COLOURING OF SUCCESSOR OF REGULAR AGAIN
SH1163

SAHARON SHELAH

Abstract. We prove that for regular cardinals $\theta < \partial$ and $\lambda = \partial^+$ the colouring property $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \theta)$ almost always holds. The only exceptions are when θ is an uncountable limit regular cardinal and λ carries a uniform θ-complete filter which is not θ^+-complete. The result is nearly optimal

Date: 2020-11-24.

2010 *Mathematics Subject Classification.* Primary: 03E02, 03E05; Secondary: 03E04, 03E75.

Key words and phrases. set theory, combinatorial set theory, colourings, partition relations.

The author thanks Alice Leonhardt for the beautiful typing. First version on May 17, 2019. References like [DS18, Th.2.2=Le8] means the label of Th.2.2 is e8. The reader should note that the version in my website is usually more updated than the one in the mathematical archive.

§ 0. Introduction

We prove a strong colouring theorem. The connection between purely combinatorial theorems and topological constructions is known for many years. Several results in general topology were proved using the property $\mathrm{Pr}_1(\lambda, \mu, \sigma, \theta)$, see recently [JS15], [She19], the later by improving the existence result on Pr_1. We continue [She19] but the proof is self contained (except in the conclusion 1.3); see history and background in [She94]. Note that [She97, §4] states more than it proved.
Recall:

Definition 0.1. 1) Assume $\lambda \geq \mu \geq \sigma + \theta_0 + \theta_1, \bar{\theta} = (\theta_0, \theta_1)$, see 0.4(1). Assume further that $\theta_0, \theta_1 \geq \aleph_0$ but σ may be finite

Let $\mathrm{Pr}_1(\lambda, \mu, \sigma, \bar{\theta})$ mean that there is $\mathbf{c} : [\lambda]^2 \to \sigma$ witnessing it, which means:

$(*)_{\mathbf{c}}$ if (a) then (b), where:

(a) for $\iota = 0, 1, \mathbf{i}_\iota < \theta_\iota$ and $\bar{\zeta}^\iota = \langle \zeta^\iota_{\alpha,i} : \alpha < \mu, i < \mathbf{i}_\iota \rangle$ are sequences of ordinals of λ without repetitions, and $\mathrm{Rang}(\bar{\zeta}^0)$, $\mathrm{Rang}(\bar{\zeta}^1)$ are disjoint and $\gamma < \sigma$

(b) there are $\alpha_0 < \alpha_1 < \mu$ such that $\forall i_0 < \mathbf{i}_0, \forall i_1 < \mathbf{i}_1, \mathbf{c}\{\zeta^0_{\alpha_0,i_0}, \zeta^1_{\alpha_1,i_1}\} = \gamma$ and $\zeta^0_{\alpha_0,i_0} < \zeta^1_{\alpha_1,i_1}$.

2) Above if $\theta_0 = \theta = \theta_1$ then we may write $\mathrm{Pr}_1(\lambda, \mu, \sigma, \theta)$.

In the previous paper [She19] we proved, e.g. $\mathrm{Pr}_1(\aleph_3, \aleph_3, 2, (\aleph_0, \aleph_1))$ which means that the sequences $\bar{\zeta}^\iota$ are finite at the first coordinate and countable, possibly infinite at the second.

In this paper we prove e.g. that $\mathrm{Pr}_1(\aleph_3, \aleph_3, \aleph_3, \aleph_1)$ holds, which means that countable infinite sequences can be taken in both coordinates. Actually, the theorem says that, in particular, $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \theta)$ holds whenever $\theta = \mathrm{cf}(\theta) > \aleph_0$ and $\lambda = \theta^{++}$.

We thank the referee for many good suggestions.

Definition 0.2. 1) A filter D on a set I is uniform <u>when</u> for every subset A of I of cardinal $< |I|$, the set $I \setminus A \in D$; all our filters will be uniform
2) A filter D on a set I is weakly θ-saturated <u>when</u> $\theta \geq |I|$ and there is no partition of I to θ sets from D^+,
3) We say the filter D on a set I is θ-saturated <u>when</u> the Boolean algebra $\mathscr{P}(I)/D$ satisfies the θ-c.c.

Fact 0.3. 1) If D is a θ-complete filter on λ and is not θ-saturated then it is not weakly θ-saturated.
2) If $\theta = \sigma^+$ and D is a θ-complete filter on θ, <u>then</u> D is not weakly θ-saturated.
3) If $n \geq 1$ and $\lambda = \sigma^n$ and D is a (uniform) σ^+-complete filter on λ <u>then</u> D is not weakly σ^{+n}-saturated

Proof. 1) Obvious and well known
2) By [Sol71],
3) Let μ be the minimal cardinal such that D is not μ^+-complete, so clearly $\mu \in [\sigma^+, \lambda]$ hence μ is a successor cardinal. So there is a function f from λ into μ such that for every subset A of μ of cardinality $< \mu$, $f^{-1}(A) = \emptyset \mod D$. Let E be the family of subsets A of μ such that $f^{-1}(A) \in D$. Clearly E is a (uniform) μ-complete filter on μ hence by part (2) is not weakly μ-saturated, let $\langle A_\varepsilon : \varepsilon < \mu \rangle$

be a partition of μ to set from E^+. Now $\langle f^{-1}(A_\varepsilon) : \varepsilon < \mu\rangle$ witnesses the desired conclusion.

