Sh:836

ON LONG EF-EQUIVALENCE IN NON-ISOMORPHIC MODELS

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Abstract. There has been a great deal of interest in constructing models which are non-isomorphic, of cardinality λ, but are equivalent under the Ehrefeuch-Fraissé game of length α, even for every $\alpha < \lambda$. So under G.C.H. particularly for λ regular we know a lot. We deal here with constructions of such pairs of models proven in ZFC, and get their existence under mild conditions.

§1. Introduction. There has been much work on constructing pairs of $\mathrm{EF}_{\alpha,\mu}$-equivalent non-isomorphic models of the same cardinality.

In the summer of 2003, Väänänen asked me whether we can provably in ZFC construct a pair of non-isomorphic models of cardinality $\aleph_1$ which are EF_α-equivalent even for α like ω^2. We try to shed light on the problem for general cardinals. We construct such models for $\lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0}$ for every $\alpha < \lambda$ simultaneously and then for singular $\lambda = \lambda^{\aleph_0}$. In subsequent work Havlin and Shelah [HvSh:866] we shall investigate further: weaken the assumption "$\lambda = \lambda^{\aleph_0}$" (e.g., $\lambda = \mathrm{cf}(\lambda) > \beth_\omega$) and generalize the results for trees with no λ-branches and investigate the case of models of a first order complete T (mainly strongly dependent). We thank Chanoch Havlin and the referee for detecting some inaccuracies.

Definition 1.1.

(1) We say that M_1, M_2 are EF_α-equivalent if M_1, M_2 are models (with same vocabulary) such that the isomorphism player has a winning strategy in the game $\partial^\alpha_1(M_1, M_2)$ defined below.

(1A) Replacing α by $< \alpha$ means: for every $\beta < \alpha$; similarly below.

(2) We say that M_1, M_2 are $\mathrm{EF}_{\alpha,\mu}$- equivalent <u>when</u> M_2, M_2 are models with the same vocabulary such that the isomorphism player has a winning strategy in the game $\partial^\alpha_\mu(M_1, M_2)$ defined below.

(3) For M_1, M_2, α, μ as above and partial isomorphism f from M_1 into M_2 we define the game $\partial^\alpha_\mu(f, M_1, M_2)$ between the player ISO and AIS as follows:

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(a) the play lasts α moves
(b) after β moves a partial isomorphism f_β from M_1 into M_2 is chosen increasing continuous with β
(c) in the $\beta+1$-th move, the player AIS chooses $A_{\beta,1} \subseteq M_1$, $A_{\beta,2} \subseteq M_2$ such that $|A_{\beta,1}| + |A_{\beta,2}| < 1 + \mu$ and then the player ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that

$$A_{\beta,1} \subseteq \mathrm{Dom}(f_{\beta+1}) \text{ and } A_{\beta,2} \subseteq \mathrm{Rang}(f_{\beta+1})$$

(d) if $\beta = 0$, ISO chooses $f_0 = f$; if β is a limit ordinal ISO chooses $f_\beta = \cup\{f_\gamma : \gamma < \beta\}$.

The ISO player loses if he had no legal move.

(4) If $f = \emptyset$ we may write $\Game^\alpha_\mu(M_1, M_2)$. If μ is 1 we may omit it. We may write $\leq \mu$ instead of μ^+. The player ISO may be restricted to choose $f_{\beta+1}$ such that$(\forall a)(a \in \mathrm{Dom}(f_{\beta+1}) \wedge a \notin \mathrm{Dom}(f_\beta) \to a \in A_{\beta,1} \vee f_{\beta+1}(a) \in A_{\beta,2})$.

§2. The case of regular $\lambda = \lambda^{\aleph_0}$.

Definition 2.1.

(1) We say that $\mathfrak{r}$ is a λ-parameter <u>if</u> $\mathfrak{r}$ consists of
(a) a cardinal λ and ordinal $\alpha^* \leq \lambda$
(b) a set I, and a set $S \subseteq I \times I$ (where we shall have compatibility demand)
(c) a function $\mathbf{u} : I \to \mathcal{P}(\lambda)$; we let $\mathbf{u}_s = \mathbf{u}(s)$ for $s \in I$
(d) a set J and a function $\mathbf{s} : J \to I$, we let $\mathbf{s}_t = \mathbf{s}(t)$ for $t \in J$ and for $s \in I$ we let $J_s = \{t \in J : \mathbf{s}_t = s\}$
(e) a set $T \subseteq J \times J$ such that $(t_1, t_2) \in T \Rightarrow (\mathbf{s}_{t_1}, \mathbf{s}_{t_2}) \in S$.

(1A) We say $\mathfrak{r}$ is a full λ-parameter <u>if</u> in addition it consists of:
(f) a function $\mathbf{g}$ with domain J such that $\mathbf{g}_t = \mathbf{g}(t)$ is a non-decreasing function from $\mathbf{u}_{\mathbf{s}(t)}$ to some $\alpha < \alpha^*$
(g) a function $\mathbf{h}$ with domain J such that $\mathbf{h}_t = \mathbf{h}(t)$ is a non-decreasing function from $\mathbf{u}_{\mathbf{s}(t)}$ to λ such that
(h) if $t_1, t_2 \in J$ and $\mathbf{s}_{t_1} = s = \mathbf{s}_{t_2}$, $\mathbf{g}_{t_1} = g = \mathbf{g}_{t_2}$ and $\mathbf{h}_{t_1} = h = \mathbf{h}_{t_2}$, $\alpha^{t_1} = \alpha = \alpha^{t_2}$ <u>then</u> $t_1 = t_2$ hence we write $t = t^\alpha_{s,g,h} = t^\alpha(s, g, h)$.

(2) We may write $\alpha^* = \alpha^*_{\mathfrak{r}}$, $\lambda = \lambda_{\mathfrak{r}}$, $I = I_{\mathfrak{r}}$, $J = J_{\mathfrak{r}}$, $J_s = J^{\mathfrak{r}}_s$, $t^\alpha(s, g, h) = t^{\alpha,\mathfrak{r}}(s, g, h)$, etc. Many times we omit $\mathfrak{r}$ when clear from the context.

Definition 2.2. Let $\mathfrak{r}$ be a λ-parameter.

(1) For $s \in I_{\mathfrak{r}}$, let $\mathbb{G}^{\mathfrak{r}}_s$ be the group[1] generated freely by $\{x_t : t \in J_s\}$.
(2) For $(s_1, s_2) \in S_{\mathfrak{r}}$ let $\mathbb{G}_{s_1,s_2} = G^{\mathfrak{r}}_{s_1,s_2}$ by the subgroup of $\mathbb{G}^{\mathfrak{r}}_{s_1} \times \mathbb{G}^{\mathfrak{r}}_{s_2}$

[1]We also could use abelian groups satisfying $\forall x(x + x = 0)$, in this case $\mathbb{G}_s$ is the family of finite subsets of J_2 with the symmetric difference operation also we could use the free abelian group.

generated by

$$\{(x_{t_1}, x_{t_2}) : (t_1, t_2) \in T_{\mathfrak{x}} \text{ and } t_1 \in J^{\mathfrak{x}}_{s_1},\ t_2 \in J^{\mathfrak{x}}_{s_2}\}.$$

(3) We say $\mathfrak{x}$ is (λ, θ)-parameter if $s \in I_{\mathfrak{x}} \Rightarrow |\mathbf{u}_s| < \theta$.

