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On a Non-vanishing Ext.

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Abstract - The existence of valuation domains admitting non-standard uniserial modules for which certain Exts do not vanish was proved in [1] under Jensen's Diamond Principle. In this note, the same is verified using the ZFC axioms alone.

In the construction of large indecomposable divisible modules over certain valuation domains R, the first author used the property that R satisfied $\mathrm{Ext}^1_R(Q, U) \neq 0$, where Q stands for the field of quotients of R (viewed as an R-module) and U denotes any uniserial divisible torsion R-module, for instance, the module $K = Q/R$; see [1]. However, both the existence of such a valuation domain R and the non-vanishing of Ext were established only under Jensen's Diamond Principle $\diamondsuit$ (which holds true, e.g., in Gödel's Constructible Universe).

Our present goal is to get rid of the Diamond Principle, that is, to verify in ZFC the existence of valuation domains R that admit divisible non-standard uniserial modules and also satisfy $\mathrm{Ext}^1_R(Q, U) \neq 0$ for several uniserial divisible torsion R-modules U. (For the proof of Corollary 3, one requires only 6 such U.)

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We start by recalling a few relevant definitions. By a *valuation domain* we mean a commutative domain R with 1 in which the ideals form a chain under inclusion. A *uniserial* R-module U is defined similarly as a module whose submodules form a chain under inclusion. $K = Q/R$ is a uniserial torsion R-module, it is *divisible* in the sense that $rK = K$ holds for all $0 \neq r \in R$. A divisible uniserial R-module is called *standard* if it is an epic image of the uniserial module Q; otherwise it is said to be *non-standard*. The existence of valuation domains admitting non-standard uniserials has been established in ZFC; see e.g. [3], [2, X.4], and the literature quoted there.

As the R-module Q is uniserial, it can be represented as the union of a well-ordered ascending chain of cyclic submodules:

$$R = Rr_0 < Rr_1^{-1} < \ldots < Rr_\alpha^{-1} < \ldots < \bigcup_{\alpha<\kappa} Rr_\alpha^{-1} = Q \qquad (\alpha < \kappa),$$

where $r_0 = 1$, $r_\alpha \in R$, and κ denotes an infinite cardinal and also the initial ordinal of the same cardinality. As a consequence, $K = \bigcup_{\alpha<\kappa} (Rr_\alpha^{-1}/R)$ where $Rr_\alpha^{-1}/R \cong R/Rr_\alpha$ are cyclically presented R-modules. We denote by $\iota_\alpha^\beta\colon Rr_\alpha^{-1}/R \to Rr_\beta^{-1}/R$ the inclusion map for $\alpha < \beta$, and may view K as the direct limit of its submodules Rr_α^{-1}/R with the monomorphisms ι_α^β as connecting maps.

A uniserial divisible torsion module U is a «*clone*» of K in the sense of Fuchs-Salce [2, VII.4], if there are units $e_\alpha^\beta \in R$ for all pairs $\alpha < \beta$ $(< \kappa)$ such that

$$e_\alpha^\beta e_\beta^\gamma - e_\alpha^\gamma \in Rr_\alpha \qquad \text{for all } \alpha < \beta < \gamma < \kappa,$$

and U is the direct limit of the direct system of the modules Rr_α^{-1}/R with connecting maps $\iota_\alpha^\beta e_\alpha^\beta\colon Rr_\alpha^{-1}/R \to Rr_\beta^{-1}/R$ $(\alpha < \beta)$; i.e. multiplication by e_α^β followed by the inclusion map. It might be helpful to point out that though K and U need not be isomorphic, they are «piecewise» isomorphic in the sense that they are unions of isomorphic pieces.

Let R denote the valuation domain constructed in the paper [1] (see also Fuchs-Salce [2, X.4.5]) that satisfies $\mathrm{Ext}_R^1(Q, K) \neq 0$ in the constructible universe L. Moreover, there are non-isomorphic clones U_n of K, for any integer $n > 0$, that satisfy $\mathrm{Ext}_R^1(Q, U_n) \neq 0$; for convenience, we let $K = U_0$.

