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## UNIFORMLY ANTISYMMETRIC FUNCTIONS WITH BOUNDED RANGE

### Abstract

The goal of this note is to construct a uniformly antisymmetric function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a bounded countable range. This answers Problem 1(b) of Ciesielski and Larson [6]. (See also the list of problems in Thomson [9] and Problem 2(b) from Ciesielski's survey [5].) A problem of existence of uniformly antisymmetric function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with finite range remains open.

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is said to be *uniformly antisymmetric* [6] (or *nowhere weakly symmetrically continuous* [9]) provided for every  $x \in \mathbb{R}$  the limit  $\lim_{n \rightarrow \infty} (f(x + s_n) - f(x - s_n))$  equals 0 for no sequence  $\{s_n\}_{n < \omega}$  converging to 0. Uniformly antisymmetric functions have been studied by Kostyrko [7], Ciesielski and Larson [6], Komjáth and Shelah [8], and Ciesielski [1, 2]. (A connection of some of these results to the paradoxical decompositions of the Euclidean space  $\mathbb{R}^n$  is described in Ciesielski [3].) In particular in [6] the authors constructed a uniformly antisymmetric function  $f: \mathbb{R} \rightarrow \mathbb{N}$  and noticed that the existence of a uniformly antisymmetric function cannot be proved without an essential use of the axiom of choice.

The terminology and notation used in this note is standard and follows [4]. In particular for a set  $X$  we will write  $|X|$  for its cardinality and  $\mathcal{P}(X)$  for its

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power set. Also  $2^\omega$  will stand for the set of all functions from  $\omega = \{0, 1, 2, \dots\}$  into  $2 = \{0, 1\}$ . We consider  $2^\omega$  as ordered lexicographically.

**Theorem 1.** *There exists a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with countable bounded range such that for every  $x \in \mathbb{R}$  there exists an  $\varepsilon_x > 0$  with the property that the set*

$$S_x = \{s \in \mathbb{R}: |f(x-s) - f(x+s)| < \varepsilon_x\}$$

*is finite. In particular  $f$  is uniformly antisymmetric.*

PROOF. First notice that it is enough to find a compact zero-dimensional metric space  $\langle T, d \rangle$  and a function  $g$  from  $\mathbb{R}$  into a countable subset  $T_0$  of  $T$  such that for every  $x \in \mathbb{R}$  there is a  $\delta_x > 0$  for which the set

$$\hat{S}_x = \{s \in \mathbb{R}: d(g(x-s), g(x+s)) < \delta_x\}$$

is finite.

To see this assume that such a function  $g: \mathbb{R} \rightarrow T$  exists and take a homeomorphic embedding  $h$  of  $T$  into  $\mathbb{R}$ . We claim that  $f = h \circ g: \mathbb{R} \rightarrow \mathbb{R}$  is as desired. Indeed,  $f[\mathbb{R}] = h[g[\mathbb{R}]]$  is countable, as it is a subset of a countable set  $h[T_0]$ , and it is bounded, since it is a subset of a compact set  $h[T]$ . So take  $x \in \mathbb{R}$  and  $\delta_x > 0$  for which  $\hat{S}_x$  is finite. Since  $h^{-1}: h[T] \rightarrow T$  is uniformly continuous, we can find an  $\varepsilon_x > 0$  such that

$$|y_1 - y_2| < \varepsilon_x \text{ implies } d(h^{-1}(y_1), h^{-1}(y_2)) < \delta_x$$

for every  $y_1, y_2 \in h[T]$ . But for such a choice of  $\varepsilon_x$  we have

$$S_x = \{s \in \mathbb{R}: |h(g(x-s)) - h(g(x+s))| < \varepsilon_x\} \subset \hat{S}_x$$

proving that  $S_x$  is finite.