REMARK 2.3. (1) We may use S a set of n-tuples from I (or $(< \omega)$-tuples) then we have to change Definitions 2.2(2) accordingly.

DEFINITION 2.4. For a λ-parameter $\mathfrak{x}$ we define a model $M = M_{\mathfrak{x}}$ as follows (where below $I = I_{\mathfrak{x}}$, etc.).

(A) its vocabulary τ consist of
  - (α) P_s, a unary predicate, for $s \in I_{\mathfrak{x}}$
  - (β) Q_{s_1,s_2}, a binary predicate for $(s_1, s_2) \in S_{\mathfrak{x}}$
  - (γ) $F_{s,a}$, a unary function for $s \in I_{\mathfrak{x}}$, $a \in \mathbb{G}^{\mathfrak{x}}_s$

(B) the universe of M is $\{(s, x) : s \in I_{\mathfrak{x}}, x \in \mathbb{G}^{\mathfrak{x}}_s\}$

(C) for $s \in I_{\mathfrak{x}}$ let $P^M_s = \{(s, x) : x \in \mathbb{G}^{\mathfrak{x}}_s\}$

(D) $Q^M_{s_1,s_2} = \{((s_1, x_1), (s_2, x_2)) : (x_1, x_2) \in \mathbb{G}^{\mathfrak{x}}_{(s_1,s_2)}\}$ for $(s_1, s_2) \in S_{\mathfrak{x}}$

(E) if $s \in I_{\mathfrak{x}}$ and $a \in \mathbb{G}^{\mathfrak{x}}_s$ then $F^M_{s,a}$ is the unary function from P^M_s to P^M_s defined by $F^M_{s,a}(y) = ay$, multiplication in $\mathbb{G}^{\mathfrak{x}}_s$ (for $y \in M \setminus P^M_s$ we can let $F^M_{s,a}(y)$ be y or undefined).

DEFINITION 2.5.

(1) For $\mathfrak{x}$ a λ-parameter and for $I' \subseteq I_{\mathfrak{x}}$ let $M^{\mathfrak{x}}_{I'} = M_{\mathfrak{x}} {\restriction} \cup\{P^{M_{\mathfrak{x}}}_s : s \in I'\}$ and let $I_\gamma = I^{\mathfrak{x}}_\gamma = \{s \in I_{\mathfrak{x}} : \sup(\mathbf{u}_s) < \gamma\}$.

(2) Assume $\mathfrak{x}$ is a full λ-parameter and $\beta < \lambda$; for $\alpha < \alpha^*_{\mathfrak{x}}$ we let $\mathcal{G}^{\mathfrak{x}}_{\alpha,\beta}$ be the set of $g : \beta \to \alpha$ which are non-decreasing; <u>then</u> for $g \in \mathcal{G}^{\mathfrak{x}}_{\alpha,\beta}$
  - (a) we define $h = h_g : \beta \to \lambda$ as follows: $h(\gamma) = \mathrm{Min}\{\beta' \leq \beta$: if $\beta' < \beta$ then $g(\beta') > g(\gamma)\}$
  - (b) we let $I_g = I^{\mathfrak{x}}_g = \{s \in I : \mathbf{u}_s \subseteq \beta$ and $t^\alpha_{s,g{\restriction}\mathbf{u}_s,h_g{\restriction}\mathbf{u}_s}$ is well defined$\}$
  - (c) we define $\bar{c}^\alpha_g = \langle c^\alpha_{g,s} : s \in I^{\mathfrak{x}}_g\rangle$ by $c^\alpha_{g,s} = x^\alpha_{t_{g,s}}$ where $t^\alpha_{g,s} = t^{\alpha,\mathfrak{x}}_{s,g{\restriction}\mathbf{u}_s,h_g{\restriction}\mathbf{u}_s}$.

(3) Let $\mathcal{G}^{\mathfrak{x}}_\alpha = \cup\{\mathcal{G}^{\mathfrak{x}}_{\alpha,\beta} : \beta < \lambda\}$ and $\mathcal{G}_{\mathfrak{x}} = \cup\{\mathcal{G}^{\mathfrak{x}}_\alpha : \alpha < \alpha^*\}$.

DEFINITION 2.6. Let $\mathfrak{x}$ be a λ-parameter.

(1) Let $\mathbf{C}_{\mathfrak{x}} = \cup\{\mathbf{C}^{\mathfrak{x}}_{I'} : I' \subseteq I_{\mathfrak{x}}\}$ where for $I' \subseteq I_{\mathfrak{x}}$ we let $\mathbf{C}^{\mathfrak{x}}_{I'} = \{\bar{c} : \bar{c} = \langle c_s : s \in I'\rangle$ satisfies $c_s \in \mathbb{G}^{\mathfrak{x}}_s$ when $s \in I'$ and $(c_{s_1}, c_{s_2}) \in \mathbb{G}_{s_1,s_2}$ when $(s_1, s_2) \in S_{\mathfrak{x}}$ and $s_1, s_2 \in I'\}$.

(2) For $\bar{c} \in \mathbf{C}^{\mathfrak{x}}_{I'}$, $I' \subseteq I_{\mathfrak{x}}$, let $f^{\mathfrak{x}}_{\bar{c}}$ be the partial function from $M_{\mathfrak{x}}$ into itself defined by $f^{\mathfrak{x}}_{\bar{c}}((s, y)) = (s, yc_s)$ for $(s, y) \in P^{M_{\mathfrak{x}}}_s$, $s \in I'$.

(3) $M_{\mathfrak{x}}$ is P_s-rigid <u>when</u> for every automorphism f of $M_{\mathfrak{x}}$, $f {\restriction} P^{M_{\mathfrak{x}}}_s$ is the identity.

OBSERVATION 2.7. (1) Let $\mathfrak{x}$ be a full λ-parameter. If $g : \gamma_2 \to \alpha$ where $\alpha < \alpha^*_{\mathfrak{x}}$, $\gamma_2 < \lambda$ and the function g is non-decreasing, $\gamma_1 < \gamma_2$ and $(\forall \gamma < \gamma_1)(g(\gamma) < g(\gamma_1))$ <u>then</u> $I_{g{\restriction}\gamma_1} \subseteq I_g$ and $h_{g{\restriction}\gamma_1} \subseteq h_g$ and $\bar{c}^\alpha_{g{\restriction}\gamma_1} = \bar{c}^\alpha_g {\restriction} I_{g{\restriction}\gamma_1}$.

(2) If $g \in \mathcal{G}^\alpha_{\mathfrak{x}}$ in Definition 2.5(3), <u>then</u> $\bar{c}^\alpha_g \in \mathbf{C}^{\mathfrak{x}}_{I^{\mathfrak{x}}_g}$.

Claim 2.8. Assume $\mathfrak{x}$ is a full λ-parameter.

(1) For $I' \subseteq I_{\mathfrak{x}}$ and $\bar{c} \in \mathbf{C}^{\mathfrak{x}}_{I'}$, $f^{\mathfrak{x}}_{\bar{c}}$ is an automorphism of $M^{\mathfrak{x}}_{I'}$ which is the identity iff $s \in I' \Rightarrow c_s = e_{\mathbb{G}_s}$.

