FUNDAMENTA
MATHEMATICAE
252 (2021)

The higher Cichoń diagram

by

Thomas Baumhauer (Wien), **Martin Goldstern** (Wien)
and **Saharon Shelah** (Jerusalem and Piscataway, NJ)

Abstract. For a strongly inacessible cardinal κ, we investigate the relationships between the following ideals:

(1) the ideal of meager sets in the $<\kappa$-box product topology,
(2) the ideal of "null" sets in the sense of [She17],
(3) the ideal of nowhere stationary subsets of a (naturally defined) stationary set $S^\kappa_{\mathrm{pr}} \subseteq \kappa$.

In particular, we analyze the provable inequalities between the cardinal characteristics for these ideals, and we give consistency results showing that certain inequalities are unprovable.

While some results from the classical case ($\kappa = \omega$) can be easily generalized to our setting, some key results (such as a Fubini property for the ideal of null sets) do not hold; this leads to the surprising inequality cov(null) $\leq$ non(null). Also, concepts that did not exist in the classical case (in particular, the notion of stationary sets) will turn out to be relevant.

We construct several models to distinguish the various cardinal characteristics; the main tools are iterations with $<\kappa$-support (and a strong "Knaster" version of κ^+-c.c.) and one iteration with $\leq\kappa$-support (and a version of κ-properness).

Contents

2020 *Mathematics Subject Classification*: Primary 03E35; Secondary 03E17, 03E55.
Key words and phrases: forcing, higher Cichoń diagram, higher reals, weakly compact cardinal.
Received 9 November 2018; revised 30 July 2019.
Published online 31 August 2020.

DOI: 10.4064/fm666-4-2020

0. Introduction. *Set theory of the reals* deals with topological, measure-theoretic and combinatorial properties of the real line, which set theorists often do not interpret as the linear continuum $\mathbb{R}$, but (often for technical or notational convenience) as the Cantor space 2^ω or the Baire space ω^ω.

We will be interested in a natural generalization of such properties to the spaces 2^κ and κ^κ for uncountable (and in this paper: always inaccessible) cardinals κ. This area of research has progressed quickly in recent years; [KL$^+$16] collected many questions inspired by workshops on generalized reals, and several recent results can be found in [BB$^+$18], [FL17], [She17], [CS19].

Concerning terminology, we suggest to use the adjective "higher" instead of the less specific "generalized". In analogy to *higher Suslin trees* (Suslin trees on cardinals larger than ω_1), *higher recursion theory* (recursion theory on ordinals greater than ω), *higher descriptive set theory* we will speak of higher reals, the higher Cantor space, higher random reals, the higher Cichoń diagram, etc.

We will occasionally refer to results or definitions involving 2^ω; to emphasize the distinction between this framework and our setup, we will use the adjective "classical" to refer to these concepts: the classical Cichoń diagram, classical random reals, etc.

Higher random reals. There exists a straightforward generalization of the meager ideal on 2^ω (or ω^ω) to an ideal on 2^κ for (regular) $\kappa > \omega$, using the $<\kappa$-box product topology and defining a set as meager if it can be covered by $\leq\kappa$-many (closed) nowhere dense sets.

In [She17] the third author introduced a generalization [1] $\mathbb{Q}_\kappa$ of the random forcing to 2^κ for inaccessible κ. The forcing $\mathbb{Q}_\kappa$ is κ-strategically closed, satisfies the κ^+-chain condition and for weakly compact κ is κ^κ-bounding. These are of course three properties that are satisfied by classical random forcing (i.e. on $\kappa = \omega$). The ideal $\mathrm{id}(\mathbb{Q}_\kappa)$ generated by all κ-Borel sets which are forced not to contain the $\mathbb{Q}_\kappa$-generic κ-real turns out to be orthogonal to the ideal Cohen_κ of all κ-meager sets.

In [CS19] it is shown how to replace the requirement of κ being weakly compact by assuming the existence of a stationary set that reflects only in inaccessibles and has a diamond sequence. A construction of a κ^+-c.c. κ^κ-bounding forcing notion using a different diamond is given in [FL17] but it implies $2^\kappa = \kappa^+$, therefore that setup does not allow us to investigate cardinal characteristics.

A different approach can be found in [BB$^+$18] where the authors use the well-known characterization of the additivity and cofinality of the null ideal

[1] Unlike [She17], we call our uncountable cardinal κ rather than λ, mainly to help us resist the temptation of calling the higher random reals "λandom reals".

by slaloms (in the classical case ($\kappa = \omega$), see for example [BJ95]) to define their versions of add(null) and cf(null) on 2^κ for inaccessible κ.

In the present paper we continue the work of [She17], and we also compare our cardinal characteristics to those derived from slaloms.

Overview of the paper

- In Section 1 we repeat some key definitions and results from [She17], introduce some notations and finally define the notion of a strengthened Galois–Tukey connection.
- In Section 2 we prove preservation theorems for iterations of $<\kappa$- and κ-support.
- In Section 3 we introduce an ideal $\mathrm{id}^-(\mathbb{Q}_\kappa) \subseteq \mathrm{id}(\mathbb{Q}_\kappa)$ whose definition is slightly simpler than that of $\mathrm{id}(\mathbb{Q}_\kappa)$; however, for weakly compact κ the ideals id and id^- coincide. We improve the characterizations of the additivity and cofinality of $\mathrm{id}(\mathbb{Q}_\kappa)$ given in [She17] and also give a new characterization of additivity and cofinality, using the additivity of the ideal of nowhere stationary sets on κ.
- In Section 4 we generalize a theorem from [She17] by introducing the notion of an anti-Fubini set.
- In Section 5 we repeat and elaborate some other results from [She17] and discuss the Bartoszyński–Raisonnier–Stern theorem for $\mathrm{id}(\mathbb{Q}_\kappa)$. We can show it for inaccessible κ only under additional assumptions, and we conjecture that it does not hold in general.
- In Section 6 we provide six models separating characteristics of the generalized Cichoń diagram using the tools developed in Section 2. Curiously we do exactly all possible vertical separations.
- In Section 7 we repeat some definitions and results from [BB$^+$18] and use a model from that paper to show that one of the generalized slalom characterizations of the additivity of null is not provably equal to the additivity of $\mathrm{id}(\mathbb{Q}_\kappa)$.

1. Preliminaries. Some familiarity with the preceding work [She17] is assumed but for the convenience of the reader we recall a few key definitions and results. For missing proofs in this section see there.

1.1. The generalized random forcing $\mathbb{Q}_\kappa$. To motivate the main definition of this section, we first give a characterization of random forcing; the definition of $\mathbb{Q}_\kappa$ can then be seen as a generalization.

Definition 1.1.1. A *positive tree on* ω is a set $T \subseteq 2^{<\omega}$ with the following properties:

- T is a tree, i.e. T is non-empty, and for all $t \in T$ and all initial segments $s \trianglelefteq t$ we also have $s \in T$.

- There is a family $(N_k : k \in \omega)$ with $N_k \subseteq 2^k$ such that:
  - The sets N_k are small, more precisely: $\sum_k |N_k|/2^k < 1$.
  - For all k, all $s \in 2^k$: $s \in T \Leftrightarrow ((\forall n < k)\ s{\restriction}n \in T$ and $s \notin N_k)$.

It is easy to see that a tree T is positive in this sense if and only if the set $[T]$ of branches of T has positive Lebesgue measure in 2^ω. Thus, the set of positive trees is isomorphic to (a dense subset of) random forcing.

It is well known and easy to see that the ideal of null sets can be defined from the random forcing in several ways:

FACT 1.1.2. *Let $A \subseteq 2^\omega$. Then each of the following properties is equivalent to the statement "A is Lebesgue measurable with measure 0":*

- *For all positive trees p there is a positive tree $q \subseteq p$ such that $[q] \cap A = \emptyset$.*
- *There is a predense set $\mathcal{C}$ of positive trees such that $A \cap \bigcup_{p\in\mathcal{C}} [p] = \emptyset$.*
- *There is a single positive tree p such that not only $[p] \cap A = \emptyset$, but also for every $s \in 2^{<\omega}$ we have $(s + [p]) \cap A = \emptyset$. Here, we write $s + X$ for the set $\{s + x : x \in X\}$, where $s + x \in 2^\omega$ is defined by $(s + x)(i) = s(i) + x(i)$ for $i \in \mathrm{dom}(s)$, and $(s + x)(i) = x(i)$ otherwise. ($s + X$ is also called a* rational translate *of X.)* ∎

DEFINITION 1.1.3. Unless stated otherwise, κ denotes a strongly inaccessible cardinal throughout this paper. When we write "inaccessible" we will always mean "strongly inaccessible" and for the set of all inaccessible cardinals below κ we write

$$S^\kappa_{\mathrm{inc}} = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}.$$

DEFINITION 1.1.4. Let $S \subseteq \kappa$. We say that S is *nowhere stationary* if for every $\delta \leq \kappa$ of uncountable cofinality the set $S \cap \delta$ is a non-stationary subset of δ. Typically we will only care about being non-stationary in $\delta \in S^\kappa_{\mathrm{inc}} \cup \{\kappa\}$.

We will now inductively define, for every inaccessible cardinal κ,

- a forcing notion $\mathbb{Q}_\kappa$ (this definition uses the ideals $\mathrm{id}(\mathbb{Q}_\delta)$ for $\delta < \kappa$),
- two ideals $\mathrm{wid}(\mathbb{Q}_\kappa) \subseteq \mathrm{id}(\mathbb{Q}_\kappa)$ on 2^κ (the ideals coincide for weakly compact κ, see 3.2.4).

DEFINITION 1.1.5. We recall the inductively defined forcing $\mathbb{Q}_\kappa$ from [She17, 1.3]. We have $p \in \mathbb{Q}_\kappa$ if there exists $(\tau, S, \vec{\Lambda} = \langle \Lambda_\delta : \delta \in S\rangle)$ (this tuple is called the *witness* for $p \in \mathbb{Q}_\kappa$) where:

(1) $p \subseteq 2^{<\kappa}$ is a tree, i.e. it is closed under initial segments.
(2) $\tau \in 2^{<\kappa}$ is the *trunk* of p, i.e. the least node which has two successors.
(3) Above τ the tree p is *fully branching*, i.e. $\tau \trianglelefteq \eta \in p \Rightarrow \eta^\frown 0, \eta^\frown 1 \in p$.
(4) $S \subseteq S^\kappa_{\mathrm{inc}}$ is nowhere stationary.
(5) For $\delta \in S$ we have $|\Lambda_\delta| \leq \delta$ and Λ_δ is a family of maximal antichains of $\mathbb{Q}_\delta$.

(6) If $\delta \notin S$ is a limit ordinal and $\eta \in 2^\delta$, then $\eta \in p$ iff $(\forall \sigma < \delta)\ \eta{\restriction}\sigma \in p$.
(7) If $\delta \in S$ is a limit ordinal and $\eta \in 2^\delta$, then $\eta \in p$ iff

 (a) $(\forall \sigma < \delta)\ \eta{\restriction}\sigma \in p$, and
 (b) $(\forall \mathcal{J} \in \Lambda_\delta)(\exists q \in \mathcal{J})\ \eta \in [q]$; note that in the notation introduced in 1.1.9 this simply means $\eta \in \mathrm{set}_1(\Lambda_\delta)$.

For $p, q \in \mathbb{Q}_\kappa$ we define q to be *stronger than* p if $q \subseteq p$. We write $q \leq p$ for "q stronger than p" throughout this paper (and we use this convention for any forcing, not just $\mathbb{Q}_\kappa$).

If G is a $\mathbb{Q}_\kappa$-generic filter then we call $\eta = \bigcup_{p \in G} \mathrm{tr}(p) \in 2^\kappa$ a *$\mathbb{Q}_\kappa$-generic real* or a *random real*, where $\mathrm{tr}(p)$ is the trunk of p. Alternatively, η is the unique element of $\bigcap_{p \in G} [p]$, where $[p]$ is the set of cofinal branches of p.

Remark 1.1.6. If we write N_δ for $\mathrm{set}_0(\Lambda_\delta)$ (see 1.1.9) we may also call a tuple $(\tau, S, \langle N_\delta : \delta \in S\rangle)$ a witness for $p \in \mathbb{Q}_\kappa$.

Remark 1.1.7. Note that the set $S \cap \lg(\tau)$ (where $\lg(\tau)$ is the order type of the predecessors of τ) is really irrelevant; if we require $\min(S) > \lg(\tau)$, then p is uniquely defined by its witness (in the sense of 1.1.6) and vice versa.

Remark 1.1.8. Let $p, q \in \mathbb{Q}_\kappa$. Then p and q are *compatible in* $\mathbb{Q}_\kappa$ iff at least one of the following holds:

$$\mathrm{tr}(p) \trianglelefteq \mathrm{tr}(q) \in p, \qquad \mathrm{tr}(q) \trianglelefteq \mathrm{tr}(p) \in q.$$

In particular, two conditions with the same stem are always compatible.

Moreover, if p and q are compatible, then $p \cap q$ is the weakest condition in $\mathbb{Q}_\kappa$ which is stronger than both.

As a consequence, any set $\mathcal{C} \subseteq \mathbb{Q}_\kappa$ with the property

$$(\forall \eta \in 2^{<\kappa})(\exists p \in \mathcal{C})\ \mathrm{tr}(p) = \eta$$

is predense in $\mathbb{Q}_\kappa$.

For inaccessible κ we now define ideals on 2^κ as follows:

Definition 1.1.9. For $\mathcal{J} \subseteq \mathbb{Q}_\kappa$ we define

$$\mathrm{set}_1(\mathcal{J}) = \bigcup_{p \in \mathcal{J}} [p], \qquad \mathrm{set}_0(\mathcal{J}) = 2^\kappa \setminus \mathrm{set}_1(\mathcal{J}).$$

For a collection Λ of subsets of $\mathbb{Q}_\kappa$ we define

$$\mathrm{set}_1(\Lambda) = \bigcap_{\mathcal{J} \in \Lambda} \mathrm{set}_1(\mathcal{J}), \qquad \mathrm{set}_0(\Lambda) = 2^\kappa \setminus \mathrm{set}_1(\Lambda).$$

Definition 1.1.10. For $A \subseteq 2^\kappa$:

(1) $A \in \mathrm{wid}(\mathbb{Q}_\kappa)$ iff there is a predense set $\mathcal{C} \subseteq \mathbb{Q}_\kappa$ such that $A \subseteq \mathrm{set}_0(\mathcal{C})$. Equivalently, $A \in \mathrm{wid}(\mathbb{Q}_\kappa)$ iff

$$(\forall p \in \mathbb{Q}_\kappa)(\exists q \in \mathbb{Q}_\kappa)\ q \leq p \text{ and } [q] \cap A = \emptyset.$$

 (We will discuss the ideal $\mathrm{wid}(\mathbb{Q}_\kappa)$ in Section 3.)

(2) $\mathrm{id}(\mathbb{Q}_\kappa)$ is the $\leq\kappa$-closure of $\mathrm{wid}(\mathbb{Q}_\kappa)$: $A \in \mathrm{id}(\mathbb{Q}_\kappa)$ iff A can be covered by the union of $\leq\kappa$-many sets in $\mathrm{wid}(\mathbb{Q}_\kappa)$. Equivalently, $A \in \mathrm{id}(\mathbb{Q}_\kappa)$ iff there is a family Λ of κ-many predense sets such that $A \subseteq \mathrm{set}_0(\Lambda)$.

THEOREM 1.1.11 ([She17, 3.2]). *The ideal* $\mathrm{id}(\mathbb{Q}_\kappa)$ *is the ideal of all sets* A *such that there exists a* κ*-Borel set* $\mathbf{B} \subseteq 2^\kappa$ *such that* $A \subseteq \mathbf{B}$ *and*

$$\mathbb{Q}_\kappa \Vdash \dot{\eta} \notin \mathbf{B}$$

where $\dot{\eta}$ *is the canonical generic* κ*-real added by* $\mathbb{Q}_\kappa$*. [More explicitly, we should say that there is a* κ*-Borel code* c *in* $\mathbf{V}$ *such that the corresponding Borel set* $\mathscr{B}_c$ *contains* A $(A \subseteq \mathscr{B}_c)$*, and that in the* $\mathbb{Q}_\kappa$*-extension,* η *will not be in the Borel set* $\mathscr{B}_c$*, computed in the extension:* $\mathbb{Q}_\kappa \Vdash \dot{\eta} \notin \mathscr{B}_c$*.]* ■

THEOREM 1.1.12 ([She17, 1.8, 1.9]).

(1) $\mathbb{Q}_\kappa$ *is* κ*-strategically closed. (See* 2.1.4.*)*
(2) $\mathbb{Q}_\kappa$ *satisfies the* κ^+*-c.c.*
(3) *If* κ *is weakly compact, then* $\mathbb{Q}_\kappa$ *is* κ^κ*-bounding.* ■

DEFINITION 1.1.13. For every $\eta \in 2^{<\kappa}$ we write $[\eta]$ for the set of $x \in 2^\kappa$ extending η; these are the basic clopen sets of the box product topology (i.e. the $<\kappa$-box product topology).

Let $\mathbf{Borel}_\kappa$ be the smallest family containing all clopen sets which is closed under complements and unions/intersections of at most κ-many sets. If $\mathbf{B} \in \mathbf{Borel}_\kappa$ then we call $\mathbf{B}$ a *κ-Borel set.*

A *Borel code* is a well-founded tree (with a unique root) with κ-many nodes whose leaves are labeled with elements of $2^{<\kappa}$; this assigns basic clopen sets to every leaf. This assignment can be naturally extended to the whole tree: if the successors of a node ν are labeled with a set $(B_i : i \in \kappa)$, then ν is labeled with $2^\kappa \setminus \bigcup_{i<\kappa} B_i$.

(Equivalently, a Borel code is an infinitary formula in the propositional language $L_{<\kappa^+}$, where the propositional variables are identified with the basic clopen sets.)

If c is a Borel code, we write $\mathscr{B}_c$ for the Borel set associated with it (i.e. the value of the assignment described above on the root of the tree c).

FACT 1.1.14. *Let* $\mathbf{V}, \mathbf{W}$ *be two universes. Let* $\eta \in 2^\kappa \cap \mathbf{V} \cap \mathbf{W}$ *and let* c *be a Borel code in* $\mathbf{V} \cap \mathbf{W}$*. Then it follows from an easy inductive argument on the rank of* c *that*

$$\mathbf{V} \models \eta \in \mathscr{B}_c \iff \mathbf{W} \models \eta \in \mathscr{B}_c. \ ■$$

This fact will allow us to speak about Borel sets when we should officially speak about Borel codes.

DEFINITION 1.1.15. Let $W \subseteq S^\kappa_{\mathrm{inc}}$ be nowhere stationary. We define $\mathbb{Q}_{\kappa,W}$ similarly to $\mathbb{Q}_\kappa$ in Definition 1.1.5 with some restrictions:

(1) In 1.1.5(4) we additionally require $S \subseteq W$.
(2) In 1.1.5(5) we additionally we require Λ_δ to be a family of maximal antichains of $\mathbb{Q}_{\delta, W\cap\delta}$.

Note that this definition is different from 3.3.8.

1.2. Quantifiers and rational translates

DEFINITION 1.2.1. Let μ be a regular cardinal. We use the following notation:

- Let $A, B \subseteq \mu$. We say $A \subseteq^*_\mu B$ if there exists $\zeta < \mu$ such that $A \setminus \zeta \subseteq B$. If μ is clear from the context we write $A \subseteq^* B$.
- "$(\exists^\mu \epsilon)\ \phi(\epsilon)$" is an abbreviation for "$\{\epsilon < \mu : \phi(\epsilon)\}$ is cofinal in μ". Similarly "$(\forall^\mu \epsilon)\ \phi(\epsilon)$" is an abbreviation for "$\{\epsilon < \mu : \neg\phi(\epsilon)\}$ is bounded in μ". If μ is clear from the context we write $\exists^\infty$ and $\forall^\infty$. Note that these quantifiers satisfy the usual equivalence

$$(\exists^\mu \epsilon)\ \phi(\epsilon) \iff \neg(\forall^\mu \epsilon)\ \neg\phi(\epsilon).$$

- For $\eta, \nu \in 2^\mu$ (or μ^μ) define:

(1) $\eta =^*_\mu \nu \Leftrightarrow (\forall^\infty i < \mu)\ \eta(i) = \nu(i)$.
(2) $\eta \leq^*_\mu \nu \Leftrightarrow (\forall^\infty i < \mu)\ \eta(i) \leq \nu(i)$.

Again we may just write $\eta =^* \nu$ and $\eta \leq^* \nu$.

DEFINITION 1.2.2. We define

$$\mathfrak{b}_\kappa = \min\{|B| : B \subseteq \kappa^\kappa \wedge (\forall \eta \in \kappa^\kappa)(\exists \nu \in B)\ \neg(\nu \leq^* \eta)\},$$
$$\mathfrak{d}_\kappa = \min\{|D| : D \subseteq \kappa^\kappa \wedge (\forall \eta \in \kappa^\kappa)(\exists \nu \in D)\ \eta \leq^* \nu)\}.$$

DEFINITION 1.2.3 (Rational translates).

- For $p \in \mathbb{Q}_\kappa$, $\alpha < \kappa$, $\nu \in 2^\alpha$, and $\eta \in p \cap 2^\alpha$ (typically $\mathrm{tr}(p) \trianglelefteq \eta$) we let $p^{[\eta,\nu]}$ be the condition obtained from p by first removing all nodes not compatible with η, and then replacing η by ν:

$$p^{[\eta,\nu]} = \{\rho : \rho \trianglelefteq \nu \vee ((\exists \varrho)\ \eta^\frown \varrho \in p \wedge \rho = \nu^\frown \varrho)\}.$$

- For $\mathcal{J} \subseteq \mathbb{Q}_\kappa$, $\alpha < \kappa$, and a permutation π of 2^α let

$$\mathcal{J}^{[\alpha,\pi]} = \{p^{[\eta,\nu]} : p \in \mathcal{J},\ \eta \in (p \cap 2^\alpha),\ \nu = \pi(\eta)\}.$$

- For a collection Λ of subsets of $\mathbb{Q}_\kappa$ and $\alpha < \kappa$ let

$$\Lambda^{[\alpha]} = \{\mathcal{J}^{[\alpha,\pi]} : \mathcal{J} \in \Lambda,\ \pi \text{ is a permutation of } 2^\alpha\}.$$

Clearly, $|\Lambda^{[\alpha]}| \leq \kappa + |\Lambda|$. If $\Lambda^{[\alpha]} = \Lambda$ for all $\alpha < \kappa$, we say that Λ is *closed under rational translates.*

1.3. The property $\mathrm{Pr}(\cdot)$ and the nowhere stationary ideal

DEFINITION 1.3.1. $\mathrm{Pr}(\kappa)$ means there exists $\Lambda = \{\Lambda_i : i < \kappa\}$ where $\Lambda_i \subseteq \mathbb{Q}_\kappa$ is a maximal antichain (or is predense) such that for no $p \in \mathbb{Q}_\kappa$ do we have

$$[p] \subseteq \mathrm{set}_1(\Lambda) = \bigcap_{i<\kappa} \mathrm{set}_1(\Lambda_i).$$

We define

$$S^\kappa_{\mathrm{pr}} = \{\lambda \in S^\kappa_{\mathrm{inc}} : \mathrm{Pr}(\lambda)\}.$$

LEMMA 1.3.2 ([She17, 4.6]). *The set of $p \in \mathbb{Q}_\kappa$ witnessed by $(\rho, S, \vec{\Lambda})$ such that $S \subseteq S^\kappa_{\mathrm{pr}}$ is dense in $\mathbb{Q}_\kappa$.* ■

LEMMA 1.3.3 ([She17, 4.4]).

(1) *If κ is inaccessible but not Mahlo then* $\mathrm{Pr}(\kappa)$.
(2) *If κ is weakly compact then* $\neg\mathrm{Pr}(\kappa)$.
(3) *If $\kappa = \sup(S^\kappa_{\mathrm{inc}})$ then $\kappa = \sup(S^\kappa_{\mathrm{pr}})$.*
(4) *If κ is Mahlo then S^κ_{pr} is a stationary subset of κ.* ■

DEFINITION 1.3.4. Define ideals

$$\mathbf{nst}_\kappa = \{S \subseteq S^\kappa_{\mathrm{inc}} : S \text{ is nowhere stationary}\},$$
$$\mathbf{nst}^{\mathrm{pr}}_\kappa = \{S \subseteq S^\kappa_{\mathrm{pr}} : S \text{ is nowhere stationary}\}.$$

The order on these ideals is $\subseteq^*$, i.e. set-inclusion modulo bounded subsets. Note that by 1.3.3(4), for every Mahlo cardinal κ the set S^κ_{pr} is stationary; so κ Mahlo is sufficient for $\mathbf{nst}^{\mathrm{pr}}_\kappa$ to be proper (i.e. $S^\kappa_{\mathrm{pr}} \notin \mathbf{nst}^{\mathrm{pr}}_\kappa$).

1.4. Ideals and strengthened Galois–Tukey connections

DEFINITION 1.4.1. Let X be a set and let $\mathbf{i} \subseteq \mathcal{P}(X)$ be an ideal. The equivalence relation $\sim_\mathbf{i}$ on $\mathcal{P}(X)$ is defined by $A \sim_\mathbf{i} B \Leftrightarrow A \triangle B \in \mathbf{i}$. We write $\mathcal{P}(X)/{\sim_\mathbf{i}}$ for the set of equivalence classes and $A/{\sim_\mathbf{i}}$ for the equivalence class of a set $A \subseteq X$.

If $\mathbf{j}$ is an ideal containing $\mathbf{i}$, we write $\mathbf{j}/\mathbf{i}$ for the naturally induced ideal on $\mathcal{P}(X)/{\sim_\mathbf{i}}$ defined as

$$\mathbf{j}/\mathbf{i} := \{A/{\sim_\mathbf{i}} : A \in \mathbf{j}\}.$$

DEFINITION 1.4.2. Let X be a set and let $\mathbf{i} \subseteq \mathcal{P}(X)$ be an ideal containing all singletons. Then

$$\mathrm{add}(\mathbf{i}) := \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{i} \wedge \bigcup \mathcal{A} \notin \mathbf{i}\Big\},$$
$$\mathrm{cov}(\mathbf{i}) := \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{i} \wedge \bigcup \mathcal{A} = X\Big\},$$
$$\mathrm{non}(\mathbf{i}) := \min\{|A| : A \in \mathcal{P}(X)\backslash\mathbf{i}\},$$
$$\mathrm{cf}(\mathbf{i}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{i} \wedge (\forall B \in \mathbf{i})(\exists A \in \mathcal{A})\ B \subseteq A\}.$$

For two ideals $\mathbf{i},\mathbf{j} \subseteq \mathcal{P}(X)$ let

$$\begin{aligned} \mathrm{add}(\mathbf{i},\mathbf{j}) &:= \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{i} \wedge \bigcup \mathcal{A} \notin \mathbf{j}\Big\}, \\ \mathrm{cf}(\mathbf{i},\mathbf{j}) &:= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{j} \wedge (\forall B \in \mathbf{i})(\exists A \in \mathcal{A})\ B \subseteq A\}. \end{aligned}$$

Fact 1.4.3. *Let X be a set and let $\mathbf{i} \subseteq \mathcal{P}(X)$ be an ideal. Then:*

(a) $\mathrm{add}(\mathbf{i}) \le \mathrm{cov}(\mathbf{i}) \le \mathrm{cf}(\mathbf{i})$.
(b) $\mathrm{add}(\mathbf{i}) \le \mathrm{non}(\mathbf{i}) \le \mathrm{cf}(\mathbf{i})$.

Fact 1.4.4. *Let X be a set and let $\mathbf{i}^- \subseteq \mathbf{i} \subseteq \mathcal{P}(X)$ be two ideals. Then:*

(a) $\mathrm{add}(\mathbf{i}) \le \mathrm{add}(\mathbf{i}^-, \mathbf{i})$.
(b) $\mathrm{add}(\mathbf{i}^-) \le \mathrm{add}(\mathbf{i}^-, \mathbf{i})$.
(c) $\mathrm{cf}(\mathbf{i}^-, \mathbf{i}) \le \mathrm{cf}(\mathbf{i})$.
(d) $\mathrm{cf}(\mathbf{i}^-, \mathbf{i}) \le \mathrm{cf}(\mathbf{i}^-)$. ■

Fact 1.4.5. *Let X be a set and let $\mathbf{i}^- \subseteq \mathbf{i} \subseteq \mathcal{P}(X)$ be two ideals. Then:*

(a) $\mathrm{add}(\mathbf{i}) \ge \min\{\mathrm{add}(\mathbf{i}^-), \mathrm{add}(\mathbf{i}/\mathbf{i}^-)\}$.
(b) $\mathrm{cf}(\mathbf{i}) \le \mathrm{cf}(\mathbf{i}^-) + \mathrm{cf}(\mathbf{i}/\mathbf{i}^-)$. ■

Definition 1.4.6. Consider ideals $\mathbf{i}^- \subseteq \mathbf{i} \subseteq \mathcal{P}(X)$, $\mathbf{j} \subseteq \mathcal{P}(U)$. We call maps

$$\phi^+ : \mathbf{i} \to \mathbf{j}, \quad \phi^- : \mathbf{j} \to \mathbf{i}^-$$

a *strengthened Galois–Tukey connection* if for all $A \in \mathbf{i}$, $B \in \mathbf{j}$,

$$\phi^-(B) \subseteq A \implies B \subseteq \phi^+(A).$$

Discussion 1.4.7. Strengthened Galois–Tukey connections are a special case of what is called a *generalized Galois–Tukey connection* in [Voj93] and a *morphism* in [Bla10].

Lemma 1.4.8. *Consider $\mathbf{i}^- \subseteq \mathbf{i} \subseteq \mathcal{P}(X)$, $\mathbf{j} \subseteq \mathcal{P}(U)$ and let ϕ^-, ϕ^+ be a strengthened Galois–Tukey connection between them. Then:*

(a) $\mathrm{add}(\mathbf{i}^-, \mathbf{i}) \le \mathrm{add}(\mathbf{j})$.
(b) $\mathrm{cf}(\mathbf{i}^-, \mathbf{i}) \ge \mathrm{cf}(\mathbf{j})$.

Proof. (a) Let $\langle B_\zeta : \zeta < \mu < \mathrm{add}(\mathbf{i}^-, \mathbf{i})\rangle$ be a family of $B_\zeta \in \mathbf{j}$. Find $A \in \mathbf{i}$ such that $\bigcup_{\zeta<\mu} \phi^-(B_\zeta) \subseteq A$, so that $\bigcup_{\zeta<\mu} B_\zeta \subseteq \phi^+(A)$.

(b) Let $\langle A_\zeta : \zeta < \mu = \mathrm{cf}(\mathbf{i}^-, \mathbf{i})\rangle$ be a family of $A_\zeta \in \mathbf{i}$ cofinal for $\mathbf{i}^-$. Then for $B \in \mathbf{j}$ we can find $\zeta < \mu$ such that $\phi^-(B) \subseteq A_\zeta$, so $B \subseteq \phi^+(A_\zeta)$, i.e. $\langle \phi^+(A_\zeta) : \zeta < \mu\rangle$ is a cofinal family of $\mathbf{j}$. ■

1.5. Miscellaneous

Definition 1.5.1. Let $X \subseteq \kappa$. Then

$$\mathrm{acc}(X) := \{\alpha < \kappa : (\exists Y \subseteq X)\ \sup(Y) = \alpha\}, \quad \mathrm{nacc}(X) := X \setminus \mathrm{acc}(X).$$

DEFINITION 1.5.2. Let id(Cohen$_\kappa$) be the ideal of meager subsets of 2^κ. We call $M \subseteq 2^\kappa$ *meager* if it is a union of κ-many nowhere dense subsets of κ, in the $<\kappa$-box product topology.

2. Tools. In this section we introduce/recall several concepts and tools that will be useful later. In particular, we give sufficient conditions for the following properties to be preserved in forcing iterations:

- 2.1: Closure properties, such as strategic closure.
- 2.2: Stationary Knaster, a property that is intermediate between the κ^+-chain condition and κ-centeredness; this property is preserved in $<\kappa$-support iterations.
- 2.3: a version of κ-centeredness. (Also, similarly to the classical case, sufficiently centered forcing notions will not add random reals, and will neither decrease non($\mathbb{Q}_\kappa$) nor increase cov($\mathbb{Q}_\kappa$).)
- 2.4 and 2.5: A property defined by a game, which allows fusion arguments in iterations with κ-support, and implies properness and κ^κ-bounding.

2.1. Closure

DEFINITION 2.1.1. Let $\mathbb{Q}$ be a forcing notion. We say that $\mathbb{Q}$ is *α-closed* if for every descending sequence $\langle p_i : i < i^* \rangle$ of length $i^* < \alpha$ (with all p_i in $\mathbb{Q}$) there is a lower bound in $\mathbb{Q}$, i.e. there exists $q \in \mathbb{Q}$ such that for every $i < i^*$ the condition q is stronger than p_i.

To avoid confusion we may write $<\alpha$-closed.

DEFINITION 2.1.2. Let $\mathbb{Q}$ be a forcing notion. We say that $\mathbb{Q}$ is *α-directed closed* if every directed set $D \subseteq \mathbb{Q}$ of cardinality $< \alpha$ has a lower bound. (A set D is called *directed* if any two elements of D are compatible and moreover have a lower bound in D.)

To avoid confusion we may write $<\alpha$-directed closed.

