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a complex problem with many parameters and degrees of freedom and allowing it to tend to its extreme value.

I have been interested in each of these questions, but particularly in the second and third. For pattern formation, some specific interests of mine have included finding a mathematical derivation of the appearance of the surprising "cross tie walls" patterns that form in micromagnetics, and to explain the distribution of vortices along triangular "Abrikosov lattices" for minimizers of the Ginzburg–Landau energy functional. In the latter case, as in many problems from quantum chemistry, one observes that nature seems to prefer regular or periodic arrangements, in the form of crystalline structures. It remains almost completely open to rigorously explain why this *crystallization* occurs, i.e., demonstrating mathematically that the arrangements which have the least energy are necessarily periodic. To me, this is a very fascinating question. While it is quite simple to pose, we still do not have much of an idea of how to address it.

Some results have been obtained in the very particular case of the sphere packing problem in two dimensions, and perturbations of it, in the works of Charles Radin and Florian Theil. Proving the same type of result in higher dimension or for more general optimization problem is a major challenge for the fundamental understanding of the structure of matter.

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Why are you interested in model theory (a branch of mathematical logic)?

In elementary school, mathematics looked (to me) like just a computational skill – how to multiply, how to find formulas for areas of squares, rectangles, triangles etc – and the natural sciences looked more attractive. Then, entering the ninth grade, Euclidean geometry captured my heart: from the

* We thank J. Baldwin, G. Cherlin, U. Hrushovski, M. Malliaris and J. Vaananen for helpful comments. This is [E70] on the author's publication list.

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bare bones of assumptions a magnificent structure is built; an intellectual endeavour in which it is enough to be right.

Undergraduate mathematics was impressive for me, but algebra considerably more so than analysis. Reading Galois theory, understanding equations in general fields, was a gem. Finding order in what looks like a chaos, not grinding water but finding natural definitions and hard theorems; generality, being able to say something from very few assumptions, was impressive. From this perspective mathematical logic was the most general direction, so I took the trouble to do my MSc thesis in mathematical logic; the thesis happened to be on the model theory of infinitary logics.

Model theory seemed the epitome of what I was looking for: rather than investigating a specific class like "the class of fields", the "class of rings with no zero divisors" or whatever, we have a class of structures, called here models. For this to be meaningful, we have to restrict somewhat the class, first by saying they are all of the same "kind", i.e. have the same function symbols (for rings: addition, multiplication; also the so-called "individual constants" 0 and 1, we may have so called predicates, i.e. symbols for relations, but we shall ignore that point; this information is called the vocabulary). We have to further restrict the classes we consider, and the classical choice in model theory is to restrict to the so called e.c., i.e. elementary classes, explained below.

Naturally model theorists start from the bottom: Consider K, an e.c. (elementary class), i.e. the class of models of a first order theory T as explained below. The class K (i.e. T) is called categorical in the infinite cardinal λ if it has a unique model up to isomorphism of cardinality (= number of elements) λ. Łoś conjectured that if an e.c. K with countable vocabulary is categorical in one uncountable cardinal then this holds for every uncountable cardinal. After more than a decade, Morley proved this, and when I started my PhD studies I thought it was wonderful (and still think so).

The point of view explained above naturally leads to the classification program. The basic thesis of the classification program is that reasonable families of classes of mathematical structures should have natural dividing lines. Here a dividing line means a partition into low, analyzable, tame classes on the one hand, and high, complicated, wild classes on the other. These partitions will generate a tameness hierarchy. For each such partition, if the class is on the tame side one should have useful structural analyses applying to all structures in the class, while if the class is on the wild side one should have strong evidence of chaotic behavior (set theoretic complexity). These results should be complementary, proving that the dividing lines are not merely sufficient conditions for being low complexity, or sufficient con-

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ditions for being high complexity. This calls for relevant test questions; we expect not to start with a picture of the meaning of "analyzable" and look for a general context, as this usually does not provide evidence for this being a dividing line. Of course, although it is hard to refute this thesis (as you may have chosen the wrong test questions; in fact this is the nature of a thesis), it may lead us to fruitful or unfruitful directions. The thesis implies the natural expectation that a success in developing a worthwhile theory will lead us also to applications in other parts of mathematics, but for me this was neither a prime motivation nor a major test, just a welcome and not surprising (in principle) side benefit and a "proof for the uninitiated", so we shall not deal with such important applications.

We still have to define what an e.c. (elementary class) is. It is a "class of structures satisfying a fixed first order theory T". For our purpose, this can be explained as follows: given a structure M, we consider subsets of M, sets of pairs of elements of M, and more generally sets of n-tuples of elements of M, *which are reasonably definable.* By this we mean the following: start with the family of the sets of n-tuples satisfying an equation (or another atomic formula if we have also relation symbols). Those we call the atomic relations. But we may also look at the set of parameters for which an equation has a solution. More generally, the set of first order definable relations on M is the closure of the atomic ones, under union (i.e. demanding at least one of two conditions, logically "or") and intersection (i.e. "and"), under complement and lastly we close under projections, which means "there is x such that..."; but we do *not* use "there is a set of elements" or even "there is a finite sequence of elements". The way we define such a set is called a first order formula, denoted by $\varphi(x_0, \dots, x_{n-1})$. If $n = 0$ this will be just true or false in the structure and such formulas are called sentences. The (complete first order) theory $Th(M)$ of M is the set of (first order) sentences it satisfies. An e.c.(=elementary class) is the class Mod_T of models of T, that is the structures M (of the relevant kind, vocabulary) such that $Th(M) = T$. Naturally, N is an elementary extension of M (and M is an elementary submodel of N) when for any of those definition, on finite sequences from the smaller model they agree. There are many natural classes which are of this form, ranging from Abelian groups and algebraically closed fields, through random graphs to Peano Arithmetic, Set Theory, and the like.