(2) In (1) for $s \in I'$, $f^{\mathfrak{x}}_{\bar{c}} \restriction P^{M_{\mathfrak{x}}}_s$ is not the identity iff $c_s \neq e_{\mathbb{G}_s}$.

(3) If f is an automorphism of $M^{\mathfrak{x}}_{I_2}$ then $f \restriction M^{\mathfrak{x}}_{I_1}$ is an automorphism of $M^{\mathfrak{x}}_{I_1}$ for every $I_1 \subseteq I_2 \subseteq I_{\mathfrak{x}}$.

(4) If $I' \subseteq I_{\mathfrak{x}}$ and f is an automorphism of $M^{\mathfrak{x}}_{I'}$, then $f = f^{\mathfrak{x}}_{\bar{c}}$ for some $\langle c_s : s \in I_{\mathfrak{x}} \rangle \in \mathbf{C}_{I'}$.

(5) If $\bar{c}_\ell \in \mathbf{C}^{\mathfrak{x}}_{I_\ell}$ for $\ell = 1, 2$ and $I_1 \subseteq I_2$ and $\bar{c}_1 = \bar{c}_2 \restriction I_1$ then $f_{\bar{c}_1} \subseteq f_{\bar{c}_2}$.

(6) The cardinality of $M_{\mathfrak{x}}$ is $|J_{\mathfrak{x}}| + \aleph_0$.

Proof. Straight, e.g. (4) For $s \in I'$ clearly $f((s, e_{\mathbb{G}_s})) \in P^{M_{\mathfrak{x}}}_s$ so it has the form $(s, c_s), c_s \in \mathbb{G}_s$ and let $\bar{c} = \langle c_s : s \in I' \rangle$. To check that $\bar{c} \in \mathbf{C}^{\mathfrak{x}}_{I'}$ assume $(s_1, s_2) \in S_{\mathfrak{x}}$; and we have to check that $(c_{s_1}, c_{s_2}) \in \mathbb{G}_{s_1,s_2}$. This holds as $((s_1, e_{\mathbb{G}_{s_1}}), (s_2, e_{\mathbb{G}_{s_2}})) \in Q^{M_{\mathfrak{x}}}_{s_1,s_2}$ by the choice of $Q^{M_{\mathfrak{x}}}_{s_1,s_2}$ hence we have $((s_1, c_{s_1}), (s_2, c_{s_2})) = (f(s_1, e_{\mathbb{G}_{s_1}}), f(s_2, e_{\mathbb{G}_{s_2}})) \in Q^{M_{\mathfrak{x}}}_{s_1,s_2}$ hence $(c_{s_1}, c_{s_2}) \in \mathbb{G}_{s_1,s_2}$. ⊣

Claim 2.9. Let $\mathfrak{x}$ be a full λ-parameter $s \in I_{\mathfrak{x}}$ and $c_1, c_2 \in P^M_s$, $c^* \in \mathbb{G}_s$ and $F^{M_{\mathfrak{x}}}_{s,c^*}(c_1) = c_2$. A sufficient condition for "$(M_{\mathfrak{x}}, c_1)$, $(M_{\mathfrak{x}}, c_2)$ are $\mathrm{EF}_{\alpha,\mu}$-equivalent" where $\alpha \leq \alpha^*_{\mathfrak{x}}$, is the existence of $R, \bar{I}, \bar{\mathbf{c}}$ such that:

⊛ (a) R is a partial order,

(b) $\bar{I} = \langle I_r : r \in R \rangle$ such that $I_r \subseteq I_{\mathfrak{x}}$ and $r_2 \leq_R r_2 \Rightarrow I_{r_1} \subseteq I_{r_2}$

(c) R is the disjoint union of $\langle R_\beta : \beta < \alpha \rangle, R_0 \neq \emptyset$

(d) $\bar{\mathbf{c}} = \langle \bar{c}^r : r \in R \rangle$ where $\bar{c}^r \in \mathbf{C}_{I_r}$ and $r_1 \leq r_2 = \bar{c}^{r_1} = \bar{c}^{r_2} \restriction I_{r_1}$ and $c^r_s = c^*$ so $s \in \bigcap \{I_r : r \in R\}$

(e) if $\langle r_\beta : \beta < \beta^* \rangle$ is $\leq_R$-increasing, $\beta < \beta^* \Rightarrow r_\beta \in R_\beta$ and $\beta^* < \alpha$ then it has an $\leq_R$-ub from R_{β^*}

(f) if $r_1 \in R_\beta$, $\beta + 1 < \alpha$ and $I' \subseteq I$, $|I'| < \mu$ then $(\exists r_2)(r_1 \leq r_2 \in R_{\beta+1} \wedge I' \subseteq I_{r_2})$.

Proof. Easy. Using Claim 2.8(1), (5). ⊣

Claim 2.10.

(1) Let $\mathfrak{x}$ be a λ-parameter and $I' \subseteq I_{\mathfrak{x}}$. A necessary and sufficient condition for "$M^{\mathfrak{x}}_{I'}$ is P_s-rigid" is:

⊛$_1$ there is no $\bar{c} \in \mathbf{C}^{\mathfrak{x}}_{I'}$ with $c_s \neq e_{\mathbb{G}_s}$.

(2) Let $\mathfrak{x}$ be a full λ-parameter and assume that $s(*) \in I_{\mathfrak{x}}$, $\alpha < \alpha^*_{\mathfrak{x}}$, $\alpha \geq \omega$ for notational simplicity and $t^* \in J^{\mathfrak{x}}_{s(*)}$. The models $M_1 = (M, (s, e_{\mathbb{G}_s}))$, $M_2 = (M, (s, x_{t^*}))$ are $\mathrm{EF}_{\alpha,\lambda}$-equivalent when:

⊛$_{2,\alpha}$ (i) λ is regular, $s \in I_{\mathfrak{x}} \Rightarrow |\mathbf{u}^{\mathfrak{x}}_s| < \lambda$

(ii) if $s \in I_{\mathfrak{x}}$ and $g \in \mathcal{G}_{\mathfrak{x}}$ and $\mathbf{u}^{\mathfrak{x}}_s \subseteq \mathrm{Dom}\,(g)$ then $t^{\alpha,\mathfrak{x}}_{s,g\restriction \mathbf{u}_s, h_g \restriction \mathbf{u}_s}$ is well defined

Sh:836

(iii) if $(s_1, s_2) \in S_{\mathfrak{x}}$ and $t_1 = t^{\alpha}_{s_1,g_1,h_1}$, $t_2 = t^{\alpha}_{s_2,g_2,h_2}$ are well defined then $(t_1, t_2) \in T_{\mathfrak{x}}$ when for some $g \in \mathcal{G}_{\mathfrak{x}}$ we have $g_{t_1} \cup g_{t_2} \subseteq g$ and $h_1 \cup h_2 \subseteq h_g$

(iv) $t^* = t^{\alpha,\mathfrak{x}}_{s(*),g,h_g}$ where $g : \mathbf{u}_{s(*)} \to \{0\}$ and h_g is constantly $\gamma^* = \cup\{\gamma + 1 : \gamma \in \mathbf{u}_{s(*)}\}$.

