ITERATED RAMSEY BOUNDS FOR THE HALES-JEWETT NUMBERS

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ABSTRACT. Consider the Hales-Jewett theorem. The k-dimensional version of it tells us that the combinatorial space $\mathcal{U}_{M,\Lambda} = \{\eta \mid \eta : M \to \Lambda\}$ has, under suitable assumptions, monochromatic k-dimensional subspaces, where by a k-dimensional subspace we mean there exist a partition $\langle N_0, N_1, \cdots, N_k\rangle$ of M such that $N_1, \cdots, N_k \neq \emptyset$ (but we allow N_0 to be empty) and some $\rho_0 : N_0 \to \Lambda$, such that the subspace consists of those $\rho \in \mathcal{U}_{M,\Lambda}$ such that for $0 < l < k+1, \rho \restriction N_l$ is constant and $\rho \restriction N_0 = \rho_0$.

It seems natural to think it is better to have each $N_l, 0 < l < k+1$ a singleton. However it is then impossible to always find monochromatic k-dimensional subspaces (for example color η by 0 if $|\eta^{-1}\{\alpha\}|$ is an even number and by 1 otherwise). But modulo restricting the sign of each $|\eta^{-1}\{\alpha\}|$, we prove the parallel theorem– whose proof is not related to the Hales-Jewett theorem. We then connect the two numbers by showing that the Hales-Jewett numbers are not too much above the present ones. This gives an alternative proof of the Hales-Jewett theorem.

1. INTRODUCTION

In [4], Graham, Rothschild and Spencer compile a list of six major theorems in Ramsey theory, one of them being the Hales-Jewett theorem. [1] The original proof of the Hales-Jewett theorem [5], proceeded by double induction and hence did not give any primitive recursive bounds. In [8], Shelah presented a completely new proof of the Hales-Jewett theorem using simple induction, and using it, he showed that the Hales-Jewett numbers belong to the class $\mathcal{E}^5$ of the Grzegorczyk hierarchy.

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[1]See Section 2 for undefined notions.

Consider the k-dimensional Hales-Jewett theorem. Roughly speaking, it says that given a positive integer k, a finite set Λ of alphabets and a finite set C of colors, we can find m such that if we color $\mathcal{U}_{m,\Lambda} = \{\eta \mid \eta : m \to \Lambda\}$ into C colors, then there are pairwise disjoint non-empty subsets $N_1, \cdots, N_k$ of m and $\rho_0 : N_0 \to \Lambda$, where $N_0 = m \setminus \bigcup_{0<l<k+1} N_l$, such that the coloring is constant on

$$\{\eta \mid \eta \restriction N_0 = \rho_0 \text{ and } \forall 0 < l < k+1, \eta \restriction N_l \text{ is constant } \}.$$

In this paper we prove some related partition theorems and obtain upper bounds for them. Suppose k, Λ and C are as above and E is one of the equivalence relations E_N or $\equiv_\alpha$, for $\alpha \in \Lambda$, given in Section 3. Then we show that there exists an integer m such that if we color $\mathcal{U}_{m,\Lambda}$ into C colors, then there are $N \subseteq m$ of size k, a one-to-one function $f : N \to m$ and a function $\rho : m \setminus \text{range}(f) \to \Lambda$ such that for all $\eta_1, \eta_2 \in \mathcal{U}_{N,\Lambda}$, if $\eta_1 E \eta_2$, then $(\eta_1 \circ f^{-1}) \cup \rho$ and $(\eta_2 \circ f^{-1}) \cup \rho$ take the same color. Furthermore if E is $\equiv_\alpha$, then we may take f to be the identity function and ρ to be the constant function α. When E is $\equiv_\alpha$, we are able to show that the least such m belongs to class $\mathcal{E}^4$ of the Grzegorczyk hierarchy, while when E is E_N, then we show that the least such m belongs to the class $\mathcal{E}^5$.

We then compare our results with those of the Hales-Jewett numbers and show that they are close to each other. As an application we obtain an alternative proof of the Hales-Jewett theorem.

The paper is organized as follows. In Section 2, we present some definitions and known results that will be used for the rest of the paper. Then is Section 3 we prove our new partition theorems. Finally in Section 4, we connect our results to the Hales-Jewett numbers and using them obtain an alternative proof for the Hales-Jewett theorem.

2. Some preliminaries

This section is devoted to some preliminary results that will be used for the rest of the paper. Let us start by fixing some notation that will be used through the paper.