A reader may well say that this setting is too general, that it is nice to deal with "everything", but if what we can say is "nothing", null or just dull, then it is not interesting. However, this is not the case. The classification program has been successfully done for the partition to stable/unstable and further subdivisions have been established on the tame side for the family of elementary classes. Critical dividing lines for the taxonomy involve

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the behavior of the Boolean algebras of parametrically first order definable sets and relations, i.e.: $\varphi(M, \bar{a}) := \{\bar{b} : M \text{ satisfies } \varphi(\bar{b}, \bar{a})\}$. E.g. $T = Th(M)$, i.e. $K = Mod_T$ is unstable iff some first order formula $\varphi(x, y)$ linearly orders some infinite set of elements (not necessarily definable itself!) in some model from K, or similarly for a set of pairs or, more generally, a set of n-tuples. A prominent test question involves the number of models from K up to isomorphism of cardinality (= number of elements) λ, called $I(\lambda, K)$. The promised "analyzable" classes include in this case notions of independence and of dimension (mainly as in the dimension of a vector space), and (first order definable) groups and fields appearing "out of nowhere".

Clearly having many non-isomorphic models is a kind of "set-theoretic witness for complexity" but certainly not a unique one.

Of course what looks like a small corner, a family of well understood classes from the present point of view, looks like a huge cosmos full of deep mysteries from another point of view, and some of these mysteries have resulted in great achievements.

In your opinion, what are the most challenging problems in model theory?

We may think that the restrictions to elementary classes is too strong, but then what takes the place of the first order definable parametrized sets? Naturally, at least a posteriori, we may generalize this concept but we may wonder can we really dispense with it? See § 1.

Dually, we may feel that as successful as the dividing line stable/unstable (and finer divisions "below" that) has been, not all unstable classes are completely wild, (and what constitutes being complicated, un-analysable depends on your yard-stick). Moreover, though many elementary classes are stable, many mathematically useful ones are not.

In fact there are provably just two "reasons" for being unstable. The two "minimal" unstable elementary classes correspond to the theory of dense linear orders and the theory of random graphs. Much attention has been given on the one hand to so-called *simple theories* which include the random graphs and also "pseudo-finite fields" (see § 1), and on the other hand to the *dependent theories*, which include the theories of dense linear order, the real field, the p-adics, and many fields of power series (see § 3).

It is tempting to look for a "maximal (somewhat) tame family of elementary classes". A natural candidate for this is the following.

We may look for an extreme condition of the form of unstability; such a condition is "K is straight maximal", which means that for some formula $\varphi(x, y)$ (or $\varphi(\bar{x}, \bar{y})$), for every n and non-empty subset $\mathcal{F}$ of $\mathcal{F}_n := \{f : f \text{ is a function from } \{0, \ldots, n-1\} \text{ to } \{0, 1\}\}$ we can find a model $M \in K$ and $b_0, \ldots, b_{n-1} \in M$ such that: if $f \in \mathcal{F}_n$ then there is $a \in M$ such that "M satisfies $\varphi(a, b_i)$

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iff $f(i) = 1$" *iff* $f \in \mathcal{F}$. Does this really define an (interesting) dividing line? I am sure it does, and that we can say many things about it; unfortunately I have neither idea what those things are, nor of any natural test problem; so we shall look instead at problems which have been somewhat clarified.

1. Non-elementary classes

We may note that the family of elementary classes is a quite restricted family of classes, and many mathematically natural classes cannot be described by first order conditions. For example "locally finite" structures, such as groups in which every finitely generated subgroup is finite, (or, similarly, is solvable or the like), or structures satisfying various chain conditions are not elementarily axiomatizable.

So a fundamental question is "have a generalization of the existing stability theory for a really wide family of classes, where the basic methods of e.c. completely fail (in particular, nothing like the family of parametrically first order definable sets), but we still have the same test questions".

A good candidate for this broader context is the family of *aec (abstract elementary classes)*, $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ where K is a class of models of a fixed vocabulary, $\leq_{\mathfrak{k}}$ is a partial order on the class refining the sub-model relation, and satisfying the obvious properties of e.c.'s (which means, that of K and $\leq_{\mathfrak{k}}$ are closed under isomorphism, for any directed system there is a $\leq_{\mathfrak{k}}$-lub, every member can be approximated by $\leq_{\mathfrak{k}}$-submodels of cardinality bounded by some $\chi = LST(\mathfrak{k})$ and if $M_1 \subseteq M_2$ are $\leq_{\mathfrak{k}}$-sub-models of some N then $M_1 \leq_{\mathfrak{k}} M_2$; for example, those defined by infinitary logics like the so-called $L_{\lambda^+,\aleph_0}$ where we allow conjunctions of λ formulas but not quantification over infinitely many variables).