Proof. (1) Toward contradiction assume that f is an automorphism of $M^{\mathfrak{x}}_{I'}$ such that $f \restriction P^{M_{\mathfrak{x}}}_s$ is not the identity. By Claim 2.8(4) for some $\bar{c} \in \mathbf{C}^{\mathfrak{x}}_{I'}$ we have $f = f_{\bar{c}}$. So $f_{\bar{c}} \restriction P^{M_{\mathfrak{x}}}_s = f \restriction P^{M_{\mathfrak{x}}}_s \neq \text{id}$ hence by Claim 2.8(1) we have $c_s \neq e_{\mathbb{G}_s}$, contradicting the assumption $\circledast_1$.

(2) We apply Claim 2.9. For every $i < \alpha$ and non-decreasing function $g \in \mathcal{G}^{\mathfrak{x}}_{\alpha}$ from some ordinal $\gamma = \gamma_g$ into i we define $\bar{c}^{\alpha}_g = \langle c^{\alpha}_{g,s} : s \in I_{g_p}\rangle$, $c^{\alpha}_{g,s} = (s, x_{t^{\alpha}_{g,s}})$, $t^{\alpha}_{g,s} = t^{\alpha}_{s,g\restriction \mathbf{u}_s,h_g\restriction \mathbf{u}_s}$. Let $R_i = \{g : g$ a non-decreasing function from some $\gamma < \lambda$ to $1 + i$ such that $\gamma^* \leq \gamma$, $g \restriction \gamma^*$ is constantly zero, $\gamma^* < \gamma \Rightarrow g(\gamma^*) = 1\}$ and let $R = \cup\{R_i : i < \alpha\}$ ordered by inclusion. Let $\bar{I} = \langle I_g : g \in R\rangle$ and $\bar{\mathbf{c}} = \langle \bar{c}^{\alpha}_g : g \in R\rangle$. It is easy to check that $(R, \bar{I}, \bar{\mathbf{c}})$ is as required. ⊣

Claim 2.11.

(1) Assume $\alpha^* \leq \lambda = \text{cf}(\lambda) = \lambda^{\aleph_0}$. Then for some full $(\lambda, \aleph_1)$-parameter $\mathfrak{x}$ we have $|I| = \lambda = |J|$, $\alpha^*_{\mathfrak{x}} = \alpha^*$ and condition $\circledast_1$ of Claim 2.10(1) holds and for every $s(*) \in I_{\mathfrak{x}} \setminus \{\emptyset\}$ condition $\circledast_{2,\alpha}$ of Claim 2.10(2) holds whenever $\alpha < \alpha^*$.

(2) Moreover, if $s \in I_{\mathfrak{x}} \setminus \{\emptyset\}$ then for some $c_1 \neq c_2 \in P^{M_{\mathfrak{x}}}_s$ and (M, c_1), (M, c_2) are $\text{EF}_{\alpha,\lambda}$-equivalent for every $\alpha < \alpha^*_{\mathfrak{x}}$ but not $\text{EF}_{\alpha^*_{\mathfrak{x}},\lambda}$-equivalent.

Claim 2.11(1) clearly implies

Conclusion 2.12.

(1) If $\lambda = \text{cf}(\lambda) = \lambda^{\aleph_0}$, $\alpha^* \leq \lambda$ then for some model M of cardinality λ we have:
  (a) M has no non-trivial automorphism
  (b) for every $\alpha < \lambda$ for some $c_1 \neq c_2 \in M$, the model (M, c_1), (M, c_2) are EF_{α}-equivalent and even $\text{EF}_{\alpha,\lambda}$-equivalent.

(2) We can strengthen clause (b) to: for some $c_1 \neq c_2$ for every $\alpha < \lambda$ the models (M, c_1), (M, c_2) are $\text{EF}_{\alpha,\lambda}$-equivalent.

Proof of Claim 2.11. (1) Assume $\alpha_* > \omega$ for notational simplicity. We define $\mathfrak{x}$ by ($\lambda_{\mathfrak{x}} = \lambda$ and):

⊠ (a) (α) $I = \{u : u \in [\lambda]^{\leq \aleph_0}\}$
  (β) the function $\mathbf{u}$ is the identity on I
  (γ) $S = \{(u_1, u_2) : u_1 \subseteq u_2 \in I\}$
  (δ) $\alpha^*_{\mathfrak{x}} = \alpha^*$

(b) (α) J is the set of quadruple (u, α, g, h) satisfying
(i) $u \in I, \alpha < \alpha^*$
(ii) h is a non-decreasing function from u to λ
(iii) g is a non-decreasing function from u to α
(iv) if $\beta_1, \beta_2 \in u$ and $g(\beta_1) = g(\beta_2)$ then $h(\beta_1) = h(\beta_2)$
(v) $h(\beta) > \beta$
(β) let $t = (u^t, \alpha^t, g^t, h^t)$ for $t \in J$ so naturally $\mathbf{s}_t = u$, $\mathbf{g}_t = g^t, \mathbf{h}_t = h^t$
(γ) $T = \{(t_1, t_2) \in J \times J : \alpha^{t_1} = \alpha^{t_2}, u^{t_1} \subseteq u^{t_2}, h^{t_1} \subseteq h^{t_2}$ and $g^{t_1} \subseteq g^{t_2}\}$.

Now

$(*)_0$ $\mathfrak{x}$ is a full $(\lambda, \aleph_1)$-parameter
[Why? Just read Definitions 2.1 and 2.2(3).]

$(*)_1$ for any $s(*) \in I \backslash \{\emptyset\}, \mathfrak{x}$ satisfies the demands for $\circledast_{2,\alpha}(i), (ii), (iii), (iv)$ from Claim 2.10(2) for every $\alpha < \alpha^*$
[Why? Just check.]

$(*)_2$ if $u_1 \subseteq u_2 \in I$, we define the function $\pi_{u_1,u_2} : J_{u_2} \to J_{u_1}$ by $\pi_{u_1,u_2}(t) = (u_1, \alpha^t, g^t \restriction u_1, h^t \restriction u_1)$ for $t \in J_{u_2}$,
[Why is π_{u_1,u_2} a function from J_{u_2} into J_{u_1}? Just check.]

$(*)_3$ for $u_1 \subseteq u_2$ we have
(α) $T \cap (J_{u_1} \times J_{u_2}) = \{(\pi_{u_1,u_2}(t_2), t_2) : t_2 \in J_{u_2}\}$ hence
(β) $\mathbb{G}_{u_1,u_2} = \{(\hat{\pi}_{u_1,u_2}(c_2), c_2) : c_2 \in \mathbb{G}_{u_2}\}$ where $\hat{\pi}_{u_1,u_2} \in \mathrm{Hom}(\mathbb{G}^{\mathfrak{x}}_{u_2}, \mathbb{G}^{\mathfrak{x}}_{u_1})$ is the unique homomorphism from $\mathbb{G}^{\mathfrak{x}}_{u_2}$ into $\mathbb{G}^{\mathfrak{x}}_{u_1}$ mapping x_{t_2} to x_{t_1} whenever $\pi_{u_1,u_2}(t_2) = t_1$
[Why? Check.]