Thus, we proceed to construct a function  $g$  described above. The value of  $g(x)$  will be defined with help of a representation of  $x$  in a Hamel basis; i.e., a linear basis of  $\mathbb{R}$  over  $\mathbb{Q}$ . For this we will use the following notation. Let  $\{y_\eta: \eta \in 2^\omega\}$  be a one-to-one enumeration of a Hamel basis  $\mathcal{H}$ . For every  $x \in \mathbb{R}$  let  $\sum_{\eta \in 2^\omega} q_{x,\eta} y_\eta$ , with  $q_{x,\eta} \in \mathbb{Q}$  for  $\eta \in 2^\omega$ , be the unique representation of  $x$  in basis  $\mathcal{H}$  and let  $w_x = \{\eta \in 2^\omega: q_{x,\eta} \neq 0\}$ . Thus  $w_x$  is finite and

$$x = \sum_{\eta \in w_x} q_{x,\eta} y_\eta.$$

The definition of the space  $T$  is considerably more technical since it reflects several different cases of the proof that the sets  $\hat{S}_x$  are indeed finite. To this

end let  $\{q_j: j < \omega\}$  be a one-to-one enumeration of  $\mathbb{Q}$  with  $q_0 = 0$ . For  $i < \omega$  let  $\mathcal{P}_i = \mathcal{P}(\{q_j: j < i\})$  and put  $P_i = \mathcal{P}(2^i \times \{0, 1\} \times \mathcal{P}_i \times \mathcal{P}_i)$ . Note that each  $P_i$  is finite; so  $T = \prod_{i < \omega} P_i$ , considered as the standard product of discrete spaces, is compact zerodimensional. We equip  $T$  with a distance function  $d$  defined between different  $s, t \in T$  by  $d(s, t) = 2^{-\min\{i < \omega: s(i) \neq t(i)\}}$  and let

$$T_0 = \{t \in T: (\exists n < \omega)(\forall i \geq n) t(i) = \emptyset\}.$$

Clearly  $T_0$  is countable.

Now we are ready to define  $g: \mathbb{R} \rightarrow T_0 \subset T$ . For this, however, we will need few more definitions. For  $x \in \mathbb{R}$ ,  $q \in \mathbb{Q}$ ,  $i < \omega$ , and  $\zeta \in 2^i$  such that  $\zeta \in \{(\eta \upharpoonright i): \eta \in w_x\}$  we define:

- $p(i) \in \{0, 1\}$  as the parity of  $i$ ; i.e.,  $p(i) = i \bmod 2$ ;
- $k_i(q) = \{q_j \in \mathbb{Q}: q_j < q \ \& \ j < i\} \in \mathcal{P}_i$ ;
- $\eta(x, \zeta)$  to be the minimum of  $\{\eta \in w_x: \zeta \subset \eta\}$  (in the lexicographical order);
- $\xi(x, \zeta)$  to be the minimum of  $\{\eta \in w_x: \zeta \subset \eta\} \setminus \{\eta(x, \zeta)\}$  provided  $|\{\eta \in w_x: \zeta \subset \eta\}| \neq 1$ ; otherwise we put  $\xi(x, \zeta) = \eta(x, \zeta)$ ;
- $n_x < \omega$  to be the smallest number  $n > 0$  such that
  - (i)  $\eta \upharpoonright n \neq \xi \upharpoonright n$  for any different  $\eta, \xi \in w_x$ , and
  - (ii)  $q_{x, \eta} \in \{q_j: j < n\}$  for every  $\eta \in w_x$ .

Consider the function  $g: \mathbb{R} \rightarrow T_0$  defined as follows. For every  $x \in \mathbb{R}$  and  $i < \omega$  we define  $g(x)(i) \in P_i$  as

$$\{\langle \zeta, p(|\{\eta \in w_x: \zeta \subset \eta\}|), k_i(q_{x, \eta(x, \zeta)}), k_i(q_{x, \xi(x, \zeta)}) \rangle: \zeta \in \{(\eta \upharpoonright i): \eta \in w_x\}\}$$

provided  $i \leq n_x$  and we put  $g(x)(i) = \emptyset$  for  $n_x < i < \omega$ . In the argument below the key role will be played by the function  $k_i$  in general, and the coordinate  $k_i(q_{x, \eta(x, \zeta)})$  in particular.