Notation 2.1.
(1) *i, j, k, l, m, n denote natural numbers.*
(2) *Λ denotes a finite non-empty set of alphabets, whose elements are usually denoted by $\alpha, \beta, \cdots$.*
(3) *C denotes a finite non-empty set; the set of colors.*
(4) *Given a natural number n, we identify it by $n = \{l \mid l < n\}$.*
(5) *$M, N, \cdots$ denote finite non-empty linear orders.*

(6) *If* $(M, \leq_M)$ *is a linear order and* $r > 0$, *then* $[M]^r$ *denotes the set of all* r*-element subsets* $\{u_0, \cdots, u_{r-1}\}$ *of* M *such that* $u_0 <_M \cdots <_M u_{r-1}$. *Also,* $[M]^{<l} = \bigcup_{0<r<l}[M]^r$.
(7) $\mathcal{U}_{M,\Lambda} = \{\eta \mid \eta : M \to \Lambda\}$.

2.1. **The Grzegorczyk hierarchy.** For each $n \in \mathbb{N}$, define the function E_n by

$$E_0(x, y) = x + y,$$
$$E_1(x) = x^2 + 2,$$
$$E_{n+2}(0) = 2,$$

and

$$E_{n+2}(x + 1) = E_{n+1}(E_{n+2}(x)).$$

Observe that each E_n is primitive recursive . Let $\mathcal{E}^0$ be the class of functions, whose initial functions are the zero function, the successor function, and the projection functions and is closed under composition and limited recursion (that is, if $g, h, j \in \mathcal{E}^0$ and f is defined by primitive recursion from g and h, has the same arity as j and is pointwise bounded by j, then f belongs to $\mathcal{E}^0$ as well). The class $\mathcal{E}^n$ is defined similarly except that the function E_n is added to the list of initial functions. $\mathcal{E}^n$ is called the n-th Grzegorczyk class. We refer to [1, Apendix A] and [7] for more details on the Grzegorczyk hierarchy.

2.2. **Van der Waerden numbers.** Given natural numbers m, r, the Van der Waerden number $W(r, m)$ is defined to be the least n such that for any coloring $\mathbf{d} : n \to r$, there exists a $\mathbf{d}$-monochromatic arithmetic progression of length m. By a celebrated theorem of Van der Waerden, such an n always exists. Van der Waerden's original proof was based on double induction on r and m and hence it did not give any primitive recursive bounds on $W(r, m)$. In [8], Shelah showed that $W(r, m)$ is primitive recursive and indeed, $W(r, m) \in \mathcal{E}^5$. The result of Shelah was later improved by Gowers [3], who proved the following.

Lemma 2.2. *(Gowers* [3]*)*

$$W(r, m) \leq 2^{2^{r^{2^{2^{m+9}}}}},$$

in particular, $W(r, m) \in \mathcal{E}^3$.

In this paper, we will work with the following generalization of the Van der Waerden numbers, where their existence is guaranteed by the Gallai-Witt theorem. We state the very special case, which is used in the proof of Theorem 4.1.

Definition 2.3. *Let $GW_C(1,h)$ be the minimal n such that if $\mathbf{d} : \mathcal{U}_{h,n} \to C$ is a C-coloring of $\mathcal{U}_{h,n}$, then we can find $d > 0$ and a sequence $\vec{a} = \langle a_e \mid e < h\rangle$ of natural numbers such that for each $e < h$, $a_e + d < n$ and $\mathbf{d}$ is constant on*

$$H = \{\vec{a}, \vec{a} + d\vec{e}_1, \cdots, \vec{a} + d\vec{e}_{h-1}\},$$

where $\vec{e}_1, \cdots, \vec{e}_h$ are the unit vectors.

We could also define $\mathrm{GW}_C(m,h)$ in a similar way, where we require $a_e + d \cdot m < n$ for all $e < h$ and $\mathbf{d}$ is constant on

$$H = \{\langle a_e + d \cdot i_e : e < h\rangle : i_0, \cdots, i_{h-1} \leq m\}$$

Remark 2.4. (1) *By results of Shelah [8], $GW_C(m,h) \in \mathcal{E}^5$.*
(2) *$W(r, m+1) = GW_C(m,1)$, where $|C| = r$.*

2.3. **The Hales-Jewett numbers.** The Hales-Jewett theorem is considered as one of the fundamental results in Ramsey theory. In the words of [4], "the Hales-Jewett theorem strips van der Waerden's theorem of its unessential elements and reveals the heart of Ramsey theory. It provides a focal point from which many results can be derived and acts as a cornerstone for much of the more advanced work".

In order to state the theorem, we need some definitions. Recall that

$$\mathcal{U}_{M,\Lambda} = \{\eta \mid \eta : M \to \Lambda\}.$$

The notions of line and m-dimensional subspace play an important role in the Hales-Jewett theorem.