$(*)_4$ if $u_1 \cup u_2 \subseteq u_3 \in I$, $t_3 \in J_{u_3}$ and $t_\ell = \pi_{u_\ell,u_3}(t_3)$ for $\ell = 1, 2$ then $\mathbf{g}_{t_1}$, $\mathbf{g}_{t_2}$ are compatible functions as well as $\mathbf{h}_{t_1}$, $\mathbf{h}_{t_2}$ and $\alpha^{t_1} = \alpha^{t_2}$ moreover $\mathbf{g}_{t_1} \cup \mathbf{g}_{t_2}$ is non-decreasing, $\mathbf{h}_{t_1} \cup \mathbf{h}_{t_2}$ is non-decreasing
[Why? Just check.]

$(*)_5$ clause $\circledast_1$ of Claim 2.10(1) holds for $I' = I, s(*) \in I \setminus \{\emptyset\}$.

[Why? Assume $\bar{c} \in C^{\mathfrak{x}}_I$ is such that $c_{s(*)} \neq e_{\mathbb{G}_{s(*)}}$. For each $u \in I$, c_u is a word in the generators $\{x_t : t \in J_u\}$ of $\mathbb{G}_u$ and let $\mathbf{n}(u)$ be the length of this word and $\mathbf{m}(u)$ the number of generators appearing in it.

Now by $(*)_3$ we have $u_1 \subseteq u_2 \Rightarrow \mathbf{n}(u_1) \leq \mathbf{n}(u_2) \wedge \mathbf{m}(u_1) \leq \mathbf{m}(u_2)$. As $(I, \subseteq)$ is $\aleph_1$-directed, for some $u_* \in I$ we have $u_* \subseteq u \in I \Rightarrow \mathbf{n}(u) = n_* \wedge \mathbf{m}(u) = m_*$ and let $c_u = (\ldots, x^{i(\ell)}_{t(u,\ell)}, \ldots)_{\ell < n_*}$ where $i(\ell) \in \{1, -1\}$ and $t(u, \ell) \in J^{\mathfrak{x}}_u$ and $t(u, \ell) = t(u, \ell + 1) \Rightarrow i(\ell) = i(\ell + 1)$. Clearly $u_* \subseteq u_1 \subseteq u_2 \in I \ \& \ \ell < n_* \Rightarrow \pi_{u_1,u_2}(t(u_2, \ell)) = t(u_1, \ell) \wedge \alpha^{t(u_2,\ell)} = \alpha^{t(u_*,\ell)}$. By our assumption toward contradiction necessarily $n_* > 0$.

As $\{u : u_* \subseteq u \in I\}$ is directed, by $(*)_4$ above, for each $\ell < n_*$ any two of the functions $\{g^{t(u,\ell)} : u_* \subseteq u \in I\}$ are compatible so $g_\ell =: \cup\{g^{t(u,\ell)} : u \in I\}$ is

Sh:836

a non-decreasing function from $\lambda = \cup\{u : u \in I\}$ to α^* and $h_\ell =: \cup\{h^{t(u,\ell)} : u_* \subseteq u \in I\}$ is similarly a non-decreasing function from λ to λ. It also follows that for some α^*_ℓ we have $\alpha^*_\ell =: \alpha^{t(u,\ell)}$ whenever $u_* \subseteq u \in I$ in fact $\alpha^*_\ell = \alpha^{t(u_*,\ell)}$ is O.K. For each $i \in \text{Rang}(g_\ell) \subseteq \alpha^*_\ell$ choose $\beta_{\ell,i} < \lambda$ such that $g_\ell(\beta_{\ell,i}) = i$ and let $E = \{\delta < \lambda : \delta$ a limit ordinal $> \sup(u_*)$ such that $i < \alpha^*_\ell \;\&\; \ell < n_* \;\&\; i \in \text{Rang}(g_\ell) \Rightarrow \beta_{\ell,i} < \delta$ and $\beta < \delta \;\&\; \ell < n \Rightarrow h_\ell(\beta) < \delta\}$, it is a club of λ. Choose u such that $u_* \subseteq u$ and $\text{Min}(u \backslash u_*) = \delta^* \in E$.

Now what can $g_\ell(\text{Min}(u\backslash u_*))$ be? It has to be i for some $i < \alpha^*_\ell < \alpha^*$ hence $i \in \text{Rang}(g_\ell)$ so for some $u_1, u_* \subseteq u_1 \subseteq \delta^*$ and $\beta_{\ell,i} \in u_1$ so $h_\ell(\beta_{\ell,i}) < \delta^*$ hence considering $u \cup u_1$ and recalling clause $(\alpha)(vi)$ of (b) from definition of $\mathfrak{x}$ in the beginning of the proof we have $h_\ell(\beta_{\ell,i}) < h_\ell(\delta^*)$ hence by (clause $(b)(\alpha)(v)$) we have $i = g_\ell(\beta_{\ell,i}) < g_\ell(\delta^*)$, contradiction.]

(2) A minor change is needed in the choice of $T^{\mathfrak{x}}$

$$T^{\mathfrak{x}} = \big\{(t_1,t_2) : (t_1,t_2) \in J \times J \text{ and } u^{t_1} \subseteq u^{t_2},\ h^{t_1} \subseteq h^{t_2},\ g^{t_1} \subseteq g^{t_2},$$
$$\gamma^{t_1} \leq \gamma^{t_2} \text{ and if } \text{Rang}(g^{t_1}) \nsubseteq \{0\} \text{ then } \alpha^{t_1} = \alpha^{t_2}\big\}. \qquad \dashv$$

§3. **The singular case.** We deal here with singular $\lambda = \lambda^{\aleph_0}$ and our aim is the parallel of Conclusion 2.12 constructing a pair of EF_α-equivalent for every $\alpha < \lambda$ non-isomorphic models of cardinality λ. But it is natural to try to construct a stronger example: This is done here:

⊛ for each $\gamma < \kappa = \text{cf}(\lambda)$, in the following game the ISO player wins.

Definition 3.1.

(1) For models M_1, M_2, λ and partial isomorphism f from M_1 to M_2 and $\gamma < \text{cf}(\lambda)$ we define a game $\eth^*_{\gamma,\lambda}(f, M_1, M_2)$. A play lasts γ moves, in the $\beta < \gamma$ move a partial isomorphism f_β was formed increasing with β, extending f, satisfying $|\text{Dom}(f_\beta)| < \lambda$. In the β-th move if $\beta = 0$, the player ISO choose $f_0 = f$, if β is a limit ordinal the ISO player chooses $f_\beta = \cup\{f_\varepsilon : \varepsilon < \beta\}$. In the $\beta + 1 < \gamma$ move the player AIS chooses $\alpha_\beta < \lambda$ and then they play a sub-game $\eth^{\alpha_\beta}_1(f_\beta, M_1, M_2)$ from Definition 1.1(3) producing an increasing sequence of partial isomorphisms $\langle f^\beta_i : i < \alpha_\beta\rangle$ and let their union be $f_{\beta+1}$. ISO wins if he always has a legal move.

(2) If ISO wins the game (i.e. has a winning strategy) then we say M_1, M_2 are $\text{EF}^*_{\gamma,\lambda}$-equivalent, we omit λ if clear from the context. If $f = \emptyset$ we may write $\eth^*_{\gamma,\lambda}(M_1, M_2)$

Remark. For $(M, c_1), (M, c_2)$ to be $\text{EF}^*_{<\alpha,\lambda}$-equivalent not $\text{EF}^*_{\alpha,\lambda}$-equivalent not just EF^*_α-equivalent not $\text{EF}^*_{\alpha+1}$-equivalent we may need a minor change.

