APPENDIX 1 [Sh 282a]

COLORINGS

In this section we present a family of "colouring" properties $\Pr_\ell(\lambda, \mu, \theta, \sigma)$ saying that there is a "colouring" of the pairs from λ with θ colours which is complicated (those properties are from [Sh282] on parts of which this section is based, such properties where earlier considered for $\lambda = \aleph_1$, see on this Juhasz [Ju]). Those properties are powerful enough to deal with the cellularity of the product of two topological spaces bigger than the product of the cellularities. We discuss the history of this problem below, and then the history of the square bracket relation which the $\Pr_\ell$ generalizes.

A Boolean algebra B satisfies the λ-chain condition (λ-c.c.) if any family of pairwise disjoint nonzero members of B has cardinality $< \lambda$. For a topological space X, by $c(X)$ we denote the cellularity of X which is the supremum of cardinalities of families of pairwise disjoint nonempty open sets in X. Clearly $c(X) \leq \lambda$ iff the Boolean algebra of regular open subsets of X satisfies λ^+-c.c. A problem is whether the λ-c.c. is a productive property (of Boolean algebras or, equivalently, of topological spaces); the first person to consider the product behaviour of cellularity was Kurepa, who proved (late 30's, probably) that it fails for a Souslin line. He also proved (late 50's, probably) that if each factor has cellularity at most κ then the product has cellularity at most 2^κ. Shanin in the 40's had results like the product of spaces of density κ has cellularity κ. Then came Martin's axiom, by which the product of factors with countable cellularity has countable cellularity (see Martin Solovay [MS]). This showed that it is independent of ZFC whether ccc is productive and it was natural to ask (for example by Archangelskii and Juhasz and Monk) what happens for higher cardinals. First ZFC examples of regular cardinals λ for which λ-c.c. is not productive were given by Todorcevic [To] (based on the existence of an entangled linear order, see 2.4 below). The cardinals of [To] are of the form $\mathrm{cf}(2^\kappa)$ where $\mathrm{ded}(\kappa) > 2^\kappa$ holds (for example $\kappa = \aleph_0$), so all of them might be weakly inaccessible. Hence [To] didn't solve the topological question: Is always $c(X \times Y) \leq c(X) \cdot c(Y)$? Note that this question is trivially equivalent to the question: Is λ^+-c.c. productive for all λ? In [To1], Todorcevic answered this question (in ZFC) by providing a class of cardinals λ for which λ^+-c.c. is not productive: the singular λ such that $(\forall \mu < \lambda)(\mu^{\mathrm{cf}\lambda} < \lambda)$. Todorcevic [To1] uses (for example, when $\lambda = (2^{\aleph_0})^{+\omega}$) [Sh-b,XIII,§5] about cofinalities of reduced products of regular cardinals. He also got negative partition relations, for example $\lambda^+ \not\rightarrow [\lambda^+]^2_{\mathrm{cf}\lambda}$ when $(\forall \mu < \lambda)[\mu^{\mathrm{cf}\lambda} < \lambda]$, and got there $\lambda - S$ and $\lambda - L$ spaces. He told me that a proof of the consistency of "λ^+-c.c. is productive" will be the real generalization of MA (unlike some soft ones; see for example [Sh80]). By [Sh280], this fails for $\lambda \geq 2^{\aleph_0}$ regular (even $\mathrm{Pr}_0(\lambda^+, \aleph_0, \aleph_0)$ fails, see Definition 1.1 and for some consequences 1.6, 1.7). This makes it more desirable to get parallel results not just for some successor of singulars, but for quite many. In [Sh282] we get the result for the folowing class of singulars: $2^{\mathrm{cf}\lambda} < \lambda$. In [Sh282] the properties $\mathrm{Pr}_\ell(\lambda, \mu, \theta, \sigma)$ were introduced; (i.e. colouring properties) those properties are stronger than negative square bracket partition relations.

Still Todorcevic says that the point is to have a simple aleph as an answer. Answering this in [Sh327] we get this for λ^+, λ regular $> \aleph_1$.