$\square_{0.3}$

Notation 0.4. 1) We denote infinite cardinals by $\lambda, \mu, \kappa, \theta, \partial$ while σ denote a finite or infinite cardinal. We denote ordinals by $\alpha, \beta, \gamma, \varepsilon, \zeta, \xi$. Natural numbers are denoted by k, ℓ, m, n and $\iota \in \{0, 1, 2\}$
1A) Let D denote a filter on an infinite set $\mathrm{dom}(D)$
2) For a set A of ordinals let $\mathrm{nacc}(A) = \{\alpha \in A; \alpha > \sup(A \cap \alpha)\}$ and $\mathrm{acc}(A) = A \setminus \mathrm{nacc}(A)$ For regular $\lambda > \kappa$ let $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

§ 1. A colouring theorem

Our aim is to prove

Theorem 1.1. $\Pr_1(\lambda, \lambda, \theta, \theta)$ *and even* $\Pr_1(\lambda, \lambda, \lambda, \theta)$ *holds provided that:*

(a) $\lambda = \partial^+, \partial = \mathrm{cf}(\partial) \geq \theta = \mathrm{cf}(\theta) > \aleph_0$
(b) *there is no* θ*-complete not* θ^+*-complete uniform weakly* θ*-saturated filter on* λ*.*

Remark 1.2. 1) The case of θ colours, i.e. proving only $\Pr_1(\lambda, \lambda, \theta, \theta)$ is easier so we prove it first.
2) If $\lambda = \aleph_2, \theta = \aleph_0$, then $\Pr_1(\aleph_2, \aleph_2, \aleph_2, \aleph_0)$ holds by [She97], so (by monotonicity in θ), the restriction $\theta > \aleph_0$ is not serious
3) We can omit the "weakly" in 1.1(b) because the filter is θ-complete by 0.3(1).
4) If θ, λ fail clause (b) of 1.1 then θ is (possibly weakly) inaccessible cardinal and is a large cardinal in some sense.
5) By monotonicity of $\Pr_1$ in θ, if clause (b) of 1.1 holds for some regular $\theta' \in (\theta, \partial)$ this suffice
6) We use $\partial > \theta$ rather then $\partial \geq \theta$ only in proving $(*)_2$ in Stage C of the proof

Proof. Stage A: We begin exactly as in earlier proofs. We let $(\kappa_1, \kappa_2) = (\theta, \lambda)$. Let $S \subseteq S^\lambda_\partial$ be stationary and $h : \lambda \to \lambda$ be such that $\alpha < \lambda \Rightarrow h(\alpha) < 1 + \alpha, h{\restriction}(\lambda \backslash S)$ is constantly zero and $S^*_\gamma := \{\delta \in S : h(\delta) = \gamma\}$ is a stationary subset of λ for every $\gamma < \lambda$. Let $F_\iota : \lambda \to \kappa_\iota$ for $\iota = 1, 2$ be such that for every $(\varepsilon_1, \varepsilon_2) \in (\kappa_1 \times \kappa_2)$ the set $W_{\varepsilon_1, \varepsilon_2}(\beta) = \{\gamma \in S^*_\beta : F_\iota(\gamma) = \varepsilon_\iota \text{ for } \iota = 1, 2\}$ is a stationary subset of λ for every $\beta < \lambda$.

For $\iota = 1, 2$ and $\rho \in {}^{\omega>}\lambda$ let $F_\iota(\rho) = \langle F_\iota(\rho(\ell)) : \ell < \ell g(\rho)\rangle$.

Let $\bar{e} = \langle e_\alpha : \alpha < \lambda\rangle$ be such that

$\odot_1$ (a) if $\alpha = 0$ then $e_\alpha = \emptyset$
(b) if $\alpha = \beta + 1$ then $e_\alpha = \{\beta\}$
(c) if α is a limit ordinal then e_α is a club of α of order type $\mathrm{cf}(\alpha)$ disjoint to S^λ_∂ hence to S.

In other cases (not here) instead h we use a sequence $\langle h_\alpha : \alpha < \lambda\rangle$ of functions, $h_\alpha : e_\alpha \to \theta$ and use e.g $\langle h_{\gamma_\ell(\beta,\alpha)}(\gamma_{\ell+1}(\beta, \alpha)) : \ell < k(\beta, \alpha)\rangle$ and ρ_h, but this is not necessary here.

Now (using $\bar{e}$) for $\alpha < \beta < \lambda$, let

$$\gamma(\beta, \alpha) := \min\{\gamma \in e_\beta : \gamma \geq \alpha\}.$$

Let us define $\gamma_\ell(\beta, \alpha)$:

$$\gamma_0(\beta, \alpha) = \beta,$$

$$\gamma_{\ell+1}(\beta, \alpha) = \gamma(\gamma_\ell(\beta, \alpha), \alpha) \text{ (if well defined).}$$

If $\alpha < \beta < \lambda$, let $k(\beta, \alpha)$ be the maximal $k < \omega$ such that $\gamma_k(\beta, \alpha)$ is defined (equivalently is equal to α) and let $\rho_{\beta,\alpha} = \rho(\beta, \alpha)$ be the sequence

$$\langle \gamma_0(\beta, \alpha), \gamma_1(\beta, \alpha), \ldots, \gamma_{k(\beta,\alpha)-1}(\beta, \alpha)\rangle.$$

Let $\gamma_{\ell t}(\beta,\alpha) = \gamma_{k(\beta,\alpha)-1}(\beta,\alpha)$ where ℓt stands for last.

Let

$$\rho_h = \langle h(\gamma_\ell(\beta,\alpha)) : \ell < k(\beta,\alpha)\rangle$$

and we let $\rho(\alpha,\alpha)$ and $\rho_h(\alpha,\alpha)$ be the empty sequences. Now clearly:

$\odot_2$ if $\alpha < \beta < \lambda$ then $\alpha \le \gamma(\beta,\alpha) < \beta$

hence

$\odot_3$ if $\alpha < \beta < \lambda, 0 < \ell < \omega$, and $\gamma_\ell(\beta,\alpha)$ is well defined, then

$$\alpha \le \gamma_\ell(\beta,\alpha) < \beta$$

and

$\odot_4$ if $\alpha < \beta < \lambda$, then $k(\beta,\alpha)$ is well defined and letting $\gamma_\ell := \gamma_\ell(\beta,\alpha)$ for $\ell \le k(\beta,\alpha)$ we have

$$\alpha = \gamma_{k(\beta,\alpha)} < \gamma_{\ell t}(\beta,\alpha) = \gamma_{k(\beta,\alpha)-1} < \cdots < \gamma_1 < \gamma_0 = \beta$$

$$\text{and } \alpha \in e_{\gamma_{\ell t}(\beta,\alpha)}$$

i.e. $\rho(\beta,\alpha)$ is a (strictly) decreasing finite sequence of ordinals, starting with β, ending with $\gamma_{\ell t}(\beta,\alpha)$ of length $k(\beta,\alpha)$.

Note that if $\alpha \in S, \alpha < \beta$ then $\gamma_{\ell t}(\beta,\alpha) = \alpha + 1$.