REMARK 2.1.3. It is customary to write κ-closed and κ-c.c. for $<\kappa$-closed and $<\kappa$-c.c., respectively.

An iteration in which the domains of the conditions have size $\leq \kappa$ should logically be called "iterations with $<\kappa^+$-supports", or abbreviated "κ^+-supports". Convention, however, dictates that such iterations are called "iterations with κ-supports"; we will follow this convention.

Most of our forcing iterations will have $<\kappa$-support and behave similarly to finite support iterations in the classical case; some of our iterations will have κ-support, in analogy to countable support iterations.

DEFINITION 2.1.4. Let $\mathbb{Q}$ be a forcing notion and let $q \in \mathbb{Q}$. Define the game $\mathfrak{C}_\kappa(\mathbb{Q}, q)$ between two players White and Black taking turns playing conditions of $\mathbb{Q}$ stronger than q, i.e. first White plays $p_0 \leq q$, then Black plays a condition $p'_0 \in \mathbb{Q}$, then White plays $p_1 \in \mathbb{Q}$ and so on. The game continues for κ-many turns and note that White plays first in limit steps. The rules of the game are:

(1) For $i < \kappa$ we require $p'_i \leq p_i$.
(2) For $i < j < \kappa$ we require $p_j \leq p'_i$.

White wins if he can follow the rules until the end.

We say that $\mathbb{Q}$ is *κ-strategically closed* if White has a winning strategy for $\mathfrak{C}_\kappa(\mathbb{Q}, q)$ for every $q \in \mathbb{Q}$.

FACT 2.1.5. *Let $\mathbb{Q}$ be a forcing notion. Consider the following statements:*

(a) $\mathbb{Q}$ *is $<\kappa$-directed closed.*
(b) $\mathbb{Q}$ *is $<\kappa$-closed.*
(c) $\mathbb{Q}$ *is κ-strategically closed.*

Then (a)$\Rightarrow$(b)$\Rightarrow$(c). ∎

FACT 2.1.6. *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ be a forcing iteration with $<\lambda$-support. If for every $\alpha < \delta$ we have $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha \models \phi$" then also $\mathbb{P} \models \phi$ where $\phi \in \{$"$<\kappa$-directed closed", "$<\kappa$-closed", "κ-strategically closed"$\}$ whenever $\lambda \geq \kappa$. In particular, these properties are preserved in $<\kappa$-support iterations and in κ-support iterations.* ∎

2.2. Stationary Knaster, preservation in $<\kappa$-support iterations

DISCUSSION 2.2.1. To obtain independence results for the classical case ($\kappa = \omega$) we often use finite support iterations of c.c.c. forcing notions. Such iterations are useful due to the well-known fact that their finite support limits will again satisfy the c.c.c.

In this section we will quote a parallel for the case of uncountable κ, first appearing in [She78].

DEFINITION 2.2.2. Let κ be a cardinal. Let $\mathbb{Q}$ be a forcing notion. We say that $\mathbb{Q}$ satisfies the *stationary κ^+-Knaster condition* if for every $\{p_i : i < \kappa^+\} \subseteq \mathbb{Q}$ there exists a club $E \subseteq \kappa^+$ and a regressive function f on $E \cap S^{\kappa^+}_\kappa$ such that for any $i, j \in E \cap S^{\kappa^+}_\kappa$ we have

$$f(i) = f(j) \implies p_i \not\perp p_j.$$

FACT 2.2.3. *The stationary κ^+-Knaster condition implies the κ^+-chain condition.*

Proof. By Fodor's pressing down lemma the stationary κ^+-Knaster condition implies that for every $\{p_i : i < \kappa^+\} \subseteq \mathbb{Q}$ there exists a stationary set $S \subseteq \kappa^+$ such for that any $i, j \in S$ the conditions p_i, p_j are compatible. ∎

DEFINITION 2.2.4. Let κ be a cardinal. Let $\mathbb{Q}$ be a forcing notion. We say that $\mathbb{Q}$ satisfies $(*_\kappa)$ if the following hold:

(a) $\mathbb{Q}$ satisfies the stationary κ^+-Knaster condition.
(b) Any decreasing sequence $\langle p_i : i < \omega \rangle$ of conditions of $\mathbb{Q}$ has a greatest lower bound.
(c) Any compatible $p, q \in \mathbb{Q}$ have a greatest lower bound.
(d) $\mathbb{Q}$ does not add elements of $(\kappa^+)^{<\kappa}$ (e.g. $\mathbb{Q}$ is κ-strategically closed).

LEMMA 2.2.5. *Let κ be a cardinal. Let $\mathbb{Q}$ be a forcing notion such that:*

(1) *$\mathbb{Q}$ satisfies the stationary κ^+-Knaster condition.*
(2) *$\mathbb{Q}$ does not add new subsets of δ for $\delta < \kappa$ (e.g. $\mathbb{Q}$ is κ-strategically closed).*

Then $\mathbb{Q}$ does not collapse cardinals. ∎

THEOREM 2.2.6. *Let κ be a cardinal. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$ be a $<\kappa$-support iteration such that for every $\alpha < \lambda$,*

$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha \text{ satisfies } (*_\kappa) \text{ from Definition 2.2.4.}$$

Then $\mathbb{P}_\lambda$ satisfies the stationary κ^+-Knaster condition.

Proof. The proof appears in [She78, 3.1] for the case $\kappa = \omega_1$ but easily generalizes to arbitrary regular κ. ∎

FACT 2.2.7. *Let κ be a cardinal. Let $\mathbb{Q}$ be a κ-linked (see 2.3.1) forcing notion. Then $\mathbb{Q}$ satisfies the stationary κ^+-Knaster condition.* ∎

2.3. κ-centered$_{<\kappa}$, preservation in $<\kappa$-support iterations

DEFINITION 2.3.1. Let κ be a cardinal, let $\mathbb{P}$ be a forcing notion and let $X \subseteq \mathbb{P}$.

(1) We say that X is *linked* if for every $p_0, p_1 \in X$ we have $p_0 \not\perp p_1$. We say that $\mathbb{P}$ is *κ-linked* if there exist $\langle X_i : i < \kappa \rangle$ such that $X_i \subseteq \mathbb{P}$ is linked and $\mathbb{P} = \bigcup_{i<\kappa} X_i$.
(2) We say that X is *centered$_{<\kappa}$* if for every $Y \in [X]^{<\kappa}$ there exists q such that $q \leq p$ for every $p \in Y$. We say that $\mathbb{P}$ is *κ-centered$_{<\kappa}$* if there exist $\langle X_i : i < \kappa \rangle$ such that each $X_i \subseteq \mathbb{P}$ is centered$_{<\kappa}$ and $\mathbb{P} = \bigcup_{i<\kappa} X_i$.

FACT 2.3.2. *Let κ be a cardinal and let $\mathbb{P}$ be a forcing notion. Consider the following statements:*

(a) *$\mathbb{P}$ is κ-centered$_{<\kappa}$.*
(b) *$\mathbb{P}$ is κ-linked.*
(c) *$\mathbb{P}$ satisfies the κ^+-c.c.*

Then (a)⇒(b)⇒(c). ∎

DEFINITION 2.3.3. Let κ be a cardinal. We say that an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \zeta \rangle$ is *κ-centered* if it has $<\kappa$-support and

$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha \text{ is } \kappa\text{-centered}_{<\kappa}.$$

Fact 2.3.4. *Let* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \zeta \rangle$ *be a* κ*-centered iteration. Then there exist sequences* $\langle \dot{C}_\alpha : \alpha < \zeta \rangle$, $\langle \dot{c}_\alpha : \alpha < \zeta \rangle$ *such that for all* $\alpha < \zeta$, $\dot{C}_\alpha$ *and* $\dot{c}_\alpha$ *are* $\mathbb{P}_\alpha$*-names such that* $\mathbb{P}_\alpha$ *forces:*

(a) $\dot{C}_\alpha$ *is a function* $\kappa \to \mathcal{P}(\dot{\mathbb{Q}}_\alpha)$.
(b) $\bigcup \operatorname{ran}(\dot{C}_\alpha) = \dot{\mathbb{Q}}_\alpha$.
(c) $i < \kappa \Rightarrow \dot{C}_\alpha(i)$ *is centered*$_{<\kappa}$.
(d) $\dot{c}_\alpha$ *is a function* $\dot{\mathbb{Q}}_\alpha \to \kappa$.
(e) $\dot{q} \in \dot{\mathbb{Q}}_\alpha \Rightarrow \dot{q} \in \dot{C}_\alpha(\dot{c}_\alpha(\dot{q}))$.

Without loss of generality we may also assume that each $\dot{C}_\alpha(n)$ *is non-empty and closed under weakening of conditions, in particular* $1_{\dot{\mathbb{Q}}_\alpha} \in \dot{C}_\alpha(n)$ *for each* n. ■

We shall use this notation throughout this section.

Definition 2.3.5. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \zeta \rangle$ be a κ-centered iteration. We call a condition $p \in \mathbb{P}$ *fine* if for each $\alpha \in \operatorname{supp}(p)$ the restriction $p{\restriction}\alpha$ decides some $n < \kappa$ such that $p{\restriction}\alpha \Vdash$ "$p(\alpha) \in \dot{C}_\alpha(n)$". Note that for $\alpha \notin \operatorname{supp}(p)$ this is trivially true because $1_{\dot{\mathbb{Q}}_\alpha}$ is in every $\dot{C}_\alpha(n)$.

Definition 2.3.6. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \zeta \rangle$ be a κ-centered iteration. We say that $\mathbb{P}$ is *finely* $<\kappa$*-closed* if for every $\alpha < \zeta$ with $\operatorname{cf}(\alpha) < \kappa$ there exist $L^1_\alpha \in \mathbf{V}$ and a $\mathbb{P}_\alpha$-name $\dot{L}^2_\alpha$ such that:

(a) L^1_α is a function $\kappa^{<\kappa} \to \kappa$.
(b) $\mathbb{P}_\alpha \Vdash$ "$\dot{L}^2_\alpha$ is a function $\dot{\mathbb{Q}}_\alpha^{<\kappa} \to \dot{\mathbb{Q}}_\alpha$".
(c) If $\vec{q} = \langle \dot{q}_i : i < i^* \rangle$ is a descending sequence of length $i^* < \kappa$ in $\dot{\mathbb{Q}}_\alpha$ then $\mathbb{P}_\alpha$ forces:
    (1) $\dot{L}^2_\alpha(\vec{q})$ is a lower bound for $\vec{q}$.
    (2) $\dot{c}_\alpha(\dot{L}^2_\alpha(\vec{q})) = L^1_\alpha(\langle \dot{c}_\alpha(\dot{q}_i) : i < i^* \rangle)$.

The typical situation here is that the coloring of the forcing is some trunk function, so if we find a lower bound $\dot{q}$ for some descending sequence $\langle \dot{q}_i : i < \alpha \rangle$, the union of the trunks of the q_i will tell us the color of $\dot{q}$.

Lemma 2.3.7. *Let* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \zeta \rangle$ *be a* κ*-centered finely* $<\kappa$*-closed iteration of length* $\zeta < (2^\kappa)^+$. *Then:*

(a) $\mathbb{P}' = \{p \in \mathbb{P} : p$ *is fine*$\}$ *is dense in* $\mathbb{P}$.
(b) $\mathbb{P}$ *is* κ*-centered*$_{<\kappa}$.

Discussion 2.3.8. The following proof closely follows [Bla11] where the result is explained for the ω-case. The only adjustment we have to make is the demand for fine closure (as defined in 2.3.6) to deal with the limit case that does not appear in the ω-version of the proof.

This lemma also appears in [BB+18].

Proof of Lemma 2.3.7. (a) Let $p \in \mathbb{P}$ be arbitrary. We are going to find a condition p' stronger than p such that p' is fine. We prove this by induction on $\delta \leq \zeta$ for $\mathbb{P}_\delta$, constructing a decreasing sequence of conditions $\langle p_i : i \leq \delta \rangle$ with $p_i \in \mathbb{P}_\delta$ such that for each $i \leq \delta$ the condition $p_i \restriction (i+1)$ is fine:

(i) $p_0 = p$.
(ii) $i = j+1$: First find q stronger than $p_i \restriction i$ such that q decides the color of $p_j(i)$. Then use the induction hypothesis to find $q' \leq q$ such that q' is fine and let $p_i = q' \wedge p$.
(iii) i a limit ordinal, $\mathrm{cf}(i) < \kappa$: Consider the condition

$$q' = \left(\dot{L}^2_j(\langle q_k(j) : k < i \rangle) : j < i\right) \in \mathbb{P}_i$$

and let $p_i = q' \wedge p$.
(iv) i a limit ordinal, $\mathrm{cf}(i) \geq \kappa$: Remember that $\mathbb{P}$ has $<\kappa$-support so this case is trivial.

(b) By the Engelking–Karłowicz theorem [EK65] there exists a family $\langle f_i : \zeta \to \kappa : i < \kappa \rangle$ of functions such that for any $A \in [\zeta]^{<\kappa}$ and every $f : A \to \kappa$ there exists $i < \kappa$ such that $f \subseteq f_i$.

For each $i < \kappa$ let

$$D(i) = \{p \in \mathbb{P}_\zeta : (\forall \alpha < \kappa)\ p \restriction \alpha \Vdash p(\alpha) \in \dot{C}_\alpha(f_i(\alpha))\}.$$

It is easy to see that each $D(i)$ is centered$_{<\kappa}$ and that every fine $p \in \mathbb{P}$ is contained in some $D(i)$. So by (a) we are done. ∎

Lemma 2.3.9. *Let κ be an inaccessible cardinal with $\sup(\kappa \cap S^{\mathrm{inc}}_\kappa) = \kappa$. Let $\mathbb{P}$ be a forcing notion that does not add new subsets of δ for $\delta < \kappa$ (e.g. $\mathbb{P}$ is κ-strategically closed). Then $\mathbb{P}$ does not add a $\mathbb{Q}_\kappa$-generic real if either*

(a) *$\mathbb{P}$ is κ-centered$_{<\kappa}$, or just*
(b) *$\mathbb{P}$ is $(2^\kappa, \kappa)$-centered$_{<\kappa}$ meaning that any set $Y \subseteq \mathbb{P}$ of cardinality at most 2^κ is included in the union of at most κ-many centered$_{<\kappa}$ subsets of $\mathbb{P}$, or just*
(c) *if $p_\rho \in \mathbb{P}$, $\rho \in 2^\kappa$, is a family of conditions, then for some non-meager $A \subseteq 2^\kappa$ we have*

$$u \in [A]^{<\kappa} \implies \{p_\rho : \rho \in u\} \text{ has a lower bound.}$$

Proof. Clearly (a)⇒(b)⇒(c). The first implication is trivial. The second implication follows from the κ^+-completeness of the meager ideal. So we shall assume (c).

Let $p^* \Vdash$ "$\dot{\nu}$ is a counterexample and thus $\dot{\nu} \restriction \epsilon \in \mathbf{V}$ for all $\epsilon < \kappa$". (Recall that $\mathbb{Q}_\kappa$ is κ-strategically closed.) Let $\langle \lambda_\epsilon : \epsilon < \kappa \rangle$ be an increasing enumeration of $\{\lambda \in S^\kappa_{\mathrm{inc}} : \lambda > \sup(\lambda \cap S^\kappa_{\mathrm{inc}})\}$. Now for $\eta \in 2^\kappa$ let

$$A_\eta = \{\rho \in 2^\kappa : (\forall^\infty \epsilon < \kappa)(\exists^\infty \alpha < \lambda_\epsilon)\ \eta(\alpha) \neq \rho(\alpha)\}.$$

Clearly $2^\kappa \backslash A_\eta \in \mathrm{id}^-(\mathbb{Q}_\kappa) \subseteq \mathrm{id}(\mathbb{Q}_\kappa)$ as defined in 3.2.1 but we may argue $2^\kappa \backslash A_\eta \in \mathrm{id}(\mathbb{Q}_\kappa)$ as follows: For $\eta \in 2^\kappa$ and $\epsilon < \kappa$ let $B_{\eta,\epsilon} = \{\rho \in 2^{\lambda_\epsilon} : \rho =^* \eta{\restriction}\lambda_\epsilon\}$ and note that $|B_{\eta,\epsilon}| = \lambda_\epsilon$, hence $B_{\eta,\epsilon} \in \mathrm{id}(\mathbb{Q}_{\lambda_\epsilon})$. Let $S = \{\lambda_\epsilon : \epsilon < \kappa\}$ and clearly S is nowhere stationary. So for every $\eta \in 2^\kappa$ the set

$$\mathcal{J}_\eta = \{p \in \mathbb{Q}_\kappa : S \subseteq S_p \wedge (\forall \epsilon < \kappa)\ [\lambda_\epsilon > \mathrm{lg}(\mathrm{tr}(p)) \Rightarrow B_{\eta,\epsilon} \in \mathrm{set}_0(\Lambda_{p,\lambda_\epsilon})]\}$$

is dense in $\mathbb{Q}_\kappa$ and $p \in \mathcal{J}_\eta \Rightarrow p \Vdash$ "$\dot{\nu} \in A_\eta$".

Now because $2^\kappa \backslash A_\zeta \in \mathrm{id}(\mathbb{Q}_\kappa)$ we have $p^* \Vdash$ "$\dot{\nu} \in A_\zeta$", hence for $\eta \in 2^\kappa$ there are (p_η, ζ_η) such that $p_\eta \leq p^*$, $\zeta_\eta < \kappa$, and

$$p_\eta \Vdash_{\mathbb{P}} \text{"if } \epsilon \in [\zeta_\eta, \kappa) \text{ then } (\exists^\infty \alpha < \lambda_\epsilon)\eta(\alpha) \neq \dot{\nu}(\alpha)\text{"}.$$

Therefore there exists a non-meager set $Y \subseteq 2^\kappa$ such that any set $\{p_\rho : \rho \in Y\}$ of cardinality $< \kappa$ has a lower bound. Because the meager ideal is κ^+-complete, there exists $\zeta^* < \kappa$ such that without loss of generality $\eta \in Y \Rightarrow \zeta_\eta = \zeta^*$. As Y is non-meager, it is somewhere dense. So there exists $\varrho^* \in 2^{<\kappa}$ such that

$$(\forall \varrho \in 2^{<\kappa})\ \varrho^* \triangleleft \varrho \in 2^{<\kappa} \implies (\exists \rho \in Y)\ \varrho \triangleleft \rho.$$

Without loss of generality $\mathrm{lg}(\varrho^*) = \zeta^*$ (we may increase either value to match the greater one). Choose $\epsilon < \kappa$ with $\lambda_\epsilon > \zeta^*$. Let $\Gamma = \{\varrho \in 2^{\lambda_\epsilon} : \varrho^* \triangleleft \varrho\}$ and for each $\varrho \in \Gamma$ let $\eta_\varrho \in Y$ be such that $\varrho \triangleleft \eta_\varrho$. Now $\{\eta_\varrho : \varrho \in \Gamma\} \in [Y]^{<\kappa}$, hence by the choice of Y there exists a lower bound q of $\{p_{\eta_\varrho} : \varrho \in \Gamma\}$.

As $p^* \Vdash$ "$\dot{\nu}{\restriction}\lambda_\epsilon \in V$", without loss of generality let q force a value to $\dot{\nu}{\restriction}\lambda_\epsilon$, so call this value ν. Now q is stronger than $p_{\eta_{\varrho^* \frown \nu \restriction [\zeta^*, \lambda_\epsilon)}}$ and forces $\lambda_\epsilon = \sup\{\alpha < \lambda_\epsilon : \varrho^* {}^\frown \nu{\restriction}[\zeta^*, \lambda_\epsilon)(\alpha) \neq \dot{\nu}(\alpha)\}$, which means $\lambda_\epsilon = \sup\{\alpha < \lambda_\epsilon : \nu(\alpha) \neq \dot{\nu}(\alpha)\}$. This contradicts the choice of ν. ■

Remark 2.3.10. Lemma 2.3.9 implies that $\mathbb{Q}_\kappa$ is not κ-centered$_{<\kappa}$. However, $\mathbb{Q}_\kappa$ has, for every $\lambda < \kappa$, a dense subset which is κ-centered$_{<\lambda}$, namely the set of conditions with trunk of length $> \lambda$. This parallels the classical case of random forcing, which is not σ-centered, but σ-n-linked for all $n \in \omega$.

Discussion 2.3.11. The following Lemma 2.3.12 is a straightforward generalization of [BJ95, 6.5.30]. We formulate it in terms of the ideal $\mathrm{id}^-(\mathbb{Q}_\kappa) \subseteq \mathrm{id}(\mathbb{Q}_\kappa)$. For the definition see 3.2.1. Note that by 3.2.5 under the assumptions of 2.3.12 we have $\mathrm{id}^-(\mathbb{Q}_\kappa) = \mathrm{id}(\mathbb{Q}_\kappa)$.

Lemma 2.3.12. *Let κ be weakly compact. Let $\mathbb{P}$ be a forcing notion such that:*

(a) $\mathbb{P}$ *is κ-centered$_{<\kappa}$.*

(b) $\mathbb{P}$ *does not add new subsets of δ for $\delta < \kappa$ (e.g. $\mathbb{P}$ is κ-strategically closed).*

Let $(\mathbf{N}, \in) \prec (H(\chi), \in)$ for some χ large enough with $\mathbf{N}^{<\kappa} \subseteq \mathbf{N}$ and $\mathbb{P} \in \mathbf{N}$. Then for $A \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ we have

$$\mathbf{N} \cap 2^\kappa \subseteq A \implies \mathbb{P} \Vdash \text{"}\mathbf{N}[G] \cap 2^\kappa \subseteq A\text{"}$$

where G is the generic filter of $\mathbb{P}$. (As usual, A is to be read as a definition of a null set, to be interpreted in $\mathbf{V}$ and $\mathbf{V}^{\mathbb{P}}$.)

Proof. Let $A \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ be witnessed by $\vec{A} = \langle A_\delta : \delta \in S\rangle$, i.e. $A = \mathrm{set}_0^-(\vec{A})$, and let $\mathbb{P} = \bigcup_{\alpha<\kappa} \mathbb{P}_\alpha$ with each $\mathbb{P}_\alpha$ centered$_{<\kappa}$.

Assume there exists a $\mathbb{P}$-name of a κ-real $\dot\eta \in \mathbf{N}$ and $p^* \in \mathbb{P}$ such that

$$p^* \Vdash \text{“}\dot\eta \notin A\text{”}$$

and without loss of generality even

$$p^* \Vdash \text{“}(\forall \delta \geq \delta_0)\ \dot\eta\restriction\delta \notin A_\delta\text{”} \tag{2.1}$$

for some $\delta_0 < \kappa$. For $\alpha < \kappa$, $\delta \in S$ we define

$$T_{\alpha,\delta} = \{\nu \in 2^\delta : (\forall p \in \mathbb{P}_\alpha)(\exists q \in \mathbb{P})\ q \leq p \text{ and } q \Vdash \text{“}\dot\eta\restriction\delta = \nu\text{”}\}.$$

Note that in general we will have $p^* \notin \mathbf{N}$. However, we will have $p^* \in \mathbb{P}_\alpha$ for some α, and the partition $(\mathbb{P}_\alpha : \alpha < \kappa)$ is in $\mathbf{N}$, as is the family $(T_{\alpha,\delta} : \alpha < \kappa,\ \delta \in S)$.

None of the sets $T_{\alpha,\delta}$ (for all $\alpha < \kappa$, $\delta \in S$) is empty. We prove this indirectly: Assume $T_{\alpha,\delta} = \emptyset$. Then for every $\nu \in 2^\delta$ there exists $p_\nu \in \mathbb{P}_\alpha$ such that $p_\nu \Vdash \nu \neq \dot\eta\restriction\delta$. Now because $\mathbb{P}_\alpha$ is centered$_{<\kappa}$, there exists a lower bound q for $\{p_\nu : \nu \in 2^\delta\}$. Thus for all $\nu \in 2^\delta$ we have $q \Vdash \nu \neq \dot\eta\restriction\delta$, contradicting our assumption that $\mathbb{P}$ does not add short sequences.

For $\alpha < \kappa$ consider the tree that is the downward closure of $\bigcup_{\delta\in S} T_{\alpha,\delta}$. Because κ is weakly compact, κ has the tree property, thus there exists a branch $\eta_\alpha \in 2^\kappa$ through this tree, i.e. for every $\delta \in S$ we have $\eta_\alpha\restriction\delta \in T_{\alpha,\delta}$. Note that η_α can be calculated from $\dot\eta$, hence $\eta_\alpha \in \mathbf{N}$, so by our assumption $\eta_\alpha \in A$, i.e. $(\exists^\infty \delta \in S)\ \eta_\alpha \in A_\delta$. Find $\alpha^* < \kappa$ such that $p^* \in \mathbb{P}_{\alpha^*}$ and find $\delta^* \geq \delta_0$ such that $\eta_{\alpha^*}\restriction\delta^* \in A_{\delta^*}$.

Let $\nu = \eta_{\alpha^*}\restriction\delta^* \in T_{\alpha^*,\delta^*}$. Then there exists $q \leq p^*$ such that

$$q \Vdash \dot\eta\restriction\delta^* = \nu = \eta_{\alpha^*}\restriction\delta^* \in A_{\delta^*},$$

a contradiction to (2.1). ■

Corollary 2.3.13. *Let κ be weakly compact. Let $\mathbb{P}$ be a forcing notion such that:*

(a) *$\mathbb{P}$ is κ-centered$_{<\kappa}$.*
(b) *$\mathbb{P}$ does not add new subsets of δ for $\delta < \kappa$ (e.g. $\mathbb{P}$ is κ-strategically closed).*

Then:

(1) *$\mathbb{P}$ does not decrease $\mathrm{non}(\mathbb{Q}_\kappa)$, i.e. if $\mathrm{non}(\mathbb{Q}_\kappa)=\lambda$, then $\mathbb{P} \Vdash$ “$\mathrm{non}(\mathbb{Q}_\kappa) \geq \lambda$”.*
(2) *$\mathbb{P}$ does not increase $\mathrm{cov}(\mathbb{Q}_\kappa)$, i.e. if $\mathrm{cov}(\mathbb{Q}_\kappa)=\lambda$, then $\mathbb{P} \Vdash$ “$\mathrm{cov}(\mathbb{Q}_\kappa) \leq \lambda$”.*

Proof. (1) Let $\mu < \lambda$ and assume $\mathbb{P} \Vdash$ “$X = \{\dot\eta_i : i < \mu\}$ is a set of size μ”. Find $\mathbf{N}$ as in 2.3.12 with $\dot\eta_i \in \mathbf{N}$ for each $i < \mu$ and $|\mathbf{N}| = \mu$. Now because κ is weakly compact by 3.2.5 we have $\mu < \mathrm{non}(\mathrm{id}^-(\mathbb{Q}_\kappa))$, so find $A \in \mathrm{id}^-(\mathbb{Q}_\kappa)$

such that $\mathbf{N} \cap 2^\kappa \subseteq A$. By 2.3.12 we have $\mathbb{P} \Vdash$ "$X \subseteq \mathbf{N}[G] \subseteq A$". That is, for any set $X \in \mathbf{V}^{\mathbb{P}}$ of size $\mu < \lambda$ we have $X \in \mathrm{id}^-(\mathbb{Q}_\kappa)$.

(2) We show: $\mathbb{P}$ does not add a $\mathbb{Q}_\kappa$-generic real. Assume $\mathbb{P} \Vdash$ "$\dot\eta$ is $\mathbb{Q}_\kappa$-generic". Find $\mathbf{N}$ as in 2.3.12 with $\dot\eta \in \mathbf{N}$ and $|\mathbf{N}| = \kappa$. Find $A \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ such that $\mathbf{N} \cap 2^\kappa \subseteq A$. Now by 2.3.12 we have $\mathbb{P} \Vdash$ "$\dot\eta \in \mathbf{N}[G] \subseteq A \in \mathrm{id}^-(\mathbb{Q}_\kappa) \subseteq \mathrm{id}(\mathbb{Q}_\kappa)$", a contradiction to $\dot\eta$ being $\mathbb{Q}_\kappa$-generic. ∎

Remark 2.3.14. Corollary 2.3.13(2) duplicates 2.3.9 but there we do not require κ to be weakly compact.

2.4. The fusion game, preservation in κ-support iterations. The work in this subsection can be considered a generalization of [Kan80, Section 6], where it is shown how to iterate κ-Sacks forcing for inaccessible κ. The games defined in this subsection and the iteration theorem 2.4.7 first appeared in [RS06] where $\mathfrak{F}^*_\kappa$, $\mathfrak{F}_\kappa$ (defined below) are called $\Game^{\mathrm{rcA}}_{\underset{\sim}{\mu}}$ and $\Game^{\mathrm{rca}}_{\underset{\sim}{\mu}}$ respectively. However $\mathfrak{F}^*_\kappa$, $\mathfrak{F}_\kappa$ are slightly more general in the sense that White may freely decide the length μ_ζ of the ζth round during the game (i.e. our games do not require an additional parameter $\vec\mu$).

Definition 2.4.1. Let $\mathbb{Q}$ be a forcing notion and let $q \in \mathbb{Q}$. We define two (very similar) games $\mathfrak{F}_\kappa(\mathbb{Q}, q), \mathfrak{F}^*_\kappa(\mathbb{Q}, q)$ between two players White and Black. A play in either of the games consists of κ-many rounds and for each $\zeta < \kappa$ the ζth round lasts μ_ζ-many moves. The rules of the ζth round of the game $\mathfrak{F}_\kappa(\mathbb{Q}, q)$ are:

(1) First White plays $0 < \mu_\zeta < \kappa$. So White decides the length of the new round.
(2) On move $i < \mu_\zeta$:
   (a) White plays $q_{\zeta,i} \leq q$.
   (b) Black responds with $q'_{\zeta,i} \leq q_{\zeta,i}$.

The rules of the ζth round of the game $\mathfrak{F}^*_\kappa(\mathbb{Q}, q)$ are:

(3) First White plays $0 < \mu_\zeta < \kappa$. For ζ a limit ordinal we additionally require $\mu_\zeta \leq \sup_{\epsilon<\zeta} \mu_\epsilon$.
(4) On move $i < \mu_\zeta$:
   (a) White plays $q_{\zeta,i} \leq q$ but without looking at any $q'_{\zeta,j}$ for $j < i$. (Equivalently: White plays all moves of the current round at once at the start of the round.)
   (b) Black responds with $q'_{\zeta,i} \leq q_{\zeta,i}$

The winning condition of both games is the same:

$$\text{White wins} \iff (\exists q^* \leq q)\ q^* \Vdash \text{“}(\forall \zeta < \kappa)\ \{q'_{\zeta,i} : i < \mu_\zeta\} \cap \dot G_{\mathbb{Q}} \neq \emptyset\text{”},$$

where $\dot G_{\mathbb{Q}}$ is a name for the generic filter of $\mathbb{Q}$.

Discussion 2.4.2. In point (3) of the definition of $\mathfrak{F}^*_\kappa(\mathbb{Q}, q)$ we could be slightly more general: instead of sup, any function $f : \kappa^{<\kappa} \to \kappa$ that gives us an upper bound for μ_ζ based on $\langle \mu_\epsilon : \epsilon < \zeta \rangle$ will do, i.e. require $\mu_\zeta \leq f(\langle \mu_\epsilon : \epsilon < \zeta \rangle)$. (This is simply a technical requirement for the proof of 2.4.7.) So we may define $\mathfrak{F}^*_{\kappa,f}(\mathbb{Q}, q)$. Now if we let $g : \kappa^{<\kappa} \to \kappa$ be such that for any $\sigma \in \kappa^{<\kappa}$ we have $g(\sigma) = \sup_{\epsilon < \lg(\sigma)} \sigma(\epsilon)$ then $\mathfrak{F}^*_\kappa(\mathbb{Q}, q) = \mathfrak{F}^*_{\kappa,g}(\mathbb{Q}, q)$.

Discussion 2.4.3. The typical forcing for which White has a winning strategy for the games defined in 2.4.1 is a tree forcing permitting fusion sequences. See 6.9.6 for an example.

Fact 2.4.4. *The game $\mathfrak{F}^*_\kappa$ is slightly harder for White than the game $\mathfrak{F}_\kappa$, hence: If White has a winning strategy for $\mathfrak{F}^*_\kappa(\mathbb{Q}, q)$ then White has a winning strategy for $\mathfrak{F}_\kappa(\mathbb{Q}, q)$.* ■

Definition 2.4.5. For technical reasons we define the game $\mathfrak{F}^*_\kappa(\mathbb{Q}, q, \lambda)$ for $\lambda < \kappa$. The rules are the same as for $\mathfrak{F}^*_\kappa(\mathbb{Q}, q)$ except the first λ rounds are skipped and the game starts with the λth round. So this is really just an index shift. Of course $\mathfrak{F}^*_\kappa(\mathbb{Q}, q) = \mathfrak{F}^*_\kappa(\mathbb{Q}, q, 0)$, and clearly for every $\lambda < \kappa$ White has a winning strategy for $\mathfrak{F}^*_\kappa(\mathbb{Q}, q)$ iff he has a winning strategy for $\mathfrak{F}^*_\kappa(\mathbb{Q}, q, \lambda)$.