The key step in the proof that  $g$  has the desired property is that for every  $x \in \mathbb{R}$  and  $s \neq 0$

$$\text{if } n_x \leq \max\{n_{x-s}, n_{x+s}\} \text{ then } g(x-s)(n_x) \neq g(x+s)(n_x). \quad (1)$$

To see (1) assume that  $n_x \leq n_{x+s}$ . If  $n_{x-s} < n_x$ , then  $g(x-s)(n_x) = \emptyset \neq g(x+s)(n_x)$ , where  $g(x+s)(n_x) \neq \emptyset$  since  $w_{x+s} \neq \emptyset$  as  $n_{x-s} < n_x \leq n_{x+s}$  implies  $x+s \neq 0$ . Thus, we can assume that  $n_x \leq \min\{n_{x-s}, n_{x+s}\}$ . Take an

$\hat{\eta} \in w_{x-s} \cup w_{x+s}$  such that  $q_{x-s,\hat{\eta}} \neq q_{x+s,\hat{\eta}}$  and let  $\zeta = \hat{\eta} \upharpoonright n_x$ . Note that, by the definition of  $n_x$ , the set  $S = \{\eta \in w_x : \zeta \subset \eta\}$  has at most one element.

If  $S = \emptyset$ , then  $\{\eta \in w_{x-s} : \zeta \subset \eta\} = \{\eta \in w_{x+s} : \zeta \subset \eta\} \neq \emptyset$  and so  $\eta(x-s, \zeta) = \eta(x+s, \zeta) \notin w_x$  while  $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 0$ . Thus  $q_0 = 0$  separates  $q_{x-s,\eta(x-s,\zeta)}$  and  $q_{x+s,\eta(x+s,\zeta)}$  implying that  $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$ . Therefore  $g(x-s)(n_x) \neq g(x+s)(n_x)$ .

So, assume that  $S \neq \emptyset$  and let  $\eta'$  be the only element of  $S$ . Then  $\eta' \in w_{x-s} \cup w_{x+s}$ . If  $\eta'$  belongs to precisely one of the sets  $w_{x+s}$  and  $w_{x-s}$ , say  $w_{x+s}$ , then  $\{\eta \in w_{x+s} : \zeta \subset \eta\} = \{\eta \in w_{x-s} : \zeta \subset \eta\} \cup \{\eta'\}$ . In particular,  $p(|\{\eta \in w_{x+s} : \zeta \subset \eta\}|) \neq p(|\{\eta \in w_{x-s} : \zeta \subset \eta\}|)$  implying that  $g(x-s)(n_x) \neq g(x+s)(n_x)$ .

So, we can assume that  $\eta' \in w_{x-s} \cap w_{x+s}$ . Then  $\{\eta \in w_{x-s} : \zeta \subset \eta\} = \{\eta \in w_{x+s} : \zeta \subset \eta\}$  and  $\eta(x-s, \zeta) = \eta(x+s, \zeta)$ . We will consider three cases.

CASE 1:  $\eta' \neq \eta(x-s, \zeta) = \eta(x+s, \zeta)$ . Then  $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 0$ ; so  $q_0 = 0$  separates  $q_{x-s,\eta(x-s,\zeta)}$  and  $q_{x+s,\eta(x+s,\zeta)}$ . Thus  $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$  and  $g(x-s)(n_x) \neq g(x+s)(n_x)$ .

CASE 2:  $\eta' = \eta(x-s, \zeta) = \eta(x+s, \zeta)$  and  $q_{x-s,\eta(x-s,\zeta)} \neq q_{x+s,\eta(x+s,\zeta)}$ . Then  $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 2q_{x,\eta'}$  and, by the definition of  $n_x$ ,  $q_{x,\eta'} \in \{q_j : j < n_x\}$ . Since  $q_{x,\eta'}$  separates  $q_{x-s,\eta(x-s,\zeta)}$  and  $q_{x+s,\eta(x+s,\zeta)}$ , we conclude that  $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$  and  $g(x-s)(n_x) \neq g(x+s)(n_x)$ .