Here a natural test question is the large scale, asymptotic behaviour of $I(\lambda, K)$, the number of models in K of cardinality λ up to isomorphism. Our dream is to prove the main gap conjecture in this case (see § 4 below).

The simplest case is the categoricity conjecture: having a unique model up to isomorphism for every large enough cardinal or failure of this in every large enough cardinal; in-spite of some advances we still do not know even this, but there are indications that a positive theory along these lines exists.

2. Unstable elementary classes – friends of random graphs

Being a *simple* e.c. can be defined similarly to stability, by "no first order formula $\varphi(\bar{x}, \bar{y})$ represents a tree" (rather than a linear order, as in the case of stability). For simple e.c.'s. we know much on analogs of the stable case, as well as something on non-structure results.

But it may well be that we should consider also different questions. Just as not whole group theory consist of generalization of the Abelian case, so also there are other natural families (extending the simple case), so-called

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$NSOP_2$ and $NSOP_3$, on which we know basically nothing.

Probably a good test problem here is the Keisler order; (an e.c. K_1 is said to be smaller than K_2 when for every so-called regular ultrafilter D in a set I, if M_2^I/D_1 is $|I|^+$-saturated for every $M_2 \in K_2$ *then* this holds for K_1.)

We have a reasonable understanding of this order for stable elementary classes, and we also know that being like the theory of linear order implies maximality. The challenge is to understand the order for simple elementary classes and for a wider family, so-called $NSOP_3$; we hope that this will shed light on those families, and lead us to a deep internal theory.

3. Unstable elementary classes: dependent theories

Simple theories include random graphs but not linear orders. On the other side we find *dependent theories*, for which the class of dense linear orders serves as a prototype (and dependent theories include many classes of fields, including many fields of formal power series).

Particularly in the last decade there has been much work on these classes, but usually in more restricted contexts.

We can count the number of so-called complete types over M_1; which can be defined by: $a, b \in M_2$ realize the same type over M_1 where M_2 is an elementary extension of M_1 if in some elementary extension N of M_2 there is an automorphism f of N over M_1 (that is, $f \restriction M_1$ is the identity) mapping a to b. Now the class is stable when for many cardinals λ, if M_1 has $\leq \lambda$ elements, then the number of those types is $\leq \lambda$, and this fails for unstable T. However, we may count the above types only up to conjugacy, that is demanding only that f maps M_1 onto M_1. Now this number may be large because M_1 has too few automorphisms, so (ignoring some points) we should restrict ourselves to M_1 with enough automorphisms, so-called saturated models. From this perspective, for stable K, the number is bounded (i.e., does not depend on the cardinal); for dependent K, we get not too many; and for independent K we get almost always the maximal values 2^λ.

A great challenge is to understand those types, and hence dependent classes.

4. Back to the stable setting

There are great challenges which remain for the stable case. *The main gap conjecture* for a family of classes, says that for a class K (from the family), the function $I(\lambda, K)$ either is usually maximal (i.e. 2^λ) or is not too large, and that there is a clear characterization. We hope that when this is not maximal every model can be represented by a graph as a "base" which is a tree with a root and the nodes are coloured by not too many colours. More specifically, every model can be described by such a tree of small models put together in a "free" (hence unique) way, the model is so called prime over this tree of models, but it is not claimed that

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the tree is unique. For general elementary classes K we still do not know it; but if the vocabulary is countable – we know. Also, even for countable vocabulary, for $\aleph_1$-saturated models we do not know.

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Why are you interested in basic structures in probability?

I wish I could say that I have understood a lot of mathematics, and then that I have chosen my areas on interest because they are the most beautiful and fundamental. The truth is more down to earth, my current interests were reached by percolation from a somewhat random starting point (which was pretty far from probability theory).

I had the privilege to be the student of a truly great person, Professor Gustave Choquet. This was somewhat wasted on me, since Choquet's genius was a sublime "geometric understanding" which I utterly lack. Nonetheless, I greatly profited from his teachings.

The first advice he gave me was as follows: when considering a problem, always formulate it in the setting that requires minimal hypotheses. This simple age-old advice has lost none of its relevance, and several of the results to which I probably owe to write this now directly benefited from it. A typical example is in the study of Gaussian processes $(X_t)_{t\in T}$. The natural distance $d(s,t)^2 = \mathsf{E}(X_s - X_t)^2$ on T induced by the process has the property that the metric space (X,d) is isometric to a subspace of a Hilbert space, and these are very special metric spaces. It turns out however that it is a deadly trap to try use this specificity, while if one forgets about it, and simply think of (X,d) as a general metric space, one in readily led to the correct approach. Of course, meeting success a few times with this type of approach develops the taste for basic structures, so Choquet's advice had a considerable influ-

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