Theorem 2.4.6. *Let $\mathbb{Q}$ be a forcing notion. If for every $q \in \mathbb{Q}$ Black does not have a winning strategy for the game $\mathfrak{F}_\kappa(\mathbb{Q}, q)$ then:*

(a) *If $\dot{A}$ is a $\mathbb{Q}$-name such that $q \Vdash$ "$|\dot{A}| \leq \kappa$" then there exist $B \in \mathbf{V}$ with $|B| \leq \kappa$ and $q^* \leq q$ such that $q^* \Vdash \dot{A} \subseteq B$. In particular $\mathbb{Q}$ does not collapse κ^+.*
(b) *$\mathbb{Q}$ does not increase* $\mathrm{cf}(\mathrm{Cohen}_\kappa)$*, and in fact if $\langle A_i : i < \mu \rangle$ are a cofinal family of meager sets in $\mathbf{V}$ then this family remains cofinal in $\mathbf{V}^{\mathbb{Q}}$.*
(c) *$\mathbb{Q}$ is κ^κ-bounding.*

Proof. (a) Like (b), just easier. But let us do it as a warm-up.

Let $\langle \dot{a}_\zeta : \zeta < \kappa \rangle$ be such that $q \Vdash \{\dot{a}_\zeta : \zeta < \kappa\} = \dot{A}$. Now consider a run of $\mathfrak{F}_\kappa(\mathbb{Q}, q)$ where Black's strategy is to play in such way that for any $\zeta < \kappa$ and $i < \mu_\zeta$ there is $b_{\zeta,i}$ such that $q'_{\zeta,i} \Vdash$ "$\dot{a}_\zeta = b_{\zeta,i}$". That is, every move Black makes during the ζth round decides $\dot{a}_\zeta$.

By our assumption, White can beat this strategy, thus there exists $q^* \leq q$ such that $q^* \Vdash \dot{A} \subseteq \{b_{\zeta,i} : \zeta < \kappa, i < \mu_\zeta < \kappa\}$.

(b) Let us show: if $\dot{M}$ is a $\mathbb{Q}$-name and $q \Vdash$ "$\dot{M}$ is nowhere dense" then there exists a nowhere dense set $N \in \mathbf{V}$ and $q^* \leq q$ such that $q^* \Vdash \dot{M} \subseteq N$. Since meager sets are unions of κ-many nowhere dense sets, we can then use (a) to conclude the proof.

We are going to find $q^* \leq q$ such that for each $s \in 2^{<\kappa}$ there exists $t_s \trianglerighteq s$ such that $q^* \Vdash$ "$\dot{M} \cap [t_s] = \emptyset$", so

$$N = 2^\kappa \setminus \bigcup_{s \in 2^{<\kappa}} [t_s]$$

is as desired.

Let $\langle s_\zeta : \zeta < \kappa \rangle$ be an enumeration of $2^{<\kappa}$. We will define a strategy for player Black. In addition to his moves $q'_{\zeta,i}$, he will construct elements $t_{\zeta,i} \in 2^{<\kappa}$ satisfying the following properties:

(1) $s_\zeta \trianglelefteq t_{\zeta,j}$.
(2) $(\bigcup_{j<i} t_{\zeta,j}) \trianglelefteq t_{\zeta,i}$.
(3) $q'_{\zeta,i} \Vdash$ "$\dot{M} \cap [t_{\zeta,i}] = \emptyset$" (and of course $q'_{\zeta,i} \leq q_{\zeta,i}$, as required by the rules of the game).

Why can Black play like that?

(1) Obvious.
(2) Obvious for i successor. For i a limit ordinal just remember that $i < \mu_\zeta < \kappa$.
(3) Remember that $q'_{\zeta,i} \leq q \Vdash$ "$\dot{M}$ is nowhere dense".

Let $t_\zeta = \bigcup_{i<\mu_\zeta} t_{\zeta,i}$. Again White can beat this strategy so there exists $q^* \leq q$ as required.

(c) This follows by (b). ■

THEOREM 2.4.7. *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \alpha^* \rangle$ be a κ-support iteration and let $p \in \mathbb{P}$ such that for all $\alpha < \alpha^*$:*

(a) *$p{\restriction}\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is κ-strategically closed".*
(b) *$p{\restriction}\alpha \Vdash$ "White has a winning strategy for $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha, q)$ for every $q \leq p(\alpha)$".*

Then:

(1) *White has a winning strategy for $\mathfrak{F}_\kappa(\mathbb{P}, p)$.*
(2) *If White plays according to his strategy from (1) in a run $\langle p_{\zeta,i}, p'_{\zeta,i} : \zeta < \kappa,\ i < \mu_\zeta \rangle$ of $\mathfrak{F}_\kappa(\mathbb{P}, p)$ then there exists p^* witnessing White's win such that for all $\alpha < \alpha^*$ we have $p^*{\restriction}\alpha \Vdash$ "$\langle p_{\zeta,i}(\alpha), p'_{\zeta,i}(\alpha) : \zeta < \kappa,\ i < \mu_\zeta \rangle$ is a run of $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha, p(\alpha))$ won by White and White's win is witnessed by $p^*(\alpha)$".*

DISCUSSION 2.4.8. Note that the proof of 2.4.7 also works for $\kappa = \omega$.

Proof of Theorem 2.4.7. Let $p \in \mathbb{P}$; we are going to show how White can win $\mathfrak{F}_\kappa(\mathbb{P}, p)$ by finding $p^* \leq p$ witnessing White's victory while also being as required by (2). We are going to construct a sequence $\langle p_\zeta : \zeta \leq \kappa \rangle$ such that:

- $\zeta < \kappa \Rightarrow p_\zeta \in \mathbb{P}$,
- $p_0 = p$,
- $\epsilon < \zeta \Rightarrow p_\epsilon \geq p_\zeta$,

of which p^* is going to be a lower bound (but remember that under our assumptions the lower bound of a κ-sequence does not exist in general, so we will have to construct p^*). We are also going to construct a sequence $\langle F_\zeta : \zeta < \kappa \rangle$ such that:

- $F_0 = \emptyset$,
- $\zeta < \kappa \Rightarrow F_\zeta \subseteq \mathrm{supp}(p_\zeta)$,
- $\zeta < \kappa \Rightarrow |F_\zeta| < \kappa$,
- $\epsilon < \zeta \Rightarrow F_\epsilon \subseteq F_\zeta$,

and we are going to use bookkeeping to ensure $F = \bigcup_{\zeta<\kappa} F_\zeta = \bigcup_{\zeta<\kappa} \mathrm{supp}(p_\zeta)$, which is also going to be the support of p^*.

Furthermore we are implicitly going to construct strategies for Black in the games $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha, p(\alpha))$ for $\alpha \in F$. Then we will choose $p^* = \langle \dot{q}^*_\alpha : \alpha \in F \rangle$ where $\dot{q}^*_\alpha$ witnesses that White can beat Black's strategy.

What does White play in the ζth round?

Let $\langle \alpha_{\zeta,\xi} : \xi < \xi^*_\zeta \rangle$ enumerate F_ζ. For $\xi < \xi^*_\zeta$ we want to play the ζth round of the game $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_{\alpha_{\zeta,\xi}}, p(\alpha_{\zeta,\xi}))$ where White plays according to the name of a winning strategy (White sticks to the same strategy throughout the proof of course). To make notation easier we do not want to keep track of when $\alpha_{\zeta,\xi}$ first appeared in F_ϵ for some $\epsilon \leq \zeta$. Instead let $\epsilon_{\zeta,\xi} = \min\{\epsilon \leq \zeta : \alpha_{\zeta,\xi} \in F_\epsilon\}$ and assume we are playing $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_{\alpha_{\zeta,\xi}}, p_{\epsilon_{\zeta,\xi}}(\alpha_{\zeta,\xi}), \epsilon_{\zeta,\xi})$, that is, we are playing in the ζth round for each $\alpha_{\zeta,\xi}$. See 2.4.5.

By induction (we are going to address this further down) we assume for each $\xi < \xi^*_\zeta$ that $p_\zeta \restriction \alpha_{\zeta,\xi} \Vdash$ "$\dot{\mu}_{\alpha_{\zeta,\xi},\zeta} \leq \mu_{\alpha_{\zeta,\xi},\zeta}$" for some $\mu_{\alpha_{\zeta,\xi},\zeta} < \kappa$ where $\dot{\mu}_{\alpha_{\zeta,\xi},\zeta}$ is the length of ζth round of $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_{\alpha_{\zeta,\xi}}, p_{\epsilon_{\zeta,\xi}}(\alpha_{\zeta,\xi}), \epsilon_{\zeta,\xi})$ as decided by the name of White's winning strategy. Then there exist (in $\mathbf{V}$ where we are trying to construct a winning strategy) not necessarily injective enumerations $\langle \dot{q}_{\alpha_{\zeta,\xi},\zeta,i} : i < \mu_{\alpha_{\zeta,\xi},\zeta} \rangle$ of the moves that White plays according to the name of his winning strategy in the ζth round of $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_{\alpha_{\zeta,\xi}}, p_{\epsilon_{\zeta,\xi}}(\alpha_{\zeta,\xi}), \epsilon_{\zeta,\xi})$. To make notation still easier we only do the proof for the special case where White always plays an antichain (but the proof works even if White does not).

Let $\mu_\zeta = |\prod_{\xi<\xi^*_\zeta} \mu_{\alpha_{\zeta,\xi},\zeta}|$ and this is what White decides to be the length of the ζth round of $\mathfrak{F}_\kappa(\mathbb{P}, p)$. Remember that κ is inaccessible so indeed $\mu_\zeta < \kappa$. Let $\langle \lambda_{\zeta,i} : i < \mu_\zeta \rangle$ enumerate $\prod_{\xi<\xi^*_\zeta} \mu_{\alpha_{\zeta,\xi},\zeta}$. Now we construct a sequence $\langle p_{\zeta,i} : i < \mu_\zeta \rangle$ (of course anything that is not explicitly stated to be done by Black is part of White's strategy that we are currently constructing):

(i) First we find $p_{\zeta,0} \leq p_\epsilon$ for every $\epsilon < \zeta$ as follows:

- If there is no $\xi < \xi^*_\zeta$ such that $\alpha = \alpha_{\zeta,\xi}$ then let $p_{\zeta,0}(\alpha)$ be such that $p_{\zeta,0} \restriction \alpha \Vdash p_{\zeta,0}(\alpha) \leq p_\epsilon(\alpha)$ according to a winning strategy for White in $\mathfrak{C}(\dot{\mathbb{Q}}_\alpha)$.

- If there is $\xi < \xi^*_\zeta$ such that $\alpha = \alpha_{\zeta,\xi}$ then let $p_{\zeta_0}(\alpha)$ be such that $p_{\zeta_0}{\restriction}\alpha \Vdash$"$p_{\zeta_0}(\alpha) \leq \bigvee_{\gamma<\mu_{\alpha,\zeta}} \dot{q}_{\alpha,\zeta,\gamma}$".

(ii) For the ith move of the ζth round White plays $p'_{\zeta,i}$ where

$$p'_{\zeta,i}(\alpha) = \begin{cases} p_{\zeta,i}(\alpha_{\zeta,\xi}) \wedge \dot{q}_{\alpha_{\zeta,\xi},\zeta,\lambda_{\zeta,i}(\xi)} & \text{if } \alpha = \alpha_{\zeta,\xi} \text{ for some } \xi < \xi^*_\zeta, \\ p_{\zeta,i}(\alpha) & \text{otherwise.} \end{cases}$$

(iii) Black responds with $p''_{\zeta,i} \leq p'_{\zeta,i}$.

(iv) Let $p'''_{\zeta,i}$ be such that for $\alpha < \alpha^*$ we have

$$p'''_{\zeta,i}{\restriction}\alpha \Vdash \text{“}p'''_{\zeta,i}(\alpha) \leq p''_{\zeta,i}(\alpha) \text{ and } p'''_{\zeta,i}(\alpha) \text{ is according to a winning strategy for White in } \mathfrak{C}(\dot{\mathbb{Q}}_\alpha)\text{”}.$$

(v) Let $p''''_{\zeta,i}$ be defined by

$$p''''_{\zeta,i}(\alpha) = \begin{cases} (p_{\zeta,i}(\alpha_{\zeta,\xi}) \backslash \dot{q}_{\alpha_{\zeta,\xi},\zeta,\lambda_{\zeta,i}(\xi)}) \vee p'''_{\zeta,i}(\alpha_{\zeta,\xi}) \\ \qquad\qquad \text{if } \alpha = \alpha_{\zeta,\xi} \text{ for some } \xi < \xi^*_\zeta, \\ p'''_{\zeta,i}(\alpha) \qquad \text{otherwise;} \end{cases}$$

we easily check $p''''_{\zeta,i} \leq p$.

(vi) If $i = j + 1$ then let $p_{\zeta,i} = p''''_{\zeta,j}$. If i is a limit ordinal, then we find $p_{\zeta,i} \leq p_{\zeta,j}$ for every $j < i$ as follows:

- If there is no $\xi < \xi^*_\zeta$ such that $\alpha = \alpha_{\zeta,\xi}$ then let $p_{\zeta,i}(\alpha)$ be such that $p_{\zeta,i}{\restriction}\alpha \Vdash$"$p_{\zeta,i}(\alpha)$ is according to a winning strategy for White in $\mathfrak{C}(\dot{\mathbb{Q}}_\alpha)$ for the sequence $\langle p_{\zeta,j}(\alpha) : j < i\rangle$".
- If there is $\xi < \xi^*_\zeta$ such that $\alpha = \alpha_{\zeta,\xi}$ then let $p_{\zeta,i}(\alpha)$ be such that

$$p_{\zeta,i}{\restriction}\alpha \Vdash \text{“}p_{\zeta,i}(\alpha) = \bigvee_{\gamma<\mu_{\alpha,\zeta}} \dot{r}_{\zeta,i,\alpha,\gamma}\text{”}$$

where $p_{\zeta,i}{\restriction}\alpha \Vdash$ "$\dot{r}_{\zeta,i,\alpha,\gamma}$ is according to a winning strategy for White in $\mathfrak{C}(\dot{\mathbb{Q}}_\alpha)$ for the sequence $\langle p_{\zeta,j}(\alpha) \wedge \dot{q}_{\alpha,\zeta,\gamma} : j < i\rangle$".

Finally let p_ζ be a lower bound for $\langle p_{\zeta,i} : i < \mu_\zeta\rangle$ as in (vi) (but not really, we have to do some preparation work for the next step first, see below). Now the strategy for Black in $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_{\alpha_{\zeta,\xi}}, p(\alpha_{\zeta,\xi}))$ is to play $p_\zeta(\alpha_{\zeta,\xi}) \wedge \dot{q}_{\alpha_{\zeta,\xi},\zeta,\lambda_{\zeta,i}(\xi)}$.

Preparation for the $(\zeta + 1)$th round.

We still have to address why the $\mu_{\alpha_{\zeta,\xi},\zeta}$ exist; but having understood the proof to this point this is now easy. Let $F_{\zeta+1} = F_\zeta \cup \{\alpha\}$ for some $\alpha \in \mathrm{supp}(p_\zeta)\backslash F_\zeta$, if such an α exists (and remember to use bookkeeping). Now for every $\alpha \in F_{\zeta+1}$ work as above on $p_\zeta{\restriction}\alpha$ and $F_\zeta \cap \alpha$ but instead of taking a response from Black in (iii) White responds to himself deciding $\mu_{\alpha,\zeta+1}$.

So we have prepared for $\zeta + 1$. But what about limit steps? Remember that the rules of $\mathfrak{F}^*_\kappa$ state that $\dot{\mu}_{\alpha,\zeta} \leq \sup_{\epsilon<\zeta} \dot{\mu}_{\alpha,\epsilon}$. So if we let $F_\zeta = \bigcup_{\epsilon<\zeta} F_\epsilon$, all is good because having an estimate for successor steps gives us an estimate for limit steps.

Why does White win?

Because for $\alpha \in F = \bigcup_{\zeta<\kappa} F_\zeta$ there exists a $\dot{\mathbb{Q}}_\alpha$-name $\dot{q}^*_\alpha$ such that $p{\restriction}\alpha \Vdash$ "$\dot{q}^*_\alpha$ witnesses that White wins if Black plays as described above in $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha, p(\alpha))$".

By construction, $p^* = \langle \dot{q}^*_\alpha : \alpha \in F\rangle$ is as required. ∎

2.5. Fusion and properness. In this section we give a sufficient condition for a limit of a $\leq\kappa$-support iteration to be κ-proper, namely, the existence of winning strategies for the games $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha)$ for all iterands $\dot{\mathbb{Q}}_\alpha$ encountered in the iteration.

We also show that if all iterands have cardinality $\leq \kappa^+$, and the length δ of the iteration is $< \kappa^{++}$, then the resulting forcing $\mathbb{P}_\delta$ has a dense set of size κ^+ and in particular will still satisfy the κ^{++}-c.c.

Definition 2.5.1. In this section we consider an iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$ with limit $\mathbb{P}_\delta$ such that:

(1) $\delta < \kappa^{++}$.
(2) $\mathbb{P}$ has κ-support.
(3) White has a winning strategy for $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha, \dot{q})$ for every $\alpha < \delta$ and $\dot{q} \in \dot{\mathbb{Q}}_\alpha$.
(4) $\mathbb{P}_\alpha \Vdash$"$\dot{\mathbb{Q}}_\alpha$ has size at most κ^+".

For $\alpha < \delta$ let $\dot{b}_\alpha$ be a $\mathbb{P}_\alpha$-name of a one-to-one map from κ^+ onto $\dot{\mathbb{Q}}_\alpha$.

Lemma 2.5.2. *Let $(\mathbf{N}, \in)$ be a model of size κ, closed under $<\kappa$-sequences; let $\mathbb{R}$ be an arbitrary forcing notion such that $\mathbb{R} \in \mathbf{N}$ and $(\mathbf{N}, \in) \prec (H(\chi), \in)$ for some χ large enough. If White has a winning strategy for $\mathfrak{F}_\kappa(\mathbb{R}, p)$ then for every $p \in \mathbb{R} \cap \mathbf{N}$ there exists $q^* \in \mathbb{R}$, $q^* \leq p$ such that q^* is $\mathbf{N}$-$\mathbb{R}$-generic. This means:*

(1) *For every maximal antichain A of $\mathbb{R}$ with $A \in \mathbf{N}$ we have*

$$q^* \Vdash A \cap \mathbf{N} \cap \dot{G}_{\mathbb{R}} \neq \emptyset.$$

(2) *Or equivalently: for every name $\dot{\tau}$ of an ordinal with $\dot{\tau} \in \mathbf{N}$ we have*

$$q^* \Vdash \dot{\tau} \in \mathbf{N}.$$

Proof. Note that because $|\mathbf{N}| = \kappa$ there are at most κ-many names of ordinals in $\mathbf{N}$. By our assumption White has a winning strategy for $\mathfrak{F}_\kappa(\mathbb{R}, p)$, and because $\mathbf{N}$ is an elementary submodel, White has a winning strategy that lies in $\mathbf{N}$. Now consider a run of the game where:

- White plays according to his winning strategy in $\mathbf{N}$. By induction all these moves are in $\mathbf{N}$ by our assumption $\mathbf{N}^{<\kappa} \subseteq \mathbf{N}$.

- Black decides all ordinals of $\mathbf{N}$ which lie in $\mathbf{N}$ by playing $p'_{\zeta,i} \in \mathbf{N}$ for $\zeta < \kappa$, $i < \mu_\zeta$.

Now q^* witnessing White's win is $\mathbf{N}$-$\mathbb{R}$-generic. ∎

DEFINITION 2.5.3. Let $\mathbb{R}$ be a forcing notion. Consider a run of the game $\mathfrak{G} \in \{\mathfrak{F}_\kappa, \mathfrak{F}^*_\kappa\}$ where:

(1) White wins.
(2) Black plays $\vec{p}' = \langle p'_{\zeta,i} : \zeta < \kappa,\ i < \mu_\zeta \rangle$.

Then we call q^* witnessing White's win a $\mathfrak{G}$-*fusion limit* of $\vec{p}'$.

COROLLARY 2.5.4. *Let $\mathbb{P}$ be as in* 2.5.1. *Then:*

(a) *For every $p \in \mathbb{P}_\delta \cap \mathbf{N}$ there exists a generic condition $q^* \leq p$ that is a $\mathfrak{F}_\kappa(\mathbb{P})$-fusion limit of $\vec{p}'$ with $p'_{\zeta,i} \in \mathbf{N}$ for all $\zeta < \kappa$, $i < \mu_\zeta$. (However, in general $q^* \notin \mathbf{N}$.)*
(b) *For $\alpha < \delta$ we have $q^* {\restriction} \alpha \Vdash$ "$q^*(\alpha)$ is a $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha)$-fusion limit".*

Proof. (a) By 2.5.1(3) and 2.4.7(1), White has a winning strategy for $\mathfrak{F}_\kappa(\mathbb{R}, p)$, so use 2.5.2.

(b) Use 2.4.7(2). ∎

DEFINITION 2.5.5. For $\alpha < \delta$ a condition $p \in \mathbb{P}_\alpha$ is called an H_{κ^+}-*condition* if for every $\beta < \alpha$ the $\mathbb{P}_\beta$-name $p(\beta)$ is an H_{κ^+}-$\mathbb{P}_\beta$-name.

For $\alpha < \delta$ we inductively define the notion of an H_{κ^+}-$\mathbb{P}_\alpha$-*name*. On the one hand we consider H_{κ^+}-names for elements of κ^+, on the other hand for elements of $\dot{\mathbb{Q}}_\alpha$.

(1) $\dot{\tau}$ is an H_{κ^+}-name for an element of κ^+ iff $\dot{b}_\alpha(\dot{\tau})$ is an H_{κ^+}-name of an element of $\dot{\mathbb{Q}}_\alpha$. ($\dot{b}_\alpha$ was defined in 2.5.1.)
(2) For every $\gamma \in \kappa^+$, the standard name $\check{\gamma}$ is an H_{κ^+}-name.
(3) For every sequence $\langle (p_i, \dot{\tau}_i) : i < \kappa \rangle$ where p_i are H_{κ^+}-conditions in $\mathbb{P}_\alpha$ and $\dot{\tau}_i$ are H_{κ^+}-$\mathbb{P}_\alpha$-names, there exists an H_{κ^+}-name $\dot{\tau}$ forced to be equal to $\dot{\tau}_i$ where i is the least index such that $p_i \in \dot{G}_{\mathbb{P}}$ if such an i exists, and $\check{0}$ otherwise.
(4) For every $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha)$-fusion sequence $\vec{p}'$ where $p'_{\zeta,i}$ are H_{κ^+}-$\mathbb{P}_\alpha$-names for elements of $\dot{\mathbb{Q}}_\alpha$, there exists an H_{κ^+}-name $\dot{\tau}$ that is forced to be equal to the condition witnessing White's win (if it exists; $\check{0}$ otherwise).

REMARK 2.5.6. If we fix a well-order of $H(\chi)$ for some χ large enough we can witness the existential statements in 2.5.5(3) and (4) by choosing the least witness according to the well-order. This makes it possible to find a set satisfying 2.5.5(1)–(4) which has size κ^+ only.

REMARK 2.5.7. The "H_{κ^+}"-names are an easy generalization of the "hereditarily countable" names appearing in [She98, 4.1]; see also [GK16].

Lemma 2.5.8. *For every condition $p \in \mathbb{P}_\delta$ there exists an H_{κ^+}-condition $q^* \leq p$.*

Proof. First let $\mathbf{N}$ be a model of size κ, closed under $<\kappa$-sequences, with $p, \mathbb{P} \in \mathbf{N}$, and let q^* be a $\mathfrak{F}_\kappa(\mathbb{P})$-fusion limit of $\vec{p}\,'$ with $p'_{\zeta,i} \in \mathbf{N}$ as in 2.5.4 (so in particular q^* is $\mathbf{N}$-generic).

Now we will try to find an H_{κ^+}-name for $p'_{\zeta,i}(\alpha)$, for all $\zeta, \alpha < \delta$, $i < \mu_\zeta$.

For $\alpha \in \mathrm{supp}(q^*)$ we define $p''_{\zeta,i}(\alpha)$ as follows. We find (in $\mathbf{N}$) a maximal antichain $A = A_{\zeta,i,\alpha}$ that decides $\dot{b}_\alpha^{-1}(p'_{\zeta,i}(\alpha))$, i.e. there exists a function $f = f_{\zeta,i,\alpha}\colon A \to \kappa^+$ such that for all $r \in A$,

$$r \Vdash p'_{\zeta,i}(\alpha) = \dot{b}_\alpha(f(r)).$$

Let $A' = A \cap \mathbf{N}$. Consider the sequence $\langle (r, \dot{b}_\alpha(f(r))) : r \in A' \rangle$. This family defines an H_{κ^+}-name $p''_{\zeta,i}(\alpha)$.

Now because $q^* \restriction \alpha$ is $\mathbf{N}$-generic,

$$q^* \restriction \alpha \Vdash p'_{\zeta,i}(\alpha) = p''_{\zeta,i}(\alpha).$$

Hence $q^* \restriction \alpha$ forces that $q^*(\alpha)$ is equal to a witness of White's win against $p''_{\zeta,i}(\alpha)$, i.e. $q^*(\alpha)$ is a $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha)$-fusion limit. Hence $q^*(\alpha)$ is an H_{κ^+}-name, so q^* is an H_{κ^+}-condition. ∎

Corollary 2.5.9. *Let $\mathbb{P}_\delta$ be as in* 2.5.1 (*so in particular $\delta < \kappa^{++}$*). *Then there exists $D \subseteq \mathbb{P}_\delta$ such that:*

(1) *D is dense.*
(2) *$|D| = \kappa^+$.*
(3) *$\mathbb{P}_\delta$ has the κ^{++}-c.c.*

Proof. Follows immediately from 2.5.8. ∎

Corollary 2.5.10. *Assume $2^\kappa = \kappa^+$, and let $\mathbb{P} = (\mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \kappa^{++})$ be an iteration with limit $\mathbb{P}_{\kappa^{++}}$ satisfying the following:*

(1) *$\mathbb{P}$ has κ-support.*
(2) *For each $\alpha < \kappa^{++}$ we have $\mathbb{P}_\alpha \Vdash |\mathbb{Q}_\alpha| = 2^\kappa$.*
(3) *For each $\alpha < \kappa^{++}$ and each name $\dot{q} \in \dot{\mathbb{Q}}_\alpha$, $\mathbb{P}_\alpha$ forces that White has a winning strategy for the fusion game $\mathfrak{F}^*_\kappa(\dot{\mathbb{Q}}_\alpha, \dot{q})$* (*defined in* 2.4.1; *check* 2.4.3 *to see for which forcings this may be the case*).

Then:

(a) *For each $\alpha < \kappa^{++}$ the forcing notion $\mathbb{P}_\alpha$ has a dense subset of cardinality κ^+.*
(b) *For each $\alpha < \kappa^{++}$, $\mathbb{P}_\alpha$ forces $2^\kappa = \kappa^+$.* (*If we assume that each Q_α adds a new κ-real, then $\mathbb{P}_\kappa \Vdash 2^\kappa = \kappa^{++}$.*)
(c) *For each $\delta \leq \kappa^{++}$, $\mathbb{P}_\delta$ has the κ^{++}-c.c.*

Proof. The κ^{++}-c.c. of $\mathbb{P}_{\kappa^{++}}$ follows by the Solovay–Tennenbaum theorem from the fact that $\mathbb{P}$ uses direct limits on a stationary set, namely, the set of ordinals of cofinality κ^+. (See [ST71].)

The rest just summarizes previous theorems. ■

3. Smaller ideals. In this section we first describe two ideals $\mathrm{wid}(\mathbb{Q}_\kappa)$ and $\mathrm{id}^-(\mathbb{Q}_\kappa)$, both of which are closely related (and often equal) to $\mathrm{id}(\mathbb{Q}_\kappa)$. We then give a more "combinatorial" characterization of $\mathrm{add}(\mathbb{Q}_\kappa)$ and $\mathrm{cf}(\mathbb{Q}_\kappa)$, involving the additivity and cofinality of the ideal $\mathbf{nst}^{\mathrm{pr}}_\kappa$ of nowhere stationary subsets of $S^\kappa_{\mathrm{pr}} \subseteq \kappa$.

3.1. The ideal $\mathrm{wid}(\mathbb{Q}_\kappa)$

DEFINITION 3.1.1. For $\mathrm{id}(\mathbb{Q}_\kappa)$ we allow κ-many antichains to define $A \in \mathrm{id}(\mathbb{Q}_\kappa)$. But we may also consider the weak ideal $\mathrm{wid}(\mathbb{Q}_\kappa)$ of all sets $A \subseteq 2^\kappa$ such that for some maximal antichain $\mathcal{A}$ (or equivalently some predense set $\mathcal{A}$) we have $A \subseteq \mathrm{set}_0(\mathcal{A})$, where $\mathrm{set}_0(\mathcal{A}) := 2^\kappa \setminus \bigcup_{p\in\mathcal{A}}[p]$.

LEMMA 3.1.2.

(a) $\mathrm{wid}(\mathbb{Q}_\kappa) \subseteq \mathrm{id}(\mathbb{Q}_\kappa)$.
(b) $\mathrm{wid}(\mathbb{Q}_\kappa) = \mathrm{id}(\mathbb{Q}_\kappa)$ *iff* $\neg\,\mathrm{Pr}(\kappa)$.
(c) $\mathrm{wid}(\mathbb{Q}_\kappa)$ *is κ-complete.*

Proof. (a) Trivial: If $\mathcal{A}$ witnesses $A \in \mathrm{wid}(\mathbb{Q}_\kappa)$ then $\Lambda = \{\mathcal{A}\}$ witnesses $A \in \mathrm{id}(\mathbb{Q}_\kappa)$.

(b) Assume $\neg\,\mathrm{Pr}(\kappa)$. Let Λ be a set of at most κ-many maximal antichains of $\mathbb{Q}_\kappa$ and without loss of generality assume that Λ is closed under rational shifts, i.e. for all $\eta_1, \eta_2 \in 2^\kappa$ we have

$$\eta_1 =^* \eta_2 \implies [\eta_1 \in \mathrm{set}_0(\Lambda) \Leftrightarrow \eta_2 \in \mathrm{set}_0(\Lambda)].$$

Let $A \subseteq \mathrm{set}_0(\Lambda)$. By our assumption about κ there exists $p \in \mathbb{Q}_\kappa$ such that $[p] \subseteq \mathrm{set}_1(\Lambda)$; let p be witnessed by $(\tau, S, \vec{\Gamma})$. Let

$$\mathcal{A} = \{q \in \mathbb{Q}_\kappa : q \text{ is witnessed by } (\rho, S, \vec{\Gamma}) \text{ for some } \rho \in 2^{<\kappa}\}$$

and check that $\mathcal{A}$ is predense. Now clearly $q \in \mathcal{A} \Rightarrow [q] \subseteq \mathrm{set}_1(\Lambda)$ hence $\mathrm{set}_1(\mathcal{A}) \subseteq \mathrm{set}_1(\Lambda)$ hence $A \subseteq \mathrm{set}_0(\mathcal{A})$, i.e. $A \in \mathrm{wid}(\mathbb{Q}_\kappa)$.

Conversely assume $\mathrm{wid}(\mathbb{Q}_\kappa) = \mathrm{id}(\mathbb{Q}_\kappa)$ and let Λ be a set of no more than κ-many maximal antichains of $\mathbb{Q}_\kappa$. By our assumption there exists a maximal antichain $\mathcal{A}$ of $\mathbb{Q}_\kappa$ such that

$$\bigcup_{p\in\mathcal{A}} [p] = \mathrm{set}_1(\mathcal{A}) \subseteq \mathrm{set}_1(\Lambda).$$

Hence for any $p \in \mathcal{A}$ we have $[p] \subseteq \mathrm{set}_1(\Lambda)$; as Λ was arbitrary, we get $\neg\,\mathrm{Pr}(\kappa)$.

(c) Because $\mathbb{Q}_\kappa$ is κ-strategically closed. ■

Lemma 3.1.3. *Consider the usual forcing ideal*

$$\operatorname{fid}(\mathbb{Q}_\kappa) = \{A \subseteq 2^\kappa : (\forall p \in \mathbb{Q}_\kappa)(\exists q \leq p)\ [q] \cap A = \emptyset\}.$$

Then $\operatorname{fid}(\mathbb{Q}_\kappa) = \operatorname{wid}(\mathbb{Q}_\kappa)$.

Proof. Let $A \in \operatorname{wid}(\mathbb{Q}_\kappa)$ be witnessed by $\mathcal{A}$. Now for any $p \in \mathbb{Q}_\kappa$ there exists $p' \in \mathcal{A}$ such that p and p' are compatible. Let $q = p \cap p'$; then clearly $A \cap [q] = \emptyset$, hence $A \in \operatorname{fid}(\mathbb{Q}_\kappa)$.

Conversely, if $A \in \operatorname{fid}(\mathbb{Q}_\kappa)$ then the set $\mathcal{D} = \{q : [q] \cap A = \emptyset\}$ is dense. Choose any maximal antichain $\mathcal{A} \subseteq \mathcal{D}$; then $\mathcal{A}$ will witness $A \in \operatorname{wid}(\mathbb{Q}_\kappa)$. ∎

3.2. The ideal $\operatorname{id}^-(\mathbb{Q}_\kappa)$**.** Recall from 3.1.1 that the ideal $\operatorname{wid}(\mathbb{Q}_\kappa)$ is generated by sets $\operatorname{set}_0(\mathcal{A})$, where $\mathcal{A} \subseteq \mathbb{Q}_\kappa$ is any predense set. Note also that the set of all rational translates (see 1.2.3) of any fixed condition is predense by 1.1.8. This suggests the following definition:

Definition 3.2.1. The ideal $\operatorname{id}^-(\mathbb{Q}_\kappa)$ consists of all sets $A \subseteq 2^\kappa$ for which there exists a condition p such that $A \subseteq \operatorname{set}_0(\{s + [p] : s \in 2^{<\kappa}\})$.