Also

$\odot_5$ if δ is a limit ordinal and $\delta < \beta < \lambda$, <u>then</u> for some $\alpha_0 < \delta$ we have: $\alpha_0 \le \alpha < \delta$ <u>implies</u>:

- (i) for $\ell < k(\beta,\delta)$ we have $\gamma_\ell(\beta,\delta) = \gamma_\ell(\beta,\alpha)$
- (ii) $\delta \in \mathrm{nacc}(e_{\gamma_{\ell t}(\beta,\delta)}) \Leftrightarrow \delta = \gamma_{k(\beta,\delta)}(\beta,\delta) = \gamma_{k(\beta,\delta)}(\beta,\alpha) \Leftrightarrow \neg[\gamma_{k(\beta,\delta)}(\beta,\delta) = \delta > \gamma_{k(\beta,\delta)}(\beta,\alpha)]$
- (iii) $\rho(\beta,\delta) \trianglelefteq \rho(\beta,\alpha)$; i.e. is an initial segment
- (iv) $\delta \in \mathrm{nacc}(e_{\gamma_{\ell t}(\beta,\delta)})$ (here always holds if $\delta \in S$) implies:
  - $\rho(\beta,\delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(\beta,\alpha)$ hence
  - $\rho_h(\beta,\delta)\hat{\ }\langle h(\beta,\delta)(\delta)\rangle \trianglelefteq \rho_h(\beta,\alpha)$.
- (v) if $\mathrm{cf}(\delta) = \partial$ <u>then</u> we have $\gamma_{\ell t}(\beta,\delta) = \delta + 1$ so $\delta \in \mathrm{nacc}(e_{\gamma_{\mathrm{lt}}(\beta,\delta)})$
- (vi) if $\mathrm{cf}(\delta) = \partial$ and $\delta \in e_\gamma$, <u>then</u> necessarily $\gamma = \delta + 1$.

Why? Just let

$$\alpha_0 = \mathrm{Max}\{\sup(e_{\gamma_\ell(\beta,\delta)} \cap \delta) + 1 : \ell < k(\beta,\delta) \text{ and } \delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})\}.$$

Notice that if $\ell < k(\beta,\delta) - 1$ then $\delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})$ follows.

Note that the outer maximum (in the choice of α_0) is well defined as it is over a finite non-empty set of ordinals. The inner sup is on the empty set (in which case we get zero) or is the maximum (which is well defined) as $e_{\gamma_\ell(\beta,\delta)}$ is a closed subset of $\gamma_\ell(\beta,\delta), \delta < \gamma_\ell(\beta,\delta)$ and $\delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})$ - as this is required. For clauses (v),

(vi) recall $\delta \in S^\lambda_\partial$ and $e_\gamma \cap S^\lambda_\partial = \emptyset$ when γ is a limit ordinal and $e_\gamma = \{\gamma - 1\}$ when γ is a successor ordinal.

$\odot_6$ (a) if $\alpha < \beta < \lambda, \ell < k(\beta,\alpha), \gamma = \gamma_\ell(\beta,\alpha)$ then $\rho(\beta,\alpha) = \rho(\beta,\gamma)\hat{}\rho(\gamma,\alpha)$ and $\rho_h(\beta,\alpha) = \rho_h(\beta,\gamma)\hat{}\rho_h(\gamma,\alpha)$

(b) if $\alpha_0 < \ldots < \alpha_k$ and $\rho(\alpha_k,\alpha_0) = \rho(\alpha_k,\alpha_{k-1})\hat{}\ldots\hat{}\rho(\alpha_1,\alpha_0)$ then this holds for any sub-sequence of $\langle \alpha_0,\ldots,\alpha_k\rangle$.

$\odot_7$ let F'_ι be $F_\iota \circ h$ for $\iota = 1,2$; so F'_1 is a function from λ into θ and F'_2 is a function from λ into λ.

Stage B:
Let

$\boxplus_2$ $\mathbf{T} = \{\bar t : \bar t = \langle t_\alpha : \alpha < \lambda\rangle$ satisfies $t_\alpha \in [\lambda]^{<\theta}$ and $t_\alpha \subseteq \lambda\backslash\alpha\}$.

$\boxplus_3$ for $\varepsilon < \theta$ and $\bar t \in \mathbf{T}$ let $A_{\bar t,\varepsilon}$ be the set of $\gamma < \lambda$ such that for some (α_0,α_1) we have:

- (a) $\alpha_0 < \alpha_1 < \lambda$ and[1] $(\zeta,\xi) \in t_{\alpha_0} \times t_{\alpha_1} \Rightarrow \zeta < \xi$
- (b) for every $(\zeta,\xi) \in t_{\alpha_0} \times t_{\alpha_1}$ for some ℓ we have:
  - (α) $\ell < k(\xi,\zeta)$
  - (β) $\gamma_\ell(\xi,\zeta) = \gamma$
  - (γ) if $k < k(\xi,\zeta)$ then $F'_1(\gamma) \geq F'_1(\gamma_k(\xi,\zeta))$ and $F'_1(\gamma) \geq \varepsilon$
  - (δ) if $k < \ell$ then $F'_1(\gamma_k(\xi,\zeta)) < F'_1(\gamma)$.

We define:

$\boxplus_4$ $D = \{A \subseteq \lambda : A$ includes $A_{\bar t,\varepsilon}$ for some $\bar t \in \mathbf{T}, \varepsilon < \theta\}$.

Now note:

$\boxplus_5$ (a) if $\bar s, \bar t \in \mathbf{T}, \varepsilon \leq \zeta < \theta$ and $(\forall \alpha < \lambda)(s_\alpha \subseteq t_\alpha)$, then $A_{\bar t,\zeta} \subseteq A_{\bar s,\varepsilon}$

(b) if $\bar s \in \mathbf{T}, \varepsilon < \theta, g$ is an increasing function from λ to λ and $\bar t = \langle t_\alpha : \alpha < \lambda\rangle$ is defined by $t_\alpha = s_{g(\alpha)}$ then $A_{\bar t,\varepsilon} \subseteq A_{\bar s,\varepsilon}$.

[Why? Read the definitions.]

