambda \rangle \in M$.

By $\lambda \rightarrow (\omega_1 \cdot \omega)_{2^\kappa}^{<\omega}$ (see [Sh 481, Claim 1.3]), we find a set $B \subseteq \lambda$ of indiscernibles in M over

$$\kappa \cup \{ \langle p_{\bar{\alpha}, l}^n : l < \omega : n \in \omega, \bar{\alpha} \in {}^n\lambda \rangle, \langle \gamma_{\bar{\alpha}, l}^n : l < \omega : n \in \omega, \bar{\alpha} \in {}^n\lambda \rangle, \chi, p \}$$

and a system $\langle N_u : u \in [B]^{<\omega} \rangle$ of elementary submodels of M such that

- (a) B is of order type $\omega_1 \cdot \omega$ and for $u, v \in [B]^{<\omega}$:
- (b) $\kappa + 1 \subseteq N_u$,
- (c) $\chi, p, \langle p_{\bar{\alpha}, l}^n : l < \omega, n < \omega, \bar{\alpha} \in {}^n\lambda \rangle, \langle \gamma_{\bar{\alpha}, l}^n : l < \omega, n < \omega, \bar{\alpha} \in {}^n\lambda \rangle \in N_u$,
- (d) $|N_u| = \kappa$, $N_u \cap B = u$,
- (e) $N_u \cap N_v = N_{u \cap v}$,
- (f) $|u| = |v| \Rightarrow N_u \cong N_v$, and let $\pi_{u,v} : N_v \rightarrow N_u$ be this (unique) isomorphism,
- (g) $\pi_{v,v} = \text{id}_{N_v}$, $\pi_{u,v}(v) = u$, $\pi_{u_0, u_1} \circ \pi_{u_1, u_2} = \pi_{u_0, u_2}$,
- (h) if $v' \subseteq v$, $|v'| = |u|$ and $u' = \pi_{u,v}(v')$ then $\pi_{u', v'} \subseteq \pi_{u,v}$.

Note that if $u \subseteq B$ is of order type ω then we may define

$$N_u = \bigcup \{ N_v : v \text{ is a finite initial segment of } u \}.$$

Then the models N_u (for $u \subseteq B$ of order type $\leq \omega$) have the properties (b)–(h) too.

Let $\langle \beta_\zeta : \zeta < \omega_1 \cdot \omega \rangle$ be the increasing enumeration of B . For a set $u \subseteq B$ of order type $\leq \omega$ let $\bar{\beta}^u$ be the increasing enumeration of u (so $\text{lg}(\bar{\beta}^u) = |u|$). Let $u^* = \{ \beta_{\omega_1 \cdot n} : n < \omega \}$. For $k \leq \omega$ and a sequence $\bar{\xi} = \langle \xi_m : m < k \rangle \subseteq \omega_1$ we define

$$u[\bar{\xi}] = \{ \beta_{\omega_1 \cdot m + \xi_m} : m < k \} \cup \{ \beta_{\omega_1 \cdot n} : n \in \omega \setminus k \}.$$

Now, working in $\mathbf{V}^{\mathbb{C}_\chi}$, we say that a sequence $\bar{\xi}$ is k -strange if

- (1) $\bar{\xi}$ is a sequence of countable ordinals greater than 0, $\text{lg}(\bar{\xi}) = k \leq \omega$,
- (2) $(\forall m < \omega)(F_m(\bar{\beta}^{u[\bar{\xi}]} \upharpoonright m) = F_m(\bar{\beta}^{u^*} \upharpoonright m))$.

CLAIM 1.1.1. In $\mathbf{V}^{\mathbb{C}_\chi}$, if $\bar{\xi}^k$ are k -strange sequences (for $k < \omega$) such that $(\forall k < \omega)(\bar{\xi}^k \triangleleft \bar{\xi}^{k+1})$ then the sequence $\bar{\xi} := \bigcup_{k < \omega} \bar{\xi}^k$ is ω -strange.

Proof. Should be clear (note that in this situation we have $\bar{\beta}^{u[\bar{\xi}]} \upharpoonright m = \bar{\beta}^{u[\bar{\xi}^m]} \upharpoonright m$). ■

CLAIM 1.1.2. $p \Vdash_{\mathbb{C}_\chi}$ “there are no ω -strange sequences”.

Proof. Assume not. Then we find a name $\bar{\xi} = \langle \xi_m : m < \omega \rangle$ for an ω -sequence and a condition $q \geq p$ such that

$$q \Vdash_{\mathbb{C}_\chi} “(\forall m < \omega)(0 < \xi_m < \omega_1 \ \& \ \underline{F}_m(\bar{\beta}^{u[\bar{\xi}]}) \upharpoonright m = \underline{F}_m(\bar{\beta}^{u^*} \upharpoonright m))”.$$

By the choice of p and \underline{F}_m we conclude that

$$q \Vdash_{\mathbb{C}_\chi} “(\forall^\infty m)(\bar{\beta}^{u[\bar{\xi}]}(m) = \bar{\beta}^{u^*}(m))”,$$

which contradicts the definition of $\bar{\beta}^{u[\bar{\xi}]}$, $\bar{\beta}^{u^*}$, Definition 0.1 and the fact that

$$q \Vdash_{\mathbb{C}_\chi} “(\forall m < \omega)(0 < \xi_m < \omega_1)”.$$
 ■

By 1.1.1, 1.1.2, any inductive attempt to construct (in $\mathbf{V}^{\mathbb{C}_\chi}$) an ω -strange sequence $\bar{\xi}$ has to fail. Consequently, we find a condition $p^* \geq p$, an integer $k < \omega$ and a sequence $\bar{\xi} = \langle \xi_l : l < k \rangle$ such that

$$p^* \Vdash_{\mathbb{C}_\chi} “\bar{\xi} \text{ is } k\text{-strange but } \neg(\exists \xi < \omega_1)(\bar{\xi} \widehat{\ } \langle \xi \rangle \text{ is } (k+1)\text{-strange})”.$$

Then in particular

$$(\boxtimes) \quad p^* \Vdash_{\mathbb{C}_\chi} “(\forall m < \omega)(\underline{F}_m(\bar{\beta}^{u[\bar{\xi}]}) \upharpoonright m = \underline{F}_m(\bar{\beta}^{u^*} \upharpoonright m))”.$$

[It may happen that $k = 0$, i.e., $\bar{\xi} = \langle \rangle$.]

For $\xi < \omega_1$ let $u_\xi = u[\bar{\xi} \widehat{\ } \langle \xi \rangle]$ and $w_\xi = u_\xi \cup (u^* \setminus \{\omega_1 \cdot k\})$. Thus $w_0 = u[\bar{\xi}] \cup u^*$ and all w_ξ have order type ω and $\pi_{w_{\xi_1}, w_{\xi_2}}$ is the identity on $N_{w_\xi \setminus \{\omega_1 \cdot k + \xi_2\}}$. Let $q := p^* \upharpoonright N_{w_0}$ and $q_\xi = \pi_{w_\xi, w_0}(q) \in N_{w_\xi}$ (so $q_0 = q$). As the isomorphism π_{w_ξ, w_0} is the identity on $N_{w_0} \cap N_{w_\xi} = N_{w_0 \cap w_\xi}$ (and by the definition of Cohen forcing), we see that the conditions q, q_ξ are compatible. Moreover, as $p^* \geq p$ and $p \in N_\emptyset$, we find that both q and q_ξ are stronger than p .