Definition 2.5. *A set $L \subseteq \mathcal{U}_{M,\Lambda}$ is a line of $\mathcal{U}_{M,\Lambda}$, if there exist $M_0 \subseteq M$ and $\rho : M \setminus M_0 \to \Lambda$ such that for every $\eta \in L$*

(1) *$\eta \restriction M \setminus M_0 = \rho$.*
(2) *$\eta \restriction M_0$ is constant.*

Definition 2.6. *A set $S \subseteq \mathcal{U}_{M,\Lambda}$ is a convex m-dimensional subspace of $\mathcal{U}_{M,\Lambda}$, if there are pairwise disjoint subsets $M_l, l < m$, of M, a map $\rho : M \setminus \bigcup_{l<m} M_l \to \Lambda$ and a function $f : \bigcup_{l<m} M_l \to m$ such that:*

(1) *For all $l_0 < l_1 < m, M_{l_0} <_M M_{l_1}$ (which means for all $j_0 \in M_{l_0}$ and $j_1 \in M_{l_1}$, $j_0 <_M j_1$).*
(2) *For every $l < m$ and $j \in M_l, f(j) = l$.*
(3) *For every $\nu \in \mathcal{U}_{M,\Lambda}, \nu \in S$ iff*
   (a) *$\nu \restriction M \setminus \bigcup_{l<m} M_l = \rho$.*
   (b) *$\exists \varrho \in \mathcal{U}_{m,\Lambda}\ \forall a \in \bigcup_{l<m} M_l$,*

$$\nu(a) = \varrho(f(a)).$$

It is evident that a line of $\mathcal{U}_{M,\Lambda}$ is just a convex 1-dimensional subspace of $\mathcal{U}_{M,\Lambda}$.

Definition 2.7. *Let $HJ_C(m, \Lambda)$ be the minimal k such that for any linear order M of size k and any C-coloring $d : \mathcal{U}_{M,\Lambda} \to C$, there exists a d-monochromatic convex m-dimensional subspace of $\mathcal{U}_{M,\Lambda}$.*

By Hales-Jewett [5], $HJ_C(m, \Lambda)$ is finite. It is worth to note that in the statement of the Hales-Jewett theorem, the convexity requirement is not needed, but the proofs of it usually give convex subspaces (cf. the proof of Theorem 4.1).

The original proof by Hales and Jewett was not primitive recursive. In [8], Shelah showed that the functions $HJ_C(m, \Lambda)$ is primitive recursive, and indeed $HJ_C(m, \Lambda) \in \mathcal{E}^5$.

We will need the following lemma, which shows that the numbers $HJ_C(m, \Lambda)$ can be bounded by $HJ_C(1, \Lambda)$.

Lemma 2.8. *([1, Page 26]) $HJ_C(m, \Lambda) \leq m \cdot HJ_C(1, {}^m\Lambda)$.*

2.4. **Ramsey numbers.** Ramsey's theorem [6] says that if $m, l > 0$ and C is a finite set of colorings, then there exists a natural number n such that if $\mathbf{d} : [n]^l \to C$ is a C-coloring of $[n]^l$, then there exists $A \in [n]^m$ which is homogeneous for $\mathbf{d}$, i.e., $\mathbf{d} \restriction [A]^l$ is constant. Let $\mathrm{R}(m, l, C)$ be the least such n. The next lemma shows that $\mathrm{R}(m, l, c)$ is primitive recursive.

Lemma 2.9. *(Erdős and Rado [2]) $R(m, l, c) \in \mathcal{E}^4$.*

In this paper, we consider the following version of Ramsey numbers as well.

Definition 2.10. *For $m, l > 0$, and a finite set C of colorings, let $RAM(m, < l, C)$ be the least n such that for any C-coloring $\boldsymbol{f} : [n]^{<l} \to C$, there exists $B \in [n]^m$ such that for each $0 < k < l$, $\boldsymbol{f} \restriction [B]^k$ is constant. Let also $RAM(< l, C) = RAM(l, < l, C)$.*

Lemma 2.11. *([1, Fact 2.20.]) $RAM(< l + 1, C) \leq R(2l, l, {}^lC)$; in particular, $RAM(< l, C) \in \mathcal{E}^4$.*

Proof. We give a proof for completeness. Let $n = \mathrm{R}(2l, l, {}^lC)$, and let $\mathbf{f} : [n]^{<l+1} \to C$. Define a coloring $\mathbf{d} : [n]^l \to {}^lC$ by

$$\mathbf{d}(u) = \langle \mathbf{f}(u \restriction k) \mid k < l + 1 \rangle,$$

where $u \restriction k$ consists of the first k elements of u. Let $A \in [n]^{2l}$ be such that $\mathbf{d} \restriction [A]^l$ is constant. Let B consists of the first $l + 1$ elements of A. Then one can easily check that, for each $0 < k < l + 1$, $\mathbf{f} \restriction [B]^k$ is constant. The result follows. $\square$

3. Some new partition theorems

In this section we introduce some new partition relations and find primitive recursive bounds for them. These partition relations are then used in the next section to obtain an alternative proof of the Hales-Jewett theorem. We refer to [9, Section 8], [10, Section 8] and [11] for some related results.