$\boxplus_6$ (a) the intersection of any $< \theta$ members of D is a member of D, equivalently includes the set $A_{\bar t,\zeta}$ for some $\bar t \in \mathbf{T}, \zeta < \theta$

(b) for every $\beta < \lambda$ for some $\bar t \in \mathbf{T}, A_{\bar t,0} \subseteq [\beta,\lambda)$

(c) if $\bar t \in \mathbf{T}$ and $\alpha < \lambda \Rightarrow t_\alpha \neq \emptyset$ then $\cap\{A_{\bar t,\varepsilon} : \varepsilon < \theta\} = \emptyset$

(d) D is upward closed.

(e) λ belongs to D

[Why? For clause (a) assume $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \theta$ then for some $\zeta_\varepsilon < \theta$ and $\bar t_\varepsilon \in \mathbf{T}$ we have $A_\varepsilon \supseteq A_{\bar t_\varepsilon,\zeta_\varepsilon}$. Define $t_\alpha = \bigcup\{t^\varepsilon_\alpha : \varepsilon < \varepsilon(*)\}$ for $\alpha < \lambda$ and $\zeta = \sup\{\zeta_\varepsilon : \varepsilon < \varepsilon(*)\}$; as the cardinal θ is regular, clearly $|t_\alpha| \leq \sum\limits_{\varepsilon<\varepsilon(*)} |t^\varepsilon_\alpha| < \theta$ and obviously $t_\alpha \subseteq [\alpha,\lambda)$ hence $\bar t = \langle t_\alpha : \alpha < \lambda\rangle \in \mathbf{T}$ and similarly $\zeta < \theta$. Easily $A_{\bar t,\zeta} \subseteq A_{\bar t_\varepsilon,\zeta_\varepsilon}$ for every $\varepsilon < \varepsilon(*)$, see $\boxplus_5(a)$ so we are done proving clause (a). For clause (b) define $t_\alpha = \{\beta + \alpha + 1\}$ and recalling $\boxplus_3(b)(\beta)$ and $\odot_4$ check that

[1] If we choose to add here "$t_{\alpha_0} \subseteq \alpha_1$", then we would a problem in proving clause $\boxplus_5(b)$.

$A_{\bar{t},0} \subseteq [\beta, \lambda)$. Also clause (c) obviously holds because $\gamma \in A_{\bar{t},\varepsilon} \Rightarrow F'_1(\gamma) \geq \varepsilon$ by $\boxplus_3(b)(\gamma)$ and $F'_1$is a function from λ to θ and clauses (d),(e) hold trivially by the definition.]

$\boxplus_7$ (a) $\emptyset \notin D$

(b) D is a filter on λ, equivalently $A_{\bar{t},\varepsilon} \neq \emptyset$ for every $\bar{t}, \varepsilon$; also D is uniform θ-complete, not θ^+-complete.

[Why? Clause (a) is a major point, proved in Stage C below. That is, by $\boxplus_6(a),(d)$ the only missing point is to show $A_{\bar{t},\zeta} \neq \emptyset$, (in fact, $|A_{\bar{t},\zeta}| = \lambda$). For clause (b) by (a) and $\boxplus_6(a),(d),(e), D$ is a θ-complete filter and the statement that D is uniform holds by $\boxplus_6(b)$ and not θ^+-complete holds by $\boxplus_6(c)$.]

Note also

$\boxplus_8$ D is not weakly θ-saturated.

[Why? By $\boxplus_7$ and clause (b) in the assumptions of the theorem.]

Stage C:

In this stage we accomplish the proof of the missing point in $\boxplus_7(a)$ from above, so we shall prove "$A_{\bar{t},\varepsilon}$ is non-empty (in fact, has cardinality λ)" when:

$\boxplus$ (a) $t_\alpha \subseteq \lambda \backslash \alpha$ for $\alpha < \lambda$

(b) $|t_\alpha| < \theta$

(c) $\varepsilon < \theta$.

To start we note that:

$(*)_1$ without loss of generality $t_\alpha \neq \emptyset$ and $\alpha < \min(t_\alpha)$.

[Why? First, recalling $\boxplus_5(a)$ we can replace $\bar{t}$ by $\bar{t} = \langle t_\alpha \cup \{\alpha\} : \alpha < \lambda \rangle$, so we may assume that each t_α is not empty. Second, let $\bar{t}' = \langle t'_\alpha : \alpha < \lambda \rangle, t'_\alpha = t_{\alpha+1}$, so easily $\bar{t}'$ satisfies $(*)_1$ and $A_{\bar{t}',\varepsilon} \subseteq A_{\bar{t},\varepsilon}$ by clause $\boxplus_5(b)$.]

Now

$(*)_2$ we can find $\mathscr{U}_1^{\mathrm{dn}}, \varepsilon^{\mathrm{dn}}$ such that:

(a) $\mathscr{U}_1^{\mathrm{dn}} \subseteq S_0^*$ is stationary in λ, see stage A on S_0^*

(b) $\alpha < \delta \in \mathscr{U}_1^{\mathrm{dn}} \Rightarrow t_\alpha \subseteq \delta$

(c) $\varepsilon^{\mathrm{dn}} < \theta$

(d) if $\delta \in \mathscr{U}_1^{\mathrm{dn}}$ then for arbitrarily large $\alpha < \delta$ we have $\zeta \in t_\alpha \Rightarrow \mathrm{Rang}(F_1(\rho_h(\delta, \zeta))) \subseteq \varepsilon^{\mathrm{dn}} < \kappa_1 = \theta$.

[Why? Clearly $E_0 = \{\delta < \lambda : \delta$ is a limit ordinal such that $\alpha < \delta \Rightarrow t_\alpha \subseteq \delta\}$ is a club of λ. For every $\delta \in S_0^* \cap E_0$ and $\alpha < \delta$ we can find $\varepsilon_{\delta,\alpha}^{\mathrm{dn}}$ as in clauses (c),(d) of $(*)_2$ and so recalling that $\mathrm{cf}(\delta) = \partial > \theta > |t_\delta|$ it follows that there is $\varepsilon_\delta^{\mathrm{dn}}$ such that $\delta = \sup\{\alpha < \delta : \varepsilon_{\delta,\alpha}^{\mathrm{dn}} = \varepsilon_\delta^{\mathrm{dn}}\}$. Then recalling $\lambda = \mathrm{cf}(\lambda) > \theta$ we can choose $\varepsilon^{\mathrm{dn}}$ such that the set $\mathscr{U}_1^{\mathrm{dn}} = \{\delta \in S_0^* \cap E_0 : \varepsilon_\delta^{\mathrm{dn}} = \varepsilon^{\mathrm{dn}}\}$ is stationary. So $(*)_2$ holds indeed.]