Equivalently, $A \in \operatorname{id}^-(\mathbb{Q}_\kappa)$ iff there are

- a nowhere stationary set $S \subseteq S^\kappa_{\mathrm{inc}}$, and
- a sequence $\vec{N} = \langle N_\delta : \delta \in S\rangle$ such that each N_δ is a "rather small" subset of 2^δ (in the sense that N_δ is in $\operatorname{id}(\mathbb{Q}_\delta)$)

such that

$$A \subseteq \operatorname{set}^-_0(\vec{N}) := \{\eta \in 2^\kappa : (\exists^\infty \delta \in S)\ \eta{\restriction}\delta \in N_\delta\}.$$

Note that we are often lazy and use the notation $\operatorname{add}(\mathbb{Q}_\kappa)$. This always means $\operatorname{add}(\operatorname{id}(\mathbb{Q}_\kappa))$, never $\operatorname{add}(\operatorname{id}^-(\mathbb{Q}_\kappa))$. The same applies for cov, non and cf.

Lemma 3.2.2. $\operatorname{id}^-(\mathbb{Q}_\kappa) \subseteq \operatorname{wid}(\mathbb{Q}_\kappa)$.

Proof. Given $S \subseteq S^\kappa_{\mathrm{inc}}$ and $\vec{\Lambda} = \langle \Lambda_\delta : \delta \in S\rangle$ let $p_\rho \in \mathbb{Q}_\kappa$ be the condition witnessed by $(\rho, S, \vec{\Lambda})$ and let $\mathcal{D} = \{p_\rho : \rho \in 2^{<\kappa}\}$. It is easy to check that $\operatorname{set}^-_0(\vec{\Lambda}) \subseteq \operatorname{set}_0(\mathcal{D})$. ∎

Lemma 3.2.3. $\operatorname{id}^-(\mathbb{Q}_\kappa)$ *is* $<\kappa^+$*-complete.*

Proof. For $i < \kappa$ let $(S_i, \vec{\Lambda^i})$ represent $A_i = \operatorname{set}^-_0(\vec{\Lambda}_i) \in \operatorname{id}^-(\mathbb{Q}_\kappa)$. Let

$$S^* = \{\delta < \kappa : (\exists i < \delta)\ \delta \in S_i\}$$

be the diagonal union of S_i and for $\delta \in S^*$ let $\Lambda^*_\delta = \bigcup\{\Lambda_{i,\delta} : i < \delta\}$. Then clearly $\bigcup_{i<\kappa} A_i \subseteq \operatorname{set}^-_0(\vec{\Lambda}^*)$. ∎

Theorem 3.2.4. *Let κ be a weakly compact cardinal. Then* $\operatorname{id}^-(\mathbb{Q}_\kappa) = \operatorname{wid}(\mathbb{Q}_\kappa)$.

Proof. Let $D = \{p_\epsilon : \epsilon < \kappa\} \subseteq \mathbb{Q}_\kappa$ be a maximal antichain witnessing $A \subseteq \mathrm{set}_0(D) \in \mathrm{wid}(\mathbb{Q}_\kappa)$. For $\epsilon < \kappa$ let p_ϵ be witnessed by $(\tau_\epsilon, S_\epsilon, \vec{\Lambda}_\epsilon)$. Using weak compactness we find a sequence $\langle \delta_\alpha : \alpha < \kappa\rangle$ such that:

- $\delta_\alpha \in S^\kappa_{\mathrm{inc}}$.
- $\delta_\alpha > \sup_{\beta<\alpha} \delta_\beta$.
- $D_\alpha = \{p_\epsilon \cap 2^{<\delta_\alpha} : \epsilon < \delta_\alpha\}$ is a maximal antichain in $\mathbb{Q}_{\delta_\alpha}$.

Let

$$S^*_\alpha = \Big(\bigcup_{\epsilon<\delta_\alpha} S_\epsilon\Big)\backslash\delta_\alpha \quad \text{and} \quad S^* = \bigcup_{\alpha<\kappa} S^*_\alpha \cup \{\delta_\alpha : \alpha < \kappa\}.$$

It is easy to check that S^* is nowhere stationary. For $\delta \in S^*$ we define

$$\Lambda^*_\delta = \bigcup_{\epsilon<\delta} \Lambda_{\epsilon,\delta} \cup \begin{cases} \{D_\alpha\} & \text{if } \delta = \delta_\alpha \text{ for some } \alpha < \kappa, \\ \emptyset & \text{otherwise.} \end{cases}$$

We claim that $\mathrm{set}_0(D) \subseteq \mathrm{set}_0^-(\vec{\Lambda}^*)$, witnessing $A \in \mathrm{id}^-(\mathbb{Q}_\kappa)$. Let $\eta \in \mathrm{set}_0(D)$.

Case 1: $(\exists^\infty \alpha < \kappa)\ \eta{\restriction}\delta_\alpha \in \mathrm{set}_0(D_\alpha)$. Thus clearly $\eta \in \mathrm{set}_0^-(\vec{\Lambda}^*)$.

Case 2: $(\forall^\infty \alpha < \kappa)\ \eta{\restriction}\delta_\alpha \in \mathrm{set}_1(D_\alpha)$. So $\eta{\restriction}\delta_\alpha \in [p_{\epsilon_\alpha} \cap 2^{<\delta_\alpha}]$ for some $\epsilon_\alpha < \delta_\alpha$ for almost all (or just infinitely many) $\alpha < \kappa$. However $\eta \in \mathrm{set}_0(D)$ implies that $\eta \notin [p_{\epsilon_\alpha}]$. Hence there exists $\delta \in S_{\epsilon_\alpha}\backslash\delta_\alpha$ such that $\eta{\restriction}\delta \in \mathrm{set}_0^-(\Lambda_{\epsilon_\alpha,\delta})$. Recall that $\Lambda_{\epsilon_\alpha,\delta} \subseteq \Lambda^*_\delta$ and thus $\eta \in \mathrm{set}_0^-(\vec{\Lambda}^*)$. ∎

Corollary 3.2.5. *Let κ be a weakly compact cardinal. Then* $\mathrm{id}^-(\mathbb{Q}_\kappa) = \mathrm{id}(\mathbb{Q}_\kappa)$.

Proof. By 3.2.4 we have $\mathrm{id}^-(\mathbb{Q}_\kappa) = \mathrm{wid}(\mathbb{Q}_\kappa)$, and by 3.2.3, $\mathrm{id}^-(\mathbb{Q}_\kappa)$ is $<\kappa^+$-complete. Of course $\mathrm{id}(\mathbb{Q}_\kappa)$ is the $<\kappa^+$-closure of $\mathrm{wid}(\mathbb{Q}_\kappa)$ so the result follows. ∎

Lemma 3.2.6. *Let $S \subseteq \kappa$ be nowhere stationary. Then we can find:*

(1) *a regressive function f on S,*
(2) *a family $\{E_\alpha : \alpha \le \kappa,\ \mathrm{cf}(\alpha) > \omega\}$ where $E_\alpha \subseteq \alpha$ is a club*

such that:

(a) $(\forall \delta \in \kappa\backslash\omega)\ |\{\lambda \in S\backslash\delta : f(\lambda) \le \delta\}| < \delta$.
(b) $(\forall \alpha \in \kappa)(\forall \lambda < \alpha)\ \mathrm{cf}(\alpha) > \omega \Rightarrow (f(\lambda), \lambda) \cap E_\alpha = \emptyset$.

Proof. We prove by induction on $\beta \le \kappa$ that we can find a regressive function f_β on $S \cap \beta$ and a family $\{E_\alpha : \alpha \le \beta,\ \mathrm{cf}(\alpha) > \omega\}$ with the required properties. (Note that for any β we will reuse the E_α from previous steps so we do not bother writing $E_{\beta,\alpha}$.) For $\beta = \kappa$ the result follows.

Case 1: β successor. Easy.

CASE 2: β limit, $\mathrm{cf}(\beta) > \omega$. Let $E_\beta = \langle \alpha_\zeta : \zeta < \mathrm{cf}(\beta)\rangle$ be an increasing continuous cofinal sequence in β, disjoint from S. Let

$$S_\zeta = S \cap [\alpha_\zeta, \alpha_{\zeta+1})$$

and let f_ζ be a function on S_ζ from the induction hypothesis. Without loss of generality $\lambda \in S_\zeta \Rightarrow f_\zeta(\lambda) \geq \alpha_\zeta$. [Why? Just round up, i.e. replace $f_\zeta(\lambda)$ by $\max(\alpha_\eta, f_\zeta(\lambda))$.] So

$$f_\beta = \bigcup_{\zeta < \mathrm{cf}(\beta)} f_\zeta$$

is as required. In particular, because S is disjoint from E_β, the function f_β is regressive. By construction, the sets E_β have the property (b).

CASE 3: β limit, $\mathrm{cf}(\beta) = \omega$. Like Case 2, but easier (no rounding up required). ∎

THEOREM 3.2.7. *Let $A \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ be represented by $\vec{\Lambda} = \langle \Lambda_\delta : \delta \in S\rangle$. Then there exists $A' \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ represented by $\vec{\Lambda}' = \langle \Lambda'_\delta : \delta \in S'\rangle$ such that:*

(1) $A \subseteq A'$.
(2) $S' \in \mathbf{nst}^{\mathrm{pr}}_\kappa$.
(3) $S \cap S^\kappa_{\mathrm{pr}} \subseteq S'$.
(4) $\delta \in S \cap S' \Rightarrow \Lambda_\delta \subseteq \Lambda'_\delta$.

Proof. First without loss of generality we assume A is closed under rational translates (see 1.2.3) and in particular Λ_δ are closed under rational translates. For $\delta \in S \setminus S^\kappa_{\mathrm{pr}}$ find $p_\delta \in \mathbb{Q}_\delta$ witnessed by $(\langle\rangle, \vec{\Gamma}_\delta, S_\delta)$ such that $[p_\delta] \subseteq \mathrm{set}_1(\Lambda_\delta)$. By 1.3.2 we may assume $S_\delta \subseteq S^\delta_{\mathrm{pr}}$.

Now let f be a regressive function on S as in 3.2.6 and let

$$S' = (S \cap S^\kappa_{\mathrm{pr}}) \cup \bigcup_{\delta \in S \setminus S^\kappa_{\mathrm{pr}}} S_\delta \setminus (f(\delta)+1)$$

and for $\delta \in S'$ let

$$\Lambda'_\delta = \bigcup \{\Gamma_{\delta^*,\delta} : \delta^* > \delta > f(\delta^*)\} \cup \begin{cases} \Lambda_\delta, & \delta \in S \cap S^\kappa_{\mathrm{pr}}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Why is S' nowhere stationary? Let $\alpha < \kappa$, $\mathrm{cf}(\alpha) > \omega$. Why is $S' \cap \alpha$ not stationary in α? Consider two cases:

- $\alpha > \sup(S \cap \alpha)$. Use 3.2.6(a).
- $\alpha = \sup(S \cap \alpha)$. For the part of $S' \cap \alpha$ that comes from S_δ with $\delta < \alpha$ use 3.2.6(b) to show that the club set E_α is disjoint from $S_\delta \setminus (f(\delta)+1)$ for all $\delta < \alpha$. For the part that comes from S_δ with $\delta > \alpha$ use 3.2.6(a) as above.

See 3.3.16 for the same argument carried out in more detail. Similarly argue that $|\Lambda'_\delta| \le \delta$.

Now check that $S', \vec{\Lambda}'$ define a set $A' \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ covering A. Indeed, if $\eta \in A$, there exists $W \subseteq S$ with $\sup(W) = \kappa$ such that $\eta{\restriction}\delta \in \mathrm{set}_0(\Lambda_\delta)$ for all $\delta \in W$. If $\delta \in W \cap S^\kappa_{\mathrm{pr}}$ we have $\Lambda'_\delta \supseteq \Lambda_\delta$ hence $\eta{\restriction}\delta \in \mathrm{set}_0(\Lambda'_\delta)$. If $\delta \in W \setminus S^\kappa_{\mathrm{pr}}$ we have $\eta{\restriction}\sigma \in \mathrm{set}_0(\Gamma)_{\delta,\sigma}$ for all $\sigma \in (f(\delta), \delta)$. Hence $(W \cap S^\kappa_{\mathrm{pr}}) \cup \bigcup_{\delta \in W \setminus S^\kappa_{\mathrm{pr}}} S_\delta$ is a cofinal subset of κ witnessing $\eta \in A'$. ∎

3.3. Characterizing additivity and cofinality

LEMMA 3.3.1 (Null set normal form theorem). *Let $\kappa = \sup(S_{\mathrm{inc}} \cap \kappa)$ and let $A \in \mathrm{id}(\mathbb{Q}_\kappa)$. For $\epsilon < \kappa$ let $W_\epsilon \subseteq \kappa = \sup(W_\epsilon)$ and otherwise arbitrary (e.g. disjoint). Then there exist S, $\vec{\Lambda} = \langle \Lambda_\delta : \delta \in S\rangle$, $\vec{p}$, $\vec{\mathcal{J}} = \langle \mathcal{J}_\epsilon : \epsilon < \kappa\rangle$ such that:*

(1) *$S \subseteq \kappa$ is nowhere stationary.*
(2) *$S \subseteq S^\kappa_{\mathrm{pr}}$.*
(3) *$\vec{p} = \{p_\rho : \rho \in 2^{<\kappa}\}$ where $p_\rho \in \mathbb{Q}_\kappa$ is witnessed by $(\rho, S, \vec{\Lambda})$.*
(4) *$\mathcal{J}_\epsilon \subseteq \{p_\rho : \rho \in 2^{<\kappa} \wedge \lg(\rho) \in W_\epsilon\}$ is predense in $\mathbb{Q}_\kappa$ (or even a maximal antichain).*
(5) *$A \subseteq \mathrm{set}_0(\vec{\mathcal{J}})$.*

DISCUSSION 3.3.2. So the idea is as follows: A general null set A is represented by κ-many antichains each consisting of κ-many conditions that are all witnessed by different nowhere stationary sets S and sequences $\vec{\Lambda}$. But using a diagonalization argument we find a representation of the null set using only a single S and $\vec{\Lambda}$.

Lemma 3.3.1 first appears in [She17, 3.16] but we repeat a sketch of the proof here for the convenience of the reader.

Proof of Lemma 3.3.1. Let $A \in \mathrm{id}(\mathbb{Q}_\kappa)$ be witnessed by $\langle \mathcal{I}_\epsilon : \epsilon < \kappa\rangle$ maximal antichains of $\mathbb{Q}_\kappa$. Let $\mathcal{I}_\epsilon = \{p_{\epsilon,i} : i < \kappa\}$ and let $p_{\epsilon,i}$ be witnessed by $(\tau_{\epsilon,i}, S_{\epsilon,i}, \vec{\Lambda}_{\epsilon,i})$. By 1.3.2 we may assume without loss of generality $S_{\epsilon,i} \subseteq S^\kappa_{\mathrm{pr}}$.

Let

$$S = \{\delta \in \kappa : (\exists \epsilon, i < \delta)\ \delta \in S_{\epsilon,i}\};$$

it is easy to see that S is nowhere stationary. For $\delta \in S$ let

$$\Lambda_\delta = \bigcup\{\Lambda_{\epsilon,i,\delta} : \epsilon < \delta,\ i < \delta,\ \delta \in S_{\epsilon,i}\};$$

it is easy to see that $|\Lambda_\delta| \le \delta$. For $\rho \in 2^{<\kappa}$ let p_ρ be the condition witnessed by $(\rho, S, \vec{\Lambda})$. Finally, let

$$\mathcal{J}_\epsilon = \{p_\rho : (\exists i < \kappa)\ i, \epsilon < \lg(\rho) \in W_\epsilon \wedge \tau_{\epsilon,i} \trianglelefteq \rho\}.$$

Now check. ∎

COROLLARY 3.3.3 (Baire's theorem for $\mathrm{id}(\mathbb{Q}_\kappa)$). *The ideal* $\mathrm{id}(\mathbb{Q}_\kappa)$ *is not trivial.*

Proof. If $\kappa > \sup(S_{\mathrm{inc}} \cap \kappa)$ then $\mathrm{id}(\mathbb{Q}_\kappa) = \mathrm{id}(\mathrm{Cohen}_\kappa)$ so the corollary follows from Baire's theorem for the meager ideal on 2^κ.

If $\kappa = \sup(S_{\mathrm{inc}} \cap \kappa)$ let S, $\vec{p}$, $\langle \mathcal{J}_\epsilon : \epsilon < \kappa\rangle$ be as in 3.3.1. Let $E \subseteq \kappa$ be a club disjoint from S. We construct a sequence $\langle \rho_\epsilon : \epsilon < \kappa\rangle$ of $\rho_\epsilon \in 2^{<\kappa}$ such that:

- $p_{\rho_\epsilon} \in \mathcal{J}_\epsilon$.
- $\zeta < \epsilon \Rightarrow \rho_\zeta \trianglelefteq \rho_\epsilon$.
- (As a consequence:) $\zeta < \epsilon \Rightarrow p_{\rho_\epsilon} \leq p_{\rho_\zeta}$, and in particular $\rho_\epsilon \in p_{\rho_\zeta}$.

We work inductively: If $\epsilon = \zeta + 1$ find $\rho_\epsilon \in \mathcal{J}_\epsilon$ such that:

$$p_{\rho_\epsilon} \not\perp p_{\rho_\zeta}, \qquad (\lg(\rho_\epsilon), \lg(\rho_\zeta)) \cap E \neq \emptyset.$$

If ϵ is a limit, then let $\rho'_\epsilon = \bigcup_{\zeta<\epsilon} \rho_\zeta$ and find $\rho_\epsilon \trianglerighteq \rho'_\epsilon$ as above. (Letting $\delta := \lg(\rho'_\epsilon)$ we have $\delta \in E$, so no branches die out in level δ, thus $\rho'_\epsilon \in p_{\rho_\zeta}$ for all $\zeta < \epsilon$.)

Finally let $\eta = \bigcup_{\epsilon<\kappa} \rho_\epsilon$; then clearly $\eta \in \mathrm{set}_1(\mathcal{J})$, i.e. $\mathrm{set}_0(\mathcal{J}) \neq 2^\kappa$. ∎

LEMMA 3.3.4. *Let κ be Mahlo (hence S^κ_{pr} is stationary by* 1.3.3(4)*). Then there exist maps*

(1) $\phi^+ : \mathrm{id}(\mathbb{Q}_\kappa) \to \mathbf{nst}^{\mathrm{pr}}_\kappa$,
(2) $\phi^- : \mathbf{nst}^{\mathrm{pr}}_\kappa \to \mathrm{id}^-(\mathbb{Q}_\kappa)$

such that for all $S \in \mathbf{nst}^{\mathrm{pr}}_\kappa$, $A \in \mathrm{id}(\mathbb{Q}_\kappa)$,

$$\phi^-(S) \subseteq A \implies S \subseteq^* \phi^+(A).$$

DISCUSSION 3.3.5. Lemma 3.3.4 first appears implicitly in [She17] but proving it in terms of the $\mathrm{id}^-(\mathbb{Q}_\kappa)$ ideal and strengthened Galois–Tukey connections may be more transparent.

Proof of Lemma 3.3.4. For $\lambda \in S^{\mathrm{pr}}_\kappa$ let Λ^*_λ witness $\lambda \in S^{\mathrm{pr}}_\kappa$. For $S \in \mathbf{nst}^{\mathrm{pr}}_\kappa$ define

$$\phi^-(S) = \{\eta \in 2^\kappa : (\exists^\infty \delta \in S)\ \eta{\restriction}\delta \in \mathrm{set}_0(\Lambda^*_\delta)\}$$

and for $A \in \mathrm{id}(\mathbb{Q}_\kappa)$ define $\phi^+(A) = S$ where S is as in 3.3.1.

Now let $A \in \mathrm{id}(\mathbb{Q}_\kappa)$ and $S^* \in \mathbf{nst}^{\mathrm{pr}}_\kappa$ be such that $S^* \not\subseteq^* \phi^+(A)$; we are going to show $\phi^-(S^*) \not\subseteq A$. So let $(S, \vec{\Lambda}, \vec{p}, \vec{\mathcal{J}})$ be as in 3.3.1 for A (hence $\phi^+(A) = S$). By our assumption $S' = S^* \backslash S$ is unbounded. We can easily find an unbounded set $S'' \subseteq S'$ with its closure E disjoint from S. (Simply take a club C disjoint from S and working inductively for $\epsilon \in C$ take $\lambda \in S'$ such that $\epsilon \leq \lambda$.)

We are going to inductively construct a $\triangleleft$-increasing sequence $\langle \eta_i : i < \kappa\rangle$ of $\eta_i \in 2^{<\kappa}$ and an increasing sequence $\langle \delta_i : i < \kappa\rangle$ of $\delta_i \in \kappa$ such that for $i < \kappa$:

(a) $|\eta_i| = \delta_i$.
(b) $\delta_i \in E$ (thus in particular $\delta_i \notin S$).
(c) $i = j+1 \Rightarrow \delta_i \in S''$ (thus in particular $\delta_i \in S^*$).
(d) $[p_{\eta_i}] \subseteq \bigcap_{j<i} \mathrm{set}_1(\mathcal{J}_j)$.
(e) $i = j+1 \Rightarrow \eta_i \in \mathrm{set}_0(\Lambda^*_{\delta_i})$.

Now let $\eta = \bigcup_{i<\kappa} \eta_i$ and note that $\eta \in \phi^-(S^*)$ by clause (e), and $\eta \notin A$ by clause (d).

It remains to prove that we can indeed carry out this induction. The case $i = 0$ is trivial. For i limit let $\eta_i = \bigcup_{j<i} \eta_j$ (remember (b)).

For $i = j+1$ consider p_{η_j}. Because $\mathcal{J}_j$ is predense, we find $\rho \in 2^{<\kappa}$ such that $p_\rho \in \mathcal{J}_j$ and p_{η_j}, p_ρ are compatible with lower bound p_ν, $\nu = \rho \cup \eta_j$. Choose $\delta_i \in S''$ such that $\delta_i > |\nu|$. Now we see that $[p_\nu \cap 2^{<\delta_i}] \not\subseteq \mathrm{set}_1(\Lambda^*_{\delta_i})$ so choose $\eta_i \in [p_\nu \cap 2^{<\delta_i}] \setminus \mathrm{set}_1(\Lambda^*_{\delta_i})$ and note that because $\delta_i \notin S$ we have $\eta_i \in p_{\eta_j}$ hence $p_{\eta_i} \subseteq p_{\eta_j}$. ∎

Theorem 3.3.6. *Let κ be Mahlo. Then:*

(1) $\mathrm{add}(\mathrm{id}^-(\mathbb{Q}_\kappa), \mathrm{id}(\mathbb{Q}_\kappa)) \leq \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$.
(2) $\mathrm{cf}(\mathrm{id}^-(\mathbb{Q}_\kappa), \mathrm{id}(\mathbb{Q}_\kappa)) \geq \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$.

Proof. By 3.3.4 and 1.4.8. ∎

Corollary 3.3.7. *Let κ be Mahlo. Then:*

(1) $\mathrm{add}(\mathrm{id}(\mathbb{Q}_\kappa)) \leq \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$.
(2) $\mathrm{add}(\mathrm{id}^-(\mathbb{Q}_\kappa)) \leq \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$.
(3) $\mathrm{cf}(\mathrm{id}(\mathbb{Q}_\kappa)) \geq \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$.
(4) $\mathrm{cf}(\mathrm{id}^-(\mathbb{Q}_\kappa)) \geq \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$. ∎

Definition 3.3.8. We define

$$\mathbb{Q}^*_{\kappa,S} = \{p \in \mathbb{Q}_\kappa : S_p \subseteq S\}.$$

Remember Definition 1.1.15 and note that $\mathbb{Q}_{\kappa,S} \subseteq \mathbb{Q}^*_{\kappa,S}$ but in general equality does not hold.

Theorem 3.3.9. *Let κ be Mahlo. Then*

$$\mathrm{add}(\mathrm{id}(\mathbb{Q}_\kappa)) = \min\{\mu_1, \mu_2\}$$

where

$$\mu_1 = \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa), \qquad \mu_2 = \min\{\mathrm{add}(\mathrm{id}(\mathbb{Q}^*_{\kappa,S}), \mathrm{id}(\mathbb{Q}_\kappa)) : S \in \mathbf{nst}^{\mathrm{pr}}_\kappa\}.$$

Proof. Let $\mu = \mathrm{add}(\mathbb{Q}_\kappa)$. Then $\mu \leq \mu_1$ follows from Theorem 3.3.6 (remember 1.4.4) and $\mu \leq \mu_2$ is trivial. So it remains to show that $\mu \geq \min\{\mu_1, \mu_2\}$.

Let $A_i \in \mathrm{id}(\mathbb{Q}_\kappa)$ for $i < i^* < \min\{\mu_1, \mu_2\}$ and let $(S_i, \vec{\Lambda}_i, \vec{\mathcal{J}}_i, \vec{p}_i)$ be as in 3.3.1. By 1.3.2 we may assume that $S_i \in \mathbf{nst}^{\mathrm{pr}}_\kappa$, and because $i^* < \mu_1$, there is $S \in \mathbf{nst}_\kappa^{\mathrm{pr}}$ such that $i < i^* \Rightarrow S_i \subseteq^* S$. Thus clearly $A_i \in \mathrm{id}(\mathbb{Q}^*_{\kappa,S})$, and because $i^* < \mu_2$ we have $\bigcup_{i<i^*} A_i \in \mathrm{id}(\mathbb{Q}_\kappa)$. ∎

THEOREM 3.3.10. *Let κ be Mahlo. Then*

$$\mathrm{cf}(\mathrm{id}(\mathbb{Q}_\kappa)) = \mu_1 + \mu_2$$

where

$$\mu_1 = \mathrm{cf}(\mathbf{nst}_\kappa^{\mathrm{pr}}), \qquad \mu_2 = \sup\{\mathrm{cf}(\mathrm{id}(\mathbb{Q}^*_{\kappa,S}), \mathrm{id}(\mathbb{Q}_\kappa)) : S \in \mathbf{nst}_\kappa^{\mathrm{pr}}\}.$$

Proof. Let $\mu = \mathrm{cf}(\mathbb{Q}_\kappa)$. Then $\mu \geq \mu_1$ follows from Theorem 3.3.6 (remember 1.4.4) and $\mu \geq \mu_2$ is trivial. So it remains to show that $\mu \leq \mu_1 + \mu_2$.

Let $\langle S_\zeta : \zeta < \mu_1\rangle$ witness μ_1 and for $\zeta < \mu_1$ let $\langle A_{\zeta,\epsilon} : \epsilon < \mu_2\rangle$ witness $\mathrm{cf}(\mathrm{id}(\mathbb{Q}^*_{\kappa,S_\zeta}), \mathrm{id}(\mathbb{Q}_\kappa)) \leq \mu_2$. We claim that

$$\{A_{\zeta,\epsilon} : \zeta < \mu_1,\ \epsilon < \mu_2\}$$

is a cofinal family of $\mathrm{id}(\mathbb{Q}_\kappa)$. Thus let $A \in \mathrm{id}(\mathbb{Q}_\kappa)$ and let $(S, \vec{\Lambda}, \vec{\mathcal{J}}, \vec{p})$ be as in 3.3.1. By 1.3.2 we may assume that $S \in \mathbf{nst}_\kappa^{\mathrm{pr}}$ and find $\zeta < \mu_1$, $\alpha^* < \kappa$ such that $S\backslash\alpha^* \subseteq S_\zeta\backslash\alpha^*$. For $\delta \in S_\zeta$ define

$$\Lambda'_\delta = \begin{cases} \Lambda_\delta & \text{if } \delta \in S\backslash\alpha^*, \\ \emptyset & \text{if } \delta \notin S \text{ or } \delta < \alpha^*. \end{cases}$$

Now for each $i < \kappa$ correct $\mathcal{J}_i$ to $\mathcal{J}'_i$ that uses only trunks of length greater than α^*. Thus we have found $A' \supseteq A$ and $A' \in \mathrm{id}(\mathbb{Q}^*_{\kappa,S_\zeta})$. Hence there exists $\epsilon < \mu_2$ such that $A' \subseteq A_{\zeta,\epsilon}$. ∎

DEFINITION 3.3.11. Let $S \subseteq \kappa$ and define

$$\Pi_S = \Big(\prod_{\delta\in S}(\mathrm{id}(\mathbb{Q}_\delta)/\mathrm{id}^-(\mathbb{Q}_\delta)), \leq^*\Big)$$

where the intended meaning of $\leq^*$ is pointwise set-inclusion for almost all places of the product. Writing $[\Lambda_\delta]$ for the id^--equivalence class of Λ_δ, for $\vec{\Lambda} = \langle[\Lambda_\delta] : \delta \in S\rangle$, $\vec{\Gamma} = \langle[\Gamma_\delta] : \delta \in S\rangle \in \Pi_S$ we define

$$\vec{\Lambda} \leq^* \vec{\Gamma} \iff (\forall^\infty \delta \in S)\ \Lambda_\delta\backslash\Gamma_\delta \in \mathrm{id}^-(\mathbb{Q}_\delta).$$

LEMMA 3.3.12. *Let $S \in \mathbf{nst}_\kappa$, $\sup(S) = \kappa$. Then there exist maps*

$$\phi^+ : \mathrm{id}(\mathbb{Q}_\kappa) \to \Pi_S, \qquad \phi^- : \Pi_S \to \mathrm{id}^-(\mathbb{Q}_\kappa)$$

such that for all $\vec{\Lambda} \in \Pi_S$ and $A \in \mathrm{id}(\mathbb{Q}_\kappa)$,

$$\phi^-(\vec{\Lambda}) \subseteq A \implies \vec{\Lambda} \leq^* \phi^+(A).$$

Proof. For $\vec{\Lambda} = \langle[\Lambda_\delta] : \delta \in S\rangle \in \Pi_S$ define $\phi^-(\vec{\Lambda}) = \mathrm{set}_0^-(\langle\Lambda_\delta : \delta \in S\rangle)$. Given $A \in \mathrm{id}(\mathbb{Q}_\kappa)$, find any $\vec{\Lambda}$ as in 3.3.1 and define $\phi^+(A) = \vec{\Lambda}{\restriction}S$.

Now assume $A \in \mathrm{id}(\mathbb{Q}_\kappa)$ and $\vec{\Lambda}^* \in \Pi_S$ are such that $\vec{\Lambda}^* \not\leq^* \phi^+(A)$; we are going to show $\phi^-(\vec{\Lambda}^*) \not\subseteq A$. Let $\vec{\Lambda}^* = \langle[\Lambda^*_\delta] : \delta \in S\rangle$; then for A there are (as in 3.3.1) S_A, $\vec{\mathcal{J}}$, $\vec{\Lambda} = \langle \Lambda_\delta : \delta \in S_A \rangle$, $\langle \Lambda_\delta : \delta \in S \rangle = \phi^+(A)$ (without loss of generality (because $S_A \supseteq S$)) such that

$$(\exists^\infty \delta \in S)\ \neg(\mathrm{set}_0(\Lambda_\delta) \supseteq \mathrm{set}_0(\Lambda^*_\delta)) \bmod \mathrm{id}^-(\mathbb{Q}_\delta).$$

Let $B_\delta = \mathrm{set}_1(\Lambda_\delta) \cap \mathrm{set}_0(\Lambda^*_\delta)$. Hence by the above we have

$$(\exists^\infty \delta \in S)\ B_\delta \notin \mathrm{id}^-(\mathbb{Q}_\delta).$$

We are going to show

(∗) there exists $\eta \in (2^\kappa \backslash A) \cap \mathrm{set}^-_0(\vec{\Lambda}^*)$ witnessing $\mathrm{set}^-_0(\vec{\Lambda}^*) \not\subseteq A$.

Without loss of generality we assume closure under rational translates, i.e. $\mathrm{set}_0(\Lambda_\delta)^{[\beta]} = \mathrm{set}_0(\Lambda_\delta)$ for $\beta < \delta \in S$, and clearly we may assume the same for $\vec{\Lambda}^*$.

Claim. *Let $p_\rho \in \mathbb{Q}_\kappa$ be the condition witnessed by $(\rho, S_A, \vec{\Lambda})$. Then for all $\rho \in 2^{<\kappa}$, there exists $\delta \in S \backslash (\lg(\rho)+1)$ such that*

$$(p_\rho \cap 2^\delta) \cap \mathrm{set}_0(\Lambda^*_\delta) \neq \emptyset.$$

To see this choose $\delta > \lg(\rho)$ such that $B_\delta \notin \mathrm{id}^-(\mathbb{Q}_\delta)$ and let

$$C_\delta = \{\eta \in 2^\delta : (\forall^\infty \sigma \in S_A \cap \delta)\ \eta{\restriction}\sigma \in \mathrm{set}_1(\Lambda_\sigma)\}.$$

The idea is that C_δ is a set of candidates for elements of $p_\rho \cap 2^\delta$. Towards a contradiction assume that

$$C_\delta \subseteq \mathrm{set}_0(\Lambda_\delta) \cup \mathrm{set}_1(\Lambda^*_\delta) = \neg B_\delta$$

i.e. every candidate either dies out at level δ by definition of p_ρ or is not in $\mathrm{set}_0(\Lambda^*_\delta)$. But clearly $C_\delta = \mathrm{set}^-_1(\vec{\Lambda}{\restriction}\delta)$ i.e. is a co-$\mathrm{id}^-(\mathbb{Q}_\delta)$-set, contradicting $B_\delta \notin \mathrm{id}^-(\mathbb{Q}_\delta)$. Hence there exists $\eta \in C_\delta \cap B_\delta$. Now use the closure under rational translates and choose $\beta \in (\lg(\rho), \delta)$ large enough such that for $\nu \in 2^\beta \cap p_\rho$ we have

$$\nu{\restriction}\beta^\frown \eta{\restriction}(\beta,\delta) \in (p_\rho \cap 2^\delta) \cap \mathrm{set}_0(\Lambda^*_\delta).$$

This concludes the proof of the Claim.