Concerning Lemma 1.7 consider the density of the topological space ${}^{\lambda}2$ with the σ-box topology (i.e. the set of points is $\{f : f$ a function from λ to $\{0,1\}\}$ with the family of basic open sets being

$$\big\{U_g = \{f : g \subseteq f \in {}^{\lambda}2\} : g \text{ a partial finite function from } \lambda \text{ to } \{0,1\},$$

$$|\mathrm{Dom}\ g| < \sigma\big\}).$$

Now the case "for $\sigma = \aleph_0$, $\lambda = 2^{\aleph_0}$, the density is $\aleph_0$" is the classical Hewitt-Marczewski-Pondiczery theorem [H], [Ma], [P]. This has been generalized by Engelking-Karlowicz [EK], and theorem from there, 1.7, is presented here; for more on box products see Comfort-Negrepontis [CN1], [CN2], Cater-Erdős-Galvin [CEG].

See more in [Sh430,§5].

The square bracket partition relation, $\lambda \rightarrow [\mu]^n_\kappa$ means that if c is a function from $[\lambda]^n = \{u \subseteq \lambda : |u| = n\}$ to κ, then for some $A \subseteq \lambda$, $|A| = \mu$ and $\mathrm{Rang}(c{\restriction}[A]^n)$ is a proper subset of κ; $\lambda \not\rightarrow [\mu]^n_\kappa$ is the negation. This was introduced by Erdös, Hajnal and Rado [EHR,p.144]. So the Ramsey theorem means $\aleph_0 \rightarrow [\aleph_0]^n_2$, and Sierpinski [Sr2] proved $2^{\aleph_0} \not\rightarrow [\aleph_1]^2_2$. Erdös, Hajnal and Rado [EHR,Thm.17,p.145] proved that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ implies $\aleph_{\alpha+1} \not\rightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}}$. Erdös and Hajnal asked [EH, Problem 15,p.25] if any of the statements $2^{\aleph_0} \not\rightarrow [\aleph_1]^2_3$, $2^{\aleph_0} \rightarrow \left[2^{\aleph_0}\right]^2_3$, $\aleph_1 \not\rightarrow [\aleph_1]^2_3$ can be proved without CH. Galvin and Shelah [GlSh23] proved the second and third, and conjecture the first is independent of ZFC (the second was proved independently by Laver, too). But they proved more: $2^{\aleph_0} \not\rightarrow \left[2^{\aleph_0}\right]^2_{\aleph_0}$ and $\aleph_1 \not\rightarrow [\aleph_1]^2_4$ and also $\aleph_1 \not\rightarrow [\aleph_1]^3_{\aleph_1}$, $\mathrm{cf}(2^{\aleph_0}) \not\rightarrow \left[\mathrm{cf}(2^{\aleph_0})\right]^2_{\aleph_0}$. By Solovay [So2] $2^{\aleph_0} \rightarrow \left[2^{\aleph_0}\right]^2_{\aleph_1}$ (in a strong sense) is consistent and by the consistency of Chang's conjecture due to Silver (see Jech [J]) $\aleph_2 \rightarrow [\aleph_1]^n_{\aleph_1}$ is consistent. Todorcevic [To2] proved $\aleph_1 \not\rightarrow [\aleph_1]^2_{\aleph_1}$, using "Todorcevic walks". He also proved $\lambda^+ \not\rightarrow [\lambda^+]^2_{\mathrm{cf}\lambda}$ if $(\forall \mu < \lambda)[\mu^{\mathrm{cf}\lambda} < \lambda]$. On the other hand, in [Sh276] the conjecture of [GlSh23] was affirmed; i.e. $2^{\aleph_0} \rightarrow [\aleph_1]^2_3$ is consistent with ZFC and more. Also [To2] was continued for example if λ is regular is not similar to a weakly compact cardinal (i.e. $\otimes$ below fails) then $\lambda \nrightarrow [\lambda]^2_{\aleph_0}$, where

$\otimes$ if $S \subseteq \lambda$ is stationary, $C_\delta \subseteq \delta$ a club of δ for $\delta \in S$ then for some club E of λ, $\delta \in S \cap E \Rightarrow \bigvee_{\alpha \in S} [C_\alpha \cap \delta \backslash C_\delta$ bounded in $\delta]$ (hence λ is quite highly Mahlo inaccessible).

(See more in [Sh365, 4.9—4.12], also [Sh288, §5].) For more consistency results see [Sh288] (for example $2^{\aleph_0} \rightarrow [\aleph_1]^7_k$ for suitable k is consistent), and stronger negative results appear here in [Sh365,§4].