$(*)_3$ We can find $\mathscr{U}_1^{\mathrm{up}}, \alpha_1^*, \varepsilon^{\mathrm{up}}$ such that:

(a) $\mathscr{U}_1^{\mathrm{up}} \subseteq S_0^*$ is stationary

(b) $h \restriction \mathscr{U}_1^{\mathrm{up}}$ is constantly 0, actually follows by (a), see Stage A

(c) $\alpha_1^* < \lambda$ satisfies $\alpha_1^* < \min(\mathscr{U}_1^{\mathrm{up}})$ and $\varepsilon^{\mathrm{up}} < \theta$

(d) if $\delta \in \mathscr{U}_1^{\rm up}$ and $\alpha \in [\alpha_1^*, \delta)$ and $\beta \in t_\delta$ then:
- $\rho_{\beta,\delta}\hat{}\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$
- $\mathrm{Rang}(F_1(\rho_h(\beta,\delta))) \subseteq \varepsilon^{\rm up}$.

[Why? For every $\delta \in S_0^* \subseteq S$ and $\zeta \in t_\delta$ let $\alpha_{1,\delta,\zeta} < \delta$ be such that $(\forall\alpha)(\alpha \in [\alpha_{1,\delta,\zeta}, \delta) \Rightarrow \rho_{\zeta,\delta}\hat{}\langle\delta\rangle \trianglelefteq \rho_{\zeta,\alpha})$, it exists by $\odot_5$ of Stage A.

Let

- $\alpha_{1,\delta} = \sup\{\alpha_{1,\delta,\zeta} : \zeta \in t_\delta\}$
- $\varepsilon_\delta^{\rm up} = \sup\{F_1'(\gamma_\rho(\zeta,\delta))(\ell)+1 : \zeta \in t_\delta \text{ and } \ell < k(\zeta,\delta)\} = \cup\{\sup \mathrm{Rang}(F_1(\rho_h(\zeta,\delta)))+1 : \zeta \in t_\delta\}$; as $\mathrm{cf}(\delta) = \partial = \mathrm{cf}(\partial) > \theta$ and $\theta = \mathrm{cf}(\theta) > |t_\delta|$, necessarily $\alpha_{1,\delta} < \delta$ and $\varepsilon_\delta^{\rm up} < \theta$.

Lastly, there are $\alpha_1^* < \lambda$ and $\varepsilon^{\rm up} < \kappa_1 = \theta$ and $\mathscr{U}_1^{\rm up} \subseteq S_0^*$ as required by using Fodor lemma. So $(*)_3$ holds indeed.]

Now let $E = \{\delta < \lambda : \delta \text{ is a limit ordinal } > \alpha_1^* \text{ such that } \delta = \sup(\mathscr{U}_1^{\rm dn} \cap \delta)$ and $\alpha < \delta \Rightarrow t_\alpha \subseteq \delta\}$, it is a club of λ because $\alpha_1^* < \lambda$ by $(*)_3(c)$ and $\mathscr{U}_1^{\rm dn}$ is an unbounded subset of λ by $(*)_2(a)$, and t_α is a subset of λ of cardinality $< \theta$ hence is bounded.

Choose $\varepsilon(*) = \max\{\varepsilon^{\rm up}+1, \varepsilon^{\rm dn}+1, \varepsilon+1\}$ where ε is from $\boxplus(c)$, so $\varepsilon(*) < \theta$ and choose $\delta_2 \in E \cap S$ such that $F_1'(\delta_2) = \varepsilon(*)$. Next choose $\alpha_2 \in \mathscr{U}_1^{\rm up} \backslash (\delta_2+1)$ and let $\alpha^* \in (\alpha_1^*, \delta_2)$ be large enough such that $\zeta \in (\alpha^*, \delta_2) \wedge \xi \in t_{\alpha_2} \Rightarrow \rho(\xi,\delta_2)\hat{}\langle\delta_2\rangle \triangleleft \rho(\xi,\zeta)$. Now choose $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap (\alpha^*, \delta_2)$ and $\alpha^{**} \in (\alpha^*, \delta_1)$ be such that $\alpha \in (\alpha^{**}, \delta_1) \wedge \xi \in t_{\alpha_2} \Rightarrow \rho(\xi,\delta_1)\hat{}\langle\delta_1\rangle \triangleleft \rho(\xi,\alpha)$.

Next let $\ell_* < \ell g(\rho(\alpha_2,\delta_1)$ be such that:

- $F_1(\rho_h(\alpha_2,\delta_1))(\ell_*) = \max \mathrm{Rang} F_1(\rho_h(\alpha_2,\delta_1))$
- under this restriction ℓ_* is minimal.

Now let $\gamma_* = \rho(\alpha_2,\delta_1)(\ell_*)$.

Lastly, choose $\alpha_1 \in (\alpha^{**}, \delta_1)$ which is as in $(*)_2(d)$ with respect to δ_1, i.e. such that:

$(*)_5$ if $\zeta \in t_{\alpha_1}$ then $\mathrm{Rang} F_1(\rho_h(\delta_1,\zeta)) \subseteq \varepsilon^{\rm dn}$.

Now we shall prove that the pair (α_1,α_2) is as required. So let $(\zeta,\xi) \in t_{\alpha_1} \times t_{\alpha_2}$; now by our choices

$(*)_6$ $\rho(\xi,\zeta) = \rho(\xi,\alpha_2)\hat{}\rho(\alpha_2,\delta_2)\hat{}\rho(\delta_2,\delta_1)\hat{}\rho(\delta_1,\zeta)$ and $\rho(\alpha_2,\delta_1) = \rho(\alpha_2,\delta_2)\hat{}\rho(\delta_2,\delta_1)$

So

$(*)_7$ $\mathrm{Rang}(F_1(\rho_h(\xi,\alpha_2)) \subseteq \varepsilon^{\rm up} \le \varepsilon(*)$

$(*)_8$ $\mathrm{Rang}(F_1(\rho_h(\delta_1,\zeta)) \subseteq \varepsilon^{\rm dn} \le \varepsilon(*)$

$(*)_9$ $\varepsilon(*) = F_1 \circ h(\delta_2) \in \mathrm{Rang}(F_1(\rho_h(\alpha_2,\delta_1)))$, see $(*)_6$ and (before and after) $\odot_1$.