Now fix a club E disjoint from S and work as in 3.3.4 constructing a $\triangleleft$-increasing sequence $\langle \eta_i : i < \kappa \rangle$ of $\eta_i \in 2^{<\kappa}$ and an increasing sequence $\langle \delta_i : i < \kappa \rangle$ of $\delta_i \in \kappa$ such that for $i < \kappa$:

(a) $|\eta_i| = \delta_i$.
(b) $i = j+1 \Rightarrow \delta_i \in S$.
(c) i limes $\Rightarrow \delta_i \in E$.
(d) $[p_{\eta_i}] \subseteq \bigcap_{j<i} \mathrm{set}_1(\mathcal{J}_j)$.
(e) $i = j+1 \Rightarrow \eta_i \in \mathrm{set}_0(\Lambda^*_{\delta_i})$.

Finally, let $\eta = \bigcup_{i<\kappa} \eta_i$ and note that $\eta \in \mathrm{set}^-_0(\vec{\Lambda}^*) = \phi^-(\vec{\Lambda}^*)$ by clause (e), and $\eta \notin A$ by clause (d). So we have shown (∗).

It remains to check that we can carry out the induction. For $i = j+1$ we find $p_\rho \in \mathcal{J}_i$ such that p_ρ and p_{η_j} are compatible. Now we let $\nu = \rho \cup \eta_j$ and find $\delta_i > |\nu|$ such that $\delta_i \in S$ with $B_{\delta_i} \notin \mathrm{id}^-(\mathbb{Q}_{\delta_i})$ and $(\delta_j, \delta_i) \cap E \neq \emptyset$. Using the claim we find $\eta_i \in p_\nu \cap 2^{\delta_i} \cap \mathrm{set}_0(\Lambda^*_{\delta_i})$. ∎

THEOREM 3.3.13. *Let $S \in \mathbf{nst}_\kappa$ with $\sup(S) = \kappa$. Then:*

(1) $\mathrm{add}(\mathrm{id}^-(\mathbb{Q}_\kappa), \mathrm{id}(\mathbb{Q}_\kappa)) \leq \mathrm{add}(\Pi_S)$.
(2) $\mathrm{cf}(\mathrm{id}^-(\mathbb{Q}_\kappa), \mathrm{id}(\mathbb{Q}_\kappa)) \geq \mathrm{cf}(\Pi_S)$.

Proof. By 3.3.12 and 1.4.8. ∎

We will use the following definition and the revised GCH theorem from [She00].

DEFINITION 3.3.14. Let μ, θ be cardinals with $\theta < \mu$ and θ regular. We define

$$\mu^{[\theta]} = \min\{|U| : U \subseteq \mathcal{P}(\mu) \wedge \varphi(U)\}$$

where $\varphi(U)$ iff:

(1) All $u \in U$ have size θ.
(2) Every $v \subseteq \mu$ of size θ is contained in the union of fewer than θ members of U.

THEOREM 3.3.15 (The revised GCH theorem). *Let α be an uncountable strong limit cardinal, i.e. $\beta < \alpha \Rightarrow 2^\beta < \alpha$. For example, $\alpha = |\mathbf{V}_{\omega+\omega}| = \beth_\omega$, the first strong limit cardinal. Then for every $\mu \geq \alpha$ for some $\epsilon < \alpha$ we have*

$$\theta \in [\epsilon, \alpha] \wedge \theta \text{ is regular} \implies \mu^{[\theta]} = \mu. \ ∎$$

THEOREM 3.3.16. *Let κ be Mahlo. Then:*

(i) $\mathrm{cf}(\mathrm{id}^-(\mathbb{Q}_\kappa)) = \mu_1 + \mu_2$,
(ii) $\mathrm{cf}(\mathrm{id}(\mathbb{Q}_\kappa)) = \mu_1 + \mu_2 + \mu_3$,

where

$$\mu_1 = \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa), \quad \mu_2 = \sup(\mathrm{cf}(\Pi_S) : S \in \mathbf{nst}^{\mathrm{pr}}_\kappa), \quad \mu_3 = \mathrm{cf}(\mathrm{id}(\mathbb{Q}_\kappa)/\mathrm{id}^-(\mathbb{Q}_\kappa)).$$

Proof. The inequality $\geq$: (i) Let $\mu^* = \mathrm{cf}(\mathrm{id}^-(\mathbb{Q}_\kappa), \mathrm{id}(\mathbb{Q}_\kappa))$. Then remembering 1.4.4 we have $\mu^* \geq \mu_1$ by 3.3.6, and $\mu^* \geq \mu_2$ by 3.3.13.

(ii) Use the same theorems. Finally, $\mathrm{cf}(\mathrm{id}(\mathbb{Q}_\kappa)) \geq \mu_3$ is trivial.

The inequality $\leq$: We only show (i), which using 1.4.5 easily implies (ii).

(1) Let $\langle S_\zeta : \zeta < \mu_1 \rangle$ witness $\mu_1 = \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$, that is:

  (a) $\zeta < \mu_1 \Rightarrow S_\zeta \in \mathbf{nst}^{\mathrm{pr}}_\kappa$.
  (b) $(\forall S \in \mathbf{nst}^{\mathrm{pr}}_\kappa)(\exists \zeta < \mu_1)\ S \subseteq^* S_\zeta$.

(2) For every $\zeta < \mu_1$ let $\langle \vec{A}_{\zeta,i} : i < \mu_2 \rangle$ witness $\mu_{2,S_\zeta} \leq \mu_2$, that is:

(a) $\vec{A}_{\zeta,i} = \langle A_{\zeta,i,\delta} : \delta \in S_\zeta \rangle$.
(b) $A_{\zeta,i,\delta} \in \mathrm{id}(\mathbb{Q}_\delta)$, representing the equivalence class $[A_{\zeta,i,\delta}] \in \mathrm{id}(\mathbb{Q}_\delta)/\mathrm{id}^-(\mathbb{Q}_\delta)$.
(c) For all $\vec{A} \in \prod_{\delta \in S_\zeta} \mathrm{id}(\mathbb{Q}_\delta)$, there is some $i < \mu_2$ such that for every δ large enough we have $A_\delta \subseteq A_{\zeta,i,\delta} \mod \mathrm{id}^-(\mathbb{Q}_\delta)$.
(d) Changing the representative of $[A_{\zeta,i,\delta}]$ (if necessary), we may assume

$$\{\eta \in 2^\delta : (\exists^\infty \sigma \in S_\zeta \cap \delta)\ \eta{\restriction}\sigma \in A_{\zeta,i,\sigma}\} \subseteq A_{\zeta,i,\delta}.$$

(3) Let

$$\theta = \min\{\theta : \theta = \mathrm{cf}(\theta) < |\mathbf{V}_{\omega+\omega}| \wedge (\mu_1 + \mu_2)^{[\theta]} = \mu_1 + \mu_2\};$$

see 3.3.14 and 3.3.15 for definition of notation and existence of θ. Then for $u \in [\mu_1 \times \mu_2]^\theta$:

(a) $S_u = \bigcup\{S_\zeta : \{\zeta\} \times \mu_2 \cap u \neq \emptyset\}$.
(b) For $\delta \in S_u$ we inductively define $A_{u,\delta} = \bigcup\{A_{\zeta,i,\delta} : (\zeta, i) \in u\} \cup \{\eta \in 2^\delta : (\exists^\infty \sigma \in S_u \cap \delta)\ \eta{\restriction}\sigma \in A_{u,\sigma}\}$.
(c) $A_u = \{\eta \in 2^\kappa : (\exists^\infty \delta \in S)\ \eta{\restriction}\delta \in A_{u,\delta}\}$.

(4) Note that in (3) (because for any $\delta \in S_{\mathrm{inc}}$ we have $\delta > |\mathbf{V}_{\omega+\omega}| > \theta$):

(a) $S_u \in \mathbf{nst}_\kappa^{\mathrm{pr}}$.
(b) $A_{u,\delta} \in \mathrm{id}(\mathbb{Q}_\delta)$.
(c) $A_u \in \mathrm{id}^-(\mathbb{Q}_\kappa)$.

(5) Remembering 3.3.14, 3.3.15 we find $\vec{u}$ such that:

(a) $\vec{u} = \langle u_\alpha : \alpha < \mu_1 + \mu_2 \rangle$.
(b) $u_\alpha \in [\mu_1 \times \mu_2]^\theta$.
(c) If $u \in [\mu_1 \times \mu_2]^\theta$ then it is the union of fewer than θ members of $\{u_\alpha : \alpha < \mu_1 + \mu_2\}$.

We claim that $\langle A_{u_\alpha} : \alpha < \mu_1 + \mu_2 \rangle$ is a cofinal family in $\mathrm{id}^-(\mathbb{Q}_\kappa)$. So let $A \in \mathrm{id}^-(\mathbb{Q}_\kappa)$, and for $\epsilon < \theta$ inductively define $A_\epsilon, \zeta_\epsilon, i_\epsilon$, etc. such that:

(A) $A \subseteq A_0$.
(B) $\epsilon' < \epsilon \Rightarrow A_{\epsilon'} \subseteq A_\epsilon$.
(C) $A_\epsilon = \mathrm{set}_0^-(\vec{\Lambda}_\epsilon^1) \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ where:

(1) $\vec{\Lambda}_\epsilon^1 = \langle \Lambda_{\epsilon,\delta}^1 : \delta \in S_\epsilon^1 \rangle$.
(2) $S_\epsilon^1 \in \mathbf{nst}_\kappa^{\mathrm{pr}}$ (remember 3.2.7).
(3) $\Lambda_{\epsilon,\delta}^1$ is a set of at most δ-many maximal antichains of $\mathbb{Q}_\delta$.

(D) $\zeta_\epsilon < \mu_1$ is minimal such that $S_\epsilon^1 \subseteq^* S_{\zeta_\epsilon}$.
(E) $\vec{\Lambda}_\epsilon^2 = \langle \Lambda_{\epsilon,\delta}^2 : \delta \in S_{\zeta_\epsilon} \rangle$ is such that $\delta \in S_\epsilon^1 \cap S_{\zeta_\epsilon} \Rightarrow \Lambda_{\epsilon,\delta}^1 = \Lambda_{\epsilon,\delta}^2$. (For instance, choose $\Lambda_{\epsilon,\delta}^2 = \emptyset$ for $\delta \in S_{\zeta_\epsilon} \setminus S_\epsilon^1$.)

(F) $i_\epsilon < \mu_2$ is minimal such that for some $S^3_\epsilon \subseteq S_{\zeta_\epsilon}$, $S^3_\epsilon =^* S_{\zeta_\epsilon}$,

$$(\forall \delta \in S^3_\epsilon)\ (\mathrm{set}_0(\Lambda^2_{\epsilon,\delta}) \subseteq A_{\zeta_\epsilon,i_\epsilon,\delta}) \bmod \mathrm{id}^-(\mathbb{Q}_\delta).$$

(G) $\vec{\Lambda}^4_\epsilon = \langle \Lambda^4_{\epsilon,\delta} : \delta \in S^4_\epsilon \rangle$ is such that:

(1) $S^3_\epsilon \subseteq S^4_\epsilon \in \mathbf{nst}^{\mathrm{pr}}_\kappa$.
(2) If $\delta \in S^3_\epsilon$ then $A_{\zeta_\epsilon,i_\epsilon,\delta} \subseteq \mathrm{set}_0(\Lambda^4_{\epsilon,\delta})$.
(3) If $\delta \in S^3_\epsilon$ then $\mathrm{set}_0(\Lambda^2_{\epsilon,\delta}) \subseteq \mathrm{set}_0(\Lambda^4_{\epsilon,\delta}) \cup \mathrm{set}^-_0(\vec{\Lambda}^4_\epsilon{\restriction}\delta)$. This point is the only non-explicit step; see below for why we can do this.

(H) If $\epsilon = \epsilon' + 1$ then $S^1_\epsilon = S^4_{\epsilon'}$, $\vec{\Lambda}^1_\epsilon = \vec{\Lambda}^4_{\epsilon'}$.
(I) If ϵ is a limit then $S^1_\epsilon = \bigcup_{\epsilon'<\epsilon} S^1_{\epsilon'}$, $\Lambda^1_{\epsilon,\delta} = \bigcup_{\epsilon'<\epsilon} \Lambda^1_{\epsilon',\delta}$.

Why is carrying out the induction enough?

Note $\{(\zeta_\epsilon, i_\epsilon) : \epsilon < \theta\} \in [\mu_1 \times \mu_2]^\theta$ so we use (5)(c) to find $\alpha < \mu_1 + \mu_2$ such that

$$(3.1) \qquad (\exists^\infty \epsilon < \theta)\ (\zeta_\epsilon, i_\epsilon) \in u_\alpha.$$

Remember $\theta < |\mathbf{V}_{\omega+\omega}| < \mathrm{cf}(\kappa)$ and find $\psi^* < \kappa$ such that

$$(\forall \epsilon < \theta)\ S^1_\epsilon \backslash \psi^* \subseteq S_{\zeta_\epsilon} \backslash \psi^* \subseteq S^3_\epsilon \backslash \psi^* \subseteq S^4_\epsilon \backslash \psi^* \subseteq S^1_{\epsilon+1} \backslash \psi^*.$$

We plan to show $A \subseteq A_{u_\alpha}$. So let $\eta \in A_0$; we will show $\eta \in A_{u_\alpha}$.

Let $W \subseteq S^1_0 \backslash \psi^*$ with $\sup(W) = \kappa$ be such that

$$(\forall \delta \in W)\ \eta{\restriction}\delta \in \mathrm{set}_0(\Lambda^1_{0,\delta}).$$

Let $S^1_\theta = \bigcup_{\epsilon<\theta} S^1_\epsilon$; we claim

$$(3.2) \qquad (\forall \delta \in S^1_\epsilon)(\forall^\infty \epsilon < \theta)\ \eta{\restriction}\delta \in A_{\zeta_\epsilon,i_\epsilon,\delta}.$$

We prove this by induction on δ:

- $\delta > \sup(\delta \cap S_{\mathrm{inc}})$. Then $\mathrm{id}^-(\mathbb{Q}_\delta)$ trivial so in (F) we always really (i.e. not just modulo $\mathrm{id}^-(\mathbb{Q}_\delta)$) cover $\mathrm{set}_0(\Lambda^2_{\epsilon,\delta})$.
- $\delta = \sup(\delta \cap S_{\mathrm{inc}})$ and $\delta = \sup(\delta \cap S^1_\theta)$. By induction hypothesis we have
$$(\forall \sigma \in S^1_\theta \cap \delta)(\exists \epsilon_\sigma < \theta)(\forall \epsilon \geq \epsilon_\sigma)\ \eta{\restriction}\sigma \in A_{\zeta_\epsilon,i_\epsilon,\sigma}.$$
Now, δ is inaccessible so in particular regular, hence there exists ϵ' such that
$$(\exists^\infty \sigma \in S^1_\theta \cap \delta)\ \epsilon_\sigma = \epsilon'$$
and for such σ we have
$$\epsilon \geq \epsilon' \Rightarrow \eta{\restriction}\sigma \in A_{\zeta_\epsilon,i_\epsilon,\sigma},$$
which by (2)(d) implies $\eta{\restriction}\delta \in A_{\zeta_\epsilon,i_\epsilon,\delta}$.
- $\delta = \sup(\delta \cap S_{\mathrm{inc}})$ but $\delta > \sup(\delta \cap S^1_\theta)$. In this case always really $A_{\zeta_\epsilon,i_\epsilon,\delta} \supseteq \mathrm{set}_0(\Lambda^2_{\epsilon,\delta})$ because otherwise δ would become a limit in S^4_ϵ by (G)(3), see below.

Note that (3.2) holds in particular for all $\delta \in W \subseteq S^1_\theta$. So combine (3.1) and (3.2) to see

$$(\forall \delta \in W)(\exists^\infty \epsilon < \theta)\ \eta{\restriction}\delta \in A_{\zeta_\epsilon, i_\epsilon, \delta} \wedge (\zeta_\epsilon, i_\epsilon) \in u_\alpha.$$

Thus $\eta \in A_{u_\alpha}$ and we are done.

How can we carry out the induction?

The only non-explicit part is how to get (G). The idea here is that in (F) we make some mistake because we only cover $\mathrm{set}_0(\Lambda^2_{\epsilon,\delta})$ modulo $\mathrm{id}^-(\mathbb{Q}_\delta)$, i.e.

$$\mathrm{set}_0(\Lambda^2_{\epsilon,\delta}) \setminus A_{\zeta_\epsilon, i_\epsilon, \delta} = X_{\epsilon,\delta} \in \mathrm{id}^-(\mathbb{Q}_\delta).$$

Let $X_{\epsilon,\delta} = \mathrm{set}^-_0(\vec{\Gamma}_{\epsilon,\delta})$ where $\vec{\Gamma}_{\epsilon,\delta} = \langle \Gamma_{\epsilon,\delta,\sigma} : \sigma \in S_{\epsilon,\delta} \subseteq \delta \rangle$. So in (G)(3) we want to fix this mistake by choosing some S^4_ϵ containing both $S_{\epsilon,\delta}$ and S^3_ϵ and then choosing $\vec{\Lambda}^4_\epsilon$ with all $\Gamma_{\epsilon,\delta,\sigma}$ added. The problem here of course is that we have to do this for all $\delta \in S^3_\epsilon$ but $|S^3_\epsilon| = \kappa$, so fixing the mistake in such a naive way will in general yield a somewhere stationary set and more than δ-many antichains at level δ. Hence we work as follows: Choose a regressive function f on S^3_ϵ as in 3.2.6, i.e. such that

$$(\forall \delta < \kappa)\ |\{\lambda \in S^3_\epsilon \setminus \delta : f(\lambda) \le \delta\}| < \delta,$$

i.e. f is regressive but in a very "lazy" way. The problem with fixing our mistakes earlier was that we tried to do it all at once so let us instead do it lazily as dictated by f. Thus let

$$S^4_\epsilon = S^3_\epsilon \cup \bigcup_{\delta \in S^3_\epsilon} S_{\epsilon,\delta} \setminus (f(\delta)+1)$$

and for $\delta \in S^4_\epsilon$ let

$$\Lambda^4_{\epsilon,\delta} = \Lambda^3_{\epsilon,\delta} \cup \{\Gamma_{\epsilon,\delta^*,\delta} : \delta^* > \delta > f(\delta^*)\}.$$

Now check that S^4_ϵ is nowhere stationary:

- $\delta > \sup(S^3_\epsilon \cap \delta)$. Then $S^3_\epsilon \cap \delta$ is disjoint from $S_{\epsilon,\delta'} \setminus (f(\delta')+1)$ for every $\delta' \in S^3_\epsilon$ with $f(\delta') > \delta$, so by 3.2.6(a) the set $S^4_\epsilon \cap \delta$ is the union of fewer than δ-many non-stationary sets.
- $\delta = \sup(S^3_\epsilon \cap \delta)$. Let

  $$S^{4*}_{\epsilon,\delta} = \bigcup_{\delta' \in S^3_\epsilon \cap \delta} S_{\epsilon,\delta'} \setminus (f(\delta')+1), \qquad S^{4**}_{\epsilon,\delta} = \bigcup_{\delta' \in S^3_\epsilon \cap (\kappa \setminus \delta)} S_{\epsilon,\delta'} \setminus (f(\delta')+1) \cap \delta.$$

  Then clearly

  $$S^4_\epsilon \cap \delta = (S^3_\epsilon \cap \delta) \cup S^{4*}_{\epsilon,\delta} \cup S^{4**}_{\epsilon,\delta}.$$

  Let E_δ be as in 3.2.6; it is easy to check using 3.2.6(b) that $S^{4*}_{\epsilon,\delta}$ is disjoint from E_δ, i.e. non-stationary. Finally, $S^{4**}_{\epsilon,\delta}$ is non-stationary by the argument from the previous point.

Similarly check $|\Lambda^4_{\epsilon,\delta}| \le \delta$. ■

THEOREM 3.3.17. *Let κ be Mahlo. Then:*

(a) $\mathrm{add}(\mathrm{id}^-(\mathbb{Q}_\kappa)) = \min\{\mu_1, \mu_2\}$,
(b) $\mathrm{add}(\mathrm{id}(\mathbb{Q}_\kappa)) = \min\{\mu_1, \mu_2, \mu_3\}$,

where

$$\mu_1 = \mathrm{add}(\mathbf{nst}_\kappa^{\mathrm{pr}}),$$
$$\mu_2 = \min(\mathrm{add}(\mathit{\Pi}_S) : S \in \mathbf{nst}_\kappa^{\mathrm{pr}}),$$
$$\mu_3 = \mathrm{add}(\mathrm{id}(\mathbb{Q}_\kappa)/\mathrm{id}^-(\mathbb{Q}_\kappa)).$$

Proof. The inequality $\le$: Same as "$\ge$" in 3.3.16.

The inequality $\ge$: We only show (a) which using 1.4.5 easily implies (b).

Let $\mu < \mu_1 + \mu_2$. We are going to show $\mu < \mathrm{add}(\mathrm{id}^-(\mathbb{Q}_\kappa))$. So let $\langle A_\zeta : \zeta < \mu\rangle$ be a family of $A_\zeta \in \mathrm{id}^-(\mathbb{Q}_\kappa)$; we are going to find $A \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ such that $\bigcup_{\zeta<\mu} A_\zeta \subseteq A$. Let A_ζ be represented by $\langle A^0_{\zeta,\delta} : \delta \in S^0_\zeta\rangle$; by 3.2.7 we may assume $S^0_\zeta \in \mathbf{nst}_\kappa^{\mathrm{pr}}$. Now work inductively for $i < \omega$:

(1) Let $S^i \in \mathbf{nst}_\kappa^{\mathrm{pr}}$ be such that $\zeta < \mu \Rightarrow S^i_\zeta \subseteq^* S^i$. (Remember $\mu < \mu_1$.)
(2) Let $\vec{A}^i \in \mathit{\Pi}_{S^i}$ be such that
$$(\forall \zeta < \mu)(\forall^\infty \delta \in S^i)\ (A^i_{\zeta,\delta} \subseteq A^i_\delta) \bmod \mathrm{id}^-(\mathbb{Q}_\delta).$$
(Remember $\mu < \mu_2$.)
(3) For each $\zeta < \mu$ work as in 3.3.16 using a regressive function to fix the error
$$X^i_{\zeta,\delta} = (A^i_{\zeta,\delta} \setminus A^i_\delta) \in \mathrm{id}^-(\mathbb{Q}_\delta)$$
for $\delta \in S^i_\zeta$. That is, we find S^{i+1}_ζ, $\langle A^{i+1}_{\zeta,\delta} : \delta \in S^{i+1}_\zeta\rangle$ such that:
   (a) $S^i \subseteq S^{i+1}_\zeta \in \mathbf{nst}_\kappa^{\mathrm{pr}}$.
   (b) $\delta \in S^{i+1}_\zeta \Rightarrow A^{i+1}_{\zeta,\delta} \in \mathrm{id}(\mathbb{Q}_\delta)$.
   (c) $\delta \in S^i_\zeta \Rightarrow A^i_{\zeta,\delta} \subseteq A^i_\delta \cup \mathrm{set}_0^-(\langle A^{i+1}_{\zeta,\epsilon} : \epsilon \in S^{i+1}_\zeta \cap \delta\rangle)$.

Let
$$S^\omega = \bigcup_{i<\omega} S^i.$$
For $\delta \in S^\omega$, $\zeta < \mu$ let
$$A^\omega_{\zeta,\delta} = \bigcup_{i<\omega} A^i_{\zeta,\delta}, \qquad A^\omega_\delta = \bigcup_{i<\omega} A^i_\delta.$$
Finally, let
$$A^\omega_\zeta = \mathrm{set}_0^-(\langle A^\omega_{\zeta,\delta} : \delta \in S^\omega\rangle), \qquad A^\omega = \mathrm{set}_0^-(\langle A^\omega_\delta : \delta \in S^\omega\rangle).$$
For $\zeta < \mu$ we claim $A^\omega_\zeta \subseteq A^\omega$. Let $W = S^\omega \setminus \alpha^*$ with $\alpha^* < \kappa$ large enough that in all ω-many steps of the construction in (1) and (2) the "almost all" quantifiers become "for all".

We now claim that

$$(3.3)\quad (\forall\delta\in W)(\forall i<\omega)\ \big(\eta\in A^i_{\zeta,\delta}\Rightarrow(\eta\in A^\omega_\delta\vee(\exists^\infty\epsilon\in W\cap\delta)\ \eta{\restriction}\epsilon\in A^\omega_\epsilon)\big),$$

clearly this suffices to show $A_\zeta\subseteq A^\omega$. So towards a contradiction assume there exists $\delta^*\in W$ such that there exists $i<\omega$, $\eta^*\in 2^{\delta^*}$ with

$$(3.4)\qquad \eta^*\in A^i_{\zeta,\delta^*}\ \wedge\ \eta^*\notin A^\omega_{\delta^*}\ \wedge\ (\forall^\infty\epsilon\in W\cap\delta^*)\ \eta^*{\restriction}\epsilon\notin A^\omega_\epsilon$$

and let δ^* be minimal with this property and without loss of generality

$$i=\min\{i:\delta^*\in S^i_\zeta\}.$$

Now because $\eta^*\in A^i_{\zeta,\delta^*}$ and $\eta^*\notin A^\omega_{\delta^*}$ (thus in particular $\eta^*\notin A^i_{\delta^*}$), we have

$$\eta^*\in X^i_{\zeta,\delta^*},\qquad \sup(W\cap\delta^*)=\delta^*.$$

By (3)(c) there exists $W^*\subseteq W\cap\delta^*$ unbounded such that

$$(\forall\epsilon\in W^*)\ \eta^*{\restriction}\epsilon\in A^{i+1}_{\zeta,\epsilon},$$

and because $W^*\subseteq\delta^*$ and we assumed δ^* to be minimal contradicting formula (3.3), we have

$$(\forall\epsilon\in W^*)\ \big(\eta^*{\restriction}\epsilon\in A^\omega_\epsilon\vee(\exists^\infty\sigma\in W\cap\epsilon)\ \eta^*{\restriction}\sigma\in A^\omega_\sigma\big),$$

contradicting the last part of the conjunction of formula (3.4) so we are done.

Intuitively the proof showed: Because κ is well ordered, we cannot keep pushing our mistakes in (2) down for ω-many steps. ∎

COROLLARY 3.3.18. *Let κ be Mahlo. We get a strengthening of the general fact about ideals from* 1.4.5*:*

(a) $\mathrm{cf}(\mathrm{id}(\mathbb{Q}_\kappa))=\mathrm{cf}(\mathrm{id}^-(\mathbb{Q}_\kappa))+\mathrm{cf}(\mathrm{id}(\mathbb{Q}_\kappa)/\mathrm{id}^-(\mathbb{Q}_\kappa))$.
(b) $\mathrm{add}(\mathrm{id}(\mathbb{Q}_\kappa))=\min\{\mathrm{add}(\mathrm{id}^-(\mathbb{Q}_\kappa)),\mathrm{add}(\mathrm{id}(\mathbb{Q}_\kappa)/\mathrm{id}^-(\mathbb{Q}_\kappa))\}$.

Proof. (a) follows by 3.3.16, and (b) by 3.3.17. ∎

4. $\mathrm{id}(\mathbb{Q}_\kappa)$ in the $\mathbb{Q}_\kappa$-extension. In this section we consider the relation between $\mathbf{V}$ and $\mathbf{V}^{\mathbb{Q}_\kappa}$, and also more generally between $\mathbf{V}$ and any extension via a strategically closed forcing.

In 4.1 we show that (in contrast to the classical case), the ideal $\mathrm{id}(\mathbb{Q}_\kappa)$ does not satisfy the Fubini theorem, and in fact violates it in a strong sense.

This allows us to to show $\mathrm{cov}(\mathbb{Q}_\kappa)\le\mathrm{non}(\mathbb{Q}_\kappa)$; the proof is similar to a proof of the classical inequality cov(null) $\le$ non(meager), in the sense that both follow from Lemma 4.1.3. It also follows that the old reals become a measure zero set in the $\mathbb{Q}_\kappa$-extension.

In 4.2, we show that $\mathbb{Q}_\kappa^{\mathbf{V}}$ is $\mathbf{V}$-completely embedded into $\mathbb{Q}_\kappa^{\mathbf{V}^{\mathbb{Q}_\kappa}}$. This parallels the classical case, but the proof is necessarily different, as we do not have a measure.

4.1. Asymmetry. In this section we elaborate on the asymmetry of $\mathrm{id}(\mathbb{Q}_\kappa)$ as promised in [She17]. Anti-Fubini sets (defined below) are called 0-1-counterexamples to the Fubini property in [RZ99].

Definition 4.1.1. Let $\mathcal{X}$, $\mathcal{Y}$ be sets and let $\mathbf{i} \subseteq \mathcal{P}(\mathcal{X})$, $\mathbf{j} \subseteq \mathcal{P}(\mathcal{Y})$ be ideals. We call a set $\mathbf{F} \subseteq \mathcal{X} \times \mathcal{Y}$ an *anti-Fubini set for* $(\mathbf{i}, \mathbf{j})$ if

(a) for all $\eta \in \mathcal{Y}$ we have $\mathcal{X} \backslash \mathbf{F}_\eta \in \mathbf{i}$,
(b) for all $\nu \in \mathcal{X}$ we have $\mathbf{F}^\nu \in \mathbf{j}$,

where

$$\mathbf{F}_\eta = \{\nu \in \mathcal{X} : (\nu, \eta) \in \mathbf{F}\}, \qquad \mathbf{F}^\nu = \{\eta \in \mathcal{Y} : (\nu, \eta) \in \mathbf{F}\}.$$

If $\mathbf{i} = \mathbf{j}$ then we simply call $\mathbf{F}$ an *anti-Fubini set for* $\mathbf{i}$.

Lemma 4.1.2. *Let $\mathcal{X}$, $\mathcal{Y}$ be sets and let $\mathbf{i} \subseteq \mathcal{P}(\mathcal{X})$, $\mathbf{j} \subseteq \mathcal{P}(\mathcal{Y})$ be ideals. Let $\mathbf{F} \subseteq \mathcal{X} \times \mathcal{Y}$ be such that:*

(a) *There exists $\mathbf{E}_1 \in \mathbf{j}$ such that for all $\eta \in \mathcal{Y} \backslash \mathbf{E}_1$ we have $\mathcal{X} \backslash \mathbf{F}_\eta \in \mathbf{i}$.*
(b) *There exists $\mathbf{E}_0 \in \mathbf{i}$ such that for all $\nu \in \mathcal{X} \backslash \mathbf{E}_0$ we have $\mathbf{F}^\nu \in \mathbf{j}$.*

Then there exists an anti-Fubini set $\mathbf{F}'$ for $(\mathbf{i}, \mathbf{j})$.

Proof. Let

$$\mathbf{F}' = (\mathbf{F} \cup (\mathcal{X} \times \mathbf{E}_1)))\backslash(\mathbf{E}_0 \times \mathcal{Y})$$

and check that $\mathbf{F}'$ is as required. ∎

Lemma 4.1.3 (Folklore). *Let $\mathbf{i}, \mathbf{j} \subseteq \mathcal{P}(\mathcal{X})$ be ideals. If there exists an anti-Fubini set $\mathbf{F}$ for $(\mathbf{i}, \mathbf{j})$ then $\mathrm{cov}(\mathbf{i}) \leq \mathrm{non}(\mathbf{j})$.*

Proof. Suppose $Y \subseteq \mathcal{Y}$, $Y \notin \mathbf{j}$. We claim that

$$\bigcup \{\mathcal{X} \backslash \mathbf{F}_\eta : \eta \in Y\} = \mathcal{X}.$$

Let $\nu \in \mathcal{X}$. Because $\mathbf{F}^\nu \in \mathbf{j}$ and $Y \notin \mathbf{j}$, we have $Y \backslash \mathbf{F}^\nu \neq \emptyset$, so choose $\eta_0 \in Y \backslash \mathbf{F}^\nu$. We conclude $\eta_0 \notin \mathbf{F}^\nu \Rightarrow (\nu, \eta_0) \notin \mathbf{F} \Rightarrow \nu \notin \mathbf{F}_{\eta_0}$, so $\nu \in \bigcup \{\mathcal{X} \backslash \mathbf{F}_\eta : \eta \in Y\}$. ∎

Lemma 4.1.4 (Folklore). *Let $\mathcal{X}$ be a set, let $\mathbf{i}, \mathbf{j} \subseteq \mathcal{P}(\mathcal{X})$ be ideals and let $\otimes : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a group operation satisfying for all $\mathbf{k} \in \{\mathbf{i}, \mathbf{j}\}$ and for all $X \in \mathbf{k}$ the conditions*

$$\eta \otimes X := \{\eta \otimes x : x \in X\} \in \mathbf{k}, \qquad X^{-1} := \{x^{-1} : x \in X\} \in \mathbf{k}$$

where x^{-1} denotes the group inverse for $\otimes$. If there exist sets $A_0, A_1 \subseteq \mathcal{X}$ such that

$$A_0 \in \mathbf{i}, \qquad A_1 \in \mathbf{j}, \qquad A_0 \cap A_1 = \emptyset, \qquad A_0 \cup A_1 = \mathcal{X},$$

then:

(1) *There exists an anti-Fubini set for $(\mathbf{i}, \mathbf{j})$.*
(2) *There exists an anti-Fubini set for $(\mathbf{j}, \mathbf{i})$.*

Proof. (1) Let

$$\mathbf{F} = \{(\nu, \eta) : \nu \in \eta \otimes A_1\}.$$

Clearly for any $\eta \in \mathcal{X}$ we have $\mathbf{F}_\eta = \eta \otimes A_1$ hence $\mathcal{X} \backslash \mathbf{F}_\eta = \eta \otimes A_0 \in \mathbf{i}$. For $\nu \in \mathcal{X}$ we have $\mathbf{F}^\nu = \{\eta : \nu \in \eta \otimes A_1\} = \{\eta : \eta \in \nu \otimes A_1^{-1}\} = \nu \otimes A_1^{-1} \in \mathbf{j}$. So $\mathbf{F}$ is an anti-Fubini set for $(\mathbf{i}, \mathbf{j})$.