Hypothesis 3.2. $j_* \leq \kappa = \text{cf}(\lambda) < \lambda, \kappa > \aleph_0, \bar{\mu} = \langle \mu_i : i < \kappa\rangle$ is increasing continuous with limit λ, $\mu_0 = 0$, $\mu_1 = \kappa (= \text{cf}(\lambda))$, μ_{i+1} is regular $> \mu_i^+$ and let $\mu_\kappa = \lambda$ and for $\alpha < \lambda$ let $\mathbf{i}(\alpha) = \text{Min}\{i : \mu_i \leq \alpha < \mu_{i+1}\}$.

DEFINITION 3.3. Under the Hypothesis 3.2 we define a λ-parameter $\mathfrak{x} = \mathfrak{x}_{j_*,\bar{\mu}}$ as follows:

(a) (α) I is the set of $u \in [\lambda \setminus \kappa]^{\leq \aleph_0}$
(β) $\mathbf{u} : I \rightarrow \mathcal{P}(\lambda \setminus \kappa)$ is the identity,
(γ) $S = \{(u_1, u_2) : u_1 \subseteq u_2 \in [\lambda \setminus \kappa]^{\leq \aleph_0}\}$
(δ) $\alpha^*_{\mathfrak{x}} = j_*$

(b) J is the set of tuples $t = (u, j, g, h) = (u^t, j^t, g^t, h^t)$ such that
(α) $u \in I$
(β) $j < j_*$
(γ) (i) g is a non-decreasing function from $u_g = u \cup v_g$ to λ where $v_g = \{\mathbf{i}(\alpha) : \alpha \in u \text{ and } g(\alpha) = \mu^+_{\mathbf{i}(\alpha)}\}$
(ii) $\alpha \in u \Rightarrow g(\alpha) \in [\mu_{\mathbf{i}(\alpha)}, \mu^+_{\mathbf{i}(\alpha)}]$
(iii) if $i \in v_g$ then $g(i) < j^t (< \kappa = \mu_1)$
(iv) v_g is an initial segment of $\{\mathbf{i}(\alpha) : \alpha \in u\}$
(δ) (i) h is a non-decreasing function with domain $u_g \cup v_g$
(ii) $\alpha \in u \Rightarrow h(\alpha) \in [\mu_{\mathbf{i}(\alpha)}, \mu_{\mathbf{i}(\alpha)+1}]$ and if $i \in v_g$ then $h(i) < \kappa$
(iii) if $\beta_1 < \beta_2$ are from $u_g \cup v_g$ and $\mathbf{i}(\beta_1) = \mathbf{i}(\beta_2)$ then $g(\beta_1) = g(\beta_2) \Leftrightarrow h(\beta_1) = h(\beta_2)$
(iv) $\alpha < h(\alpha)$ for $\alpha \in u_g \cup v_g$ and $g(\alpha) = \mu^+_{\mathbf{i}(\alpha)} \Leftrightarrow h(\alpha) = \mu^+_{\mathbf{i}(\alpha)}$ for $\alpha \in u$

(c) T is the set of pairs $(t_1, t_2) \in J \times J$ satisfying
(i) $u^{t_1} \subseteq u^{t_2} \in I$ and
(ii) $g^{t_1} \subseteq g^{t_2}$, $h^{t_1} \subseteq h^{t_2}$, $j^{t_1} = j^{t_2}$

OBSERVATION 3.4. $\mathfrak{x}_\lambda = \mathfrak{x}_{j_*,\bar{\mu}}$ is a full λ-parameter.

PROOF. Read the Definition 2.1(1) + 2.1(1A). ⊣

CLAIM 3.5. Assume $s \in I_{\mathfrak{x}}$, $c_1 = (s, e_{\mathbb{G}_s})$, $c_2 = (s, x_t)$, $t \in J_s$, and for simplicity $\mathrm{Rang}(g^t \restriction [\mu_{1+i}, \mu_{1+i+1})) \subseteq \{\mu_{1+i}\}$, $\mathrm{Rang}(g^t \restriction \kappa) = \{0\}$ and $\omega < j^t < j_*$. <u>Then</u> $(M_{\mathfrak{x}}, c_1)$, $(M_{\mathfrak{x}}, c_2)$ are $\mathrm{EF}^*_{\lambda,j^t}$-equivalent.

PROOF. So t, j^t are fixed. For $i_* < \kappa$, $j < j_*$ let

(a) $B_{i_*} = \{\bar{\beta} : \bar{\beta} = \langle \beta_i : i < \kappa \rangle$ and $\mu_i \leq \beta_i \leq \mu_{i+1}$ and $\beta_0 = i_*$ and $(\beta_{1+i} = \mu_{1+i+1} \equiv 1 + i < i_*)\}$
(b) for $\bar{\beta} \in B_{i_*}$ let $A_{\bar{\beta}} = \cup\{[\mu_i, \beta_i) : i < \kappa\}$ which by our conventions is equal to $i_* \cup \bigcup\{[\mu_j, \mu_{j+1}) : 1 \leq j < i_*\} \cup \bigcup\{[\mu_i, \beta_i) : i \in [i_*, \kappa)\}$
(c) for $\bar{\beta} \in B_{i_*}$ let $\mathcal{G}_{j,i_*,\bar{\beta}} = \{g : g$ is a function from $A_{\bar{\beta}}$ to λ, non-decreasing and the function $g \restriction \kappa$ is into j and the function $g \restriction [\mu_{1+i}, \mu_{1+i+1})$ is into $[\mu_i, \mu^+_i]$ and $1 \leq i < i_* \Leftrightarrow (\exists \alpha)(\mu_i \leq \alpha < \mu_{i+1} \wedge g(\alpha) = \mu^+_i)\}$
(d) for $g \in \mathcal{G}_{j,i_*\bar{\beta}}, \bar{\beta} \in B_{i_*}$ we define $h_g : A_{\bar{\beta}} \rightarrow \lambda$ as follows: if $\gamma \in A_{\bar{\beta}}$ then $h(\gamma) = \mathrm{Min}\{\beta' \leq \beta_{\mathbf{i}(\gamma)}$: $i(\gamma) > 0 \wedge g(\gamma) = \mu^+_{\mathbf{i}(\gamma)}$ then $\beta' = \mu_{\mathbf{i}(\gamma)+1}$, otherwise $\beta' \in [\mu_{\mathbf{i}(\gamma)}, \beta_{\mathbf{i}(\gamma)}]$ and $\beta' \neq \beta_{\mathbf{i}(\gamma)} \Rightarrow g(\gamma) < g(\beta')\}$
(e) $\mathcal{G}_{j,i_*} = \cup\{\mathcal{G}_{j,i_*,\bar{\beta}} : \bar{\beta} \in B_{i_*}\}$ and $\mathcal{G}_j = \cup\{\mathcal{G}_{j,i_*} : i_* < \kappa\}$.

Let $R = \mathcal{G}_{j^t}$ and for $g \in R$ let $i_*(g)$ be the unique $i_* < \kappa$ such that $g \in \mathcal{G}_{j^t,i_*}$ and $\bar{\beta}_g$ the unique $\bar{\beta} \in B_{i_*}$ such that $g \in \mathcal{G}_{j^t,i_*(g),\bar{\beta}}$ and $\bar{\beta} = \langle \beta_i(g) : i < \kappa \rangle$.