Now fix $\xi_0 \in (0, \omega_1)$ (e.g. $\xi_0 = 1$) and look at the sequences $\bar{\beta}^{u_{\xi_0}}$ and $\bar{\beta}^{u^*}$. They are eventually equal and hence

$$p \Vdash_{\mathbb{C}_\chi} “(\forall^\infty m)(\underline{F}_m(\bar{\beta}^{u_{\xi_0}} \upharpoonright m) = \underline{F}_m(\bar{\beta}^{u^*} \upharpoonright m))”.$$

So we find $m^* < \omega$ and a condition $q'_{\xi_0} \geq q_{\xi_0}, q$ such that

$$(\otimes_{q'_{\xi_0}}^{\xi_0, m^*}) \quad q'_{\xi_0} \Vdash_{\mathbb{C}_\chi} “(\forall m \geq m^*)(\underline{F}_m(\bar{\beta}^{u_{\xi_0}} \upharpoonright m) = \underline{F}_m(\bar{\beta}^{u^*} \upharpoonright m))”$$

and (as we can increase q'_{ξ_0})

$$(\oplus_{q'_{\xi_0}}^{\xi_0, m^*}) \quad q'_{\xi_0} \text{ decides the values of } \underline{F}_m(\bar{\beta}^{u_{\xi_0}} \upharpoonright m) \text{ and } \underline{F}_m(\bar{\beta}^{u^*} \upharpoonright m) \text{ for all } m \leq m^*.$$

Note that the condition $(\otimes_{q'_{\xi_0}}^{\xi_0, m^*})$ means that there are NO $m \geq m^*, l_0, l_1 < \omega$ with $\gamma_{\bar{\beta}^{u_{\xi_0}} \upharpoonright m, l_0}^m \neq \gamma_{\bar{\beta}^{u^*} \upharpoonright m, l_1}^m$ and the three conditions $q'_{\xi_0}, p_{\bar{\beta}^{u_{\xi_0}} \upharpoonright m, l_0}^m$ and $p_{\bar{\beta}^{u^*} \upharpoonright m, l_1}^m$ have a common upper bound in \mathbb{C}_χ (remember the choice of the $p_{\bar{\alpha}, l}^n$'s and $\gamma_{\bar{\alpha}, l}^n$'s). Similarly, the condition $(\oplus_{q'_{\xi_0}}^{\xi_0, m^*})$ means there are

NO $m \leq m^*$, $l_0, l_1 < \omega$ with either $\gamma_{\bar{\beta}^{u_{\xi_0}} \upharpoonright m, l_0}^m \neq \gamma_{\bar{\beta}^{u_{\xi_0}} \upharpoonright m, l_1}^m$ and both q'_{ξ_0} and $p_{\bar{\beta}^{u_{\xi_0}} \upharpoonright m, l_0}^m$, and q'_{ξ_0} and $p_{\bar{\beta}^{u_{\xi_0}} \upharpoonright m, l_1}^m$ are compatible in \mathbb{C}_χ , or $\gamma_{\bar{\beta}^{u^*} \upharpoonright m, l_0}^m \neq \gamma_{\bar{\beta}^{u^*} \upharpoonright m, l_1}^m$ and both q'_{ξ_0} and $p_{\bar{\beta}^{u^*} \upharpoonright m, l_0}^m$, and q'_{ξ_0} and $p_{\bar{\beta}^{u^*} \upharpoonright m, l_1}^m$ are compatible in \mathbb{C}_χ .

Consequently, the condition $q_{\xi_0}^* := q'_{\xi_0} \upharpoonright N_{w_0 \cup w_{\xi_0}}$ has both properties $(\otimes_{q_{\xi_0}^*}^{\xi_0, m^*})$ and $(\oplus_{q_{\xi_0}^*}^{\xi_0, m^*})$ (and it is stronger than both q and q_{ξ_0}).

Now, for $0 < \xi < \omega_1$ let

$$q_\xi^* := \pi_{w_0 \cup w_\xi, w_0 \cup w_{\xi_0}}(q_{\xi_0}^*) \in N_{w_0 \cup w_\xi}.$$

Then (for $\xi \in (0, \omega_1)$) the condition q_ξ^* is stronger than

$$\text{both } q = \pi_{w_0 \cup w_\xi, w_0 \cup w_{\xi_0}}(q) \text{ and } q_\xi = \pi_{w_0 \cup w_\xi, w_0 \cup w_{\xi_0}}(q_{\xi_0})$$

and it has the properties $(\otimes_{q_\xi^*}^{\xi, m^*})$ and $(\oplus_{q_\xi^*}^{\xi, m^*})$. Moreover for all ξ_1, ξ_2 the conditions $q_{\xi_1}^*, q_{\xi_2}^*$ are compatible. [Why? By the definition of Cohen forcing, and $\pi_{w_0 \cup w_{\xi_2}, w_0 \cup w_{\xi_1}}(q_{\xi_1}^*) = q_{\xi_2}^*$ (chasing arrows) and $\pi_{w_0 \cup w_{\xi_2}, w_0 \cup w_{\xi_1}}$ is the identity on $N_{w_0 \cup w_{\xi_2}} \cap N_{w_0 \cup w_{\xi_1}} = N_{(w_0 \cup w_{\xi_2}) \cap (w_0 \cup w_{\xi_1})}$ (see clauses (e), (f), (h) above).]

CLAIM 1.1.3. For each $\xi_1, \xi_2 \in (0, \omega_1)$ the condition $q_{\xi_1}^* \cup q_{\xi_2}^*$ forces in \mathbb{C}_χ that

$$(\forall m < \omega)(\underline{F}_m(\bar{\beta}^{u_{\xi_1}} \upharpoonright m) = \underline{F}_m(\bar{\beta}^{u_{\xi_2}} \upharpoonright m)).$$

Proof. If $m \geq m^*$ then, by $(\otimes_{q_{\xi_1}^*}^{\xi_1, m^*})$ and $(\otimes_{q_{\xi_2}^*}^{\xi_2, m^*})$ (passing through $\underline{F}(\bar{\beta}^{u^*} \upharpoonright m)$), we get

$$q_{\xi_1}^* \cup q_{\xi_2}^* \Vdash_{\mathbb{C}_\chi} \text{“}\underline{F}_m(\bar{\beta}^{u_{\xi_1}} \upharpoonright m) = \underline{F}_m(\bar{\beta}^{u_{\xi_2}} \upharpoonright m)\text{”}.$$

If $m < m^*$ then we use $(\oplus_{q_{\xi_1}^*}^{\xi_1, m^*})$ and $(\oplus_{q_{\xi_2}^*}^{\xi_2, m^*})$ and the isomorphism: the values assigned by $q_{\xi_1}^*, q_{\xi_2}^*$ to $\underline{F}_m(\bar{\beta}^{u_{\xi_1}} \upharpoonright m)$ and $\underline{F}_m(\bar{\beta}^{u_{\xi_2}} \upharpoonright m)$ have to be equal (remember $\kappa \subseteq N_\emptyset$, so the isomorphism is the identity on κ). ■

Look at the conditions

$$q_{\xi_1, \xi_2} := q_{\xi_1}^* \upharpoonright N_{w_{\xi_1}} \cup q_{\xi_2}^* \upharpoonright N_{w_{\xi_2}} \in N_{w_{\xi_1} \cup w_{\xi_2}}.$$

It should be clear that for each $\xi_1, \xi_2 \in (0, \omega_1)$,

$$q_{\xi_1, \xi_2} \Vdash_{\mathbb{C}_\chi} \text{“}(\forall m < \omega)(\underline{F}_m(\bar{\beta}^{u_{\xi_1}} \upharpoonright m) = \underline{F}_m(\bar{\beta}^{u_{\xi_2}} \upharpoonright m))\text{”}.$$

Now choose $\xi \in (0, \omega_1)$ so large that

$$\text{dom}(p^*) \cap (N_{w_\xi} \setminus N_{w_0}) = \emptyset$$

(possible as $\text{dom}(p^*)$ is finite, use (e)). Take any $0 < \xi_1 < \xi_2 < \omega_1$ and put

$$q^* := \pi_{w_0 \cup w_\xi, w_{\xi_1} \cup w_{\xi_2}}(q_{\xi_1, \xi_2}).$$

(Note: $\pi_{w_0, w_{\xi_1}} \subseteq \pi_{w_0 \cup w_{\xi_1}, w_{\xi_1} \cup w_{\xi_2}}$ and $\pi_{w_{\xi_1}, w_{\xi_2}} \subseteq \pi_{w_0 \cup w_{\xi_1}, w_{\xi_1} \cup w_{\xi_2}}$.) By the isomorphism we get

$$q^* \Vdash_{\mathbb{C}_\chi} “(\forall m < \omega)(F_m(\bar{\beta}^{u_\xi} \upharpoonright m) = F_m(\bar{\beta}^{u[\bar{\xi}]}) \upharpoonright m)”.$$