[Why? Recall that δ_2 was chosen in $E \cap S$ such that $F_1'(\delta_2) = \varepsilon(*)$.]

Hence

$(*)_{10}$ in $\boxplus_3(b)$ for our $\bar{t}$ and the pair (α_1, α_2), our γ_* (chosen before $(*)_5$) is gotten, witnessing $\gamma_* \in A_{\bar{t},\varepsilon(*)} \subseteq A_{\bar{t},\varepsilon}$ as first $\varepsilon < \varepsilon(*)$, by the choice of $\epsilon(*)$, and second if $(\zeta, \xi) \in t_{\alpha_1} \times t_{\alpha_2}$ then $\ell = \ell g(\rho(\xi, \alpha_2)) + \ell_*$ is as required in $\boxplus_3(b)$ for $\bar{t}$ by $(*)_6 - (*)_9$

So we are done proving $\boxplus_7(a)$.

Stage D: By $\boxplus_8$

$\circledast_1$ there is $F_* : \lambda \to \theta$ such that $\varepsilon < \theta \Rightarrow F_*^{-1}(\{\varepsilon\}) \neq \emptyset \mod D$.

We first deal with the easier version with θ colours, i.e. proving $\text{Pr}_1(\lambda, \lambda, \theta, \theta)$.

We now define the colouring $\mathbf{c}_1 : [\lambda]^2 \to \theta$ by:

$\circledast_2$ if $\alpha < \beta < \lambda$ then $\mathbf{c}_1\{\alpha, \beta\}$ is $F_*(\gamma_{\ell(\beta,\alpha)}(\beta, \alpha))$ where $\ell(\beta, \alpha) = \min\{\ell < k(\beta, \alpha) : F_1'(\gamma_\ell(\beta, \alpha)) = \max \text{Rang}(F_1'(\rho(\beta, \alpha)))\}$.

To prove that the colouring $\mathbf{c}_1$ really witnesses $\text{Pr}_1(\lambda, \lambda, \theta, \theta)$, our task is to prove:

$\circledast_3$ given $\bar{t} \in \mathbf{T}$ and $\iota < \theta$ there are $\alpha < \beta$ such that:
- $\zeta \in t_\alpha \wedge \xi \in t_\beta \Rightarrow \mathbf{c}_1\{\zeta, \xi\} = \iota$.

[Why does $\circledast_3$ holds? Let $B_\iota = \{\gamma < \lambda : F_*(\gamma) = \iota\}$. By the choice of F_* we know that $B_\varepsilon \neq \emptyset \mod D$. Focus on $A_{\bar{t},\varepsilon}$ for the specific $\bar{t} \in \mathbf{T}$ and any $\varepsilon < \theta$. Since $A_{\bar{t},\varepsilon} \in D$ we conclude that $B_\varepsilon \cap A_{\bar{t},\varepsilon} \neq \emptyset$.

Fix an ordinal $\gamma \in B_\iota \cap A_{\bar{t},\varepsilon}$. By the very definition of $A_{\bar{t},\varepsilon}$ in $\boxplus_3$ we choose $\alpha < \beta < \lambda$ and $\gamma \in B_\iota$ such that for every $(\zeta, \xi) \in t_\alpha \times t_\beta$ there exists $\ell < k(\xi, \zeta)$ for which $\gamma_\ell(\xi, \zeta) = \gamma$ and $F_1'(\gamma) \geq F_1'(\gamma_k(\xi, \zeta))$ whenever $k < k(\xi, \zeta)$ and $F_1(\gamma) \geq \varepsilon$ and $F_1'(\gamma) > F_1'(\gamma_k(\xi, \zeta))$ whenever $k < \ell$. Let $\ell(\xi, \zeta)$ be this ℓ, in fact, this ℓ is unique (for the pair (ζ, ξ)).

Now $\mathbf{c}_1\{\zeta, \xi\} = F_*(\gamma_{\ell(\xi,\zeta)}(\xi, \zeta))$ (by $\circledast_2$) which equals $F_*(\gamma)$ (by the choice of $\ell(\xi, \zeta)$) which equals ι (since $\gamma \in B_\iota$). Hence $\circledast_3$ holds and we finish Stage D.]

Stage E: The full theorem: the case of λ colors

Let h', h'' be functions from θ into θ, ω respectively such that the mapping $\zeta \mapsto (h'(\zeta), h''(\zeta))$ is onto $\theta \times \omega$ and moreover each such pair is gotten θ times.

We have to define a colouring $\mathbf{c}_2 : [\lambda]^2 \to \lambda$ exemplifying $\text{Pr}_1(\lambda, \lambda, \lambda, \theta)$.

This is done as follows using h', h'' and F_* from $\circledast_1$:

$\oplus_1$ for $\alpha < \beta < \lambda$ we let
- $\bullet_1$ $\zeta = \zeta(\beta, \alpha) := h'(\mathbf{c}_1\{\beta, \alpha\})$, necessarily $< \theta$
- $\bullet_2$ $n = n(\beta, \alpha) := h''(\mathbf{c}_1\{\beta, \alpha\})$, necessarily $< \omega$
- $\bullet_3$ $m = m(\beta, \alpha)$ is the n-th member of $\{k < k(\beta, \alpha) : F_1'(\gamma_k(\beta, \alpha)) = \zeta\}$ when there is such m and is zero otherwise
- $\bullet_4$ we define $\mathbf{c}_2$ as follows: for $\alpha < \beta, \mathbf{c}_2\{\alpha, \beta\}$ is $F_2'(\gamma_{m(\beta,\alpha)}(\beta, \alpha))$ recalling that F_2', a function from $\lambda to \lambda$ is from $\odot_2$ from the end of stage A.