On R we define a partial order $g_1 \leq g_2 \Leftrightarrow g_1 \subseteq g_2 \wedge h_{g_1} \subseteq h_{g_2}$.

For $g \in R$ we define I_g, $\bar{c}_g$ as follows

⊛ (a) $I_g = \{u \in I : u \subseteq \mathrm{Dom}(g) \setminus \kappa\}$
(b) $\bar{c}_g = \langle c_{g,s} : s \in I_g \rangle$
(c) $c_{g,s} = x_{t_g(s)}$ where $t_g(s) = (s, j, g{\restriction}u_{g,s}, h_g{\restriction}u_{g,s})$ where $u_{g,s} = u \cup \{\mathbf{i}(\alpha) : \alpha \in u \text{ and } g(\alpha) = \mu^+_{\mathbf{i}(\alpha)}\}$.

Let $g_* \in \mathcal{G}_1$ be chosen such that for $i > 0$, $\beta_i(g_*) = \sup(\{g^t(\alpha) : \alpha \in u^t \cap [\mu_i, \mu_{i+1})\} \cup \{\mu_i\})$ and $\beta_0(g_*) = \cup\{\mathbf{i}(\alpha) + 1 : \alpha \in u^t \text{ and } g^t(\alpha) = \mu^+_{\mathbf{i}(\alpha)}\} \cup \{1\}$. Let $\bar{c}_* = \bar{c}_{g_*}$ and $f_* = f^{\mathfrak{x}}_{\bar{c}_*}$ is the partial automorphism of M_γ with domain $\cup\{P^{M_{\mathfrak{x}}}_u : u \in I_{g_*}\}$ from Definition 2.6. We prove that the player ISO wins in the game $\Game^*_{\lambda,j}(f_*, M_1, M_1)$, as $f_*(c_1) = c_2 (\in P^{M_{\mathfrak{x}}}_{u^t})$ this is enough. Recall that a play last j moves; now the player ISO commit himself to choose in the $\beta < j$ move on the side a function $g_\beta \in \mathcal{G}_{1+\beta}$, increasing with β, $g_0 = g_*$ and his actual move f_β is $f^{\mathfrak{x}}_{\bar{c}_\beta}$ where $\bar{c}_\beta = \bar{c}_{g_\beta}$. For the β-th move if $\beta = 0$ or β limit let $g_\beta = \cup\{g_\varepsilon : \varepsilon < \beta\} \cup g_* \in \mathcal{G}_{1+\beta}$. In the $(\beta+1)$-th move let the AIS player choose $\alpha_\beta < \lambda$. Now the player ISO, on the side, first choose $i_\beta < \kappa$ such that $i_*(g_\beta) < i_\beta$, and $\mu_{i_\beta} > \alpha_\beta$, second he chooses $g^+_\beta \in \mathcal{G}_{i_\beta 1+\beta+1}, i_\beta$ satisfying:

⊛ (a) g^+_β extends g_β,
(b) $\mathrm{Dom}(g^+_\beta) \cap \kappa = i_\beta$
(c) $g^+_\beta{\restriction}(i_\beta \setminus \mathrm{Dom}(g_\beta))$ is constantly $1 + \beta$
(d) if $0 < i \in \mathrm{Dom}(g_\beta) \cap \kappa$ then $g^+_\beta{\restriction}[\mu_i, \mu_{i+1}) = g_\beta{\restriction}[\mu_i, \mu_{i+1})$
(e) if $i \notin \mathrm{Dom}(g_\beta) \cap \kappa$ and $i\, \mathrm{Dom}(g^+_\beta) \cap \kappa$ then $\mathrm{Dom}(g^+_\beta{\restriction}[\mu_i, \mu_{i+1})) = [\mu_i, \mu_{i+1})$ and $\varepsilon \in [\mu_i, \mu_{i+1}) \setminus \mathrm{Dom}(g_\beta) \Rightarrow g^+_\beta(\varepsilon) = \mu^+_i$
(f) if $i < \kappa, i \notin \mathrm{Dom}(g^+_\beta)$ then $g^+_\beta{\restriction}[\mu_i, \mu_{i+1}) = g_\beta{\restriction}[\mu_i, \mu_{i+1})$

Now ISO and AIS has to play the sub-game $\Game^{\alpha_\beta}_1(f_\beta, M_1, M_2)$. The player ISO has to play $f_{\beta,\alpha}$ in the α-th move for $\alpha \leq \alpha_\beta$ and on the side he chooses $g_{\beta,\alpha} \in \mathcal{G}_{1+\beta+1}$ with large enough domain and range, to make it a legal move, increasing with α, and $g_{\beta,0} = g^+_\beta$ and $g_{\beta,\alpha}{\restriction}\mu_{i_\beta} = g^+_\beta{\restriction}\mu_{i_\beta}$. Now obviously $\{g : g \in \mathcal{G}_{1+\beta+1}, g^+_\beta \subseteq g\}$ is closed under increasing union of length $< \mu_{i_\beta}$, it is enough to show that he can make the $(\alpha+1)$-th move which is trivial so we are done. ⊣

CLAIM 3.6. $M_{\mathfrak{x}}$ is P_s-rigid for $s \in I^*$.

PROOF. We imitate the proof of Claim 2.11.

$(*)_0$ $\mathfrak{x}$ is a full $(\lambda, \aleph_1)$-parameter

$(*)_1$ if $u_1 \subseteq u_2 \in I$, we define the function $\pi_{u_1,u_2} : J_{u_2} \to J_{u_1}$ by $F_{u_1,u_2}(t) = (u_1, j^t, g^t \restriction u_1, h^t \restriction u_1)$ for $t \in J_{u_2}$,

$(*)_2$ if $u_1 \subseteq u_2 \subseteq u_3$ are from I then $\pi_{u_1,u_3} = \pi_{u_1,u_2} \circ \pi_{u_2,u_3}$ that is $\pi_{u_1,u_2}(t) = \pi_{u_1,u_2}(\pi_{u_2,u_3}(t))$

$(*)_3$ for $u_1 \subseteq u_2$ we have

- (α) $T \cap (J_{u_1} \times J_{u_2}) = \{(\pi_{u_1,u_2}(t_2), t_2) : t_2 \in J_{u_2}\}$
- (β) $\mathbb{G}_{u_1,u_2} = \{(\hat{\pi}_{u_1,u_2}(c_2), c_2) : c_2 \in \mathbb{G}_{u_2}\}$ where $\hat{\pi}_{u_1,u_2} \in \mathrm{Hom}(\mathbb{G}^{\mathfrak{x}}_{u_2}, \mathbb{G}^{\mathfrak{x}}_{u_1})$ is the unique homomorphism from $\mathbb{G}^{\mathfrak{x}}_{u_2}$ into $\mathbb{G}^{\mathfrak{x}}_{u_1}$ mapping x_{t_2} to x_{t_1} whenever $\pi_{u_1,u_2}(t_2) = t_1$
  [Why? Check.]