Now look back:

$$\begin{aligned} q_{\xi_1}^* &\geq q_{\xi_1} = \pi_{w_0 \cup w_{\xi_1}, w_0 \cup w_{\xi_0}}(q_{\xi_0}) = \pi_{w_{\xi_1}, w_{\xi_0}}(q_{\xi_0}) \\ &= \pi_{w_{\xi_1}, w_{\xi_0}}(\pi_{w_{\xi_0}, w_0}(q)) = \pi_{w_{\xi_1}, w_0}(q) \end{aligned}$$

and hence

$$q_{\xi_1}^* \upharpoonright N_{w_{\xi_1}} \geq \pi_{w_{\xi_1}, w_0}(q)$$

and thus

$$q^* \upharpoonright N_{w_0} \geq \pi_{w_0, w_{\xi_1}}(q_{\xi_1}^* \upharpoonright N_{w_{\xi_1}}) \geq q = p^* \upharpoonright N_{w_0}.$$

Consequently, by the choice of ξ , the conditions q^* and p^* are compatible (remember the definition of q_{ξ_1, ξ_2} and q^*). Now use (\boxtimes) to conclude that

$$q^* \cup p^* \Vdash_{\mathbb{C}_\chi} “(\forall m < \omega)(F_m(\bar{\beta}^{u^*} \upharpoonright m) = F_m(\bar{\beta}^{u[\bar{\xi}]} \upharpoonright m) = F_m(\bar{\beta}^{u_\xi} \upharpoonright m))”,$$

which implies that $q^* \cup p^* \Vdash_{\mathbb{C}_\chi} “\bar{\xi} \frown \langle \xi \rangle$ is $(k+1)$ -strange”, a contradiction. ■

REMARK 1.2. About the proof of 1.1:

- (1) No harm is done by forgetting 0 and replacing it by ξ_1, ξ_2 .
- (2) A small modification of the proof shows that in $\mathbf{V}^{\mathbb{C}_\chi}$: If $F_n : {}^n \lambda \rightarrow \kappa$ ($n \in \omega$) are such that

$$(\forall \eta, \nu \in {}^\omega \lambda)[(\forall^\infty n)(\eta(n) = \nu(n)) \Rightarrow (\forall^\infty n)(F_n(\eta \upharpoonright n) = F_n(\nu \upharpoonright n))]$$

then there are infinite sets $X_n \subseteq \lambda$ (for $n < \omega$) such that

$$(\forall n < \omega) \left(\forall \nu, \eta \in \prod_{l < n} X_l \right) (F_n(\nu) = F_n(\eta)).$$

Say we shall have $X_n = \{\gamma_{n,i} : i < \omega\}$. Starting we have $\gamma_0^*, \dots, \gamma_n^*, \dots$. In the proof at stage n we have determined $\gamma_{l,i}$ ($l, i < n$) and $p \in G$, $p \in N_{\{\gamma_{l,i} : l, i < \omega\} \cup \{\gamma_n^*, \gamma_{n+1}^*, \dots\}}$. For $n = 0, 1, 2$ as before. For $n+1 > 2$ first $\gamma_{0,n}, \dots, \gamma_{n-1,n}$ are easy by transitivity of equalities. Then find $\gamma_{n,0}, \gamma_{n,1}$ as before, and then again duplicate.

(3) In the proof it is enough to use $\{\beta_{\omega \cdot n + l} : n < \omega, l < \omega\}$. Hence, by 1.2 of [Sh 481] it is enough to assume $\lambda \rightarrow (\omega^3)_{2^\kappa}^{< \omega}$. This condition is compatible with $\mathbf{V} = \mathbf{L}$.

(4) We can use only $\lambda \rightarrow (\omega^2)_{2^\kappa}^{< \omega}$.

DEFINITION 1.3. (1) For a sequence $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$ of cardinals we define the property $(\otimes)_{\bar{\lambda}}$:

- $(\otimes)_{\bar{\lambda}}$ for every model M of a countable language with universe $\sup_{n \in \omega} \lambda_n$ and Skolem functions (for simplicity) there is a sequence $\langle X_n : n < \omega \rangle$ such that

- (a) $X_n \in [\lambda_n]^{\lambda_n}$ (actually $X_n \in [\lambda_n]^{\omega_1}$ suffices)
 (b) for every $n < \omega$ and $\bar{\alpha} = \langle \alpha_l : l \in [n+1, \omega) \rangle \in \prod_{l \geq n+1} X_l$,
 letting (for $\xi \in X_n$)

$$M_{\bar{\alpha}}^{\xi} = \text{Sk} \left(\bigcup_{l < n} X_l \cup \{\xi\} \cup \{\alpha_l : l \in [n+1, \omega)\} \right)$$

we have:

- (\oplus) the sequence $\langle M_{\bar{\alpha}}^{\xi} : \xi \in X_n \rangle$ forms a Δ -system with heart $N_{\bar{\alpha}}$ and its elements are pairwise isomorphic over the heart $N_{\bar{\alpha}}$.

(2) For a cardinal λ the condition (\otimes) $^{\lambda}$ is:

- (\otimes) $^{\lambda}$ there exists a sequence $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$ such that $\sum_{n < \omega} \lambda_n = \lambda$ and the condition (\otimes) $_{\bar{\lambda}}$ holds true.

In [Sh 76] a condition $(*)_{\lambda}$, weaker than (\otimes) $^{\lambda}$, was considered. Now, [Sh 124] continues [Sh 76] to get stronger indiscernibility. But by the same proof (using ω -measurable) one can show the consistency of (\otimes) $^{\aleph_{\omega}} + \text{GCH}$.

Note that to carry out the proof of 1.1 we need even less than (\otimes) $^{\lambda}$: the $\bigcup_{l < n} X_l$ (in (b) of 1.3) is much more than needed; it suffices to have $\bar{\beta}^0 \cup \bar{\beta}^1$ where $\bar{\beta}^0, \bar{\beta}^1 \in \prod_{l < n} X_l$.

CONCLUSION 1.4. It is consistent that

$$2^{\aleph_0} = \aleph_{\omega+1} \quad \text{and} \quad \bigwedge_{n < \omega} \neg \mathcal{KL}(\aleph_{\omega}, \aleph_n) \quad \text{so} \quad \neg \mathcal{KL}(2^{\aleph_0}, 2).$$

REMARK 1.5. Koepke [Ko84] continues [Sh 76] to get equiconsistency. His refinement of [Sh 76] (for the upper bound) works below too.

2. The positive result. For an algebra M on λ and a set $X \subseteq \lambda$ the closure of X under functions of M is denoted by $\text{cl}_M(X)$. Before proving our result (2.6) we remind the reader of some definitions and propositions.

PROPOSITION 2.1. For an algebra M on λ the following conditions are equivalent:

- (\star) $_M^0$ for each sequence $\langle \alpha_n : n \in \omega \rangle \subseteq \lambda$ we have

$$(\forall^{\infty} n)(\alpha_n \in \text{cl}_M(\{\alpha_k : n < k < \omega\})),$$

- (\star) $_M^1$ there is no sequence $\langle A_n : n \in \omega \rangle \subseteq [\lambda]^{\aleph_0}$ such that

$$(\forall n \in \omega)(\text{cl}_M(A_{n+1}) \not\subseteq \text{cl}_M(A_n)),$$

- (\star) $_M^2$ $(\forall A \in [\lambda]^{\aleph_0})(\exists B \in [A]^{\aleph_0})(\forall C \in [B]^{\aleph_0})(\text{cl}_M(B) = \text{cl}_M(C))$.