Definition 1.1 $\mathrm{Pr}_0(\lambda, \mu, \kappa, \theta)$ where $\lambda \geq \mu \geq \kappa + \theta$, λ an infinite cardinal, means that there is a symmetric two place function c from λ to κ which witnesses it, which means:

(*) <u>if</u> $\xi < \theta$ and for $i < \mu$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $< \lambda$, the $\alpha_{i,\zeta}$ distinct and h is a two place function from $\xi = \{\zeta : \zeta < \xi\}$ to κ, <u>then</u> there are $i < j < \mu$ such that
$\otimes$ $\zeta_1 < \xi \wedge \zeta_2 < \xi \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = h(\zeta_1, \zeta_2)$.

Definition 1.2 (1) $\mathrm{Pr}_1(\lambda, \mu, \kappa, \theta)$ where $\lambda \geq \mu \geq \kappa + \theta$, λ an infinite cardinal, means that there is a symmetric two place function c from λ to κ which witnesses it, which means:

(*) <u>if</u> $\xi < \theta$ and for $i < \mu$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequenceof ordinals $< \lambda$, the $\alpha_{i,\zeta}$ distinct and $\gamma < \kappa$, then there are $i < j < \mu$ such that
$\otimes$ $\zeta_1 < \xi \wedge \zeta_2 < \xi \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = \gamma$.

(2) $\mathrm{Pr}_1^-(\lambda, \mu, \kappa, \theta)$ is defined similarly but we add to the assumption of (*) : $[i < \lambda, \zeta_1 < \zeta_2 < \xi \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = \gamma]$.

Definition 1.3 (1) $\mathrm{Pr}_2(\lambda, \mu, \kappa, \theta)$ where $\lambda \geq \mu \geq \kappa + \theta$, λ an infinite cardinal, means that there is a symmetric two place function c from λ to κ which witnesses it, which means:

(*) <u>if</u> $\xi < \theta$ and for $i < \mu$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $< \lambda$, the $\alpha_{i,\zeta}$ distinct and $\gamma < \kappa$, then there are $i < j < \lambda$, such that for ζ_1, $\zeta_2 < \xi$:
(i) $\zeta_1 = \zeta_2 \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = \gamma$,
(ii) $\zeta_1 \neq \zeta_2 \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2})$.

(2) $\mathrm{Pr}_2(\lambda, \mu, \kappa, \theta; \chi)$ is defined as in (1), but we add to the assumption of the statement (*) an ordinal $\gamma_0 < \chi$ and get a set $u \subseteq \mu$ of order type γ_0 such that any $i < j$ from u satisfies the conclusion of (*).

Definition 1.4 (1) $\mathrm{Pr}_3^S(\lambda,\mu,\kappa,\theta)$ where $\lambda \geq \mu \geq \kappa+\theta$, λ an infinite cardinal, means that there is a symmetric two place function c from λ to κ which witnesses it, which means:

(*) if $\xi < \theta$ and for $i < \mu$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is an increasing sequence of ordinals $< \lambda$, the $\alpha_{i,\zeta}$ distinct and $2\gamma+1 < \kappa$ and $\zeta(*) < \xi$, then there are $i < j < \lambda$ such that:

(i) $\zeta < \zeta(*) \Rightarrow c\big(\alpha_{i,\zeta(*)}, \alpha_{j,\zeta}\big) = 2\gamma$,

(ii) $\zeta(*) \leq \zeta < \xi \Rightarrow c\big(\alpha_{i,\zeta(*)}, \alpha_{j,\zeta}\big) = 2\gamma+1$.

(2) $\mathrm{Pr}_3^L(\lambda,\mu,\kappa,\theta)$ is defined similarly replacing (i) , (ii) by:

(i)′ $\zeta < \zeta(*) \Rightarrow c\big(\alpha_{i,\zeta}, \alpha_{j,\zeta(*)}\big) = 2\gamma$,

(ii)′ $\zeta(*) \leq \zeta < \xi \Rightarrow c\big(\alpha_{i,\zeta}, \alpha_{j,\zeta(*)}\big) = 2\gamma+1$,

Definition 1.5 In Definitions 1.1 — 1.4 omitting μ means $\mu = \lambda$, if we also omit θ we mean that it is $\aleph_0$.