(2) Same proof, interchanging A_0 and A_1. ∎

THEOREM 4.1.5. *Suppose that:*

(a) $\mathbf{i} = (\mathbb{Q}, \dot\eta)$ *is an ideal case, that is:*

- $\mathbb{Q}$ *is a κ-strategically closed forcing notion (or at least does not add bounded subsets of κ).*
- $\dot\eta$ *is a $\mathbb{Q}$-name for a κ-real.*
- *The name $\dot\eta$ determines $\mathbf{i}$ in the following sense: $A \in \mathbf{i}$ iff there exists a (definition of a) κ-Borel set $\mathbf{B} \supseteq A$ such that $\mathbb{Q} \Vdash$ "$\dot\eta \notin \mathbf{B}$".*

(b) *There exists a Borel set $\mathbf{F} \subseteq 2^\kappa \times 2^\kappa$ that is anti-Fubini for $\mathbf{i}$ both in $\mathbf{V}$ and in $\mathbf{V}^{\mathbb{Q}}$.*

Then:

(1) $\mathbb{Q} \Vdash$ "$(2^\kappa)^{\mathbf{V}} \in \mathbf{i}$".
(2) $\mathbb{Q}$ *is asymmetric, i.e. if η_1 is $\mathbb{Q}$-generic over $\mathbf{V}$ and η_2 is $\mathbb{Q}^{\mathbf{V}[\eta_1]}$-generic over $\mathbf{V}[\eta_1]$, then η_1 is not $\mathbb{Q}$-generic over $\mathbf{V}[\eta_2]$.*
(3) $\mathrm{cov}(\mathbf{i}) \le \mathrm{non}(\mathbf{i})$.

Proof. (1) We want to show $\mathbb{Q} \Vdash \mathbf{V} \cap \mathbf{F}_{\dot\eta} = \emptyset$. So let $\nu \in 2^\kappa \cap \mathbf{V}$. Consider $\mathbf{F}^\nu = \{\eta : \nu \in \mathbf{F}_\eta\}$. Now because $\mathbf{F}^\nu \in \mathbf{i}$, we see that $\dot\eta \notin \mathbf{F}^\nu$ thus $\nu \notin \mathbf{F}_{\dot\eta}$.

(2) By (1) we have $\mathbf{V}[\eta_1, \eta_2] \models \eta_1 \in 2^\kappa \backslash \mathbf{F}_{\eta_2}$.

(3) By 4.1.3. ∎

LEMMA 4.1.6. *Assume $\kappa = \sup(S^\kappa_{\mathrm{inc}})$. Then there exists an anti-Fubini set for $(\mathrm{id}^-(\mathbb{Q}_\kappa), \mathrm{id}(\mathbb{Q}_\kappa))$.*

DISCUSSION 4.1.7. This is implicitly shown in [She17] but we repeat it here for the convenience of the reader.

Proof of Lemma 4.1.6. Let $\langle \delta_\epsilon : \epsilon < \kappa \rangle$ enumerate S^κ_{inc} and let $S = \{\delta_{\epsilon+1} : \epsilon < \kappa\}$. For $\eta \in 2^\kappa$, $\delta \in S$ define

$$\mathbf{F}_{\eta,\delta} = \{\rho \in 2^\delta : (\forall^\infty \zeta < \delta)\ \rho(\zeta) = \eta(\delta + \zeta)\}.$$

Then clearly $\mathbf{F}_{\eta,\delta} \in \mathrm{id}(\mathbb{Q}_\delta)$. Let

$$\mathbf{F}_\eta = \mathrm{set}_1^-(\langle \mathbf{F}_{\eta,\delta} : \delta \in S \rangle)$$

so $2^\kappa \backslash \mathbf{F}_\eta \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ by definition. Let

$$\mathbf{F} = \{(\nu, \eta) \in 2^\kappa \times 2^\kappa : \nu \in \mathbf{F}_\eta\}.$$

It remains to check that $\mathbf{F}^\nu \in \mathrm{id}(\mathbb{Q}_\kappa)$. Thus let $\nu \in 2^\kappa$ and consider $\mathbf{F}^\nu = \{\eta \in 2^\kappa : \nu \in \mathbf{F}_\eta\}$; we want to show $\mathbb{Q}_\kappa \Vdash$ "$\nu \notin \mathbf{F}_{\dot\eta}$". Clearly for every $\zeta < \kappa$ the set

$$\{p \in \mathbb{Q}_\kappa : (\exists \delta \in S \backslash \zeta)(\forall \eta \in [p])\ \nu{\restriction}\delta \in \mathbf{F}_{\eta,\delta}\}$$

is a dense subset of $\mathbb{Q}_\kappa$ so we are done. ∎

4.2. Absoluteness of $\mathrm{id}(\mathbb{Q}_\kappa)$

Lemma 4.2.1. *If $\mathbb{P}$ is a κ-strategically closed forcing notion, then:*

(1) *$\boldsymbol{\Pi}^1_1$-formulas are absolute between $\mathbf{V}$ and $\mathbf{V}^{\mathbb{P}}$.*
(2) *If c is a Borel code such that $\mathscr{B}_c = \emptyset$, then also $\mathbb{P} \Vdash \mathscr{B}_c = \emptyset$.*
(3) *If c, d are Borel codes such that $\mathscr{B}_c \subseteq \mathscr{B}_d$, then also $\mathbb{P} \Vdash \mathscr{B}_c \subseteq \mathscr{B}_d$.*

Proof. (1) This was proved in [FKK16, 2.7] for $<\kappa$-closed forcings, but essentially the same proof works for κ-strategically closed forcing notions.

Now (2) and (3) easily follow. ∎

Lemma 4.2.2. *Let $\mathcal{J} = \{q_i : i < \kappa\} \subseteq \mathbb{Q}_\kappa$ be a maximal antichain and let $\mathbb{P}$ be a κ-strategically closed forcing notion. Then*

$$\mathbb{P} \Vdash \text{“}\check{\mathcal{J}} \text{ is a maximal antichain of } \mathbb{Q}_\kappa\text{”}.$$

Proof. We easily check that "$\mathcal{J}$ is a maximal antichain" is a $\mathbf{\Pi}^1_1$-property. So the lemma follows from 4.2.1(1). ∎

Corollary 4.2.3. *As a consequence, if c is a Borel code such that $\mathscr{B}_c \in \mathrm{id}(\mathbb{Q}_\kappa)$, then also $\mathbb{P} \Vdash \mathscr{B}_c \in \mathrm{id}(\mathbb{Q}_\kappa)$ for any $<\kappa$-strategically closed forcing $\mathbb{P}$.*

Proof. Let $\mathscr{B}_c \subseteq \mathrm{set}_0(\Lambda)$ where Λ is a family of κ-many maximal antichains of $\mathbb{Q}_\kappa$. Now work in $\mathbf{V}^{\mathbb{P}}$: Any $\mathcal{J} \in \Lambda$ is still maximal by 4.2.2, hence the Borel set $\mathrm{set}_0(\Lambda)$ is still in $\mathrm{id}(\mathbb{Q}_\kappa)$, and contains $\mathscr{B}_c$ by Lemma 4.2.1(3). ∎

Remark 4.2.4. It is easy to see that a similar fact is true for $\mathrm{id}^-(\mathbb{Q}_\kappa)$: If $\mathrm{set}^-_0(\langle A_\delta : \delta \in S\rangle)$ is a definition of an id^--set in $\mathbf{V}$, and $\mathbb{P}$ does not add bounded subsets of κ, then in $\mathbf{V}^{\mathbb{P}}$ the set S is still nowhere stationary, and the sequence $\langle A_\delta : \delta \in S\rangle$ still defines an $\mathrm{id}^-(\mathbb{Q}_\kappa)$-set, i.e. $\mathrm{id}^-(\mathbb{Q}_\kappa)$-sets remain $\mathrm{id}^-(\mathbb{Q}_\kappa)$-sets.

Fact 4.2.5 ([Bau19]). *For κ weakly compact every positive Borel set contains a random condition.* ∎

Corollary 4.2.6. *For weakly compact κ, also the converse of 4.2.3 is true: If c is a Borel code with $\mathscr{B}_c \notin \mathrm{id}(\mathbb{Q}_\kappa)$, then $\mathbb{P} \Vdash \mathscr{B}_c \notin \mathrm{id}(\mathbb{Q}_\kappa)$ for $\mathbb{P}$ $<\kappa$-strategically closed.*

Proof. By 4.2.5 and 4.2.1(3). ∎

5. ZFC-results

5.1. Cichoń's diagram

DISCUSSION 5.1.1. In this subsection we establish some results about the relation between $\mathrm{id}(\mathbb{Q}_\kappa)$ and the ideal of meager sets $\mathrm{id}(\mathrm{Cohen}_\kappa)$. These theorems are either quotes of or promised elaborations on results first appearing in [She17].

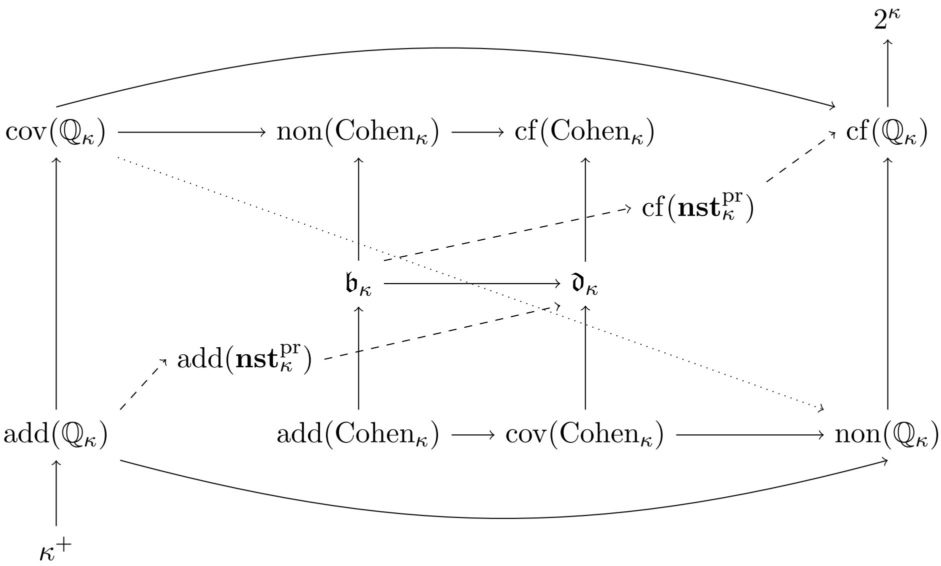

Fig. 1. The general diagram including $\mathbf{nst}^{\mathrm{pr}}_\kappa$, showing results established in this section. Dashed or dotted arrows have the same meaning as the solid ones but are intended to make the crossing arrows visually less confusing. To prove the implications represented by the dashed arrows (those involving $\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$ and $\mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$) we need to assume that κ is Mahlo.

FACT 5.1.2 (Folklore?).

(1) $\mathrm{add}(\mathrm{Cohen}_\kappa) = \min(\mathfrak{b}_\kappa, \mathrm{cov}(\mathrm{Cohen}_\kappa))$.
(2) $\mathrm{cf}(\mathrm{Cohen}_\kappa) = \max(\mathfrak{d}_\kappa, \mathrm{non}(\mathrm{Cohen}_\kappa))$.

Proof. See for example [She17]. ■

The following theorem first appears in [She17, 3.8], but we repeat it here for the convenience of the reader.

THEOREM 5.1.3. *Let $\kappa = \sup(S^\kappa_{\mathrm{inc}})$. Then there exist sets $A_0, A_1 \subseteq 2^\kappa$ such that $A_0 \in \mathrm{id}(\mathbb{Q}_\kappa)$, $A_1 \in \mathrm{id}(\mathrm{Cohen}_\kappa)$, $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 = 2^\kappa$.*

Proof. Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing enumeration of S^κ_{inc}. For $i < \lambda$ let

$$\mathcal{J}_{\lambda_{i+1}} = \{p \in \mathbb{Q}_{\lambda_{i+1}} : \lg(\mathrm{tr}(p)) > \lambda_i \wedge \mathrm{tr}(p){\restriction}[\lambda_i, \lg(\mathrm{tr}(p))) \text{ is not constantly } 0\}.$$

For $\eta \in 2^{<\kappa}$ let $p_\eta \in \mathbb{Q}_\kappa$ be the condition witnessed by

$$(\eta, \{\lambda_{i+1} : i < \kappa,\ \lambda_{i+1} > \lg(\eta)\}, \langle\{D_{\lambda_{i+1}}\} : i < \kappa,\ \lambda_{i+1} > \lg(\eta)\rangle).$$

It is easy to see that $[p_\eta]$ is a nowhere dense subset of 2^κ. Hence for

$$A_1 = \bigcup_{\eta \in 2^{<\kappa}} [p_\eta]$$

we have $A_1 \in \mathrm{id}(\mathrm{Cohen}_\kappa)$.

Let $A_0 = 2^\kappa \setminus A_1$. It remains to check that $A_0 \in \mathrm{id}(\mathbb{Q}_\kappa)$. Indeed, for any $p \in \mathbb{Q}_\kappa$ let $\eta = \mathrm{tr}(p)$ and let q be a lower bound for p, p_η. Now $q \Vdash$ "$\dot\eta \in [q] \subseteq [p_\eta] \subseteq A_1$", i.e. $q \Vdash$ "$\dot\eta \notin A_0$". ∎

Corollary 5.1.4. *Let* $\kappa = \sup(S^\kappa_{\mathrm{inc}})$. *Then:*

(1) $\mathrm{cov}(\mathrm{Cohen}_\kappa) \le \mathrm{non}(\mathbb{Q}_\kappa)$.
(2) $\mathrm{cov}(\mathbb{Q}_\kappa) \le \mathrm{non}(\mathrm{Cohen}_\kappa)$.

Proof. Let $\oplus$ be pointwise addition modulo 2. In 5.1.3 it is shown there exist sets $A_0 \in \mathrm{id}(\mathbb{Q}_\kappa)$, $A_1 \in \mathrm{id}(\mathrm{Cohen}_\kappa)$ satisfying 4.1.4(a)–(d) for $\kappa = \sup(S^\kappa_{\mathrm{inc}})$, so the conclusion follows by 4.1.3. ∎

Corollary 5.1.5. *Let* $\kappa = \sup(S^\kappa_{\mathrm{inc}})$. *Then:*

(1) $\mathrm{cov}(\mathrm{id}^-(\mathbb{Q}_\kappa)) \le \mathrm{non}(\mathrm{id}(\mathbb{Q}_\kappa))$.
(2) *In particular* $\mathrm{cov}(\mathbb{Q}_\kappa) \le \mathrm{non}(\mathbb{Q}_\kappa)$.

Proof. By 4.1.5 and 4.1.6. ∎

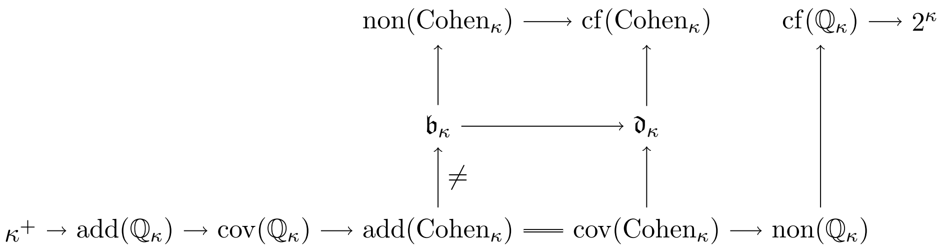

Fig. 2. The diagram for $\mathrm{add}(\mathrm{Cohen}_\kappa) < \mathfrak{b}_\kappa$

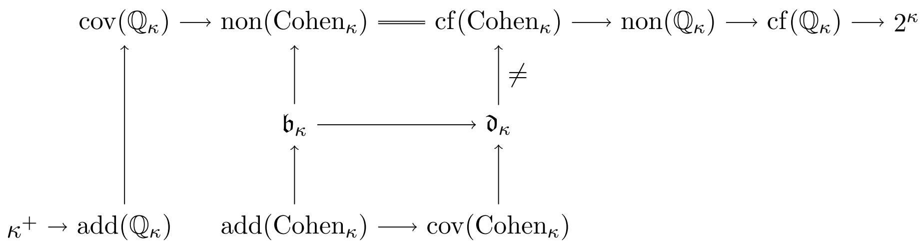

Fig. 3. The diagram for $\mathfrak{d}_\kappa < \mathrm{cf}(\mathrm{Cohen})_\kappa$

Theorem 5.1.6.

(1) *If* $\mathfrak{b}_\kappa > \mathrm{add}(\mathrm{Cohen}_\kappa)$ *then* $\mathrm{cov}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathrm{Cohen}_\kappa)$.
(2) *If* $\mathfrak{d}_\kappa < \mathrm{cf}(\mathrm{Cohen}_\kappa)$ *then* $\mathrm{cf}(\mathrm{Cohen}_\kappa) \leq \mathrm{non}(\mathbb{Q}_\kappa)$.

Proof. See [She17, 5.5 and 5.7]. ∎

5.2. On $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathrm{Cohen}_\kappa)$

Discussion 5.2.1. For the classical case ($\kappa = \omega$) the Bartoszyński–Raisonnier–Stern theorem states that add(null) $\leq$ add(meager). By 5.1.6 we know that $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathrm{Cohen}_\kappa)$ for large $\mathfrak{b}_\kappa$ and dually $\mathrm{cf}(\mathrm{Cohen}_\kappa) \leq \mathrm{add}(\mathbb{Q}_\kappa)$ for small $\mathfrak{d}_\kappa$. But what about small $\mathfrak{b}_\kappa$, i.e. $\mathrm{add}(\mathrm{Cohen}_\kappa) = \mathfrak{b}_\kappa$ and large $\mathfrak{d}_\kappa$, i.e. $\mathfrak{d}_\kappa = \mathrm{cf}(\mathrm{Cohen}_\kappa)$?

The original plan for this case was to first prove $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$ (see 3.3.6) and show that $\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa) \leq \mathfrak{b}_\kappa$. We conjecture that this second inequality does not hold (see 5.2.12). In [She17] it was shown that it does at least for sufficiently weak κ (there exists a stationary non-reflecting subset of κ) and here we elaborate on this result as promised.

Furthermore we offer a consolation prize: we show that at least $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathfrak{d}_\kappa$ for κ Mahlo and dually $\mathfrak{b}_\kappa \leq \mathrm{cf}(\mathbb{Q}_\kappa)$.

We begin by establishing a characterization of $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ via characteristics of the club filter of κ.

Lemma 5.2.2.

(1) *Let* $\langle E_\alpha : \alpha < \mu < \mathfrak{b}_\kappa\rangle$ *be a sequence of clubs of* κ. *Then there exists a club* E *of* κ *such that* $\alpha < \mu \Rightarrow E \subseteq^* E_\alpha$.
(2) *There exists a sequence* $\langle E_\alpha : \alpha < \mathfrak{b}_\kappa\rangle$ *of clubs of* κ *such that for no club* E *of* κ *do we have* $\alpha < \mathfrak{b}_\kappa \Rightarrow E \subseteq^* E_\alpha$.
(3) $\mathfrak{b}_\kappa = \mathrm{add}(\mathbf{NS}_\kappa)$, *where* $\mathbf{NS}_\kappa$ *is the ideal of non-stationary subsets of* κ, *ordered by eventual containment* $\subseteq^*$.

Proof. (1) Let $\langle E_\alpha : \alpha < \mu < \mathfrak{b}_\kappa\rangle$ be a sequence of clubs of κ. We define

$$f_\alpha(i) = \lceil i+1\rceil^{E_\alpha} = \min(E_\alpha \backslash (i+2)).$$

and find f such that $\alpha < \mu \Rightarrow f_\alpha \leq^* f$. Now let $E = \{\delta : f[\delta] \subseteq \delta\}$ and check that indeed $\alpha < \mu \Rightarrow E \subseteq^* E_\alpha$.

(2) Let $\langle f_\alpha : \alpha < \mathfrak{b}_\kappa\rangle$ witness $\mathfrak{b}_\kappa$ and let

$$E_\alpha = \{\delta : f_\alpha[\delta] \subseteq \delta\}.$$

Assume there exists a club E of κ such that $\alpha < \mathfrak{b}_\kappa \Rightarrow E \subseteq^* E_\alpha$. Let

$$f(i) = \lceil i+1\rceil^E$$

and check that $\alpha < \mathfrak{b}_\kappa \Rightarrow f_\alpha \leq^* f$, a contradiction.

(3) By (1) and (2). ∎

Lemma 5.2.3.

(1) *Let* $\langle E_\alpha : \alpha < \mu < \mathfrak{d}_\kappa \rangle$ *be a sequence of clubs of* κ. *Then there exists a club* E *of* κ *such that* $E_\alpha \subseteq^* E$ *for no* $\alpha < \mathfrak{d}_\kappa$.
(2) *There exists a sequence* $\langle E_\alpha : \alpha < \mathfrak{d}_\kappa \rangle$ *of clubs of* κ *such that for all clubs* E *of* κ *there exists* $\alpha < \mathfrak{d}_\kappa$ *such that* $E_\alpha \subseteq^* E$.
(3) $\mathfrak{d}_\kappa = \mathrm{cf}(\mathbf{NS}_\kappa)$.

Proof. Dual of that of 5.2.2. ∎

Theorem 5.2.4. *Let* κ *be Mahlo (hence* S^κ_{pr} *is stationary by* 1.3.3(4)). *Then*

$$\mathfrak{b}_\kappa \leq \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa).$$

Proof. Towards a contradiction assume $\mu = \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa) < \mathfrak{b}_\kappa$ and let $\langle W_\alpha : \alpha < \mu \rangle$ be a sequence of nowhere stationary subsets of S^{pr}_κ witnessing $\mu = \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$. For $\alpha < \mu$ let $E_\alpha \subseteq \kappa$ be a club disjoint from W_α. Now we use 5.2.2 to find a club E such that $E \subseteq^* E_\alpha$ for every α. Because S^κ_{pr} is stationary, the closure of $E \cap S^\kappa_{\mathrm{pr}}$ is a club too, so without loss of generality $W = \mathrm{nacc}(E) \subseteq S^\kappa_{\mathrm{pr}}$. Clearly W is nowhere stationary so there exists $\alpha < \mu$ such that $W \subseteq^* W_\alpha$.

Now because $E \subseteq^* E_\alpha$ and $W_\alpha \cap E_\alpha = \emptyset$, we see that $W_\alpha \cap E$ is bounded. On the other hand, because W is an unbounded subset of E and $W \subseteq^* W_\alpha$, we see that $W_\alpha \cap E$ is unbounded, a contradiction. ∎

Corollary 5.2.5. $\mathfrak{b}_\kappa \leq \mathrm{cf}(\mathbb{Q}_\kappa)$.

Proof. Combine 5.2.4 and 3.3.7. ∎

Theorem 5.2.6. *Let* κ *be Mahlo. Then*

$$\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa) \leq \mathfrak{d}_\kappa.$$

Proof. Let $\langle E_\alpha : \alpha < \mu \rangle$ witness $\mathfrak{d}_\kappa = \mu$ in the sense of 5.2.3, i.e. for every club E of κ there is $\alpha < \mu$ such that $E_\alpha \subseteq^* E$. If we restrict ourselves to clubs E such that $\mathrm{nacc}(E) \subseteq S^\kappa_{\mathrm{pr}}$ then we may also assume $W_\alpha = \mathrm{nacc}(E_\alpha) \subseteq S^\kappa_{\mathrm{pr}}$. Towards a contradiction assume $\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa) > \mu$ and let $W \in \mathbf{nst}^{\mathrm{pr}}_\kappa$ be such that $\alpha < \mu \Rightarrow W_\alpha \subseteq^* W$. Choose a club E disjoint from W such that $\mathrm{nacc}(E) \subseteq S^\kappa_{\mathrm{pr}}$. Now there exists $\alpha < \mu$ such that $E_\alpha \subseteq^* E$ hence

$$\sup(E_\alpha \backslash E) + \sup(W_\alpha \backslash W) < \delta \in W_\alpha \subseteq E_\alpha \implies \delta \in E \implies \delta \notin W_\alpha,$$

a contradiction. ∎

Corollary 5.2.7. $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathfrak{d}_\kappa$.

Proof. Combine 5.2.6 and 3.3.7. ∎

Theorem 5.2.8. *Let κ be inaccessible and let $S \subseteq S^\kappa_{\rm pr}$ be stationary non-reflecting. Then:*

(1) $\mathrm{add}(\mathbf{nst}^{\rm pr}_\kappa) \leq \mathfrak{b}_\kappa$.
(2) $\mathrm{add}(\mathbf{nst}^{\rm pr}_{\kappa,S}) = \mathfrak{b}_\kappa$.

Remark 5.2.9. Note that under these assumptions, by [She17, Claim 6.9] the forcing $\mathbb{Q}_\kappa$ adds a κ-Cohen real.

Proof of Theorem 5.2.8. First note that because S is not reflecting, it follows that $W \subseteq S$ is nowhere stationary iff W is not stationary.

Recall 5.2.2 and let $\langle E_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ be a set of clubs of κ such that for every club E of κ there exists $\alpha < \mathfrak{b}_\kappa$ such that $\neg(E \subseteq^* E_\alpha)$. So the family $\langle S \backslash E_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ is a set of nowhere stationary subsets of $S^\kappa_{\rm pr}$ with no upper bound in $\mathbf{nst}^{\rm pr}_{\kappa,S}$ (and in particular not in $\mathbf{nst}^{\rm pr}_\kappa$).

Conversely, let $\langle W_\alpha : \alpha < \mu \rangle$ witness $\mathrm{add}(\mathbf{nst}^{\rm pr}_{\kappa,S}) = \mu$ and let E_α be club disjoint from W_α. Then $\langle E_\alpha : \alpha < \mu \rangle$ is an unbounded family in the sense of 5.2.2. ∎

Theorem 5.2.10. *Let κ be inaccessible and let $S \subseteq S^\kappa_{\rm pr}$ be stationary non-reflecting. Then:*

(1) $\mathfrak{d}_\kappa \leq \mathrm{cf}(\mathbf{nst}^{\rm pr}_\kappa)$.
(2) $\mathfrak{d}_\kappa = \mathrm{cf}(\mathbf{nst}^{\rm pr}_{\kappa,S})$.

Proof. Dual to that of 5.2.8. ∎

We summarize the results of this section in the following corollary.

Corollary 5.2.11. *If at least one of the following conditions is satisfied:*

(1) $\kappa > \sup(S^\kappa_{\rm inc})$, *or*
(2) *there exists a stationary non-reflecting* $S \subseteq S^\kappa_{\rm pr}$, *or*
(3) $\mathfrak{b}_\kappa > \mathrm{add}(\mathrm{Cohen}_\kappa)$,

then the Bartoszyński–Raisonnier–Stern theorem holds, i.e.

$$\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathrm{Cohen}_\kappa).$$

Likewise, if we let

(3′) $\mathfrak{d}_\kappa < \mathrm{cf}(\mathrm{Cohen}_\kappa)$,

then $(1) \vee (2) \vee (3')$ *implies*

$$\mathrm{cf}(\mathrm{Cohen}_\kappa) \leq \mathrm{cf}(\mathbb{Q}_\kappa).$$

Finally, if $(1) \vee (2) \vee ((3) \wedge (3'))$, *then the Cichoń diagram for* $\mathrm{id}(\mathbb{Q}_\kappa)$ *and* $\mathrm{id}(\mathrm{Cohen}_\kappa)$ *looks like the classical diagram.* ∎

Conjecture 5.2.12. *There exists a model* $\mathbf{V}$ *such that*

$$\mathbf{V} \models \mathrm{add}(\mathbb{Q}_\kappa) > \mathrm{add}(\mathrm{Cohen}_\kappa)$$

for some sufficiently strong cardinal κ. *Note that by* 5.1.6 *we necessarily have*

$$\mathbf{V} \models \mathfrak{b}_\kappa = \operatorname{add}(\mathrm{Cohen}_\kappa)$$

so we really conjecture

$$\mathrm{CON}(\operatorname{add}(\mathbb{Q}_\kappa) > \mathfrak{b}_\kappa).$$

6. Models. We follow the notation of [BJ95]: Let $\square = \kappa^+$, $\blacksquare = \kappa^{++}$. This will allow us to graphically represent the values of the cardinal characteristics in Figure 1. For instance, $\square$ in the top left corner means $\operatorname{cov}(\mathbb{Q}_\kappa) = \square$. Note that in all diagrams of this section we have $2^\kappa = \blacksquare = \kappa^{++}$.

For visual clarity we omit the diagonal arrow from $\operatorname{cov}(\mathbb{Q}_\kappa)$ to $\operatorname{non}(\mathbb{Q}_\kappa)$; see 5.1.4. Note again that the dashed arrows representing $\operatorname{add}(\mathbb{Q}_\kappa) \leq \mathfrak{d}_\kappa$ and $\mathfrak{b}_\kappa \leq \operatorname{cf}(\mathbb{Q}_\kappa)$ require κ to be Mahlo.

If we would like $\mathbb{Q}_\kappa$ to be κ^κ-bounding, i.e. want κ weakly compact, we may use Laver preparation to preserve supercompactness (so in particular weak compactness) in the forcing extension (see [Lav78]). Note that all forcing notions in this section, with the exception of Amoeba forcing, are $<\kappa$-directed closed, and Amoeba forcing may be included in the preparation as well by 6.6.4.

6.1. The Cohen model. The Cohen forcing $\mathbb{C}_\kappa$ and model $\mathbf{V}^{\mathbb{C}_{\kappa,\mu}}$ are well known (see e.g. [Eas70]). As a warm-up for subsequent constructions we show how to apply the tools developed in Section 2 for analyzing this model.

Definition 6.1.1. Let

$$\mathbb{C}_\kappa = 2^{<\kappa}$$

and for $p, q \in \mathbb{C}_\kappa$ define q to be stronger than p if $p \trianglelefteq q$. We call $\mathbb{C}_\kappa$ the *κ-Cohen forcing*. If G is a $\mathbb{C}_\kappa$-generic filter then we call $\eta = \bigcup_{s\in G} s$ the *generic κ-Cohen real* (*of* $\mathbf{V}[G]$). Conversely, we say $\nu \in 2^\kappa$ is a *κ-Cohen real* (*over* $\mathbf{V}$) if $G = \{s \in 2^{<\kappa} : s \triangleleft \nu\}$ is a $\mathbb{C}_\kappa$-generic filter.

Fact 6.1.2. *Let* $\nu \in 2^\kappa$. *Then* ν *is a* κ*-Cohen real over* $\mathbf{V}$ *iff it is not contained in any meager set of* $\mathbf{V}$. ■

Lemma 6.1.3.

(1) $\mathbb{C}_\kappa$ *is* $<\kappa$*-directed closed.*
(2) $\mathbb{C}_\kappa$ *is* κ*-centered*$_{<\kappa}$.
(3) $\mathbb{C}_\kappa$ *satisfies* $(*)_\kappa$.

Proof. (1) and (2) are trivial; (3) easily follows from 2.1.5, 2.3.2, 2.2.7. ■

Definition 6.1.4. Let μ be an ordinal. Let $\mathbb{C}_{\kappa,\mu}$ be the limit of the $<\kappa$-support iteration $\langle \mathbb{C}_{\kappa,\alpha}, \dot{\mathbb{R}}_\alpha : \alpha < \mu\rangle$ where $\mathbb{C}_{\kappa,\alpha} \Vdash$ "$\dot{\mathbb{R}}_\alpha = \mathbb{C}_\kappa$" for every $\alpha < \mu$.

It is easy to check that the $<\kappa$-support product $\prod_{i<\mu} \mathbb{C}_\kappa$ can be canonically embedded as a dense subset into $\mathbb{C}_{\kappa,\mu}$.

LEMMA 6.1.5. *Let μ be an ordinal. Then $\mathbb{C}_{\kappa,\mu}$ satisfies the stationary κ^+-Knaster condition and in particular $\mathbb{C}_{\kappa,\mu}$ satisfies the κ^+-c.c.*

Proof. By 6.1.3, 2.2.6, 2.2.3. ∎

THEOREM 6.1.6. *Let $\mathbf{V} \models 2^\kappa = \kappa^+$. Then $\mathbf{V}^{\mathbb{C}_{\kappa,\kappa^{++}}}$ satisfies:*

(1) $\mathrm{non}(\mathrm{Cohen}_\kappa) = \kappa^+$.
(2) $\mathrm{cov}(\mathrm{Cohen}_\kappa) = \kappa^{++}$.
(3) $2^\kappa = \kappa^{++}$.