DEFINITION 2.2. We say that a cardinal λ has the (\star)-property for κ (and then we write $\text{Pr}^{\star}(\lambda, \kappa)$) if there is an algebra M on λ with vocabulary

of cardinality $\leq \kappa$ satisfying one (equivalently: all) of the conditions $(\star)_M^i$ ($i < 3$) of 2.1. If $\kappa = \aleph_0$ we may omit it.

Remember

PROPOSITION 2.3. *If $\mathbf{V}_0 \subseteq \mathbf{V}_1$ are universes of set theory and $\mathbf{V}_1 \models \neg \text{Pr}^\star(\lambda)$ then $\mathbf{V}_0 \models \neg \text{Pr}^\star(\lambda)$.*

PROOF. By absoluteness of the existence of an ω -branch of a tree. ■

REMARK 2.4. The property $\neg \text{Pr}^\star(\lambda)$ is a kind of large cardinal property. It was clarified in \mathbf{L} (remember that it is inherited from \mathbf{V} to \mathbf{L}) by Silver [Si70] to be equiconsistent with “there is a beautiful cardinal” (terminology of 2.3 of [Sh 110]), another partition property inherited by \mathbf{L} . More in [Sh 513].

PROPOSITION 2.5. *For each $n \in \omega$, $\text{Pr}^\star(\aleph_n)$.*

PROOF. This was done in [Sh:b, Chapter XIII], see [Sh:g, Chapter VII] too, and probably earlier by Silver. However, for the sake of completeness we will give the proof.

First note that clearly $\text{Pr}^\star(\aleph_0)$ and thus we have to deal with the case when $n > 0$. Let $f, g : \aleph_n \rightarrow \aleph_n$ be two functions such that if $m < n$, $\alpha \in [\aleph_m, \aleph_{m+1})$ then $f(\alpha, \cdot) \upharpoonright \alpha : \alpha \xrightarrow{1-1} \aleph_m$, $g(\alpha, \cdot) \upharpoonright \aleph_m : \aleph_m \xrightarrow{1-1} \alpha$ are functions inverse to each other.

Let M be the following algebra on \aleph_n :

$$M = (\aleph_n, f, g, m)_{m \in \omega}.$$

We want to check the condition $(\star)_M^1$: assume that a sequence $\langle A_k : k < \omega \rangle \subseteq [\aleph_n]^{\aleph_0}$ is such that for each $k < \omega$,

$$\text{cl}_M(A_{k+1}) \not\subseteq \text{cl}_M(A_k).$$

For each $m < n$, the sequence $\langle \sup(\text{cl}_M(A_k) \cap \aleph_{m+1}) : k < \omega \rangle$ is non-increasing and therefore it is eventually constant. Consequently, we find k^* such that

$$(\forall m < n)(\sup(\text{cl}_M(A_{k^*+1}) \cap \aleph_{m+1}) = \sup(\text{cl}_M(A_{k^*}) \cap \aleph_{m+1})).$$

By the choice of $\langle A_k : k < \omega \rangle$ we have $\text{cl}_M(A_{k^*+1}) \not\subseteq \text{cl}_M(A_{k^*})$. Let

$$\alpha_0 := \min(\text{cl}_M(A_{k^*}) \setminus \text{cl}_M(A_{k^*+1})).$$

As the model M contains individual constants m (for $m \in \omega$) we know that $\aleph_0 \subseteq \text{cl}_M(\emptyset)$ and hence $\aleph_0 \leq \alpha_0$. Let $m < n$ be such that $\aleph_m \leq \alpha_0 < \aleph_{m+1}$. By the choice of k^* we find $\beta \in \text{cl}_M(A_{k^*+1}) \cap \aleph_{m+1}$ such that $\alpha_0 \leq \beta$. Then necessarily $\alpha_0 < \beta$. Look at $f(\beta, \alpha_0)$: we know that $\alpha_0, \beta \in \text{cl}_M(A_{k^*})$ and therefore $f(\beta, \alpha_0) \in \text{cl}_M(A_{k^*}) \cap \aleph_m$ and $f(\beta, \alpha_0) < \alpha_0$. The minimality of α_0 implies that $f(\beta, \alpha_0) \in \text{cl}_M(A_{k^*+1})$ and hence

$$\alpha_0 = g(\beta, f(\beta, \alpha_0)) \in \text{cl}_M(A_{k^*+1}),$$

a contradiction. ■

EXPLANATION. Better think of the proof below from the end. Let $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle \in {}^\omega \lambda$. So for some $n(*), n(*) \leq n < \omega \Rightarrow \alpha_n \in \text{cl}_M(\alpha_l : l > n)$. So for some $m_n > n$, $\{\alpha_{n(*)}, \dots, \alpha_{m_n-1}\} \subseteq \text{cl}_M(\alpha_n, \dots, \alpha_{m_n-1})$ and

$$(\forall l < n(*))(\alpha_l \in \text{cl}_M(\alpha_k : k > n(*))) \Rightarrow \alpha_l \in \text{cl}_M(\alpha_k : k \in [n, m_n]).$$

Let $w = \{l < n(*) : \alpha_l \in \text{cl}_M(\alpha_n : n \geq n(*)\}$. It is natural to aim at:

- (*) for n large enough (say $n > m_{n(*)}$), $F_n(\langle \alpha_l : l < n \rangle)$ depends just on $\{\alpha_l : l \in [n(*), n] \text{ or } l \in w\}$ and $\langle F_m(\bar{\alpha} \upharpoonright m) : m \geq n \rangle$ codes $\bar{\alpha} \upharpoonright (w \cup [n(*), \omega])$.

Of course, we are given an n and we do not know how to compute the real $n(*)$, but we can approximate. Then we look at a late enough end segment where we compute down.

THEOREM 2.6. *Assume that $\lambda \leq 2^{\aleph_0}$ is such that $\text{Pr}^\star(\lambda)$ holds. Then $\mathcal{KL}(\lambda, \omega)$ (and hence $\mathcal{KL}(\lambda, 2)$).*

PROOF. We have to construct functions $F_n : {}^n \lambda \rightarrow \omega$ witnessing $\mathcal{KL}(\lambda, \omega)$. For this we will introduce functions \mathbf{k} and \mathbf{l} such that for $\bar{\alpha} \in {}^n \lambda$ the value of $\mathbf{k}(\bar{\alpha})$ will say which initial segment of $\bar{\alpha}$ will be irrelevant for $F_n(\bar{\alpha})$ and $\mathbf{l}(\bar{\alpha})$ will be such that (under certain circumstances) elements α_i (for $\mathbf{k}(\bar{\alpha}) \leq i < \mathbf{l}(\bar{\alpha})$) will be encoded by $\langle \alpha_j : j \in [\mathbf{l}(\bar{\alpha}), n] \rangle$.

Fix a sequence $\langle \eta_\alpha : \alpha < \lambda \rangle \subseteq {}^\omega 2$ with no repetitions.

Let M be an algebra on λ such that $(\star)_M^0$ holds true. We may assume that there are no individual constants in M (so $\text{cl}_M(\emptyset) = \emptyset$).