Let M be a finite linear order and let $\pi \in \mathrm{Sym}(M)$, where $\mathrm{Sym}(M)$ is the set of permutations of M. Then π induces $\hat{\pi} \in \mathrm{Sym}(\mathcal{U}_{M,\Lambda})$, defined by $\hat{\pi}(\rho) = \rho \circ \pi$. Furthermore, if H is a subgroup of $\mathrm{Sym}(M)$, then H induces an equivalence relation $E_{H,M}$ on $\mathcal{U}_{M,\Lambda}$ defined by

$$\rho_1 E_{H,M} \rho_2 \iff \exists \pi \in H,\ \rho_2 = \rho_1 \circ \pi.$$

When $H = \mathrm{Sym}(M)$, we simply write E_M for $E_{H,M}$.

Definition 3.1. *Let H be a subgroup of Sym(M).*

1. *A C-coloring $\boldsymbol{d} : \mathcal{U}_{M,\Lambda} \to C$ is called H-invariant if for all ν_1, ν_2 in $\mathcal{U}_{M,\Lambda}$, if $\nu_1 E_{H,M} \nu_2$, then $\boldsymbol{d}(\nu_1) = \boldsymbol{d}(\nu_2)$.*
2. *Suppose $N \subseteq M$. Then $Par_{C,H}(M, \Lambda, N)$ is the statement: if $\boldsymbol{d} : \mathcal{U}_{M,\Lambda} \to C$ is a C-coloring of $\mathcal{U}_{M,\Lambda}$, then there exists a one-to-one function $f : N \to M$ and a function $\varrho : M \setminus \mathrm{range}(f) \to \Lambda$ such that for all $\eta_1, \eta_2 \in \mathcal{U}_{N,\Lambda}$,*

   $$\eta_1 E_{H,N} \eta_2 \implies \boldsymbol{d}((\eta_1 \circ f^{-1}) \cup \varrho) = \boldsymbol{d}((\eta_2 \circ f^{-1}) \cup \varrho).$$

   When $H = Sym(M)$, we remove H and write $Par_C(M, \Lambda, N)$ for $Par_{C,H}(M, \Lambda, N)$.
3. *$f^{13}_{\Lambda,H}(m, C)$ is the minimal k such that $Par_{C,H}(M, \Lambda, N)$ holds for any linear order M of size k and some $N \subseteq M$ of size m. We write $f^{13}_{\Lambda}(m, C)$ for $f^{13}_{\Lambda,Sym(M)}(m, C)$.*

Given $\alpha \in \Lambda$, $u \subseteq M$ and $\bar{\beta} \in {}^{|u|}\Lambda$, we define $\eta_{\alpha,\bar{\beta},u} \in \mathcal{U}_{M,\Lambda}$, by

$$\eta_{\alpha,\bar{\beta},u}(a) = \begin{cases} \alpha & \text{if } a \in M \setminus u, \\ \bar{\beta}(l) & \text{if } a \text{ is the } l\text{-th element of } u. \end{cases}$$

The next lemma shows that every $\eta \in \mathcal{U}_{M,\Lambda}$ is of the above form.

Lemma 3.2. *Suppose $\alpha \in \Lambda$ and $\eta \in \mathcal{U}_{M,\Lambda}$. Then there are $u \subseteq M$ and $\bar{\beta} \in {}^{|u|}\Lambda$ such that*

1. *$\eta = \eta_{\alpha,\bar{\beta},u}$,*
2. *For all $l < |u|$, $\bar{\beta}(l) \neq \alpha$.*

Proof. Let $u = \{a \in M \mid \eta(a) \neq \alpha\}$. Let also $\{a_0, \cdots, a_{|u|-1}\}$ be an increasing enumeration of u, and define $\bar{\beta} : |u| \to \Lambda$ by $\bar{\beta}(l) = \eta(a_l)$. Then u and $\bar{\beta}$ are as required. □

When writing some $\eta \in \mathcal{U}_{M,\Lambda}$ as $\eta = \eta_{\alpha,\bar{\beta},u}$, we always assume that it also satisfies $\bar{\beta}(l) \neq \alpha$, for $l < |u|$, in particular, $u = \{a \in M \mid \eta(a) \neq \alpha\}$.