*We call $\mathbf{V}^{\mathbb{C}_{\kappa,\kappa^{++}}}$ the κ-*Cohen model.

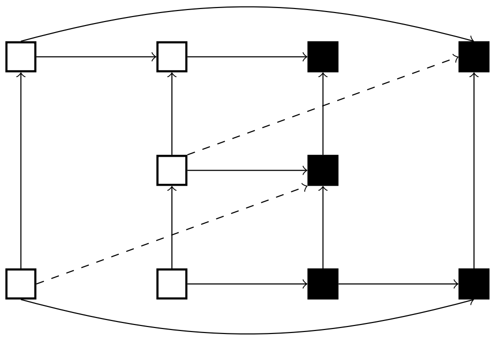

Fig. 4. The Cohen model

Proof. (1) This is a standard argument from the classical case but we give the details.

Let $\dot{M} = \{\dot{\eta}_\alpha : \alpha < \kappa^+\}$ where $\dot{\eta}_\alpha$ is a name for the κ-Cohen real added by $\dot{\mathbb{R}}_\alpha$. We claim $\mathbb{C}_{\kappa,\kappa^{++}} \Vdash$ "$\dot{M}$ is a non-meager set". Towards a contradiction assume that there are $\langle \dot{A}_i : i < \kappa \rangle$ where $\dot{A}_i$ is a $\mathbb{C}_{\kappa,\kappa^{++}}$-name for a closed, nowhere dense set and there exists $p \in \mathbb{C}_{\kappa,\kappa^{++}}$ such that $p \Vdash$ "$\dot{M} \subseteq \bigcup_{i<\kappa} \dot{A}_i$". It is easy to see that any closed nowhere dense set $\dot{A}_i \in \mathbf{V}^{\mathbb{C}_{\kappa,\kappa^{++}}}$ is decided by $|2^{<\kappa}| = \kappa$-many antichains $\langle \mathcal{J}_{i,s} : s \in 2^{<\kappa} \rangle$ where $\mathcal{J}_{i,s}$ decides the whole of $\dot{A}_i$ above s, i.e. decides $\dot{t}_{i,s} \trianglerighteq s$ such that $[\dot{t}_{i,s}] \cap \dot{A}_i = \emptyset$. Remember 6.1.5 and let

$$\alpha \in \kappa^+ \setminus \bigcup_{i<\kappa} \bigcup_{s \in 2^{<\kappa}} \bigcup_{p \in \mathcal{J}_{i,s}} \mathrm{supp}(p).$$

Remember 6.1.4 and let Π be the range of the dense embedding of $\prod_{i<\kappa^{++}} \mathbb{C}_\kappa$ into $\mathbb{C}_{\kappa,\kappa^{++}}$. Without loss of generality $\mathcal{J}_{i,s} \subseteq \Pi$ for all $i < \kappa$ and all $s \in 2^{<\kappa}$. Find $p' \leq p$ such that $p' \in \Pi$ and let $s = p(\alpha)$. Now for arbitrary $i < \kappa$ we can find $r \in \mathcal{J}_{i,s}$, $r \not\perp p'$ and let $p'' = r \wedge p'$. Because $p', r \in \Pi$, we have $p''(\alpha) = s$ and p'' decides $t_s \trianglerighteq s$ to be missing from $\dot{A}_i$. Thus define $p''' \le p''$ such that $p'''(\alpha) = t_s$ and $p'''(\beta) = p''(\beta)$ for $\beta \in \kappa^{++} \setminus \{\alpha\}$. Clearly $\dot{\eta}_\alpha \trianglerighteq t_s$ thus $p''' \Vdash$ "$\dot{\eta}_\alpha \notin \dot{A}_i$". Clearly $p''' \le p$, contradicting $p \Vdash$ "$\dot{M} \subseteq \bigcup_{i<\kappa} \dot{A}_i$".

(2) Same argument as in 6.2.7 below.

(3) Should be clear using nice names. ∎

6.2. The Hechler model. The Hechler forcing $\mathbb{H}_\kappa$ and model $\mathbf{V}^{\mathbb{H}_{\kappa,\mu}}$ are well known (see e.g. [BB+18] where 6.2.7(2)–(5) are also shown).

Definition 6.2.1. Let

$$\mathbb{H}_\kappa = \kappa^{<\kappa} \times [\kappa^\kappa]^{<\kappa}$$

and for $p_1 = (\rho_1, X_1)$, $p_2 = (\rho_2, X_2) \in \mathbb{H}_\kappa$ define p_2 to be stronger than p_1 if:

(1) $\rho_2 \trianglerighteq \rho_1$.
(2) $X_2 \supseteq X_1$.
(3) For all $i \in \operatorname{dom}(\rho_2) \setminus \operatorname{dom}(\rho_1)$ and all $f \in X_1$ we have $\rho_2(i) > f(i)$.

We call $\mathbb{H}_\kappa$ the *κ-Hechler forcing.* If G is a $\mathbb{H}_\kappa$-generic filter then we call $\eta = \bigcup_{(\rho,X)\in G} \rho$ the *generic κ-Hechler real.*

The intended meaning of a condition (ρ, X) is the promise that the κ-Hechler real will start with ρ and from now on (i.e. past the length of ρ) dominate all functions in X.

Fact 6.2.2. *Let η be a κ-Hechler real over $\mathbf{V}$. Then for every $\nu \in \kappa^\kappa \cap \mathbf{V}$ we have $\nu \le^* \eta$.* ∎

Fact 6.2.3. *Let η be a κ-Hechler real over $\mathbf{V}$. Let $\nu \in 2^\kappa$ be such that for all $i < \kappa$,*

$$\nu(i) \equiv \eta(i) \bmod 2.$$

Then ν is a κ-Cohen real over $\mathbf{V}$. ∎

Lemma 6.2.4.

(1) *$\mathbb{H}_\kappa$ is $<\kappa$-directed closed.*
(2) *$\mathbb{H}_\kappa$ is κ-centered$_{<\kappa}$.*
(3) *$\mathbb{H}_\kappa$ satisfies $(*)_\kappa$.* ∎

Proof. (1) Let $D \subseteq \mathbb{H}_\kappa$, $|D| < \kappa$, $p, q \in D \Rightarrow p \not\perp q$. If $p = (\rho_1, X_1)$, $q = (\rho_2, X_2) \in D$ then because p, q are compatible we have $\rho_1 \trianglelefteq \rho_2 \vee \rho_2 \trianglelefteq \rho_1$. Hence (ρ^*, X^*) is a lower bound for D where $\rho^* = \bigcup_{(\rho,X)\in D} \rho$, $X^* = \bigcup_{(\rho,X)\in D} X$.

(2) $\mathbb{H}_\kappa = \bigcup_{\rho\in\kappa^{<\kappa}} (\{\rho\} \times [\kappa^\kappa]^{<\kappa})$.

(3) By (1), (2), 2.1.5, 2.3.2, 2.2.7. ∎

Definition 6.2.5. Let μ be an ordinal. Let $\mathbb{H}_{\kappa,\mu}$ be the limit of the $<\kappa$-support iteration $\langle \mathbb{H}_{\kappa,\alpha}, \dot{\mathbb{R}}_\alpha : \alpha < \mu\rangle$ where $\mathbb{H}_{\kappa,\alpha} \Vdash$ "$\dot{\mathbb{R}}_\alpha = \mathbb{H}_\kappa$" for every $\alpha < \mu$.

Lemma 6.2.6. *Let μ be an ordinal. Then:*

(1) $\mathbb{H}_{\kappa,\mu}$ *satisfies the stationary κ^+-Knaster condition and in particular $\mathbb{H}_{\kappa,\mu}$ satisfies the κ^+-c.c.*
(2) *If $\mu < (2^\kappa)^+$ then $\mathbb{H}_{\kappa,\mu}$ is κ-centered$_{<\kappa}$.*

Proof. (1) By 6.2.4, 2.2.6, 2.2.3.

(2) Remember 6.2.4(2). We easily check that $\mathbb{H}_{\kappa,\mu}$ is finely $<\kappa$-closed so we can use 2.3.7. ∎

Theorem 6.2.7. *Let $\mathbf{V} \models 2^\kappa = \kappa^+$. Then $\mathbf{V}^{\mathbb{H}_{\kappa,\kappa^{++}}}$ satisfies:*

(1) $\mathrm{cov}(\mathbb{Q}_\kappa) = \kappa^+$.
(2) $\mathfrak{b}_\kappa = \kappa^{++}$.
(3) $\mathrm{cov}(\mathrm{Cohen}_\kappa) = \kappa^{++}$.
(4) $\mathrm{add}(\mathrm{Cohen}_\kappa) = \kappa^{++}$.
(5) $2^\kappa = \kappa^{++}$.

*We call $\mathbf{V}^{\mathbb{H}_{\kappa,\kappa^{++}}}$ the κ-*Hechler model.

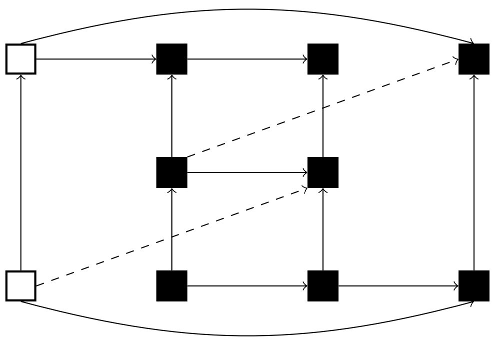

Fig. 5. The Hechler model

Proof. We use the iteration theorems from Section 2 so the following proofs become standard arguments from the classical case.

(1) We claim that $\mathbb{H}_{\kappa,\kappa^{++}}$ does not add $\mathbb{Q}_\kappa$-generic reals. Remembering 6.2.6(1), if we have a nice $\mathbb{H}_{\kappa,\kappa^{++}}$-name $\dot{\eta}$ for a κ-real, then the antichains deciding $\dot{\eta}$ are already antichains of $\mathbb{H}_{\kappa,\alpha}$ for some $\alpha < \kappa^{++}$. Note that if we show that $\mathbb{H}_{\kappa,\alpha}$ does not add $\mathbb{Q}_\kappa$-generic reals for any $\alpha < \kappa^{++}$ we are done:

If $\eta \in \mathbf{V}^{\mathbb{H}_{\kappa,\alpha}}$ is not $\mathbb{Q}_\kappa$-generic over $\mathbf{V}$ then there is a Borel code $c \in \mathbf{V}$ of an $\mathrm{id}(\mathbb{Q}_\kappa)$-set $\mathscr{B}_c$ such that $\eta \in \mathscr{B}_c$. The same is still true in $\mathbf{V}^{\mathbb{H}_{\kappa,\kappa^{++}}}$ (see 1.1.14).

By 6.2.6(2), $\mathbb{H}_{\kappa,\alpha}$ is a κ-centered$_{<\kappa}$ forcing notion for each $\alpha < \kappa^{++}$ and thus by 2.3.9 does not add a $\mathbb{Q}_\kappa$-generic real. In $\mathbf{V}$ there exists a covering of $\mathrm{id}(\mathbb{Q}_\kappa)$ of size κ^+, and because $\mathbb{H}_{\kappa,\kappa^{++}}$ does not add $\mathbb{Q}_\kappa$-generic reals, this covering remains a covering in $\mathbf{V}^{\mathbb{H}_{\kappa,\kappa^{++}}}$.

(2) Assume there exists an unbounded family of size κ^+ in $\mathbf{V}^{\mathbb{H}_{\kappa,\kappa^{++}}}$. Argue as above to see that this family already appears in some $\mathbf{V}^{\mathbb{H}_{\kappa,\alpha}}$. But by 6.2.2, $\mathbb{R}_\alpha$ adds a bound, a contradiction.

(3) Assume there exists an covering of $\mathrm{id}(\mathrm{Cohen}_\kappa)$ of size κ^+ in $\mathbf{V}^{\mathbb{H}_{\kappa,\kappa^{++}}}$. Again this family already appears in some $\mathbf{V}^{\mathbb{H}_{\kappa,\alpha}}$. But by 6.2.3, $\dot{\mathbb{R}}_\alpha$ adds a κ-Cohen real, hence the covering is destroyed, a contradiction.

(4) Remember 5.1.2 so this follows from (2) and (3).

(5) Should be clear. ∎

6.3. The short Hechler model

THEOREM 6.3.1. *Let* $\mathbf{V} \models \kappa$ *is weakly compact. Let* $\mathbf{V} \models \mathrm{non}(\mathbb{Q}_\kappa) = \kappa^{++}$ *(e.g.* $\mathbf{V} = \mathbf{V}_0^{\mathbb{H}_{\kappa,\kappa^{++}}}$*). Let* $\mathbb{H}_{\kappa,\kappa^+}$ *be the* $<\kappa$*-support iteration of length* κ^+ *of Hechler reals (see* 6.2.5*). Then* $\mathbf{V}^{\mathbb{H}_{\kappa,\kappa^+}}$ *satisfies:*

(1) $\mathrm{non}(\mathbb{Q}_\kappa) = \kappa^{++}$.
(2) $\mathfrak{d}_\kappa = \kappa^+$.
(3) $\mathrm{non}(\mathrm{Cohen}_\kappa) = \kappa^+$.
(4) $\mathrm{cf}(\mathrm{Cohen}_\kappa) = \kappa^+$.
(5) $2^\kappa = \kappa^{++}$.

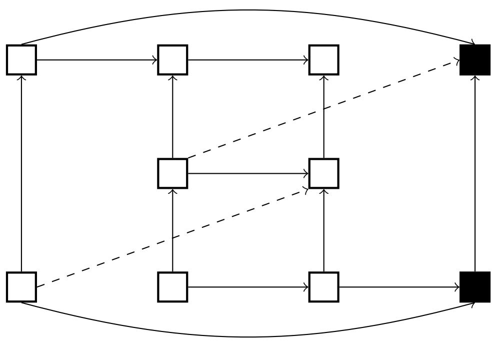

Fig. 6. The short Hechler model

Proof. (1) Follows by 2.3.7 and 2.3.13. (Note that 2.3.13 requires κ to be weakly compact.)

(2) Remember 6.2.2 so $\{\eta_\epsilon : \epsilon < \kappa^+\}$ is a dominating family where η_ϵ is the κ-Hechler real added by $\mathbb{R}_\epsilon$.

(3) We claim $\{\nu_\epsilon : \epsilon < \kappa^+\} \notin \mathrm{id}(\mathrm{Cohen}_\kappa)$ where $\nu_\epsilon \in 2^\kappa$ is the canonical κ-Cohen real added by $\mathbb{R}_\epsilon$ (see 6.2.3). Argue as in 6.1.6 but instead of using the product we find α greater than the support of all antichains.

(4) Remember 5.1.2 so this follows from (2) and (3).

(5) Should be clear. ■

6.4. Amoeba forcing, part 1

DEFINITION 6.4.1. Let $\mathbb{Q}^{\mathrm{am},1}_\kappa$ be the forcing consisting of tuples (ϵ, S, E) where:

- $\epsilon \in S^\kappa_{\mathrm{inc}}$.
- $S \subseteq S^\kappa_{\mathrm{inc}}$ is nowhere stationary.
- $E \subseteq \kappa$ is a club disjoint from S.

For $p \in \mathbb{Q}^{\mathrm{am},1}_\kappa$ we will write ϵ_p, S_p, E_p for the respective components of p.

For $p = (\epsilon_p, S_p, E_p)$, $q = (\epsilon_q, S_q, E_q)$ we define $q \leq p$ (q *stronger than* p) iff either $q = p$, or:

- $\epsilon_p < \epsilon_q$, and moreover the set E_q meets the interval (ϵ_p, ϵ_q).
- $S_p \cap \epsilon_p = S_q \cap \epsilon_p$.
- $S_p \backslash \epsilon_p \subseteq S_q \backslash \epsilon_p$.
- $E_p \cap \epsilon_p = E_q \cap \epsilon_p$.
- $E_p \supseteq E_q$.

The intended meaning of a condition (ϵ, S, E) is the promise to cover S from now on above ϵ but not tamper with it below ϵ (to preserve the fact that $S \cap \epsilon$ is nowhere stationary in ϵ). The purpose of E is to ensure that the generic set will not be stationary in κ.

LEMMA 6.4.2. *Let G be a $\mathbb{Q}^{\mathrm{am},1}_\kappa$-generic filter and let*

$$S^* = \bigcup \{S : (\exists p \in G)\ S = S^p\}, \qquad E^* = \bigcap \{E : (\exists p \in G)\ E = E^p\}.$$

Then:

(1) *E^* is a club of κ disjoint from S^*.*
(2) *S^* is a nowhere stationary subset of κ.*
(3) *For any nowhere stationary set $S \subseteq \kappa$ with $S \in \mathbf{V}$ we have $\mathbf{V}^{\mathbb{Q}^{\mathrm{am}1}_\kappa} \models S \subseteq^* S^*$ (i.e. the set $S \backslash S^*$ is bounded).*

We call S^* the generic nowhere stationary set.

Proof. (1) Assume that $(\epsilon, S, E) \Vdash$ "$E^* \subseteq \alpha < \kappa$". Find $\beta \in E$, $\gamma \in S^\kappa_{\mathrm{inc}}$ with $\alpha < \beta < \gamma$. Then $(\gamma, S, E) \leq (\alpha, S, E)$ and $(\gamma, S, E) \Vdash \beta \in E^*$, contradicting what (ϵ, S, E) forced. So E^* is unbounded.

As an intersection of closed sets, E^* must be closed; and it is disjoint from S^* by definition.

(2) To see $S^* \cap \alpha$ is non-stationary for $\alpha \in S^\kappa_{\mathrm{inc}}$ argue as in (1). To see S^* is non-stationary in κ, remember that E^* is a club disjoint from S^* by (1).

(3) Let $p = (\epsilon, S, E) \in \mathbb{Q}^{\mathrm{am},1}_\kappa$ and let $S' \in \mathbf{V}$ be nowhere stationary and let E' be a club disjoint from S'. Then $(\epsilon, S \cup (S' \setminus \epsilon), E \cap (E' \cup \epsilon)) \le p$ forces $S' \subseteq S^* \cup \epsilon$, hence also $S' \subseteq^* S^*$. As p was arbitrary, we are done. ∎

Lemma 6.4.3.

(1) $\mathbb{Q}^{\mathrm{am},1}_\kappa$ *is* $<\kappa$*-closed.*
(2) $\mathbb{Q}^{\mathrm{am},1}_\kappa$ *is* κ*-linked.*
(3) $\mathbb{Q}^{\mathrm{am},1}_\kappa$ *satisfies* $(*)_\kappa$.

Proof. (1) Let $\langle p_i : i < \delta \rangle$ be a strictly decreasing sequence, $\delta < \kappa$ a limit ordinal, and let $p_i = (\epsilon_i, S_i, E_i)$. Hence the sequence $\langle \epsilon_i : i < \delta \rangle$ is strictly increasing, so in particular $\epsilon_i \ge i$.

We define a condition $p^* = (\epsilon^*, S^*, E^*)$ as follows:

$$\epsilon^* = \sup_{j<\delta} \epsilon_j \quad (\text{so } \epsilon^* \ge \delta), \qquad S^* = \bigcup_{j<\delta} S_j, \qquad E^* = \bigcap_{j<\delta} E_j.$$

Clearly, E^* is a club in κ and is disjoint from S^*, so S^* is non-stationary.

For $\delta' < \delta$ the sequence $\langle S_i \cap \delta' : i < \delta \rangle$ is eventually constant with value $S_{\delta'} \cap \delta'$, so $S^* \cap \delta'$ is non-stationary in δ'.

For $\delta' > \delta$ the set $S^* \cap \delta'$ is the union of a small number of non-stationary sets, hence is non-stationary.

We have to check that $S^* \cap \delta$ is non-stationary in δ (if δ is inaccessible).

Case 1: $\epsilon^* = \delta$. Then $E^* \cap (\epsilon_i, \epsilon_{i+1}) = E_{i+1} \cap (\epsilon_i, \epsilon_{i+1})$ is non-empty for all $i < \delta$, so E^* is unbounded (hence club) in ϵ^*. Hence S^* is non-stationary in ϵ^*.

Case 2: $\epsilon^* > \delta$. Then we can find $i < \delta$ with $\epsilon_i > \delta$, and we see that $S^* \cap \epsilon_i = S_i \cap \epsilon_i$, so also $S^* \cap \delta = S_i \cap \delta$ is non-stationary.

Finally, we show that $p^* \le p_i$: the main point is that $(\forall j \ge i)\ S_j \cap \epsilon_i = S_i \cap \epsilon_i$, so also $S^* \cap \epsilon_i = S_i \cap \epsilon_i$.

(2) Consider $f : \mathbb{Q}^{\mathrm{am},1}_\kappa \to \kappa \times 2^{<\kappa} \times 2^{<\kappa}$ where $f(\epsilon, S, E) = (\epsilon, S \cap \epsilon, E \cap \epsilon)$. Now check that for $p, q \in \mathbb{Q}^{\mathrm{am},1}_\kappa$ we have $f(p) = f(q) \Rightarrow p \not\perp q$.

(3) By (1), (2) and 2.2.7. ∎

We want to iterate Amoeba forcing (together with the forcing in the next subsection, and possibly other forcings) and not lose the weak compactness of κ. So we will start in a model where κ is supercompact, and this supercompactness is not destroyed by $<\kappa$-directed closed forcing, nor by our Amoeba forcings.

As Amoeba forcing is not $<\kappa$-directed closed, we cannot use Laver's theorem directly. However, it is well known that a slightly weaker property is also sufficient.

The following definition is copied from [Kön06].

Definition 6.4.4. If P is a partial ordering, then we always let $\theta = \theta_P$ be the least regular cardinal such that $P \in H_\theta$. We say that a set $X \in \mathcal{P}_\kappa(H_\theta)$ is *P-complete* if every (X, P)-generic filter has a lower bound in P.

Define $\mathcal{H}(P) := \{X \in \mathcal{P}_\kappa(H_\theta) : X \text{ is } P\text{-complete}\}$. Then a partial ordering P is called *almost κ-directed closed* if P is κ-strategically closed and $\mathcal{H}(P)$ is in every supercompact ultrafilter on $\mathcal{P}_\kappa(H_\theta)$.

We will show that for the forcings P we consider, the set $\mathcal{H}(P)$ contains all small elementary submodels of H_θ, is therefore closed unbounded, hence an element of every (fine) normal ultrafilter on $\mathcal{P}_\kappa(H_\theta)$. (See [Kan94, Sect. 22 and 25.4].)

Definition 6.4.5. Let $G_1 \subseteq \mathbb{Q}_\kappa^{\mathrm{am},1}$. We call a triple (δ_1, S_1, E_1) a *pivot* for G_1 if the following hold (where δ_2 denotes the first inaccessible above δ_1):

- $\delta_1 < \kappa$ (usually a limit ordinal).
- S_1, E_1 are disjoint subsets of δ_1, E_1 is a club in δ_1, S_1 is nowhere stationary in δ_1.
- $G_1 \subseteq \mathbb{Q}_\kappa^{\mathrm{am},1}$, $|G_1| < \delta_2$, G_1 is a filter.
- For all $p = (\epsilon, S, E) \in G_1$, (S_1, E_1) is "stronger" than p in the following sense:
  - $\epsilon < \delta_1$.
  - $S \cap \epsilon = S_1 \cap \epsilon$, $E \cap \epsilon = E_1 \cap \epsilon$.
  - $S \cap \delta_1 \subseteq S_1$.
  - $E \cap \delta_1 \supseteq E_1$.

Note. When we say that G_1 has a pivot, it is implied that G_1 is a filter of small cardinality.

Lemma 6.4.6 (Master conditions in $\mathbb{Q}_\kappa^{\mathrm{am},1}$). *Assume that $G_1 \subseteq \mathbb{Q}_\kappa^{\mathrm{am},1}$ has a pivot. Then G_1 has a lower bound in $\mathbb{Q}_\kappa^{\mathrm{am},1}$, i.e. $(\exists p^* \in \mathbb{Q}_\kappa^{\mathrm{am},1})(\forall p \in G_1)$ $p^* \leq p$.*

Proof. Let (δ_1, S_1, E_1) be a pivot for G_1. We let $p^* := (\delta_1, S^*, E^*)$, where:

- $S^* \cap \delta_1 := S_1 \cap \delta_1$.
- $E^* \cap \delta_1 := E_1 \cap \delta_1$.
- $S^* \setminus \delta_1 := \bigcup_{(\epsilon,S,E) \in G_1} S \setminus \delta_1$.
- $E^* \setminus \delta_1 := \bigcap_{(\epsilon,S,E) \in G_1} E \setminus \delta_1$.

Note that the ideal of nowhere stationary subsets of $[\delta_1, \kappa)$ is δ_2-closed, so S^* is indeed nowhere stationary above δ_1 (also nowhere stationary below and up to δ_1, because S_1 had this property). Hence p^* is indeed a condition. It is clear that p^* is stronger than all $p \in G_1$. ∎

COROLLARY 6.4.7. *Let* $N \prec H_\theta$, $N \in \mathcal{P}_\kappa(H_\theta)$, $\mathbb{Q}^{\mathrm{am},1}_\kappa \in N$, $N \cap \kappa \in \kappa$. *Then* $N \in \mathcal{H}(\mathbb{Q}^{\mathrm{am},1}_\kappa)$ (*see Definition* 6.4.4).

Proof. Let $G \subseteq \mathbb{Q}^{\mathrm{am},1}_\kappa \cap N$ be $(N, \mathbb{Q}^{\mathrm{am},1}_\kappa)$-generic. Let $\delta_1 := N \cap \kappa$, and let (S_1, E_1) be the generic object determined by G as in 6.4.2. Then (δ_1, S_1, E_1) is a pivot for G, so by 6.4.6 we can find a lower bound for G in $\mathbb{Q}^{\mathrm{am},1}_\kappa$. ∎

6.5. Amoeba forcing, part 2

DEFINITION 6.5.1. Let $S \subseteq S^\kappa_{\mathrm{inc}}$. Let $\mathbb{Q}^{\mathrm{am},2}_{\kappa,S}$ be the forcing consisting of pairs $(\epsilon, \vec{A})$ where

$$\epsilon < \kappa, \qquad \vec{A} = (A_\delta : \delta \in S) \in \prod_{\delta \in S} \mathrm{id}(\mathbb{Q}_\delta).$$

For $p = (\epsilon_p, \vec{A}_p)$, $q = (\epsilon_q, \vec{A}_q)$ we define $q \le p$ iff either $q = p$, or

$$\epsilon_p < \epsilon_q, \qquad \vec{A}_p \restriction (S \cap \epsilon_p) = \vec{A}_q \restriction (S \cap \epsilon_p).$$

For all $\delta \in S$, $A_p(\delta) \subseteq A_q(\delta)$.

LEMMA 6.5.2. *Let G be a $\mathbb{Q}^{\mathrm{am},2}_{\kappa,S}$-generic filter, and let*

$$\vec{A}^* = (A^*_\delta : \delta \in S) = \bigcup_{(\epsilon, \vec{A}) \in G} \vec{A} \restriction \epsilon \in \prod_{\delta \in S} \mathrm{id}(\mathbb{Q}_\delta).$$

Then:

(1) *For all* $(B_\delta : \delta \in S)$, *where each* $B_\delta \subseteq 2^\delta$ *is in* $\mathrm{id}(\mathbb{Q}_\delta)$, *we have* $\Vdash (\forall^\infty \delta)\ B_\delta \subseteq A^*_\delta$.
(2) *For all* $B \in \mathrm{id}^-_0(\mathbb{Q}_{\kappa,S})$ *we have* $B \subseteq \mathrm{set}^-_0(\vec{A}^*)$.

Proof. (1) Let $p = (\epsilon, \vec{A}) \in \mathbb{Q}^{\mathrm{am},2}_{\kappa,S}$. Find $(\epsilon, \vec{A}') \in \mathbb{Q}^{\mathrm{am},2}_{\kappa,S}$ such that:

(a) $\vec{A} \restriction (S \cap \epsilon) = \vec{A}' \restriction (S \cap \epsilon)$.
(b) For all $\delta \in S$ with $\delta \ge \epsilon$ we have $A'_\delta = A_\delta \cup B_\delta$.

Because p was arbitrary, we are done.

(2) Let $B \subseteq \mathrm{set}^-_0(\langle B_\delta : \delta \in S\rangle)$. By (1) we have $B \subseteq \mathrm{set}^-_0(\vec{A}') \subseteq \mathrm{set}^-_0(\vec{A}^*)$. ∎

LEMMA 6.5.3. *Let $S \subseteq S^\kappa_{\mathrm{inc}}$. Then:*

(1) $\mathbb{Q}^{\mathrm{am},2}_\kappa$ *is $<\kappa$-closed.*
(2) $\mathbb{Q}^{\mathrm{am},2}_\kappa$ *is κ-linked.*
(3) $\mathbb{Q}^{\mathrm{am},2}_\kappa$ *satisfies* $(*)_\kappa$.

Proof. Similar to 6.4.3. ∎

DEFINITION 6.5.4. Let $\mathbb{Q}^{\mathrm{am}}_\kappa := \mathbb{Q}^{\mathrm{am},1}_\kappa * \mathbb{Q}^{\mathrm{am},2}_{\kappa,S^*}$ where S^* is the generic object from $\mathbb{Q}^{\mathrm{am},1}_\kappa$ as in 6.4.2.

Discussion 6.5.5. Note that $\mathbb{Q}^{\rm am}_\kappa$ here is not the same as the Amoeba forcing $\mathbb{Q}^{\rm am}_\kappa$ defined in [She17]. But as we see in 6.5.6, it is a modularized variant.

Lemma 6.5.6. *There exists $A^* \in \mathrm{id}^-(\mathbb{Q}) \cap \mathbf{V}^{\mathbb{Q}^{\rm am}_\kappa}$ such that:*

(1) *For every $A \in \mathbf{V} \cap \mathrm{id}^-(\mathbb{Q}_\kappa)$ we have $A \subseteq A^*$.*
(2) *If κ is weakly compact then $A \subseteq A^*$ for every $A \in \mathbf{V} \cap \mathrm{id}(\mathbb{Q}_\kappa)$.*

Proof. (1) Combine 6.4.2 and 6.5.2 and check that $A^* = \mathrm{set}_0^-(\langle A^*_\delta : \delta \in S^*\rangle)$ is as required.

(2) By (1) and 3.2.5. ∎

The generic null set added by Amoeba forcing will cover all ground model sets in $\mathrm{id}^-(\mathbb{Q}_\kappa)$. If κ is weakly compact, then we also cover all id-sets. So we are interested in keeping κ weakly compact after our Amoeba iteration.

Definition 6.5.7. Let $S \subseteq S^\kappa_{\rm inc}$ be nowhere stationary, and let $G_1 \subseteq \mathbb{Q}^{\rm am,2}_{\kappa,S}$. We call a pair $(\delta_1, \vec{A}_1)$ a *pivot* for G_1 if the following hold:

- $\delta_1 \in S^\kappa_{\rm inc} \backslash S$.
- $\vec{A}_1 = (A_{1,\delta} : \delta \in S \cap \delta_1) \in \prod_{\delta \in S \cap \delta_1} \mathrm{id}(\mathbb{Q}_\delta)$.
- $G_1 \subseteq \mathbb{Q}^{\rm am,2}_{\kappa,S}$, $|G_1| < \delta_2$, G_1 is a filter (where again δ_2 is the smallest inaccessible $> \delta_1$).
- For all $p := (\epsilon, \vec{B}) \in G_1$, we have $\epsilon < \delta_1$, and $(\delta_1, \vec{A}_1)$ is "stronger" than p in the sense that

$$(\forall \delta < \delta_1)\ B_\delta \subseteq A_{1,\delta}, \qquad (\forall \delta < \epsilon)\ B_\delta = A_{1,\delta}.$$

Lemma 6.5.8 (Master conditions in $\mathbb{Q}^{\rm am,2}_{\kappa,S}$). *Assume that S is nowhere stationary, and $G_1 \subseteq \mathbb{Q}^{\rm am,2}_{\kappa,S}$ has a pivot. Then the set G_1 has a lower bound in $\mathbb{Q}^{\rm am,2}_{\kappa,S}$, i.e. $(\exists p^* \in \mathbb{Q}^{\rm am,2}_{\kappa,S})(\forall p \in G_1)\ p^* \leq p$.*

Proof. The reasoning is similar to the proof of Lemma 6.4.6.

Let $(\delta_1, \vec{A}_1)$ be a pivot. We define a condition $p^* = (\delta_1, \vec{A}^*)$ as follows:

- $(\forall \delta \in S \cap \delta_1)\ A^*_\delta := A_{1,\delta}$.
- $(\forall \delta \in S \backslash \delta_1)\ A^*_\delta := \bigcup_{(\epsilon, \vec{A}) \in G_1} A_\delta$.

Why is p a condition? Because for all $\delta \in \kappa \backslash \delta_1$, the ideal $\mathrm{id}(\mathbb{Q}_\delta)$ is δ_1-complete, so the set $\bigcup_{(\epsilon,\nu) \in G_1} A_\delta$ is in the ideal.