This R has the value group $\Gamma = \oplus_{\alpha<\omega_1}\mathbb{Z}$, ordered anti-lexicographically, and its quotient field Q consists of all formal rational functions of u^γ with coefficients in an arbitrarily chosen, but fixed field, where u is an indeterminate and $\gamma \in \Gamma$. It is shown in [2, X.4] that such an R admits divisible non-standard uniserials (i.e. clones of K non-isomorphic to K), and under the additional hypothesis of $\diamondsuit_{\aleph_1}$, $\mathrm{Ext}^1_R(Q, U_n) \neq 0$ holds; in other words, there is a non-splitting exact sequence

$$0 \to U_n \to H_n \xrightarrow{\phi} Q \to 0.$$

Using the elements $r_\alpha \in R$ introduced above, for each $n < \omega$ we define a tree T_n of length κ whose set of vertices at level α is given by

$$T_{n\alpha} = \{x \in H_n \mid \phi(x) = r_\alpha^{-1}\}.$$

The partial order $<_T$ is defined in the following way: $x <_T y$ in T_n if and only if, for some $\alpha < \beta$, we have $\phi x = r_\alpha^{-1}$ and $\phi y = r_\beta^{-1}$ such that

$$x = r_\alpha^{-1} r_\beta y \quad \text{ in } H_n,$$

where evidently $r_\alpha^{-1} r_\beta \in R$.

Fix an integer $n > 0$, and define T as the union of the trees $T_0, T_1, \ldots, T_n$ with a minimum element z adjoined. It is straightforward to check that $(T, <_T)$ is indeed a tree with κ levels, and the inequalities $\mathrm{Ext}^1_R(Q, U_i) \neq 0$ $(i = 0, 1, \ldots, n)$ guarantee that T has no branch of length κ.

We now define a multisorted model **M** as follows. Its universe is the union of the universes of R, Q, U_i, H_i $(i = 0, \ldots, n)$, and it has the following relations:

(i) unary relations R, Q, U_i, H_i, T, and $S = \{r_\alpha \mid \alpha < \kappa\}$,

(ii) binary relation $<_T$, and $<_S$ (which is the natural well-ordering on S),

(iii) individual constants $0_R, 0_Q, 0_{U_i}, 1_R$, and

(iv) functions: the operations in R, Q, U_i, H_i, where R is a domain, Q, U_i, H_i are R-modules, ϕ_i is an R-homomorphism from H_i onto Q $(i = 0, \ldots, n$, and $\psi : Q \to K$ with $\mathrm{Ker}\,\psi = R$ is the canonical map.

We argue that even though our universe V does not satisfy V = L, the class L does satisfy it, and so in L we can define the model **M** as above. Let **T** be the first order theory of **M**. So the first order (countable) theory **T** has in L a model in which

$(^\star)_{\mathbf{M}}$ the tree $(T, <_T)$ with set of levels $(S, <_S)$ and with the function $\phi = \cup \phi_i$ giving the levels, as interpreted in **M**, has no full branch (this means that there is no function from S to T increasing in the natural sense and inverting ϕ, or any ϕ_i).

Hence we conclude as in [3] (by making use of Shelah [4]) that there is a model $\mathbf{M}'$ with those properties in V (in fact, one of cardinality $\aleph_1$).

Note that $(S, <_S)$ as interpreted in $\mathbf{M}'$ is not well ordered, but it is still a linear order of uncountable cofinality (in fact, of cofinality $\aleph_1$), the property $(^\star)_{\mathbf{M}'}$ still holds, and it is a model of $\mathbf{T}$. This shows that all relevant properties of $\mathbf{M}$ in L hold for $\mathbf{M}'$ in V, just as indicated in [3].

It should be pointed out that, as an alternative, instead of using a smaller universe of set theory L, we could use a generic extension not adding new subsets of the natural numbers (hence essentially not adding new countable first order theories like $\mathbf{T}$).

If we continue with the same argument as in [3], then using [4] we can claim that we have proved in ZFC the following theorem:

THEOREM 1. *There exist valuation domains R admitting non-standard uniserial torsion divisible modules such that* $\mathrm{Ext}^1_R(Q, U_i) \neq 0$ *for various clones U_i of K.* ■

Hence we derive at once that the following two corollaries are true statements in ZFC; for their proofs we refer to [2, VII.5].

COROLLARY 2. *There exist valuation domains R such that if U, V are non-isomorphic clones of K, then* $\mathrm{Ext}^1_R(U, V)$ *satisfies:*

(i) *it is a divisible mixed R-module;*

(ii) *its torsion submodule is uniserial.* ■

More relevant consequences are stated in the following corollaries; they solve Problem 27 stated in [2, p. 272].

COROLLARY 3. *There exist valuation domains admitting indecomposable divisible modules of cardinality larger than any prescribed cardinal.* ■

COROLLARY 4. *There exist valuation domains with superdecomposable divisible modules of countable Goldie dimension.* ■

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