Definition 3.3. *Suppose $\alpha \in \Lambda$.*

(1) *Let $\eta_1 = \eta_{\alpha,\bar{\beta}_1,u_1}$ and $\eta_2 = \eta_{\alpha,\bar{\beta}_2,u_2}$. We say η_1 and η_2 are α-isomorphic, denoted $\eta_1 \equiv_\alpha \eta_2$, if there exists an order preserving bijection $h : u_1 \to u_2$ such that $h(a) = b$ implies $\eta_1(a) = \eta_2(b)$.*
(2) *Suppose $N \subseteq M$ and $\alpha \in \Lambda$. Then $Par_{C,\alpha}(M, \Lambda, N)$ is the statement: if $\boldsymbol{d} : \mathcal{U}_{M,\Lambda} \to C$ is a C-coloring of $\mathcal{U}_{M,\Lambda}$, then for $\eta_1, \eta_2 \in \mathcal{U}_{N,\Lambda}$,*
$$\eta_1 \equiv_\alpha \eta_2 \implies \boldsymbol{d}(\eta_1 \cup \varrho) = \boldsymbol{d}(\eta_2 \cup \varrho),$$
where $\varrho : M \setminus N \to \Lambda$ is the constant function α.
(3) *$f^{13}_{\Lambda,\alpha}(m, C)$ is the minimal k such that $Par_{C,\alpha}(M, \Lambda, N)$ holds for any linear order M of size k and some $N \subseteq M$ of size m.*

The next theorem gives an upper bound for $f^{13}_{\Lambda,\alpha}(m, C)$ in terms of Ramsey numbers.

Theorem 3.4. *Let $\alpha \in \Lambda$. Then $f^{13}_{\Lambda,\alpha}(m, C) \leq RAM(< m, m \cdot |C|^{(|\Lambda|^m)})$. In particular $f^{13}_{\Lambda,\alpha}(m, C) \in \mathcal{E}^4$.*

Proof. Let $n = \mathrm{RAM}(< m, m \cdot |C|^{(|\Lambda|^m)})$ and let $\mathbf{d} : \mathcal{U}_{n,\Lambda} \to C$ be a C-coloring of $\mathcal{U}_{n,\Lambda}$. Define f on $[n]^{<m}$ by
$$f(u) = \{(\bar{\beta}, i) \in {}^{|u|}\Lambda \times C \mid \mathbf{d}(\eta_{\alpha,\bar{\beta},u}) = i\}.$$
We can identify $f(u)$ by a function $f(u) : \mathcal{U}_{|u|,\Lambda} \to C$, which is defined, for each $\bar{\beta} \in {}^{|u|}\Lambda$, by
$$f(u)(\bar{\beta}) = i \iff \mathbf{d}(\eta_{\alpha,\bar{\beta},u}) = i.$$
It follows that $\mathrm{range}(f) \subseteq \bigcup_{k<m} \mathcal{U}_{\mathcal{U}_{k,\Lambda},C}$, and hence
$$|\,\mathrm{range}(f)| \leq \sum_{k<m} |C|^{(|\Lambda|^k)} \leq m \cdot |C|^{(|\Lambda|^m)}.$$
Let $N \in [n]^m$ be such that, for each $0 < l < m$, $f \restriction [N]^l$ is constant. Let also $\varrho : n \setminus N \to \Lambda$ be the constant function α. We show that if $\eta_1, \eta_2 : N \to \Lambda$ and $\eta_1 \equiv_\alpha \eta_2$, then $\mathbf{d}(\eta_1 \cup \varrho) = \mathbf{d}(\eta_2 \cup \varrho)$.

Let $u_1, u_2 \subseteq N$, $\bar{\beta}_1 : |u_1| \to \Lambda$ and $\bar{\beta}_2 : |u_2| \to \Lambda$ be such that $\eta_1 \cup \varrho = \eta_{\alpha,\bar{\beta}_1,u_1}$ and $\eta_2 \cup \varrho = \eta_{\alpha,\bar{\beta}_2,u_2}$. Let also $h : u_1 \to u_2$ witness $\eta_1 \equiv_\alpha \eta_2$. Note that $|u_1| = |u_2|$.

Claim 3.5. $\bar{\beta}_1 = \bar{\beta}_2$.

Proof. Let $\{a^1_0, \cdots, a^1_{|u_1|-1}\}$ and $\{a^2_0, \cdots, a^2_{|u_2|-1}\}$ be the increasing enumerations of u_1 and u_2 respectively. For $l < |u_1|$, we have

$$\bar\beta_1(l) = \eta_1(a^1_l) = \eta_2(h(a^1_l)) = \eta_2(a^2_l) = \bar\beta_2(l),$$

as required. □

Thus $|u_1| = |u_2|$ and $\bar\beta_1 = \bar\beta_2$. But then $f(u_1) = f(u_2)$, in particular, $(\bar\beta_1, \mathbf{d}(\eta_{\alpha,\bar\beta_1,u_1})) \in f(u_2)$, which means

$$\mathbf{d}(\eta_1 \cup \varrho) = \mathbf{d}(\eta_{\alpha,\bar\beta_1,u_1}) = \mathbf{d}(\eta_{\alpha,\bar\beta_2,u_2}) = \mathbf{d}(\eta_2 \cup \varrho).$$

The theorem follows. □

The next theorem show that $f^{13}_\Lambda(m, C)$ is primitive recursive as well.