To prove that $\mathbf{c}_2$ indeed exemplifies $\text{Pr}_1(\lambda, \lambda, \lambda, \theta)$ it suffice to prove (this is the task of the rest of the proof)

$\oplus_2$ assume $\bar{t} \in \mathbf{T}$ and $j_* < \lambda$ and we shall find $\alpha < \beta$ such that $t_\alpha \subseteq \beta$ and $(\zeta, \xi) \in t_\alpha \times t_\beta \Rightarrow \mathbf{c}_2\{\zeta, \xi\} = j_*$.

Toward this:

$\oplus_3$ (a) we apply $(*)_3$ to our $\bar{t}$, getting $\varepsilon^{\rm up}, \mathscr{U}_1^{\rm up}, \alpha_1^*$ as there
(b) we apply $(*)_2$ to our $\bar{t}$ getting $\mathscr{U}_1^{\rm dn}, \varepsilon^{\rm dn}$
(c) let $\varepsilon^{\rm md} = \max\{\varepsilon^{\rm up}+1, \varepsilon^{\rm dn}+1\}$.

We can find $g_2, \mathscr{U}_2^{\rm up}, \gamma_*, \alpha_2^*, m_2^*$ such that:

$\oplus_4$ (a) $\gamma_* < \lambda$ satisfies $F_2(\gamma_*) = j_*$ and $F_1(\gamma_*) = \varepsilon^{\rm md}$
(b) $\mathscr{U}_2^{\rm up} \subseteq S^*_{\gamma_*}$ is stationary such that $\delta \in \mathscr{U}_2^{\rm up} \Rightarrow F_2'(\delta) = F_2(h(\delta)) = F_2(\gamma_*) = j_* \wedge F_1'(\delta) = F_1(h(\delta)) = F_1(\gamma_*) = \varepsilon^{\rm md}$
(c) g_2 is a function with domain $\mathscr{U}_2^{\rm up}$ such that $\delta \in \mathscr{U}_2^{\rm up} \Rightarrow \delta < g_2(\delta) \in \mathscr{U}_1^{\rm up}$
(d) α_2^* satisfies $\alpha_1^* < \alpha_2^* < \min(\mathscr{U}_2^{\rm up})$
(e) if $\delta \in \mathscr{U}_2^{\rm up}$ and $\alpha \in [\alpha_2^*, \delta)$ and $\beta \in t_{g_2(\delta)}$ then
- $\rho(g_2(\delta),\delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(g_2(\delta),\alpha)$ hence
- $\rho_{\beta,\delta}\hat{\ }\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$

(f) m_2^* satisfies: for every $\delta \in \mathscr{U}_2^{\rm up}$, the cardinality of the set $\{\ell < k(g_2(\delta),\delta) : F_1'(\gamma_\ell(g_2(\delta),\delta)) = \varepsilon^{\rm md}\}$ which may be zero.

[Why? First choose γ_* as in clause (a) of $\oplus_4$ (possible by the choice of F_0, F_1, F_2 in the beginning of Stage A; hence $\delta \in S_{\gamma_*} \Rightarrow F_2'(\delta) = F_2(h(\delta))F_2(\gamma_*) = j_*$ and $F_1'(\delta) = F_1(h(\delta)) = F_1(\gamma_*) = \varepsilon^{\rm md}$ (by the choice of F_1' in $\odot_7$ recalling the definitions of h, F_1'). Second, define $g' : S^*_{\gamma_*} \to \mathscr{U}_1^{\rm up}$ such that $\delta \in S^*_{\gamma_*} \Rightarrow \delta < g'(\delta) \in \mathscr{U}_1^{\rm up}$. Third, for each $\delta \in S^*_{\gamma_*} \backslash (\alpha_1^*+1)$, find $\alpha'_{2,\delta} < \delta$ above α_1^* and $m_{2,\delta}$ such that the parallel of clauses (e),(f) (with g' here instead of g_2 there) of $\oplus_4$ holds. Fourth, use Fodor lemma to get a stationary $\mathscr{U}_2^{\rm up} \subseteq S^*_{\gamma_*}$ such that $\langle(\alpha'_{2,\delta}, m_{2,\delta}) : \delta \in \mathscr{U}_2^{\rm up}\rangle$ is constantly (α_2^*, m_2^*) and lastly let $g_2 = g' {\restriction} \mathscr{U}_2^{\rm up} \backslash (\alpha_2^*+1)$. Now it is easy to check that $\oplus_4$ holds indeed.]

Next

$\oplus_5$ if $\delta \in \mathscr{U}_2^{\rm up}$ <u>then</u>:
(a) $F_1'(\delta) = \varepsilon^{\rm md}$
(b) if $\alpha \in [\alpha_2^*, \delta), \xi \in t_{g_2(\delta)}$ <u>then</u> $u = \{\ell < k(\xi,\alpha) : F_1'(\gamma_\ell(\xi,\alpha)) = \varepsilon^{\rm md}\}$ has $> m_2^*$ members and if ℓ is the m_2^*-th member of u then $\gamma_\ell(\xi,\alpha) = \delta$.

Why? Clause (a) holds by $\oplus_4(a),(b)$. For clause (b) use clause (a) and the demands on m_2^*. That is

(a) $\rho(\xi,\alpha) = \rho(\xi, g_2(\delta))\hat{\ }\rho(g_2(\delta),\delta)\hat{\ }\rho(\delta,\alpha)$
[Why? by $(*)_3, \oplus_4(e)$]
(b) $\mathrm{Rang}(\rho_h(\alpha, g_2(\delta))) \subseteq \epsilon^{\rm up} \subseteq \epsilon^{\rm md}$
[Why? by $(*)_2$]
(c) the set $\{\ell < k(g_2(\delta),\delta) : F_1'(\gamma_\ell(g_2(\delta),\delta)) = \epsilon^{\rm md}\}$ has m_2^* members
[why? by $\oplus_4(f)$]
(d) $F_1'(\gamma_0(\delta,\alpha)) = F_1'(\delta) = \epsilon^{\rm md}$
[Why? by $\oplus_4(a),(b)$]]
(e) if ℓ_* is the m_2^*-th member of $\{\ell : F_1(\gamma_\ell(\xi,\alpha)) = \epsilon^{\rm md}\}$ then $\gamma_{\ell_*}(\xi,\alpha) = \delta$
[Why? putting the above together]

So $\oplus_5$ holds indeed.