CASE 3:  $\eta' = \eta(x-s, \zeta) = \eta(x+s, \zeta)$  and  $q_{x-s,\eta(x-s,\zeta)} = q_{x+s,\eta(x+s,\zeta)}$ . Then  $Z = \{\eta \in w_{x-s} : \zeta \subset \eta\} \setminus \{\eta(x-s, \zeta)\} = \{\eta \in w_{x+s} : \zeta \subset \eta\} \setminus \{\eta(x+s, \zeta)\}$  is non-empty, since it contains  $\hat{\eta}$ , and so  $\xi(x-s, \zeta) = \xi(x+s, \zeta) \notin w_x$ . Therefore, as in Case 1,  $q_{x-s,\xi(x-s,\zeta)} + q_{x+s,\xi(x+s,\zeta)} = 0$ ; so  $q_0 = 0$  separates  $q_{x-s,\xi(x-s,\zeta)}$  and  $q_{x+s,\xi(x+s,\zeta)}$ . Thus  $k_{n_x}(q_{x-s,\xi(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\xi(x+s,\zeta)})$  and  $g(x-s)(n_x) \neq g(x+s)(n_x)$ .

This finishes the proof of (1).

Next, for every  $x \in \mathbb{R}$  put  $\delta_x = 2^{-n_x}$ . To finish the proof of the theorem it is enough to show that every  $\hat{S}_x$  defined for such a choice of  $\delta_x$  is a subset of a finite set

$$Z_x = \left\{ s \in \mathbb{R} : w_{x+s} \subset w_x \ \& \ n_{x+s} < n_x \right\} = \left\{ \sum_{\eta \in w_x} p_\eta y_\eta : p_\eta \in \{q_j : j < n_x\} \right\}.$$

Indeed, take an  $s \in \hat{S}_x$ . Then, by (1) and the definition of the distance function  $d$ , we have  $\max\{n_{x-s}, n_{x+s}\} < n_x$ . Notice also that if  $n_{x-s} \neq n_{x+s}$ , say  $n_{x-s} < n_{x+s}$ , then  $g(x-s)(n_{x+s}) = \emptyset \neq g(x+s)(n_{x+s})$  implying that  $d(g(x+s), g(x-s)) \geq 2^{-n_{x+s}} > 2^{-n_x} = \delta_x$ , which contradicts  $s \in \hat{S}_x$ . So, we have  $n_{x-s} = n_{x+s}$ . To prove that  $s \in Z_x$  it is enough to show that

$w_{x+s} \subset w_x$ . But if it is not the case, then there exists an  $\eta \in w_{x+s} \setminus w_x$ . Moreover,  $q_{x+s,\eta} = -q_{x-s,\eta} \neq 0$  and  $\eta = \eta(x+s, \zeta) = \eta(x-s, \zeta)$ , where  $\zeta = \eta \upharpoonright n_{x+s}$ . In particular,  $q_0 = 0$  separates  $q_{x+s,\eta(x+s,\zeta)}$  and  $q_{x-s,\eta(x-s,\zeta)}$ . Therefore  $k_{n_{x+s}}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_{x+s}}(q_{x+s,\eta(x+s,\zeta)})$  and  $g(x-s)(n_{x+s}) \neq g(x+s)(n_{x+s})$ . So  $d(g(x+s), g(x-s)) \geq 2^{-n_{x+s}} > 2^{-n_x} = \delta_x$  again contradicting  $s \in \dot{S}_x$ . Thus,  $w_{x+s} \subset w_x$  and  $s \in Z_x$ .  $\square$

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<sup>1</sup>Preprints marked by \* are available in electronic form from *Set Theoretic Analysis Web Page*: <http://www.math.wvu.edu/homepages/kcies/STA/STA.html>