For simplicity, we use $\mu = \lambda$ below.

Observation 1.6 (1) $\mathrm{Pr}_0(\lambda,\kappa,\theta)$ implies $\mathrm{Pr}_1(\lambda,\kappa,\theta)$, $\mathrm{Pr}_2(\lambda,\kappa,\theta)$.

(2) $\mathrm{Pr}_0(\lambda,\kappa,\theta)$ implies $\mathrm{Pr}_3^S(\lambda,\kappa,\theta)$ and $\mathrm{Pr}_3^L(\lambda,\kappa,\theta)$.

(3) $\mathrm{Pr}_1(\lambda,\kappa,\theta)$ implies $\mathrm{Pr}_1^-(\lambda,\kappa,\theta)$.

(4) $\mathrm{Pr}_2(\lambda,\kappa,\theta)$ implies $\mathrm{Pr}_1^-(\lambda,\kappa,\theta)$.

(5) $\mathrm{Pr}_2(\lambda,\kappa,\theta;3)$ is equivalent to $\mathrm{Pr}_2(\lambda,\kappa,\theta)$.

(6) If $\kappa_1 \leq \kappa_2$, $\theta_1 \leq \theta_2$, $\mu_2 \leq \mu_1$ and $\lambda_1 \leq \lambda_2$ then

$\mathrm{Pr}_0(\lambda_2,\mu_2,\kappa_2,\theta_2) \Rightarrow \mathrm{Pr}_0(\lambda_1,\mu_1,\kappa_1,\theta_1)$,

$\mathrm{Pr}_1(\lambda_2,\mu_2,\kappa_2,\theta_2) \Rightarrow \mathrm{Pr}_1(\lambda_1,\mu_1,\kappa_1,\theta_1)$,

$\mathrm{Pr}_1^-(\lambda_2,\mu_2,\kappa_2,\theta_2) \Rightarrow \mathrm{Pr}_1^-(\lambda_1,\mu_1,\kappa_1,\theta_1)$,

$\mathrm{Pr}_2(\lambda_2,\mu_2,\kappa_2,\theta_2,\mu_2) \Rightarrow \mathrm{Pr}_2(\lambda_1,\mu_1,\kappa_1,\theta_1,\mu_1)$,

$\mathrm{Pr}_3(\lambda_2,\mu_2,\kappa_2,\theta_2) \Rightarrow \mathrm{Pr}_3(\lambda_1,\mu_1,\kappa_1,\theta_1)$.

Proof: Check.

Claim 1.6A Let $\lambda > \aleph_0$ be regular

(7) $\mathrm{Pr}_1^-(\lambda,2)$ implies that the λ-c.c. is not productive for Boolean algebras.

(8) $\mathrm{Pr}_3^X(\lambda,2)$ implies that there are $\lambda - X$ spaces for $X = L, S$ (Hausdorff with a basis of clopen sets).

(9) $\mathrm{Pr}_1(\lambda,n)$ implies that there are Boolean algebras $B_\ell(\ell < n)$ such that $\prod_{\ell<n} B_\ell$ does not satisfy the λ-c.c. but $\prod_{\ell<n,\ell\neq m} B_\ell$ satisfies it for $m < n$.

(10) If $\mathrm{Pr}_3^S(\lambda,2)$ or $\mathrm{Pr}_3^L(\lambda,2)$ then for some Hausdorff spaces X_1, X_2 with clopen basis, neither has a discrete subspace of cardinality λ but $X_1 \times X_2$ has.

Proof: 7) So assume $\Pr_1^-(\lambda, 2)$ and let c be a two-place function from λ to $\{0,1\}$ exemplifying this. We define two Boolean algebras, B_0, B_1. For $\ell = 0,1$, B_ℓ is the Boolean algebra generated freely by $\{x^\ell_\alpha : \alpha < \lambda\}$ except for the relations $x^\ell_\alpha \cap x^\ell_\beta = 0$ when $c(\alpha, \beta) = \ell$ (for $\alpha < \beta < \lambda$). Clearly in B_ℓ, the x^ℓ_α 's are distinct and $B_\ell \models x^\ell_\alpha \cap x^\ell_\beta = 0$ iff $c(\alpha, \beta) = \ell$. Now $B_0 \times B_1$ does not satisfy the λ-c.c. as $\left\{\left(x^0_\alpha, x^1_\alpha\right) : \alpha < \lambda\right\}$ exemplifies this. Why does B_ℓ satisfy the λ-c.c.? Suppose $a_\zeta \in B_\ell$ for $\zeta < \lambda$, $a_\zeta \neq 0$, there are $n(\zeta)$, $\alpha(\zeta, 0) < ... < \alpha(\zeta, n(\zeta)) < \lambda$, $\sigma(\zeta, 0), ..., \sigma(\zeta, n(\zeta)) \in \{0,1\}$