Theorem 3.6. *If $|\Lambda| = k$, then $f^{13}_\Lambda(m, C) \le m_k$, where m_k is defined by recursion as $m_0 = m$ and for $l < k$*

$$m_{l+1} = RAM(< m_l, |C|^{(|\Lambda|^{m_l})}).$$

In particular $f^{13}_\Lambda(m, C) \in \mathcal{E}^5$

Proof. Let $\{\alpha_0, \cdots, \alpha_{k-1}\}$ enumerate Λ. Suppose $\mathbf{c} : \mathcal{U}_{m_k,\Lambda} \to C$ is a C-coloring of $\mathcal{U}_{m_k,\Lambda}$. Set $M = m_k$. By downward induction on $l \le k$ we choose a pair (N_l, ρ_l) such that:

- $(a)_l$ $N_l \in [M]^{m_l}$.
- $(b)_l$ $\rho_l : M \setminus N_l \to \Lambda$.
- $(c)_l$ If $i \in [l, h)$ and $\eta_1, \eta_2 : N_l \to \Lambda$ are α_i-isomorphism, then $\mathbf{c}(\eta_1 \cup \rho_l) = \mathbf{c}(\eta_2 \cup \rho_l)$.

For $l = k$, let $N_k = M$ and let ρ_k be the empty function. It is evident that clauses $(a)_k$ and $(b)_k$ hold and clause $(c)_k$ is vacuous as $[h, h) = \emptyset$.

Now suppose that the pair (N_{l+1}, ρ_{l+1}) has been chosen. Let $\mathbf{c}_l : \mathcal{U}_{N_{l+1},\Lambda} \to C$ be defined by $\mathbf{c}_l(\eta) = \mathbf{c}(\eta \cup \rho_{l+1})$.

By Lemma 3.4, $m_{l+1} = \mathrm{RAM}(< m_l, |C|^{(|\Lambda|^{m_l})}) \ge f^{13}_{\Lambda,\alpha_l}(m_l, C)$, and hence we can find a set $N \in [N_{l+1}]^{m_l}$ such that for $\eta_1, \eta_2 \in \mathcal{U}_{N,\Lambda}$,

$$\eta_1 \equiv_{\alpha_l} \eta_2 \implies \mathbf{c}_l(\eta_1 \cup \tau_l) = \mathbf{c}_l(\eta_2 \cup \tau_l),$$

where $\tau_l : N_{l+1} \setminus N \to \Lambda$ is the constant function α_l.

Set $N_l = N$ and define $\rho_l : M \setminus N_l \to \Lambda$ by

$$\rho_l(a) = \begin{cases} \rho_{l+1}(a) & \text{if } a \in M \setminus N_{l+1}, \\ \alpha_l & \text{if } a \in N_{l+1} \setminus N_l. \end{cases}$$

Let us show that the pair (N_l, ρ_l) satisfies clauses $(a)_l$, $(b)_l$ and $(c)_l$. The first two clauses are clear from the construction, so let $i \in [l, h)$ and suppose $\eta_1, \eta_2 : N_l \to \Lambda$ are α_i-isomorphism. For $j = 1, 2$, set $\tilde\eta_j = \eta_j \cup (\rho_l \restriction N_i \setminus N_l)$. Then $\tilde\eta_1, \tilde\eta_2 \in \mathcal{U}_{N_i,\Lambda}$ and are α_i-isomorphic, and hence $\mathbf{c}_i(\tilde{\eta}_1 \cup \tau_i) = \mathbf{c}_i(\tilde{\eta}_2 \cup \tau_i)$, where $\tau_i : N_{l+1} \setminus N \to \Lambda$ is the constant function α_i. It then follows that

$$\mathbf{c}(\eta_1 \cup \rho_l) = \mathbf{c}_i(\tilde{\eta}_i \cup \tau_i) = \mathbf{c}_i(\tilde{\eta}_2 \cup \tau_i) = \mathbf{c}(\eta_2 \cup \rho_l).$$

We now show that $\mathrm{Par}_C(M, \Lambda, N_0)$ holds as witnessed by the identity function $\mathrm{id} : N_0 \to M$ and $\rho_0 : M \setminus N_0 \to \Lambda$. Thus suppose that $\eta_1, \eta_2 \in \mathcal{U}_{N_0,\Lambda}$ and $\eta_2 E_{N_0} \eta_1$. We have to show that $\mathbf{c}(\eta_1 \cup \rho_0) = \mathbf{c}(\eta_2 \cup \rho_0)$.