Let $\langle \tau_l^n(x_0, \dots, x_{n-1}) : l < \omega \rangle$ list all n -place terms of the language of the algebra M (and $\tau_0^1(x)$ is x) when $0 < n < \omega$. For $\bar{\alpha} \in {}^\omega \lambda$ (with α_j the j th element in $\bar{\alpha}$) let

$$u(\bar{\alpha}) = \{l < \text{lg}(\bar{\alpha}) : \alpha_l \notin \text{cl}_M(\bar{\alpha} \upharpoonright (l, \text{lg}(\bar{\alpha})))\} \cup \{0\}$$

and for $l \notin u(\bar{\alpha})$, $l < \text{lg}(\bar{\alpha})$ let

$$f_l(\bar{\alpha}) = \min\{j : \alpha_l \in \text{cl}_M(\bar{\alpha} \upharpoonright (l, j))\},$$

$$g_l(\bar{\alpha}) = \min\{i : \alpha_i = \tau_i^{f_l(\bar{\alpha})-l-1}(\bar{\alpha} \upharpoonright (l, f_l(\bar{\alpha})))\}.$$

For $\bar{\alpha} \in {}^n \lambda$ ($1 < n < \omega$) put

$$k_1(\bar{\alpha}) = \min((u(\bar{\alpha} \upharpoonright (n-1)) \setminus u(\bar{\alpha})) \cup \{n-1\}),$$

$$k_0(\bar{\alpha}) = \max(u(\bar{\alpha}) \cap k_1(\bar{\alpha})).$$

Note that if ($n > 1$ and) $\bar{\alpha} \in {}^n \lambda$ then $n-1 \in u(\bar{\alpha})$ (as $\text{cl}_M(\emptyset) = \emptyset$) and $k_1(\bar{\alpha}) > 0$ (as always $0 \in u(\bar{\beta})$) and $k_0(\bar{\alpha})$ is well defined (as $0 \in u(\bar{\alpha}) \cap k_1(\bar{\alpha})$) and $k_0(\bar{\alpha}) < k_1(\bar{\alpha}) < n$. Moreover, for all $l \in (k_0(\bar{\alpha}), k_1(\bar{\alpha}))$ we have $\alpha_l \notin u(\bar{\alpha})$ by the choice of $k_0(\bar{\alpha})$, hence $\alpha_l \notin u(\bar{\alpha} \upharpoonright (n-1))$ by the choice of $k_1(\bar{\alpha})$ and thus $\alpha_l \in \text{cl}_M(\bar{\alpha} \upharpoonright (l, n-1))$. Now, for $\bar{\alpha} \in {}^\omega \lambda$, $\text{lg}(\bar{\alpha}) > 1$ we define

$$\mathbf{l}(\bar{\alpha}) = \max\{j \leq k_1(\bar{\alpha}) : j > k_0(\bar{\alpha}) \Rightarrow (\forall i \in (k_0(\bar{\alpha}), j))(g_i(\bar{\alpha}) \leq \text{lg}(\bar{\alpha}))\},$$

$$\mathbf{m}(\bar{\alpha}) = \max\{j \leq \mathbf{l}(\bar{\alpha}) : j > \max\{1, k_0(\bar{\alpha})\} \Rightarrow k_0(\bar{\alpha} \upharpoonright j) = k_0(\bar{\alpha})\},$$

$$\mathbf{k}(\bar{\alpha}) = \mathbf{l}(\bar{\alpha} \upharpoonright \mathbf{m}(\bar{\alpha})) \quad (\text{if } \mathbf{m}(\bar{\alpha}) \leq 1 \text{ then put } \mathbf{k}(\bar{\alpha}) = -1).$$

Clearly $\mathbf{k}(\bar{\alpha}) < \mathbf{m}(\bar{\alpha}) \leq \mathbf{l}(\bar{\alpha}) \leq k_1(\bar{\alpha}) < \lg(\bar{\alpha})$.

CLAIM 2.6.1. *For each $\bar{\alpha} \in {}^\omega \lambda$, the set $u(\bar{\alpha})$ is finite and:*

- (1) *The sequence $\langle k_1(\bar{\alpha} \upharpoonright n) : n < \omega \rangle$ diverges to ∞ .*
- (2) *The sequence $\langle k_0(\bar{\alpha} \upharpoonright n) : n < \omega \ \& \ k_0(\bar{\alpha} \upharpoonright n) \neq \max u(\bar{\alpha}) \rangle$, if infinite, diverges to ∞ . There are infinitely many $n < \omega$ with $k_0(\bar{\alpha} \upharpoonright n) = \max u(\bar{\alpha})$.*
- (3) *The sequence $\langle \mathbf{l}(\bar{\alpha} \upharpoonright n) : n < \omega \rangle$ diverges to ∞ .*
- (4) *The sequences $\langle \mathbf{m}(\bar{\alpha} \upharpoonright n) : n < \omega \rangle$ and $\langle \mathbf{k}(\bar{\alpha} \upharpoonright n) : n < \omega \rangle$ diverge to ∞ .*

PROOF. Let $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle \in {}^\omega \lambda$. By the property $(\star)_M^0$ we find $n^* < \omega$ such that $u(\bar{\alpha}) \subseteq n^*$. Fix $n_0 > n^*$ and define

$$n_1 = \max\{f_n(\bar{\alpha}) + g_n(\bar{\alpha}) + 2 : n \in (n_0 + 1) \setminus u(\bar{\alpha})\}$$

(so as $\text{cl}_M(\emptyset) = \emptyset$ we have $n_1 \geq f_{n_0}(\bar{\alpha}) + 2 > n_0 + 3$ and for $l \in (n_0 + 1) \setminus u(\bar{\alpha})$, $\alpha_l \in \text{cl}_M(\alpha_{l+1}, \dots, \alpha_{n_1-1})$ is witnessed by $\tau_{g_l(\bar{\alpha})}^{f_l(\bar{\alpha})-l-1}(\alpha_{l+1}, \dots, \alpha_{f_l(\bar{\alpha})-1})$ with $f_l(\bar{\alpha}), g_l(\bar{\alpha}) < n_1 - 1$).

(1) Note that $u(\bar{\alpha} \upharpoonright n) \cap (n_0 + 1) = u(\bar{\alpha})$ for all $n \geq n_1 - 1$ and hence for $n \geq n_1$,

$$u(\bar{\alpha} \upharpoonright n) \cap (n_0 + 1) = u(\bar{\alpha} \upharpoonright (n - 1)) \cap (n_0 + 1).$$

Consequently, for all $n \geq n_1$ we have $k_1(\bar{\alpha} \upharpoonright n) > n_0$. As we could have chosen n_0 arbitrarily large we may conclude that $\lim_{n \rightarrow \infty} k_1(\bar{\alpha} \upharpoonright n) = \infty$.

(2) Note that for all $n \geq n_1$,

$$\text{either } k_0(\bar{\alpha} \upharpoonright n) = \max(u(\bar{\alpha})) \quad \text{or } k_0(\bar{\alpha} \upharpoonright n) > n_0.$$

Hence, by the arbitrariness of n_0 , we get the first part of (2).

Let $l^* = \min(u(\bar{\alpha} \upharpoonright n_1) \setminus u(\bar{\alpha}))$ (note that $n_1 - 1 \in u(\bar{\alpha} \upharpoonright n_1) \setminus u(\bar{\alpha})$). Clearly $l^* > n_0$ and $\alpha_{l^*} \notin u(\bar{\alpha})$. Consider $n = f_{l^*}(\bar{\alpha})$ (so $l^* \leq n - 2$, $n_1 \leq n - 1$). Then $l^* \in u(\bar{\alpha} \upharpoonright (n - 1)) \setminus u(\bar{\alpha} \upharpoonright n)$. As

$$l^* \cap u(\bar{\alpha} \upharpoonright n_1) = l^* \cap u(\bar{\alpha} \upharpoonright (n - 1)) = u(\bar{\alpha})$$

(remember the choice of l^*) we conclude that

$$l^* = k_1(\bar{\alpha} \upharpoonright n) \quad \text{and} \quad k_0(\bar{\alpha} \upharpoonright n) = \max u(\bar{\alpha}).$$

Now, since n_0 was arbitrarily large, we find that for infinitely many n , $k_0(\bar{\alpha} \upharpoonright n) = \max u(\bar{\alpha})$.