and

$$y_{\zeta,m} = \begin{cases} x^\ell_{\alpha(\zeta,m)} & \text{if } \sigma(\zeta,m) = 0 \\ 1 - x^\ell_{\zeta(\zeta,m)} & \text{if } \sigma(\zeta,m) = 1 \end{cases}$$

such that

$$0 \neq \bigcap_{m=0}^{n(\zeta)} y_{\zeta,m} \subseteq a_\zeta.$$

Without loss of generality for some m_0, n, $\sigma(0), ..., \sigma(n)$ for every ζ, $n(\zeta) = n$, $\sigma(\zeta, m) = \sigma(m)$ for $m \leq n$, $\alpha(\zeta, m) = \alpha(0, m)$ for $m < m_0$ and $\{\alpha(\zeta, m_0), ..., \alpha(\zeta, n)\}$ are pairwise disjoint for $\zeta < \lambda$. Let $u = \{m : m_0 \leq m \leq n \text{ and } \sigma(m) = 0\}$. As $\bigcap_{m=0}^{n(\zeta)} y_{\zeta,m} \neq 0$ in B_ℓ, by the definition of B_ℓ:

$(*)$ $\zeta < \lambda \& m \in u \& k \in u \& m \neq k \Rightarrow c(\alpha(\zeta, m), \alpha(\zeta, k)) = 1 - \ell)$.

As c exemplifies $\Pr_2^-(\lambda, 2)$ for some $\zeta < \xi < \lambda$

$$\bigwedge_{m \in u} \bigwedge_{k \in u} c(\alpha(\zeta, m), \alpha(\xi, k)) = 1 - \ell.$$

Clearly $B_\ell \models a_\zeta \cap a_\xi \neq 0$ and we finish.

8) Let c be a symmetric two place function from λ to $2 = \{0,1\}$. We define a topological space Y, with set of points λ, and basis the set of Boolean combinations of the sets $\mathbf{u}_\alpha (\alpha < \lambda)$ where

Case 1: if $X = L$, $\mathbf{u}_\alpha = \{\alpha\} \cup \{\beta : \beta < \lambda, \beta > \alpha, c(\alpha, \beta) = 0\}$

Case 2: if $X = S$, $\mathbf{u}_\alpha = \{\alpha\} \cup \{\beta : \beta < \lambda, \beta < \alpha, c(\alpha, \beta) = 0\}$.

Clearly it is Hausdorff: let for $\beta < \alpha < \lambda$ and $\ell = 0,1$:

$$v_\ell = \begin{cases} \mathbf{u}_\alpha & \ell = 0 \\ \lambda \setminus \mathbf{u}_\alpha & \ell = 1 \end{cases} \quad \text{if } X = L$$

$$v_\ell = \begin{cases} \mathbf{u}_\beta & \ell = 0 \\ \lambda \setminus \mathbf{u}_\beta & \ell = 1 \end{cases} \quad \text{if } X = S$$

so v_0, v_1 are disjoint open neighborhoods of α, β respectively.

Suppose $A \subseteq \lambda$, $|A| = \lambda$ and $\langle v_\alpha : \alpha \in A \rangle$ is a sequence of open subsets of Y such that:

$$v_\alpha \cap A \cap (\lambda \backslash \alpha) = \{\alpha\} \text{ if } \quad X = L$$

$$v_\alpha \cap A \cap (\alpha + 1) = \{\alpha\} \text{ if } \quad X = S.$$

and we shall eventually get a contradiction. By Boolean algebra rules without loss of generality

$$v_\alpha = \mathbf{u}_{\zeta(\alpha,0)} \cap \cdots \cap \mathbf{u}_{\zeta(\alpha,n_\alpha - 1)} \backslash \mathbf{u}_{\zeta(\alpha,n_\alpha)} \backslash \cdots \backslash \mathbf{u}_{\zeta(\alpha,m_\alpha - 1)}.$$