$(*)_4$ if $u_1 \cup u_2 \subseteq u_3 \in I$, $t_3 \in J_{u_3}$ and $t_\ell = \pi_{u_\ell,u_3}(t_3)$ for $\ell = 1,2$ then, recalling Definition 2.1 (1A)(h), g^{t_1}, g^{t_2} are compatible functions as well as h^{t_1}, h^{t_2} and $j^{t_1} = j^{t_2}$ moreover $g^{t_1} \cup g^{t_2}$ is non-decreasing, $h^{t_1} \cup h^{t_2}$ is non-decreasing
  [Why? Just check.]

$(*)_5$ clause $\circledast_1$ of Claim 2.10(1) holds for $I' = I (= I_{\mathfrak{x}})$.

Why? Assume $\bar{c} \in C^{\mathfrak{x}}_I$ is such that $c_{s(*)} \neq e_{\mathbb{G}_{s(*)}}$ for some $s(*) \in I$. For each $u \in I, c_u$ is a word in the generators $\{x_t : t \in J_u\}$ of $\mathbb{G}_u$ and let $\mathbf{n}(u)$ be the length of this word and $\mathbf{m}(u)$ the number of generators appearing in it.

Now by clause (β) of $(*)_3$ we have $u_1 \subseteq u_2 \Rightarrow \mathbf{n}(u_1) \leq \mathbf{n}(u_2) \wedge \mathbf{m}(u_1) \leq \mathbf{m}(u_2)$. As $(I, \subseteq)$ is $\aleph_1$-directed, for some $u_* \in I$, $n_* < \omega$ and $m_* < \omega$ we have $u_* \subseteq u \in I \Rightarrow \mathbf{n}(u) = n_* \wedge \mathbf{m}(u) = m_*$ and let $c_u = (\ldots, x^{i(u,\ell)}_{t(u,\ell)}, \ldots)_{\ell < n_*}$ where $k(u,\ell) \in \{1,-1\}$ and $t(u,\ell) \in J^{\mathfrak{x}}_u$ and $t(u,\ell) = t(u,\ell+1) \Rightarrow k(\ell) = k(u_1+1)$. Clearly $u_* \subseteq u_1 \subseteq u_2 \in I \;\&\; \ell < n_* \Rightarrow \pi_{u_1,u_2}(t(u_2,\ell)) = t(u_1,\ell) \wedge k(u_1,\ell) = k(u_2,\ell) = k(u_*,\ell)$ hence $j^{t(u_2,\ell)} = j^{t(u_*,\ell)} \wedge j^{t(u_2,\ell)} = j^{t(u_*,\ell)}$. By our assumption toward contradiction necessarily $n_* > 0$ and let $k(\ell) = k(u_*,\ell)$.

As $\{u : u_* \subseteq u \in I\}$ is directed, by $(*)_4$ above, for each $\ell < n_*$ any two of the functions $\{g^{t(u,\ell)} : u_* \subseteq u \in I\}$ are compatible so $g_\ell =: \cup\{g^{t(u,\ell)} : u \in I\}$ is a non-decreasing function from $Y_{i_\ell(*)}$ to λ where $Y_{i_\ell(*)} = (\lambda \setminus \kappa) \cup i_\ell(*)$ for some $i_\ell(*) \leq \kappa$ and $h_\ell =: \cup\{h^{t(u,\ell)} : u_* \subseteq u \in I\}$ is similarly a non-decreasing function from $Y_{i_\ell(*)}$ to λ. Also g_ℓ maps $[\mu_i, \mu_{i+1})$ into $[\mu_i, \mu_i^+]$ for $i < \kappa$ and maps κ to κ.

CASE 1. $i_\ell(*) = \kappa$.

It also follows that for some j^*_ℓ we have $j^*_\ell =: j^{t(u,\ell)}$ whenever $u_* \subseteq u \in I$ in fact $j^*_\ell = j^{t(u_*,\ell)}$ is O.K. and $j^*_\ell < j_* \leq \kappa$. For each $i \in \mathrm{Rang}(g_\ell \restriction \kappa)$ choose $\beta_{\ell,i} < \kappa$ such that $g_\ell(\beta_{\ell,i}) = i$ and let $E = \{\delta < \kappa : \delta$ a limit ordinal $> \sup(u_* \cap \kappa)$ such that $i < j^*_\ell \;\&\; \ell < n_* \;\&\; i \in \mathrm{Rang}(g_\ell) \Rightarrow \beta_{\ell,i} < \delta$ and $\beta < \delta \;\&\; \ell < n \Rightarrow h_\ell(\beta) < \delta\}$, it is a club of κ. Choose u such that $u_* \subseteq u$ and $\mathrm{Min}(u \cap \kappa \backslash u_*) = \delta^* \in E$.

Now what can $g^{t(u,\ell)}(\mathrm{Min}(u \backslash u_*))$ be?

It has to be i for some $i < j^*_\ell < j^*$ hence $i \in \mathrm{Rang}(g_\ell)$ so for some $u_1, u_* \subseteq u_1 \subseteq \delta^*$ and $\beta_{\ell,i} \in u_1$ so $h_\ell(\beta_{\ell,i}) < \delta^*$ hence considering $u \cup u_1$ and recalling clause $(\delta)(iv)$ of (b) from Definition 3.3 of $\mathfrak{x}$ we have $h_\ell(\beta_{\ell,i}) < h_\ell(\delta^*)$ hence by (clause $(b)(\alpha)(iii)$) we have $i = g_\ell(\beta_{\ell,i}) < g_\ell(\delta^*)$, contradiction.

CASE 2. $i_\ell(*) \neq \kappa$ so $i_\ell(*) < \kappa$.

Clearly if $i \in (i_\ell(*), \kappa)$ and $\alpha \in [\mu_i, \mu_{i+1})$ then $g_\ell(\alpha) \neq \mu_i^+$ (see clause (b)(γ)(iii) of Definition 3.3) hence $g_\ell \restriction [\mu_i, \mu_{i+1})$ is a non-decreasing function from $[\mu_i, \mu_{i+1})$ to μ_i^+, but μ_{i+1} is regular $> \mu_i^+$ (see Hypothesis 3.2) hence $g_\ell \restriction [\mu_i, \mu_{i+1})$ is eventually constant say $\gamma_i \in [\mu_i, \mu_{i+1})$ and $g_\ell \restriction [\gamma_i, \mu_{i+1})$ is constantly $\varepsilon_i \in [\mu_i, \mu_i^+)$. So also $h_\ell \restriction [\gamma_i, \mu_{i+1}^+)$ is constant and its value is $< \mu_{i+1}$, and we get contradiction as in Case 1. $\dashv$

CONCLUSION 3.7. If $\lambda = \lambda^{\aleph_0} > \text{cf}(\lambda) > \aleph_0$ then for every $\alpha < \text{cf}(\lambda)$ there are non-isomorphic models M_1, M_2 of cardinality λ which are $\text{EF}^*_{\alpha,\lambda}$-equivalent.

PROOF. By Claim 3.5 + 3.6 as the cardinality of $M_{\mathfrak{x}}$ is λ. $\dashv$

REMARK 3.8. By minor changes, for some $t \in P_u^M$, $u = \emptyset$ letting $c_1 = e_{\mathbb{G}_u}$, $c_2 = x_t$ we have: $(M_{\mathfrak{x}}, c_1), (M_{\mathfrak{x}}, c_2)$ are non-isomorphism but $\text{EF}^*_{\lambda,j}$-equivalent for every $j < \kappa = \text{cf}(\lambda)$. This is similar to the parallel remark in the end of §1.

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