Let $\pi \in \mathrm{Sym}(N_0)$ be such that $\eta_1 = \eta_2 \circ \pi$.

Claim 3.7. *There exists a unique $\nu \in \mathcal{U}_{N_0,\Lambda}$ such that*

(1) *For every $\alpha \in \Lambda$,*

$$|\nu^{-1}\{\alpha\}| = |\eta_1^{-1}\{\alpha\}| = |\eta_2^{-1}\{\alpha\}|,$$

(2) *If $a <_M b$ are from N_0, $\nu(a) = \alpha_i$ and $\nu(b) = \alpha_j$, then $i \leq j$,*

Proof. We define $\nu_0, \cdots, \nu_k$ as follows:

(a) $\nu_0 = \eta_1$.
(b) $\nu_l \in \mathcal{U}_{N_0,\Lambda}$ and for $\alpha \in \Lambda, |\nu_l^{-1}\{\alpha\}| = |\nu_0^{-1}\{\alpha\}|$.
(c) If ν_l is defined, then
  (α) $\{a \in N_0 \mid \nu_l(a) \in \{\alpha_0, \cdots, \alpha_{l-1}\}$ is an initial segment of N_0,
  (β) If $a <_{N_0} b$, $\nu_l(a) = \alpha_i$, $\nu_l(b) = \alpha_j$ and $i, j < l$, then $i \leq j$.
(d) If ν_l is defined and $l < k - 1$, then ν_{l+1} will satisfy (c)(α) and (c)(β) and ρ_l is the unique $<_{N_0}$-increasing function from

$$\{a \in N_0 \mid \nu_l(a) \in \{\alpha_{l+1}, \cdots, \alpha_{k-1}\}\}$$

onto

$$\{a \in N_0 \mid \nu_{l+1}(a) \in \{\alpha_{l+1}, \cdots, \alpha_{k-1}\}\}.$$

It is easy to carry the induction. Finally $\nu = \nu_k$ is as required, which is easily seen to be unique as well. □

Note that for each $l < k$, and by the choice of the pair (N_l, ρ_l),

$$\mathbf{c}(\nu_l \cup \rho_0) = \mathbf{c}(\nu_{l+1} \cup \rho_0),$$

and hence

$$\mathbf{c}(\eta_1 \cup \rho_0) = \mathbf{c}(\nu_0 \cup \rho_0) = \mathbf{c}(\nu_1 \cup \rho_0) = \cdots = \mathbf{c}(\nu_k \cup \rho_0) = \mathbf{c}(\nu \cup \rho_0).$$

By symmetry $\mathbf{c}(\eta_2 \cup \rho_0) = \mathbf{c}(\nu \cup \rho_0)$ and hence

$$\mathbf{c}(\eta_1 \cup \rho_0) = \mathbf{c}(\nu \cup \rho_0) = \mathbf{c}(\eta_2 \cup \rho_0).$$

The theorem follows. □

4. Iterated Ramsey bounds for $HJ_C(m, \Lambda)$

In this section we relate the results of the previous section to the Hales-Jewett theorem. The main technical result of this section is the following theorem which gives an upper bound for the Hales-Jewett numbers using the function f^{13}_{Λ}.

Due to Lemma 2.8, we just consider the one-dimensional version of the Hales-Jewett theorem.

Theorem 4.1. $HJ_C(1, \Lambda) \leq f^{13}_{\Lambda}(|\Lambda| \cdot GW_C(1, |\Lambda|), C)$.

Proof. Let $n_1 = \mathrm{GW}_C(1, |\Lambda|), n_2 = |\Lambda| \cdot n_1$ and $n_3 = f^{13}_{\Lambda}(n_2, C)$. We have to show that $HJ_C(1, \Lambda) \leq n_3$. We may assume that $|\Lambda| > 1$.

So assume $\mathbf{c} : \mathcal{U}_{n_3,\Lambda} \to C$ is a C-coloring of $\mathcal{U}_{n_3,\Lambda}$. By the choice of n_3, we can find $N \in [n_3]^{n_2}$ and $\rho_* : n_3 \setminus N \to \Lambda$ such that for any $\rho_1, \rho_2 : N \to \Lambda$, if $\rho_1 E_N \rho_2$ then $\mathbf{c}(\rho_* \cup \rho_1) = \mathbf{c}(\rho_* \cup \rho_2)$.