(3) Suppose that $n \geq n_1$. Then we know that $k_1(\bar{\alpha} \upharpoonright n) > n_0$ and either $k_0(\bar{\alpha} \upharpoonright n) = \max u(\bar{\alpha})$ or $k_0(\bar{\alpha} \upharpoonright n) > n_0$ (see above). If the first possibility takes place then, as $n \geq n_1$, we may use $j = n_0 + 1$ to witness that $\mathbf{l}(\bar{\alpha} \upharpoonright n) > n_0$ (remember the choice of n_1). If $k_0(\bar{\alpha} \upharpoonright n) > n_0$ then clearly $\mathbf{l}(\bar{\alpha} \upharpoonright n) > n_0$. As n_0 could be arbitrarily large we are done.

(4) Suppose we are given $m_0 < \omega$. Take $m_1 > m_0$ such that for all $n \geq m_1$,

$$\text{either } k_0(\bar{\alpha} \upharpoonright n) = \max u(\bar{\alpha}) \quad \text{or} \quad k_0(\bar{\alpha} \upharpoonright n) > m_0$$

(possible by (2)) and then choose $m_2 > m_1$ such that $k_0(\bar{\alpha} \upharpoonright m_2) = \max u(\bar{\alpha})$ (by (2)). Due to (3) we find $m_3 > m_2$ such that for all $n \geq m_3$, $\mathbf{l}(\bar{\alpha} \upharpoonright n) > m_2$. Now suppose that $n \geq m_3$. If $k_0(\bar{\alpha} \upharpoonright n) = \max u(\bar{\alpha})$ then, as $\mathbf{l}(\bar{\alpha} \upharpoonright n) > m_2$, we get $\mathbf{m}(\bar{\alpha} \upharpoonright n) \geq m_2 > m_0$. Otherwise $k_0(\bar{\alpha} \upharpoonright n) > m_0$ (as $n > m_1$) and hence $\mathbf{m}(\bar{\alpha} \upharpoonright n) > m_0$. This shows that $\lim_{n \rightarrow \infty} \mathbf{m}(\bar{\alpha} \upharpoonright n) = \infty$. Now, immediately by the definition of \mathbf{k} and (3) above we conclude that $\lim_{n \rightarrow \infty} \mathbf{k}(\bar{\alpha} \upharpoonright n) = \infty$. ■

CLAIM 2.6.2. *If $\bar{\alpha}^1, \bar{\alpha}^2 \in {}^\omega \lambda$ are such that $(\forall^\infty n)(\alpha_n^1 = \alpha_n^2)$ then*

$$(\forall^\infty n)(\mathbf{l}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{l}(\bar{\alpha}^2 \upharpoonright n) \ \& \ \mathbf{m}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{m}(\bar{\alpha}^2 \upharpoonright n) \ \& \ \mathbf{k}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{k}(\bar{\alpha}^2 \upharpoonright n)).$$

Proof. Let n_0 be greater than $\max(u(\bar{\alpha}^1) \cup u(\bar{\alpha}^2))$ and such that

$$\bar{\alpha}^1 \upharpoonright [n_0, \omega) = \bar{\alpha}^2 \upharpoonright [n_0, \omega).$$

For $k = 1, 2, 3$ define n_k by

$$n_{k+1} = \max\{f_n(\bar{\alpha}^i) + g_n(\bar{\alpha}^i) + 2 : n \in (n_k + 1) \setminus u(\bar{\alpha}^i), \ i < 2\}.$$

As in the proof of 2.6.1, for $i = 1, 2$ and $j < 3$ we have:

$$(\otimes^1) \quad (\forall n \geq n_{j+1})(k_0(\bar{\alpha}^i \upharpoonright n) = \max u(\bar{\alpha}^i) \ \text{or} \ k_0(\bar{\alpha}^i \upharpoonright n) > n_j),$$

$$(\otimes^2) \quad (\forall n \geq n_{j+1})(k_1(\bar{\alpha}^i \upharpoonright n) > n_j \ \& \ \mathbf{l}(\bar{\alpha}^i \upharpoonright n) > n_j \ \& \ \mathbf{m}(\bar{\alpha}^i \upharpoonright n) > n_j \ \& \ \mathbf{h}(\bar{\alpha}^i \upharpoonright n) > n_j),$$

$$(\otimes^3) \quad (\exists n' \in (n_1, n_2))(k_0(\bar{\alpha}^1 \upharpoonright n') = \max u(\bar{\alpha}^1) \ \& \ k_0(\bar{\alpha}^2 \upharpoonright n') = \max u(\bar{\alpha}^2))$$

(for (\otimes^3) repeat arguments from 2.6.1(2) and use the fact that $\bar{\alpha}^1 \upharpoonright [n_0, \omega) = \bar{\alpha}^2 \upharpoonright [n_0, \omega)$). Clearly

$$(\otimes^4) \quad (\forall n > n_0)(u(\bar{\alpha}^1 \upharpoonright n) \setminus n_0 = u(\bar{\alpha}^2 \upharpoonright n) \setminus n_0).$$

Hence, applying $(\otimes^4) + (\otimes^2) +$ the definition of $k_1(-)$, we conclude that

$$(\otimes^5) \quad (\forall n \geq n_1)(k_1(\bar{\alpha}^1 \upharpoonright n) = k_1(\bar{\alpha}^2 \upharpoonright n)),$$

and then applying $(\otimes^4) + (\otimes^2) + (\otimes^5) +$ the definition of $k_0(-)$, we get

$$(\otimes^6) \quad \text{for all } n \geq n_1: \text{ either } k_0(\bar{\alpha}^1 \upharpoonright n) = \max u(\bar{\alpha}^1) \ \text{and} \ k_0(\bar{\alpha}^2 \upharpoonright n) = \max u(\bar{\alpha}^2), \ \text{or} \ k_0(\bar{\alpha}^1 \upharpoonright n) = k_0(\bar{\alpha}^2 \upharpoonright n).$$

Since

$$(\forall n \geq n_0)(f_n(\bar{\alpha}^1) = f_n(\bar{\alpha}^2) \ \& \ g_n(\bar{\alpha}^1) = g_n(\bar{\alpha}^2))$$

and by $(\otimes^2) + (\otimes^5) +$ the choice of $n_0 +$ the definition of $\mathbf{l}(-)$, we get (compare the proof of 2.6.1)

$$(\otimes^7) \quad (\forall n \geq n_1)(\mathbf{l}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{l}(\bar{\alpha}^2 \upharpoonright n))$$

and by $(\otimes^2) + (\otimes^7) + (\otimes^6) +$ the definition of $\mathbf{m}(-)$,

$$(\forall n \geq n_3)(\mathbf{m}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{m}(\bar{\alpha}^2 \upharpoonright n) \geq n_2).$$

Moreover, now we easily get

$$(\forall n \geq n_3)(\mathbf{k}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{k}(\bar{\alpha}^2 \upharpoonright n)). \blacksquare$$

For integers $n_0 \leq n_1 \leq n_2$ we define functions $F_{n_0, n_1, n_2}^0 : {}^{n_2}\lambda \rightarrow \mathcal{H}(\aleph_0)$ by letting $F_{n_0, n_1, n_2}^0(\alpha_0, \dots, \alpha_{n_2-1})$ (for $\langle \alpha_0, \dots, \alpha_{n_2-1} \rangle \in {}^{n_2}\lambda$) be the sequence consisting of:

- (a) $\langle n_0, n_1, n_2 \rangle$,
- (b) the set T_{n_1, n_2} of all terms τ_l^n such that $n \leq n_2 - n_1$ and either $l \leq n_2$ (we will call it *the simple case*) or τ_l^n is a composition of depth at most n_2 of such terms,
- (c) $\langle \eta_\alpha \upharpoonright n_2, n, l, \langle i_0, \dots, i_{n-1} \rangle \rangle$ for $n \leq n_2 - n_1$, $i_0, \dots, i_{n-1} \in [n_1, n_2)$ and l such that $\tau_l^n \in T_{n_1, n_2}$ and $\alpha = \tau_l^n(\alpha_{i_0}, \dots, \alpha_{i_{n-1}})$,
- (d) $\langle n, l, \langle i_0, \dots, i_{n-1} \rangle, i \rangle$ for $n \leq n_2 - n_1$, $i_0, \dots, i_{n-1} \in [n_1, n_2)$, $i \in [n_0, n_1)$ and l such that $\tau_l^n \in T_{n_1, n_2}$ and $\alpha_i = \tau_l^n(\alpha_{i_0}, \dots, \alpha_{i_{n-1}})$,
- (e) equalities among appropriate terms, i.e. all tuples

$$\langle n', l', n'', l'', \langle i'_0, \dots, i'_{n'-1} \rangle, \langle i''_0, \dots, i''_{n''-1} \rangle \rangle$$

such that $n_1 \leq i'_0 < \dots < i'_{n'-1} < n_2$, $n_1 \leq i''_0 < \dots < i''_{n''-1} < n_2$, $n', n'' \leq n_2 - n_1$, l', l'' are such that $\tau_{l'}^{n'}, \tau_{l''}^{n''} \in T_{n_1, n_2}$ and

$$\tau_{l'}^{n'}(\alpha_{i'_0}, \dots, \alpha_{i'_{n'-1}}) = \tau_{l''}^{n''}(\alpha_{i''_0}, \dots, \alpha_{i''_{n''-1}}).$$

(Note that the value of $F_{n_0, n_1, n_2}^0(\bar{\alpha})$ does not depend on $\bar{\alpha} \upharpoonright n_0$.)

Finally we define functions $F_n : {}^n\lambda \rightarrow \mathcal{H}(\aleph_0)$ (for $1 < n < \omega$) by:

$$\text{if } \bar{\alpha} \in {}^n\lambda \text{ then } F_n(\bar{\alpha}) = F_{\mathbf{k}(\bar{\alpha}), \mathbf{l}(\bar{\alpha}), n}^0(\bar{\alpha}).$$

As $\mathcal{H}(\aleph_0)$ is countable we may think that these functions are into ω . We are going to show that they witness $\mathcal{KL}(\lambda, \omega)$.

CLAIM 2.6.3. *If $\bar{\alpha}^1, \bar{\alpha}^2 \in {}^\omega\lambda$ are such that $(\forall^\infty n)(\alpha_n^1 = \alpha_n^2)$ then*

$$(\forall^\infty n)(F_n(\bar{\alpha}^1 \upharpoonright n) = F_n(\bar{\alpha}^2 \upharpoonright n)).$$

Proof. Take $m_0 < \omega$ such that for all $n \in [m_0, \omega)$ we have

$$\alpha_n^1 = \alpha_n^2, \quad \mathbf{l}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{l}(\bar{\alpha}^2 \upharpoonright n), \quad \mathbf{k}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{k}(\bar{\alpha}^2 \upharpoonright n)$$

(possible by 2.6.2). Let $m_1 > m_0$ be such that for all $n \geq m_1$,

$$\mathbf{k}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{k}(\bar{\alpha}^2 \upharpoonright n) > m_0$$

(use 2.6.1). Then, for $n \geq m_1$, $i = 1, 2$ we have

$$F_n(\bar{\alpha}^i \upharpoonright n) = F_{\mathbf{k}(\bar{\alpha}^i \upharpoonright n), \mathbf{l}(\bar{\alpha}^i \upharpoonright n), n}^0(\bar{\alpha}^i \upharpoonright n) = F_{\mathbf{k}(\bar{\alpha}^1 \upharpoonright n), \mathbf{l}(\bar{\alpha}^1 \upharpoonright n), n}^0(\bar{\alpha}^i \upharpoonright n).$$

Since the value of $F_{n_0, n_1, n_2}^0(\bar{\beta})$ does not depend on $\bar{\beta} \upharpoonright n_0$ and the sequences $\bar{\alpha}^1 \upharpoonright n$, $\bar{\alpha}^2 \upharpoonright n$ agree on $[m_0, \omega)$, we get

$$F_{\mathbf{k}(\bar{\alpha}^1 \upharpoonright n), \mathbf{l}(\bar{\alpha}^1 \upharpoonright n), n}^0(\bar{\alpha}^1 \upharpoonright n) = F_{\mathbf{k}(\bar{\alpha}^1 \upharpoonright n), \mathbf{l}(\bar{\alpha}^1 \upharpoonright n), n}^0(\bar{\alpha}^2 \upharpoonright n) = F_{\mathbf{k}(\bar{\alpha}^2 \upharpoonright n), \mathbf{l}(\bar{\alpha}^2 \upharpoonright n), n}^0(\bar{\alpha}^2 \upharpoonright n),$$

and hence

$$(\forall n \geq m_1)(F_n(\bar{\alpha}^1 \upharpoonright n) = F_n(\bar{\alpha}^2 \upharpoonright n)). \blacksquare$$

CLAIM 2.6.4. *If $\bar{\alpha}^1, \bar{\alpha}^2 \in {}^\omega \lambda$ and $(\forall^\infty n)(F_n(\bar{\alpha}^1 \upharpoonright n) = F_n(\bar{\alpha}^2 \upharpoonright n))$ then $(\forall^\infty n)(\alpha_n^1 = \alpha_n^2)$.*

Proof. Take $n_0 < \omega$ such that

$$u(\bar{\alpha}^1) \cup u(\bar{\alpha}^2) \subseteq n_0 \quad \text{and} \quad (\forall n \geq n_0)(F_n(\bar{\alpha}^1 \upharpoonright n) = F_n(\bar{\alpha}^2 \upharpoonright n)).$$

Then for all $n \geq n_0$ we have (by clause (a) of the definition of F_{n_0, n_1, n_2}^0)

$$\mathbf{l}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{l}(\bar{\alpha}^2 \upharpoonright n) \quad \text{and} \quad \mathbf{k}(\bar{\alpha}^1 \upharpoonright n) = \mathbf{k}(\bar{\alpha}^2 \upharpoonright n).$$

Further, let $n_1 > n_0$ be such that for all $n \geq n_1$, $\mathbf{k}(\bar{\alpha}^1 \upharpoonright n) > n_0$ and $k_0(\bar{\alpha}^1 \upharpoonright n_1) = \max u(\bar{\alpha}^1)$ (exists by 2.6.1) and choose $n_2 > n_1$ such that $n \geq n_2$ implies $\mathbf{m}(\bar{\alpha}^1 \upharpoonright n) > n_2$.

We are going to show that $\alpha_n^1 = \alpha_n^2$ for all $n > n_1$. Assume not. Then we have $n > n_1$ with $\alpha_n^1 \neq \alpha_n^2$ and thus $\eta_{\alpha_n^1} \neq \eta_{\alpha_n^2}$. Take $n' > n$ such that $\eta_{\alpha_n^1} \upharpoonright n' \neq \eta_{\alpha_n^2} \upharpoonright n'$. Applying 2.6.1(2) and (4) choose $n'' > n'$ such that

$$\mathbf{m}(\bar{\alpha}^1 \upharpoonright n'') > n' \quad \text{and} \quad k_0(\bar{\alpha}^1 \upharpoonright n'') = \max u(\bar{\alpha}^1).$$

Now define inductively: $m_0 = n''$, $m_{k+1} = \mathbf{m}(\bar{\alpha}^1 \upharpoonright m_k)$. Thus

$$n'' = m_0 > \mathbf{l}(\bar{\alpha}^1 \upharpoonright m_0) \geq m_1 > \mathbf{l}(\bar{\alpha}^1 \upharpoonright m_1) \geq m_2 > \dots$$

and (by induction on k)

$$m_k > \max u(\bar{\alpha}^1) \Rightarrow k_0(\bar{\alpha}^1 \upharpoonright m_k) = \max u(\bar{\alpha}^1)$$

(see the definition of \mathbf{m}). Let k^* be the first such that $n \geq m_{k^*}$ (so $k^* \geq 2$, exists by the choice of n_1). Note that by the choice of n_1 above we necessarily have

$$m_{k^*} > \mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*}) = \mathbf{k}(\bar{\alpha}^1 \upharpoonright m_{k^*-1}) > n_0.$$