As $v_\alpha \neq 0$, clearly w.l.o.g.

$$k \neq \ell \Rightarrow \zeta(\alpha, \ell) \neq \zeta(\alpha, k);$$

without loss of generality α appears in $\{\zeta(\alpha, \ell) : \ell < m_\alpha\}$, as $\alpha \in \mathbf{u}_\alpha$ necessarily $\alpha \in \{\zeta(\alpha, \ell) : \ell < n_\alpha\}$ so without loss of generality $\zeta(\alpha, 0) = \alpha$. As λ regular we can replace $A \subseteq \lambda$ by any unbounded $A' \subseteq A$. So without loss of generality for every $\alpha \in A$ we have $n_\alpha = n(*)$ and $m_\alpha = m(*)$ and

$$\zeta(\alpha, \ell) = \zeta(\alpha', \ell') \Rightarrow \ell = \ell' \ \& \ \bigwedge_\beta \zeta(\beta, \ell) = \zeta(\alpha, \ell)$$

and let $t = \{\ell : 1 \leq \ell < m(*) \text{ and for } \alpha \neq \beta \ \ \zeta(\alpha, \ell) \neq \zeta(\beta, \ell)\}$, and again thinning A without loss of generality $\alpha < \beta$ implies:

$$\bigwedge_{\ell, k \in t} \zeta(\alpha, \ell) < \zeta(\beta, k) \text{ and } \bigwedge_{\ell \in t} \zeta(\alpha, \ell) < \beta \text{ and } \bigwedge_{k \in t} \alpha < \zeta(\beta, k)$$

and (by Fodor's Lemma) $\bigwedge_{k \in t} \alpha \leq \zeta(\alpha, k)$. If $X = S$ as $\alpha \in v_\alpha$ necessarily

$$\bigwedge_{\ell \in [1, n(*))} \zeta(\alpha, \ell) > \alpha \ \& \bigwedge_{\ell \in [n(*), m(*))} \zeta(\alpha, \ell) < \alpha$$

and we look for $\alpha < \beta$ such that $\alpha \in v_\beta$, i.e.

$$\bigwedge_{\ell < n(*)} \alpha \in \mathbf{u}_{\zeta(\beta, \ell)} \ \& \bigwedge_{n(*) \leq \ell < m(*)} \alpha \notin \mathbf{u}_{\zeta(\beta, \ell)}.$$

Now for $\ell \in m(*) \setminus t \setminus \{0\}$ this always holds. So without loss of generality $t = [1, m(*))$. Now we apply Definition 1.4(1) with $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ there corresponding to $\langle \zeta(\alpha, \ell) : \ell < n(*) \rangle \hat{} \langle \zeta(\alpha, \ell) : n(*) \leq \ell < m(*) \rangle$ here and $\zeta(*)$ there corresponding to $n(*)$ here. For $X = L$ the proof is similar using

Definition 1.4(2).
9) So assume $\Pr_1(\lambda, n)$ and let c be a two-place function from λ to $\{0, \dots, n+1\}$ exemplifying this. For $e = 0, ..., n-1$, B_e is the Boolean algebra generated freely by $\{x^e_\alpha : \alpha < \lambda\}$ except for the relations $x^e_\alpha \cap x^e_\beta = 0$ when $c(\alpha, \beta) = e$ (for $\alpha < \beta < \lambda$). Clearly in B_e, the x^e_α's are distinct and $B_e \models x^e_\alpha \cap x^e_\beta = 0$ iff $c(\alpha, \beta) = e$. Now $B_0 \times ... \times B_{n-1}$ does not satisfy the λ-c.c. as $\{\langle x^e_\alpha : e < n\rangle : \alpha < \lambda\}$ exemplifies this. Let $m < n$; why does $B^m = \prod_{e<n, e\neq m} B_e$ satisfy the λ-c.c.? Suppose $a_\zeta \in B^m$ for $\zeta < \lambda$, $a_\zeta \neq 0$, then there are (for $e < n$, $e \neq m$ and $\zeta < \lambda$) natural number $n(\zeta, e)$, and $\alpha(\zeta, 0, e) < \cdots < \alpha(\zeta, n(\zeta), e) < \lambda$, and $\sigma(\zeta, 0, e), \dots, \sigma(\zeta, n(\zeta), e) \in \{0, 1\}$