Let $\{\alpha_0, \cdots, \alpha_h\}$, where $h > 0$, enumerate Λ with no repetitions. Let $\mathbf{c}_1 : \mathcal{U}_{h+1,n_1} \to C$ be defined by $\mathbf{c}_1(\eta) = x$ iff

$$\forall \rho \in \mathcal{U}_{N,\Lambda}(\bigwedge_{e<h} |\rho^{-1}\{\alpha_e\}| = \eta(e) \implies \mathbf{c}(\rho_* \cup \rho) = x). \tag{4.1}$$

Claim 4.2. *For each $\eta \in \mathcal{U}_{h+1,n_1}$, there exists at least one $\rho \in \mathcal{U}_{N,\Lambda}$ satisfying $\bigwedge_{e<h} |\rho^{-1}\{\alpha_e\}| = \eta(e)$.*

Proof. Let $\langle A_e \mid e < h+1 \rangle$ be a partition of N such that for each $e < h, |A_e| = \eta(e)$. This is possible since, $\sum_{e<h} \eta(e) \leq h \cdot (n_1 - 1) < n_2 = |N|$. Define $\rho : N \to h$ by

$$\rho(m) = \alpha_e \iff m \in A_e.$$

Then ρ is as required. □

Claim 4.3. *Suppose $\rho_0, \rho_1 \in \mathcal{U}_{N,\Lambda}$, and for each $i < 2$ and each $e < h$, $|\rho_i^{-1}\{\alpha_e\}| = \eta(e)$. Then $\mathbf{c}(\rho_1 \cup \rho_*) = \mathbf{c}(\rho_2 \cup \rho_*)$.*

Proof. Let $\rho_1, \rho_2 \in \mathcal{U}_{N,\Lambda}$ be as above. Then one can easily show that $\rho_1 E_N \rho_2$, and hence $\mathbf{c}(\rho_* \cup \rho_1) = \mathbf{c}(\rho_* \cup \rho_2)$. □

The above two claims show that $\mathbf{c}_1$ is well-defined. By the choice of n_1, we can find some $d > 0$ and a sequence $\vec{a} = \langle a_e \mid e < h+1 \rangle$ of natural numbers, such that for each $e < h+1$, $a_e + d < n_1$, and $\mathbf{c}_1$ is constant on

$$H = \{\vec{a}, \vec{a} + d\vec{e}_1, \cdots, \vec{a} + d\vec{e}_h\},$$

where $\vec{e}_1, \cdots, \vec{e}_h$ are the unit vectors. Let $x \in C$ be this constant value. Let $\{N_{e,0} \mid e < h\} \cup \{N_*\}$ be a collection of pairwise disjoint subsets of N such that each $N_{e,0}$ has size m_e and N_* has size d. Define $F : \mathcal{U}_{1,\Lambda} \to \mathcal{U}_{n_3,\Lambda}$ by

$$F(v)(m) = \begin{cases} \rho_*(m) & \text{if } m \notin N, \\ \alpha_e & \text{if } m \in N_{e,0} \text{ with } e < h, \\ v(0) & \text{if } m \in N_*, \\ \alpha_h & \text{otherwise.} \end{cases}$$

Let $S = F''[\mathcal{U}_{n,\Lambda}]$. Then S is a line of $\mathcal{U}_{n_3,\Lambda}$, as witnesses by

- The set N_*.
- The function $\rho : n_3 \setminus N_* \to \Lambda$, which is defined by
$$\rho(m) = \begin{cases} \rho_*(m) & \text{if } m \notin N, \\ \alpha_e & \text{if } m \in N_{e,0}, \\ \alpha_h & \text{otherwise.} \end{cases}$$
- The constant function $f : N_* \to 1$.

Claim 4.4. *For each $v \in \mathcal{U}_{n,\Lambda}$, $\mathbf{c}(F(v)) = x$.*

Proof. Let $F(v) = \rho_* \cup \rho_1$, where $\rho_1 : N \to \Lambda$. Define $\eta : h+1 \to n_1$ in such a way that

$$\bigwedge_{e<h} \eta(e) = |\rho_1^{-1}\{\alpha_e\}|.$$

One can easily show, using the definition of F, that there exists $i \le h$ such that $\eta \restriction h = \vec{a} + d\vec{e}_i$ (where by convention, $\vec{e}_0$ is the zero vector, so that $\vec{a} = \vec{a} + d\vec{e}_0$). It then follows that $\mathbf{c}_1(\eta) = x$, and hence $\mathbf{c}(F(v)) = x$, as required. □

The theorem follows. □

We are now ready to relate the results of the paper to the Hales-Jewett numbers.

Theorem 4.5. *$HJ_C(1, \Lambda) \le m_k$, where $k = |\Lambda|$ and the sequence $\langle m_l : l \le k\rangle$ is defined by recursion as $m_0 = |\Lambda| \cdot GW_C(1, |\Lambda|)$, and for $l < k$, $m_{l+1} = RAM(< m_l, |C|^{(|\Lambda|^{m_l})})$.*

Proof. By Theorems 3.6 and 4.1. □

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