Hence for all $k < k^*$:

$$F_{m_k}(\bar{\alpha}^1 \upharpoonright m_k) = F_{m_k}(\bar{\alpha}^2 \upharpoonright m_k),$$

$$\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k+1}) = \mathbf{l}(\bar{\alpha}^2 \upharpoonright m_{k+1}) = \mathbf{k}(\bar{\alpha}^1 \upharpoonright m_k) = \mathbf{k}(\bar{\alpha}^2 \upharpoonright m_k).$$

By the definition of the functions $\mathbf{l}, \mathbf{m}, \mathbf{k}$ and the choice of m_0 (remember $k_0(\bar{\alpha}^1 \upharpoonright m_0) = \max u(\bar{\alpha}^1)$) we know that for each $i \in [\mathbf{k}(\bar{\alpha}^1 \upharpoonright m_k), \mathbf{l}(\bar{\alpha}^1 \upharpoonright m_k))$ and $k < k^*$, for some $\tau_l^m \in T_{\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_k), m_k}$ and $i_0, \dots, i_{m-1} \in [\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_k), m_k)$ we have $\alpha_i^1 = \tau_l^m(\alpha_{i_0}^1, \dots, \alpha_{i_{m-1}}^1)$. Moreover we may demand that τ_l^m is a composition of depth at most $\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_k) - i$ of simple case terms. Since

$$F_{\mathbf{k}(\bar{\alpha}^1 \upharpoonright m_k), \mathbf{l}(\bar{\alpha}^1 \upharpoonright m_k), m_k}^0(\bar{\alpha}^1 \upharpoonright m_k) = F_{\mathbf{k}(\bar{\alpha}^2 \upharpoonright m_k), \mathbf{l}(\bar{\alpha}^2 \upharpoonright m_k), m_k}^0(\bar{\alpha}^2 \upharpoonright m_k)$$

we conclude that (by clause (d) of the definition of the functions F_{n_0, n_1, n_2}^0)

$$\alpha_i^2 = \tau_l^m(\alpha_{i_0}^2, \dots, \alpha_{i_{m-1}}^2).$$

Now look at our n . If $\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-1}) > n$ then $\mathbf{k}(\bar{\alpha}^1 \upharpoonright m_{k^*-1}) \leq n < \mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-1})$ and thus we find $i_0, \dots, i_{m-1} \in [\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-1}), m_{k^*-1}]$ and $\tau_l^m \in T_{\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-1}), m_{k^*-1}}$ such that

$$\alpha_n^1 = \tau_l^m(\alpha_{i_0}^1, \dots, \alpha_{i_{m-1}}^1) \quad \text{and} \quad \alpha_n^2 = \tau_l^m(\alpha_{i_0}^2, \dots, \alpha_{i_{m-1}}^2).$$

If $\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-1}) \leq n$ then $n \in [\mathbf{k}(\bar{\alpha}^1 \upharpoonright m_{k^*-2}), \mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-2})]$ (as $\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-1}) = \mathbf{k}(\bar{\alpha}^1 \upharpoonright m_{k^*-2})$ and $n < m_{k^*-1} \leq \mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-2})$). Hence, for some $i_0, \dots, i_{m-1} \in [\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-2}), m_{k^*-2}]$ and $\tau_l^m \in T_{\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-2}), m_{k^*-2}}$, we have

$$\alpha_n^1 = \tau_l^m(\alpha_{i_0}^1, \dots, \alpha_{i_{m-1}}^1) \quad \text{and} \quad \alpha_n^2 = \tau_l^m(\alpha_{i_0}^2, \dots, \alpha_{i_{m-1}}^2).$$

In both cases we may additionally demand that the term τ_l^m is a composition of depth $\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-1}) - n$ (or $\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_{k^*-2}) - n$, respectively) of terms of the simple case. Now we proceed inductively (taking care of the depth of the terms involved) and we find a term $\tau \in T_{\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_0), m_0}$ (which is a composition of depth at most $\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_0) - n$ of terms of the simple case) and $i_0, \dots, i_{m-1} \in [\mathbf{l}(\bar{\alpha}^1 \upharpoonright m_0), m_0]$ such that

$$\alpha_n^1 = \tau(\alpha_{i_0}^1, \dots, \alpha_{i_{m-1}}^1) \quad \text{and} \quad \alpha_n^2 = \tau(\alpha_{i_0}^2, \dots, \alpha_{i_{m-1}}^2).$$

But now applying clause (c) of the definition of the functions F_{n_0, n_1, n_2}^0 we conclude that $\eta_{\alpha_n^1} \upharpoonright m_0 = \eta_{\alpha_n^2} \upharpoonright m_0$, contradicting the choice of n' and the fact that $m_0 > n'$. ■

The last two claims finish the proof of the theorem. ■

REMARK 2.7. If the model M has $\kappa < \lambda$ functions (so $\langle \tau_i^n(x_0, \dots, x_{n-1}) : i < \kappa \rangle$ lists the n -place terms) we can prove $\mathcal{KL}(\lambda, \kappa)$ and the proof is similar.

FINAL REMARKS 2.8. (1) Now we phrase exactly what is needed to carry out the proof of Theorem 1.1 for $\lambda > \kappa$. It is:

- (⊠) for every model M with universe λ and Skolem functions and with countable vocabulary, we can find pairwise distinct $\alpha_{n,l} < \lambda$ (for $n < \omega$, $l < \omega$) such that
- (⊗) if $m_0 < m_1 < \omega$ and $l'_i < l''_i$ for $i < m_0$ and $l_i < \omega$ for $i \in [m_0, m_1]$ and $k_0 < k_1 < k_2 < \omega$ then the models

$$(\text{Sk}(\{\alpha_{i,l'_i}, \alpha_{i,l''_i} : i < m_0\} \cup \{\alpha_{m_0,k_0}, \alpha_{m_0,k_1}\} \cup \{\alpha_{i,l_i} : i \in (m_0, m_1)\})),$$

$$\alpha_0, l'_0, \alpha_0, l''_0, \alpha_1, l'_1, \alpha_1, l''_1, \dots, \alpha_{m_0-1}, l'_{m_0-1}, \alpha_{m_0-1}, l''_{m_0-1}, \alpha_{m_0, k_0},$$

$$\alpha_{m_0, k_1}, \alpha_{m_0+1, l_{m_0+1}}, \dots, \alpha_{m_1-1, l_{m_1-1}})$$

and

$$(\text{Sk}(\{\alpha_{i,l'_i}, \alpha_{i,l''_i} : i < m_0\} \cup \{\alpha_{m_0,k_0}, \alpha_{m_0,k_2}\} \cup \{\alpha_{i,l_i} : i \in (m_0, m_1)\})),$$

$$\alpha_0, l'_0, \alpha_0, l''_0, \alpha_1, l'_1, \alpha_1, l''_1, \dots, \alpha_{m_0-1}, l'_{m_0-1}, \alpha_{m_0-1}, l''_{m_0-1}, \alpha_{m_0, k_0},$$

$$\alpha_{m_0, k_2}, \alpha_{m_0+1, l_{m_0+1}}, \dots, \alpha_{m_1-1, l_{m_1-1}})$$

are isomorphic and the isomorphism is the identity on their intersection and they have the same intersection with κ .

For more details and more related results we refer the reader to [Sh:F254].

(2) Together with 1.5, 2.7 this gives a good bound on the consistency strength of $\neg\mathcal{KL}(\lambda, \kappa)$.

(3) What if we ask $F_n : {}^n\lambda \rightarrow {}^\omega\kappa$ such that $F_n(\eta) \subseteq F_{n+1}(\eta)$ and $\eta \in {}^\omega\lambda \Rightarrow F(\eta) = \bigcup F_n(\eta|n) \in {}^\omega\kappa$? No real change.

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