Now choose $\varepsilon(*) < \theta$ such that $h'(\varepsilon(*)) = \varepsilon^{\rm md}$ and $h''(\varepsilon(*)) = m_2^*$.

Next, let $E = \{\delta < \lambda : \delta$ limit ordinal $> \alpha_2^*$ such that $\delta = \sup(\mathscr{U}_1^{\rm dn} \cap \delta)$ and $\alpha < \delta \Rightarrow g_2(\alpha) < \delta\}$.

Lastly,

$\oplus_6$ choose $\delta_1 < \delta_2$ such that
- (a) $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap E$
- (b) $\delta_2 \in \mathscr{U}_2^{\rm up} \cap E \backslash (\delta_1 + 1)$
- (c) $\mathbf{c}_1\{\delta_2, \delta_1\} = \varepsilon(*)$,

[Why does such a pair (δ_1, δ_2) exist? By Stage D applied to $\bar{s} = \langle s_\alpha : \alpha < \lambda\rangle$ where $s_\alpha = \{\min(\mathscr{U}_s^{\rm dn} \cap E \backslash \alpha), \min(\mathscr{U}_2^{\rm up} \cap E \backslash \alpha)\}$.

That is, we can find ordinals $\alpha < \beta < \lambda$ such that: for every $(\zeta, \xi) \in (s_\alpha \times s_\beta)$ we have $\mathbf{c}_1\{\xi, \zeta\} = \epsilon^{\rm md}$.

Let $\delta_1 = \min(\mathscr{U}_1^{\rm dn} \cap E \setminus \alpha$ and let $\delta_2 = \min(\mathscr{U}_1^{\rm up} \cap E \setminus \beta$.

So $(\delta_1, \delta_2) \in (s_\alpha \times s_\beta)$ hence clearly $\delta_1 < \delta_2$, $\mathbf{c}_1\{\delta_1, \delta_2\} = \epsilon(*)$, $\delta_1 \in \mathscr{U}_1^{\rm dn} \cap E$ and $\delta_2 \in \mathscr{U}_1^{\rm up} \cap E$. So the pair (δ_1, δ_2) is as promised in in $\oplus_6$]

Now let $\beta = g_2(\delta_2)$ and choose $\alpha \in \mathscr{U}_2^{\rm dn} \cap \delta_1 \backslash (\alpha_2^* + 1)$. Easy to check that α, β are as required.

So we have finished proving Theorem 1.1. $\square_{1.1}$

Conclusion 1.3. *If $\lambda = \partial^+, \partial > \theta$ are regular, <u>then</u> $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \theta)$ except possibly when the statement $\boxplus$ holds where:*

$\boxplus$
- (a) *there is a θ-complete not θ^+-complete uniform filter on λ and*
- (b) $\partial = \theta^+$
- (c) *θ is a regular limit uncountable cardinal.*

Proof. <u>Case 1</u>: $\partial = \chi^{++}$ so $> \aleph_1$

By monotonicity of Pr_1 in θ, without loss of generality $\theta = \chi^+$, hence $\theta \geq \aleph_1$ and there is no θ-complete weakly θ-saturated filter on $\theta^{+2} = \chi^{+3} = \partial^+ = \lambda$ by 0.3(3) so we can use Theorem 1.1 noting $\theta = \mathrm{cf}(\theta) > \aleph_0$.

<u>Case 2</u>: $\partial = \aleph_1$

In this case necessarily $\theta = \aleph_0, \partial = \aleph_1, \lambda = \aleph_2$ and the result hold by [She97].

<u>Case 3</u>: $\partial = \chi^+, \chi$ singular

As θ is regular, necessarily $\theta < \chi$ and we can apply Theorem [She19, 2.5=Le16] with $(\lambda, \theta^+, \theta)$ here standing for $(\lambda, \theta_1, \theta_0)$ there, (λ is indeed a successor of regular because ∂ is regular, and θ is regular; if we like to allow θ singular we should have used $(\lambda, \theta^{++}, \theta^+)$ here standing for $(\lambda, \theta_1, \theta_0)$ there). So we get that $\mathrm{Pr}_1(\lambda, \lambda, \lambda, (\theta^+, \theta))$ holds. Hence by monotonicity $\mathrm{Pr}_1(\lambda, \lambda, \lambda, (\theta, \theta))$ holds which means that also $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \theta)$ holds as promised.

<u>Case 4</u>: $\partial = \chi^+ > \aleph_1, \chi$ a regular limit cardinal.

If $\chi > \theta$ we can repeat the proof of case 3 relying on [She19], so without loss of generality we have $\theta = \chi$. Now in $\boxplus$, clauses (b) and (c) hold but it should fail hence by $\boxplus(a)$ fail, which give clause (b) in Theorem 1.1 hence we can apply 1.1 as in Case 1.

<u>Case 5</u>: ∂ is a limit cardinal

Still ∂ is regular $> \theta$ and we continue as in Case 3. $\square_{1.3}$

References

[DS18] Alan Stewart Dow and Saharon Shelah, *On the cofinality of the splitting number*, Indag. Math. (N.S.) **29** (2018), no. 1, 382–395, arXiv: 1801.02517. MR 3739621

[JS15] István Juhász and Saharon Shelah, *Strong colorings yield κ-bounded spaces with discretely untouchable points*, Proc. Amer. Math. Soc. **143** (2015), no. 5, 2241–2247, arXiv: 1307.1989. MR 3314130

[She94] Saharon Shelah, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994. MR 1318912

[She97] ______, *Colouring and non-productivity of $\aleph_2$-c.c*, Ann. Pure Appl. Logic **84** (1997), no. 2, 153–174, arXiv: math/9609218. MR 1437644

[She19] ______, *The colouring existence theorem revisited*, Acta Math. Hungar. **159** (2019), no. 1, 1–26, arXiv: 1311.1026. MR 4003692

[Sol71] Robert M. Solovay, *Real valued measurable cardinals*, Axiomatic set theory (Providence R.I), Proc. of Symposia in Pure Math., vol. XIII, Part 1, A.M.S., 1971, pp. 397–428.

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: http://shelah.logic.at