and

$$y_{\zeta,m,e} = \begin{cases} x^e_{\alpha(\zeta,m,e)} & \text{if } \sigma(\zeta, m, e) = 0 \\ 1 - x^e_{\alpha(\zeta,m)} & \text{if } \sigma(\zeta, m, e) = 1 \end{cases}$$

such that $a_\zeta = \langle a^e_\zeta : e < n, e \neq m\rangle$ and for $e < n$, $e \neq m$ we have

$$0 \neq \bigcap_{m=0}^{n(\zeta)} y_{\zeta,m,e} \subseteq a^e_\zeta.$$

Without loss of generality for some $m_0(e)$, $n(e)$, $\sigma(0, e), ..., \sigma(n(e), e)$ for every ζ, $n(\zeta, e) = n(e)$, $\sigma(\zeta, m, e) = \sigma(m, e)$ for $m \leq n$, $\alpha(\zeta, m, e) = \alpha(0, m, e)$ for $m < m_0(e)$ and $\{\alpha(\zeta, m_0, e), ..., \alpha(\zeta, n, e)\}$ are pairwise disjoint for $\zeta < \lambda$. As c exemplifies $\Pr_1(\lambda, n)$ for some $\zeta < \xi < \lambda$

$$\bigwedge_{\substack{\ell<n \\ \ell\neq m}} \bigwedge_{m=m_0(e)}^{n(e)} \bigwedge_{k=m_0(e)}^{n(e)} c(\alpha(\zeta, m, e), \alpha(\xi, k, e)) = m.$$

Clearly $B^m \models a_\zeta \cap a_\xi \neq 0$ and we finish.
10) Like 8). $\square_{1.6A}$

Remark 1.6B If we want in the conclusion of 1.6A(9) to allow n to be an infinite cardinal called σ, we need to assume $\Pr_1(\lambda, \sigma, \sigma)$ and to demand $\bigwedge_{\alpha<\lambda} |\alpha|^n < \lambda$ and [Sh365,§4] gives cases of $\Pr_1(\lambda, \sigma, \sigma)$.

Lemma 1.7 *If $\mu = \mu^{<\sigma} < \lambda \leq 2^\mu$ then there are functions $f_\alpha : \lambda \to \mu$ for $\alpha < \mu$ such that for every partial function f from λ to μ of cardinality $< \sigma$, for some $\alpha < \mu$, $f \subseteq f_\alpha$.*

Proof: Let $\langle A_\alpha : \alpha < \lambda\rangle$ be a sequence of distinct subsets of μ, let

$$W = \big\{(u, \mathcal{P}, h) : u \text{ a subset of } \mu \text{ of cardinality } < \sigma, \mathcal{P} \text{ a family of subsets of } u \text{ of cardinality } < \sigma, h \text{ a function from } \mathcal{P} \text{ to } \mu\big\}.$$

Clearly W has cardinality $\mu^{<\sigma} = \mu$, so let

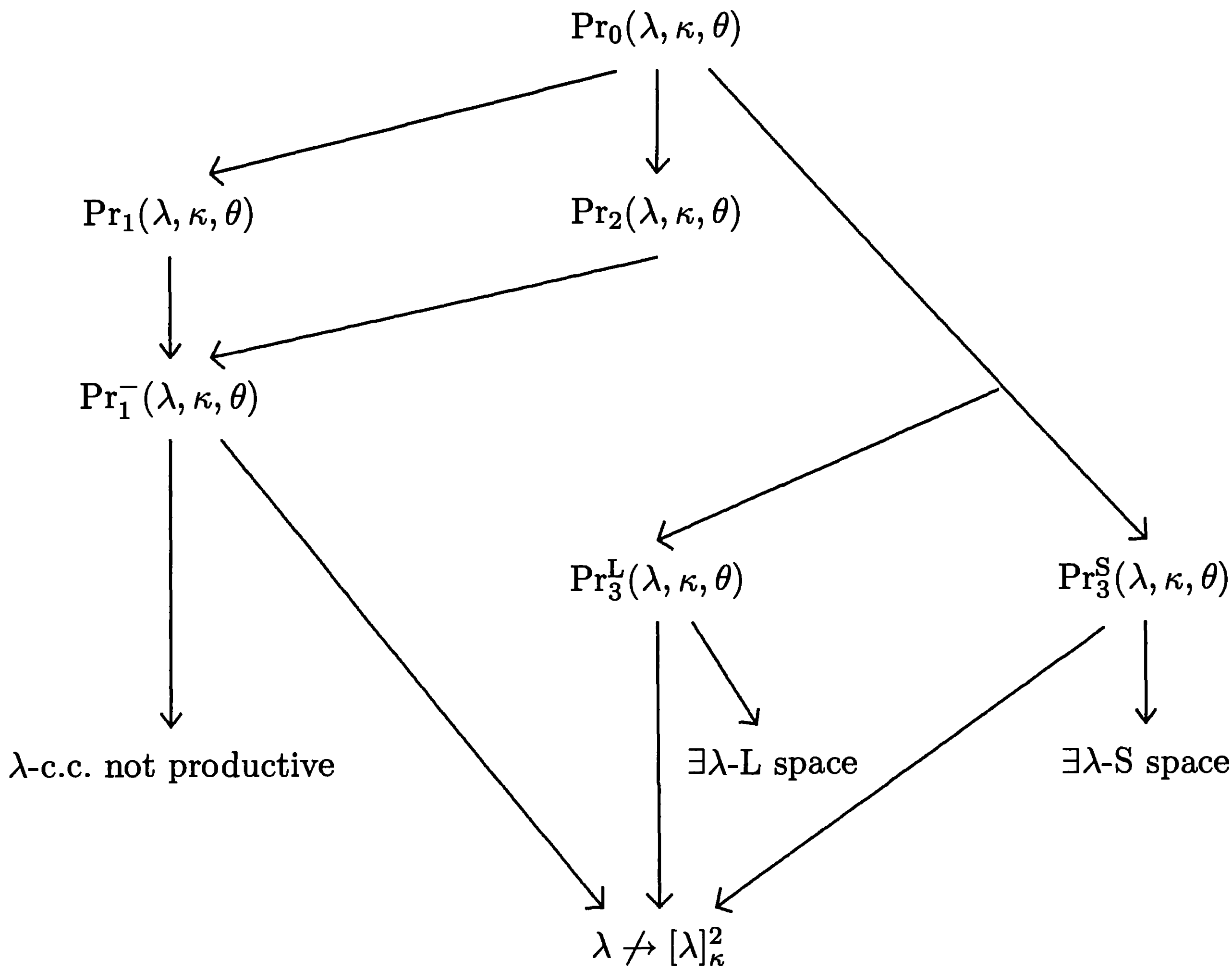
$$W = \{(u_i, \mathcal{P}_i, h_i) : i < \mu\}.$$

Now we define for $i < \mu$ the function $f_i : \lambda \to \mu$ by

$$f_i(\alpha) = \begin{cases} h(A_\alpha \cap \mu) & \text{if } A_\alpha \cap \mu \in \mathcal{P}_i \\ 0 & \text{otherwise} \end{cases}$$

Why the f_i's are as required? Let $v \subseteq \lambda$, $|v| < \sigma$, $f : v \to \mu$, choose for $\alpha \neq \beta$ in v, $j(\alpha,\beta) < \mu$ such that $j(\alpha,\beta) \in A_\alpha \equiv j(\alpha,\beta) \notin A_\beta$. Let $u = \{j(\alpha,\beta) : \alpha \neq \beta \text{ in } v\}$, $\mathcal{P} = \{A_\alpha \cap u : \alpha \in v\}$, let $h : \mathcal{P} \to \mu$ be $h(A_\alpha \cap u) = f(\alpha)$; h is well defined as for $\alpha \neq \beta$ in v we have $A_\alpha \cap u \neq A_\beta \cap \beta$. Clearly $(u, \mathcal{P}, h) \in W$ so for some i, $(u, \mathcal{P}, h) = (u_i, \mathcal{P}_i, h_i)$ hence $f \subseteq f_i$.

Diagram 1.8 (For $\kappa \geq 2$, $\theta \geq 2$) we sum the claims above:



Remark: On Pr_4 see [Sh365, 4.3, 4.4, 4.5].