It is clear that $p^* \leq p$ for all $p \in G_1$. ∎

Corollary 6.5.9. *For S nowhere stationary let $N \prec H_\theta$, $N \in \mathcal{P}_\kappa(H_\theta)$, $\mathbb{Q}^{\rm am,2}_{\kappa,S} \in N$, $N \cap \kappa \in \kappa$. Then $N \in \mathcal{H}(\mathbb{Q}^{\rm am,2}_{\kappa,S})$ (see Definition 6.4.4).*

Proof. Like 6.4.7. ∎

6.6. Iterated Amoeba forcing

NOTATION 6.6.1. For every forcing notion $\mathbb{P}$ we write $\Gamma_{\mathbb{P}}$ for the canonical name of the generic filter on $\mathbb{P}$.

DEFINITION 6.6.2.

(1) Let μ be an ordinal and let $\mathbb{P}$ be the limit of a $<\kappa$-support iteration $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{R}}_\alpha : \alpha < \mu \rangle$. We call the iteration $\vec{\mathbb{P}}$ and its limit $\mathbb{P}$ *relevant* if for every $\alpha < \mu$ we have either

  (a) $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{R}}_\alpha = \mathbb{Q}^{\mathrm{am},1}_\kappa$", or
  (b) $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{R}}_\alpha = \mathbb{Q}^{\mathrm{am},2}_{\kappa,S}$ for some nowhere stationary $S \subseteq S^\kappa_{\mathrm{inc}}$", or
  (c) $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{R}}_\alpha$ is $<\kappa$-directed closed".

  (In particular, any $<\kappa$-directed closed forcing is an example of a relevant iteration.)

(2) Let $G_0 \subseteq \mathbb{P}$ be a filter. For $\alpha < \mu$ we will write $G_0{\restriction}\alpha$ for the set $\{p{\restriction}\alpha : p \in G_0\}$, and $G_0(\alpha)$ will be a $\mathbb{P}_\alpha$-name for the set $\{p(\alpha) : p \in G_0\}$. We remark that $G_0{\restriction}(\alpha+1)$ is a subset of $\mathbb{P}_\alpha * \dot{\mathbb{R}}_\alpha$, so the empty condition of $\mathbb{P}_\alpha$ forces "if $G_0{\restriction}\alpha \subseteq \Gamma_{\mathbb{P}_\alpha}$, then $G_0(\alpha) \subseteq \dot{\mathbb{R}}_\alpha$".

(3) Let $G_0 \subseteq \mathbb{P}$ be a filter. A sequence $\langle \eta_\alpha : \alpha < \mu \rangle$ (where each η_α is a $\mathbb{P}_\alpha$-name) is called a *pivot* for G_0 if for all $\alpha < \mu$ the following statement is forced:

  If $G_0{\restriction}\mathbb{P}_\alpha \subseteq \Gamma_{\mathbb{P}_\alpha}$, then either
  - $\dot{\mathbb{R}}_\alpha$ is $<\kappa$-directed closed, $\eta_\alpha = 0$, or
  - η_α is a pivot (in the sense of Definition 6.4.5 or 6.5.7, respectively) for $G_0(\alpha) \subseteq \dot{\mathbb{R}}_\alpha$.

LEMMA 6.6.3 (Existence of master conditions in iterations). *Assume that $\mathbb{P}$ is the limit of a relevant iteration. Let $G_0 \subseteq \mathbb{P}$ be a filter, and assume that there is a pivot for G_0. Then there exists $p^* \in \mathbb{P}$ such that*

$$(\forall p \in G_0)\ p^* \leq p.$$

Proof. We define p^* by induction, in each coordinate appealing to Lemma 6.4.6 or 6.5.8, as appropriate. (Note that fewer than κ coordinates appear in the conditions in G_0, so the resulting condition will have support of size $< \kappa$.) ■

COROLLARY 6.6.4. *Let $N \prec H_\theta$, $N \in \mathcal{P}_\kappa(H_\theta)$, $N \cap \kappa \in \kappa$. Let $P \in N$ be a relevant iteration. Then $N \in \mathcal{H}(P)$ (see Definition 6.4.4).*

Hence by [Kön06, Theorem 9]*: If κ is supercompact, then after forcing with a modified Laver preparation we obtain a model in which κ is not only supercompact, but moreover this supercompactness cannot be destroyed by almost κ-directed closed forcing, so in particular not by relevant iterations.* ■

Definition 6.6.5. Let μ be an ordinal. Let $\mathbb{A}_{\kappa,\mu}$ be the limit of the $<\kappa$-support iteration $\langle \mathbb{A}_{\kappa,\alpha}, \dot{\mathbb{R}}_\alpha : \alpha < \mu\rangle$ where for every $\alpha < \mu$ we have

$$\mathbb{A}_{\kappa,\alpha} \Vdash \dot{\mathbb{R}}_\alpha = \begin{cases} \mathbb{Q}^{\mathrm{am}}_\kappa, & \alpha \text{ even},\\ \mathbb{H}_\kappa, & \alpha \text{ odd}.\end{cases}$$

Fact 6.6.6. *$\mathbb{A}_{\kappa,\mu}$ is an iteration satisfying the requirements of* 6.6.3. ∎

Lemma 6.6.7. *Let μ be an ordinal. Then $\mathbb{A}_{\kappa,\mu}$ satisfies the stationary κ^+-Knaster condition and in particular $\mathbb{A}_{\kappa,\mu}$ satisfies the κ^+-c.c.*

Proof. By 6.4.3, 6.5.3, 2.2.6, 2.2.3. ∎

Theorem 6.6.8. *Let $\mathbf{V} \models 2^\kappa = \kappa^+$ and let κ be supercompact, indestructible in the sense of* 6.6.4. *Then $\mathbf{V}^{\mathbb{A}_{\kappa,\kappa^{++}}}$ satisfies:*

(1) $2^\kappa = \kappa^{++}$.
(2) $\mathrm{add}(\mathbb{Q}_\kappa) = \kappa^{++}$.
(3) $\mathrm{add}(\mathrm{Cohen}_\kappa) = \kappa^{++}$.

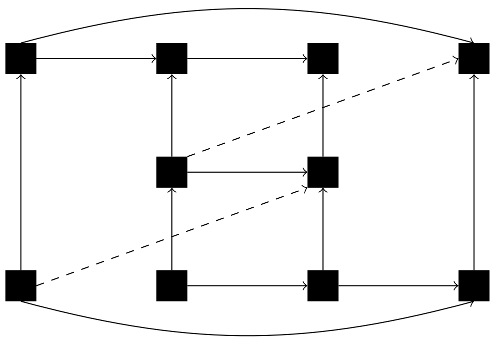

Fig. 7. The Amoeba model

Proof. (1) Should be clear.

(2) By (1) it suffices to show $\mathrm{add}(\mathbb{Q}_\kappa) \ge \kappa^{++}$. So towards a contradiction assume $\mathrm{add}(\mathbb{Q}_\kappa) = \kappa^+$ and let $\langle B_i : i < \kappa^+\rangle$ witness it. Remember $\mathbb{A}_{\kappa,\kappa^{++}}$ satisfies the κ^+-c.c. by 6.6.7. So there exists $\alpha < \kappa^{++}$ such that $B_i \in \mathbf{V}^{\mathbb{P}_\alpha}$ for every $i < \kappa^+$. But by 6.5.6 there exists $A \in V^{\mathbb{P}_{\alpha+2}} \cap \mathrm{id}(\mathbb{Q}_\kappa)$ such that $B_i \subseteq A$ for every $i < \kappa^+$. By 4.2.2 also $\mathbf{V}^{\mathbb{A}_{\kappa,\kappa^{++}}} \models A \in \mathrm{id}(\mathbb{Q}_\kappa)$, a contradiction.

(3) Argue as in 6.2.7. ∎

6.7. The short Amoeba model

Theorem 6.7.1. *Let $\mathbf{V} \models 2^\kappa = \kappa^+$ and let κ be supercompact, indestructible in the sense of* 6.4.4. *Let $\mu = \kappa^{++} \cdot \kappa^+$. Then $\mathbf{V}^{\mathbb{A}_{\kappa,\mu}}$ satisfies:*

(1) $2^\kappa = \kappa^{++}$.
(2) $\mathrm{cf}(\mathbb{Q}_\kappa) = \kappa^+$.

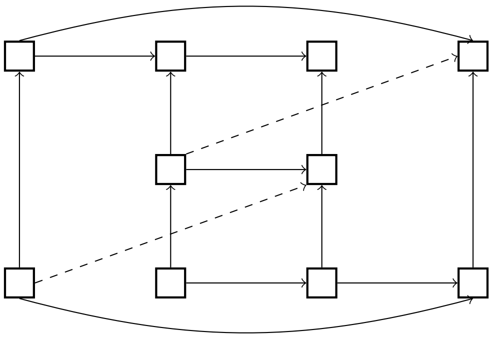

Fig. 8. The short Amoeba model

(3) $\mathfrak{d}_\kappa = \kappa^+$.
(4) $\mathrm{cf}(\mathrm{Cohen}_\kappa) = \kappa^+$.

Proof. (1) Should be clear.

(2) Let $\langle \mu_i : i < \kappa^+ \rangle$ be a cofinal sequence in μ such that each μ_i is even. Let A_i be the null set added by $\dot{\mathbb{R}}_{\mu_i}$. Clearly by 6.5.6 the sequence $\langle A_i : i < \kappa^+ \rangle$ is cofinal in $\mathrm{id}(\mathbb{Q}_\kappa)$.

(3) Let η_i be the Hechler real added by $\dot{\mathbb{R}}_{\mu_i+1}$. Clearly by 6.2.2 the sequence $\langle \eta_i : i < \kappa^+ \rangle$ is dominating.

(4) Assume $\mathrm{cf}(\mathrm{Cohen}_\kappa) > \kappa^+$. Then by (3) and 5.1.6 and (2), $\mathrm{cf}(\mathrm{Cohen}_\kappa) \leq \mathrm{non}(\mathbb{Q}_\kappa) \leq \mathrm{cf}(\mathbb{Q}_\kappa) = \kappa^+$, a contradiction. ■

6.8. Cohen-Amoeba forcing

Definition 6.8.1. Let $\mathbb{C}_\kappa^{\mathrm{am}}$ be the set of all pairs (α, A) such that:

- $\alpha < \kappa$.
- $A \subseteq 2^{<\kappa}$ is a tree.
- $[A] \subseteq 2^\kappa$ is non-empty nowhere dense.

For $p = (\alpha_p, A_p)$, $q = (\alpha_q, A_q)$, $p, q \in \mathbb{C}_\kappa^{\mathrm{am}}$ we define q to be stronger than p if:

- $\alpha_q \geq \alpha_p$.
- $A_q \supseteq A_p$.
- $A_q \restriction \alpha_p = A_p \restriction \alpha_p$.

We call $\mathbb{C}_\kappa^{\mathrm{am}}$ the *Cohen-Amoeba forcing.*

Note that $\mathbb{C}_\kappa^{\mathrm{am}}$ is a straightforward generalization of the universal meager forcing defined in [BJ95, 3.1.9].

Lemma 6.8.2. *Let $\langle A_i : i < i^* < \kappa \rangle$ be a family of nowhere dense subsets of 2^κ. Then $A = \bigcup_{i<i^*} A_i$ is nowhere dense.*

Proof. For $i < i^*$, $s \in 2^{<\kappa}$ let $t(i,s) \in 2^{<\kappa}$ be such that:

$$s \trianglelefteq t(i,s), \qquad A_i \cap [t(i,s)] = \emptyset.$$

Let $s \in A$ and define an increasing sequence $\langle \eta_i : i < i^* \rangle$ such that:

- $\eta_0 = s$.
- $i = j+1 \Rightarrow \eta_i = t(j, \eta_j)$.
- If i is a limit ordinal then $\eta_i = \bigcup_{j<i} \eta_j$.

Let $\eta = \bigcup_{i<i^*} \eta_i$ and check that

$$s \trianglelefteq \eta, \qquad A \cap [\eta] = \emptyset.$$

Because s was arbitrary, we are done. ■

Lemma 6.8.3.

(1) $\mathbb{C}_\kappa^{\mathrm{am}}$ *is* $<\kappa$*-directed closed.*
(2) $\mathbb{C}_\kappa^{\mathrm{am}}$ *is* κ*-centered*$_{<\kappa}$.
(3) $\mathbb{C}_\kappa^{\mathrm{am}}$ *satisfies* $(*)_\kappa$.

Proof. (1) Easy using 6.8.2.

(2) By 6.8.2.

(3) By (1), (2), and 2.2.7. ■

Lemma 6.8.4. *Let G be generic for $\mathbb{C}_\kappa^{\mathrm{am}}$ and let $N = \bigcup_{(\alpha,A)\in G} A$. Then for the set*

$$M = \{\eta \in 2^\kappa : (\exists \nu \in N)\ \nu =^* \eta\}$$

we have:

(1) *M is meager.*
(2) *M covers every meager set $X \in \mathbf{V}$. More precisely: For every family $(X_i : i < \kappa) \in \mathbf{V}$ of nowhere dense trees it is forced that $(\forall i < \kappa)\ [X_i] \subseteq M$.*

Proof. (1) It suffices to show that M is nowhere dense. We check that for each $s \in 2^{<\kappa}$ the set

$$D_s = \{q \in \mathbb{C}_\kappa^{\mathrm{am}} : (\exists t \trianglerighteq s)\ q \Vdash \text{“}N \cap [t] = \emptyset\text{”}\}$$

is dense in $\mathbb{C}_\kappa^{\mathrm{am}}$. Indeed for any $(\alpha, A) \in \mathbb{C}_\kappa^{\mathrm{am}}$ there exists $t \trianglerighteq s$ such that $A \cap [t] = \emptyset$. Now clearly $(\max(\alpha, |t|), A) \in D_s$.

(2) Let $X \subseteq 2^{<\kappa}$ be such that $[X]$ is nowhere dense and let $(\alpha, A) \in \mathbb{C}_\kappa^{\mathrm{am}}$. Without loss of generality we may assume $|X \cap 2^\alpha| = 1$ (otherwise we just split up X). Now find $\rho \in A \cap 2^\alpha$ and let

$$X' = \{\eta \in 2^\kappa : (\exists \nu \in X)\ \eta{\restriction}[\alpha,\kappa) = \nu{\restriction}[\alpha,\kappa),\ \eta{\restriction}\alpha = \rho\}.$$

Clearly, $q = (\alpha, A \cup X') \in \mathbb{C}_\kappa^{\mathrm{am}}$ and q forces X to be covered by M. ■

THEOREM 6.8.5. *Let* $\mathbf{V} \models 2^\kappa = \kappa^+$. *Let* $\mathbb{P} = \langle \mathbb{P}_i, \dot{\mathbb{R}}_i : i < \mu \rangle$ *be the limit of a* $<\kappa$*-support iteration such that* $\mathbb{P}_i \Vdash$ "$\dot{\mathbb{R}}_i = \mathbb{C}^{\mathrm{am}}_\kappa$" *for each* $i < \mu$. *Then* $\mathbf{V}^{\mathbb{P}}$ *satisfies:*

(1) *If* $\mu = \kappa^{++}$ *then* $\mathrm{add}(\mathrm{Cohen}_\kappa) = \kappa^{++}$.
(2) *If* $\mathrm{cf}(\mu) = \kappa^+$ *then* $\mathrm{cf}(\mathrm{Cohen}_\kappa) = \kappa^+$.

Proof. (1) Use 6.8.4 and argue as in 6.6.8(2).
(2) Use 6.8.4 and argue as in 6.7.1(2). ∎

COROLLARY 6.8.6. *We could use* $\mathbb{C}^{\mathrm{am}}_\kappa$ *instead of* $\mathbb{H}_\kappa$ *for odd iterands in the definition of* $\mathbb{A}_{\kappa,\mu}$ *in* 6.6.5 *to achieve the same results in* 6.6.8 *and* 6.7.1 *in regard to the characteristics of the diagram.*

THEOREM 6.8.7. *Let* $\mathbf{V} \models 2^\kappa = \kappa^+$. *Then* $\mathbf{V}^{\mathbb{C}^{\mathrm{am}}_{\kappa,\kappa^{++}}}$ *satisfies:*

(1) $\mathrm{cov}(\mathbb{Q}_\kappa) = \kappa^+$.
(2) $\mathrm{add}(\mathrm{Cohen}_\kappa) = \kappa^{++}$.
(3) $2^\kappa = \kappa^{++}$.

Proof. (1) Remember 6.8.3(2) and argue as in 6.2.7(1).
(2) Use 6.8.5(1).
(3) Should be clear. ∎

COROLLARY 6.8.8. $\mathbb{C}^{\mathrm{am}}_{\kappa,\kappa^{++}}$ *could be used as an alternative for* $\mathbb{H}_{\kappa,\kappa^{++}}$ *in* 6.2.7, *in the sense that the effect on the characteristics of the diagram is the same.*

THEOREM 6.8.9. *Let* $\mathbf{V} \models \kappa$ *is weakly compact. Let* $\mathbf{V} \models \mathrm{non}(\mathbb{Q}_\kappa) = \kappa^{++}$ (*e.g.* $\mathbf{V} = \mathbf{V}_0^{\mathbb{C}^{\mathrm{am}}\kappa,\kappa^{++}}$). *Then* $\mathbf{V}^{\mathbb{C}^{\mathrm{am}}_{\kappa,\kappa^+}}$ *satisfies:*

(1) $\mathrm{non}(\mathbb{Q}_\kappa) = \kappa^{++}$.
(2) $\mathrm{cf}(\mathrm{Cohen}_\kappa) = \kappa^+$.
(3) $2^\kappa = \kappa^{++}$.

Proof. (1) Remember 6.8.3(2) and argue as in 6.3.1(1).
(2) Use 6.8.5(2).
(3) Should be clear. ∎

COROLLARY 6.8.10. $\mathbb{C}^{\mathrm{am}}_{\kappa,\kappa^+}$ *could be used as an alternative for* $\mathbb{H}_{\kappa,\kappa^+}$ *in* 6.3.1, *in the sense that the effect on the characteristics of the diagram is the same.*

6.9. Bounded perfect tree forcing. We give a κ-support alternative to the short Hechler model.

DEFINITION 6.9.1. Let:

(1) $S \subseteq \kappa \cap S_{\mathrm{inc}}$, $\sup(S) = \kappa$, $\partial \in S \Rightarrow \partial > \sup(\partial \cap S_{\mathrm{inc}})$.
(2) $\langle \partial_\epsilon : \epsilon < \kappa \rangle$ enumerate S in increasing order.

(3) $\theta_\epsilon = 2^{\partial_\epsilon}$ for $\epsilon < \kappa$.
(4) $T = \bigcup_{\zeta<\kappa} T_\zeta$ where $T_\zeta = \prod_{\epsilon<\zeta} \theta_\epsilon$.

We define $\mathbb{T}^S_\kappa$ to be the set of all $p \subseteq T$ such that:

(a) For all $\eta \in p$ we have $\nu \trianglelefteq \eta \Rightarrow \nu \in p$.
(b) There exists a club $E \subseteq \kappa$ such that for all $\eta \in p$,

$$\begin{aligned} \mathrm{suc}_p(\eta) &= \{i < \theta_{\mathrm{lg}(\eta)} : \eta^\frown i \in p\} \\ &= \begin{cases} \theta_{\mathrm{lg}(\eta)} & \text{if } \mathrm{lg}(\eta) \in E, \\ \{i^*\} & \text{if } \mathrm{lg}(\eta) \notin E \text{ for some } i^* < \theta_{\mathrm{lg}(\eta)}. \end{cases} \end{aligned}$$

(c) No branches die out in p, that is, if ζ is a limit ordinal and $\eta \in T_\zeta$ then

$$\eta \in p \iff (\forall \epsilon < \zeta)\ \eta{\restriction}\epsilon \in p.$$

So $\mathbb{T}^S_\kappa$ is the forcing of all subtrees of T that split fully on a club $E \subseteq \kappa$ of levels and otherwise do not split. The order is defined the usual way, i.e. for $p, q \in \mathbb{T}^S_\kappa$ we have q stronger than p iff $q \subseteq p$. Because for our purposes every S works we will simply write $\mathbb{T}_\kappa$ instead of $\mathbb{T}^S_\kappa$.

We call $\mathbb{T}_\kappa$ *bounded perfect tree forcing* in analogy with [BJ95, 7.3.3, 7.3.43]. Note that $\mathbb{T}_\kappa$ is not the natural analogue of Sacks forcing $\mathbb{S}_\kappa$ from [Kan80] but $\mathbb{S}_\kappa$ and $\mathbb{T}_\kappa$ both have a natural notion of fusion and are κ^κ-bounding.

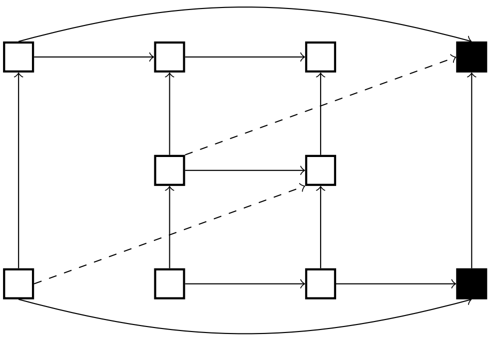

Fig. 9. The bounded perfect tree model

Definition 6.9.2. Let $\mathbb{T}_{\kappa,\mu}$ be the limit of the κ-support iteration $\langle \mathbb{T}_{\kappa,\alpha}, \dot{\mathbb{R}}_\alpha : \alpha < \mu \rangle$ where $\mathbb{T}_{\kappa,\alpha} \Vdash$ "$\dot{\mathbb{R}}_\alpha = \mathbb{T}_\kappa$" for every $\alpha < \mu$.

Lemma 6.9.3.

(1) $\mathbb{T}_\kappa$ *is* $<\kappa$*-directed closed.*
(2) $\mathbb{T}_{\kappa,\kappa^{++}}$ *is* $<\kappa$*-directed closed.*

Proof. (1) Let D be a directed subset of $\mathbb{T}_\kappa$ of size $< \kappa$. Intersecting the club sets associated with each $p \in D$ will give us a club set E. Letting q be the intersection of all $p \in D$, we claim that q is a condition. It is then clear that q is a lower bound for D.

Clearly q is non-empty and satisfies conditions (a), (c) of 6.9.1. It remains to verify (b). Let $\eta \in q$.

Case 1: $\lg(\eta) \in E$. So $\lg(\eta) \in E_p$ for all $p \in D$, hence $\mathrm{suc}_q(\eta) = \bigcap_{p\in D} \mathrm{suc}_p(\eta) = \theta_{\lg(\eta)}$.

Case 2: $\lg(\eta) \notin E$. So there is some $p^* \in D$ and some i^* such that $\mathrm{suc}_{p^*}(\eta) = \{i^*\}$. As D is directed, and $\eta \in p$ for all $p \in D$, we also have $\eta^\frown i^* \in p$ for all $p \in D$. Hence $\mathrm{suc}_q(\eta) = \bigcap_{p\in D} \mathrm{suc}_p(\eta) = \{i^*\}$, as required.

(2) By 2.1.6. ■

Definition 6.9.4. Let $\alpha < \kappa$, $p, q \in \mathbb{T}_\kappa$ and let $\langle e_i : i < \kappa\rangle$ be an enumeration of the club of splitting levels of p. We define

$$q \leq_\alpha p \quad \text{iff} \quad q \leq p \wedge q \cap \prod_{\zeta<e_\alpha} \theta_\zeta = p \cap \prod_{\zeta<e_\alpha} \theta_\zeta.$$

Lemma 6.9.5. *Let $\vec{p} = \langle p_i : i < \kappa\rangle$ be a sequence of conditions in $\mathbb{T}_\kappa$ such that $i < j < \kappa \Rightarrow p_j \leq_i p_i$. Then $\vec{p}$ has a lower bound $q \in \mathbb{T}_\kappa$.*

Proof. It is easy to check that $q = \bigcap_{i<\kappa} p_i$ is a condition in $\mathbb{T}_\kappa$ and a lower bound for $\vec{p}$. ■

Definition 6.9.6. We refer to sequences as in 6.9.5 as *fusion sequences.*

Lemma 6.9.7.

(a) *White has a winning strategy for $\mathfrak{F}^*_\kappa(\mathbb{T}_\kappa, p)$ for every $p \in \mathbb{T}_\kappa$.*
(b) *White has a winning strategy for $\mathfrak{F}_\kappa(\mathbb{T}_{\kappa,\kappa^{++}}, p)$ for every $p \in \mathbb{T}_{\kappa,\kappa^{++}}$.*

Proof. (a) We are going to construct a fusion sequence $\langle p_\zeta : \zeta < \kappa\rangle$ and a winning strategy for White such that:

(1) $p_0 = p$.
(2) In the ζ-round White plays $\mu_\zeta = |p_\zeta \cap T_\beta|$ and $p_{\zeta,i} = p^{[\eta_{\zeta,i}]}$, where $p_\zeta \cap T_\beta = \{\eta_{\zeta,i} : i < \mu_\zeta\}$ and β is the ζth splitting level of p_ζ.
(3) $p_{\zeta+1} = \bigcup_{i<\mu_\zeta} p'_{\zeta,i}$ where $p'_{\zeta,i}$ are the moves played by Black.
(4) For δ a limit ordinal $p_\delta = \bigcap_{\zeta<\delta} p_\zeta$.

Now use 6.9.5 and check that $q = \bigcap_{\zeta<\kappa} p_\zeta$ witnesses that White wins.

(b) By 2.4.7. ■

Lemma 6.9.8.

(a) *$\mathbb{T}_{\kappa,\kappa^{++}}$ does not collapse κ^+.*
(b) *Let N be a κ-meager set in $\mathbf{V}^{\mathbb{T}_{\kappa,\kappa^{++}}}$. Then there exists a κ-meager set $M \in \mathbf{V}$ such that $N \subseteq M$.*
(c) *In particular: If $\mathbf{V} \models 2^\kappa = \kappa^+$ then $\mathbf{V}^{\mathbb{T}_{\kappa,\kappa^{++}}} \models \mathrm{cf}(\mathrm{Cohen}_\kappa) = \kappa^+$.*

Proof. By 6.9.7, 2.4.6. ■

LEMMA 6.9.9. *If* $\mathbf{V} \models 2^\kappa = \kappa^+$ *then:*

(a) $\mathbb{T}_\kappa$ *satisfies the* κ^{++}*-c.c.*
(b) $\mathbb{T}_{\kappa,\kappa^{++}}$ *satisfies the* κ^{++}*-c.c.*

Proof. (a) By our assumption, $|\mathbb{T}_\kappa| = \kappa^+$.

(b) By 6.9.7, player White has a winning strategy in the game $\mathfrak{F}^*_\kappa(\mathbb{T}_\kappa, p)$. As each iterand $\dot{\mathbb{R}}_\alpha$ is forced to be equal to $\mathbb{T}_\kappa$, the assumptions of 2.5.9 are satisfied, so the limit of this iteration satisfies the κ^{++}-c.c. ∎

LEMMA 6.9.10.

(a) $\mathbb{T}_\kappa \Vdash (2^\kappa)^V \in \mathrm{id}^-(\mathbb{Q}_\kappa)$.
(b) $\mathbf{V}^{\mathbb{T}_{\kappa,\kappa^{++}}} \models \mathrm{non}(\mathrm{id}^-(\mathbb{Q}_\kappa)) \geq \kappa^{++}$.
(c) $\mathbf{V}^{\mathbb{T}_{\kappa,\kappa^{++}}} \models \mathrm{non}(\mathrm{id}(\mathbb{Q}_\kappa)) \geq \kappa^{++}$.

Proof. (a) Let $\langle A_{\epsilon,i} : i < \theta_\epsilon \rangle$ be a covering sequence in $\mathrm{id}(\mathbb{Q}_{\partial_\epsilon})$. Let $\dot\nu$ be a name for the generic κ-real added by $\mathbb{T}_\kappa$ and define $\vec{\Lambda} = \langle \Lambda_\partial : \partial \in S \rangle$ such that $\mathrm{set}_0(\Lambda_{\partial_\epsilon}) = A_{\epsilon,\dot\nu(\epsilon)}$. Now Λ witnesses $(2^\kappa)^{\mathbf{V}} \in \mathrm{id}^-(\mathbb{Q}_\kappa)$ in $\mathbf{V}^{\mathbb{T}_\kappa}$.

(b) Remember that by 6.9.9 all Borel sets appear in $\mathbf{V}^{\mathbb{T}_{\kappa,\alpha}}$ for some $\alpha < \kappa^{++}$. So (b) follows from (a), remembering 6.9.3, 2.1.5, 4.2.2.

(c) Remember $\mathrm{id}^-(\mathbb{Q}_\kappa) \subseteq \mathrm{id}(\mathbb{Q}_\kappa)$ hence $\mathrm{non}(\mathrm{id}^-(\mathbb{Q}_\kappa)) \leq \mathrm{non}(\mathrm{id}(\mathbb{Q}_\kappa))$. So this follows from (b). ∎

DISCUSSION 6.9.11. The coverings in 6.9.10 could just be sequences of singletons. So we could say that the lemma speaks about some ideal id^{--} that is defined similar to id^- just with singletons (or maybe sets of size at most κ) instead of $\mathrm{id}(\mathbb{Q}_\delta)$-sets on each level. So we really show $\mathrm{non}(\mathrm{id}^{--}(\mathbb{Q}_\kappa)) \geq \kappa^{++}$.

THEOREM 6.9.12. *If* $\mathbf{V} \models 2^\kappa = \kappa^+$ *then* $\mathbf{V}^{\mathbb{T}_{\kappa,\kappa^{++}}} \models 2^\kappa = \kappa^{++}$. ∎

6.10. On $\mathbb{Q}_\kappa$-models

QUESTION 6.10.1. Finding the "right" way to iterate $\mathbb{Q}_\kappa$ is an open problem. Ideally we wish for a κ^κ-bounding forcing adding many κ-random reals.

It is not clear how to iterate $\mathbb{Q}_\kappa$ with κ-support because we lack a fusion/properness argument. In particular it is not clear how to preserve κ^κ-boundedness.

Starting with $2^\kappa = \kappa^+$ and iterating $\mathbb{Q}_\kappa$ with $<\kappa$-support for κ^{++}-many steps, we get $\mathrm{cov}(\mathbb{Q}_\kappa) = \mathrm{cov}(\mathrm{Cohen}_\kappa) = \kappa^{++}$ (2). It is not clear what happens to $\mathfrak{b}_\kappa$ in this model but there may be some hope to control it by a so-called *corrected iteration* developed in [Shea] (which is a tool used in [Sheb]). It seems reasonable to conjecture that $\mathrm{add}(\mathbb{Q}_\kappa)$ remains κ^+ in this model.

(2) Note that $\mathrm{cov}(\mathbb{Q}_\kappa) = \kappa^{++}$ needs 4.2.6.

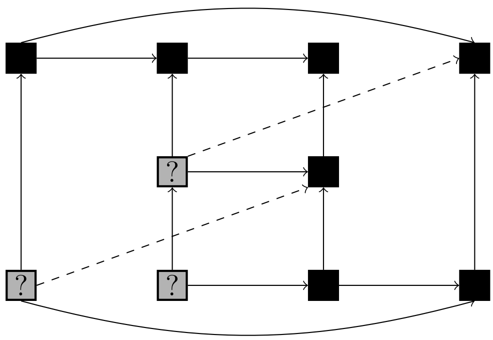

Fig. 10. The $<\kappa$-support random model

Starting with $\mathrm{cov}(\mathrm{Cohen}_\kappa) = 2^\kappa = \kappa^{++}$ and iterating $\mathbb{Q}_\kappa$ with $<\kappa$-support for κ^+-many steps we get $\mathrm{cov}(\mathbb{Q}_\kappa) = \mathrm{cov}(\mathrm{Cohen}_\kappa) = \kappa^+$. Again it is not clear what happens to $\mathfrak{d}_\kappa$.

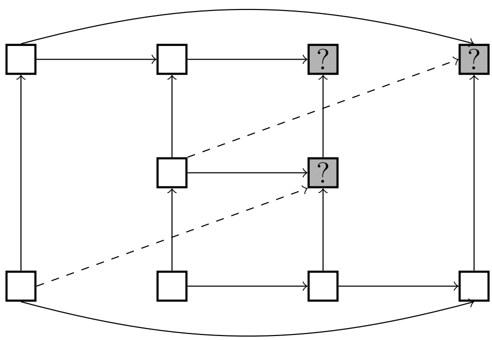

Fig. 11. The short $<\kappa$-support random model

7. Slaloms. It is well known that slaloms can be used to characterize the additivity and cofinality of measure in the classical case (see for example [BJ95, Section 2.3.A]). In [BB+18] this result motivates a definition: the cardinals add(null) and cof(null) are replaced by the appropriate additivity and covering numbers for slaloms.

This raises the question how the characteristics introduced there related to the characteristics of $\mathrm{id}(\mathbb{Q}_\kappa)$ discussed here. In particular one might wonder if the generalized characterization of the additivity of null by slaloms is equal to the additivity of $\mathrm{id}(\mathbb{Q}_\kappa)$. It turns out that for partial slaloms the answer is negative. We conjecture that for total slaloms the answer is negative too, see 7.2.5 and 7.3.1 respectively.

7.1. Recapitulation. Let us start with some results and definitions from [BB+18] (for more details and proofs see there). Since also successor cardinals κ are considered there, let us remind the reader that in this paper the cardinal κ is always (at least) inaccessible.

Definition 7.1.1. Let $h \in \kappa^\kappa$ be an unbounded function. We define

$$\mathcal{C}_h = \{\phi \in ([\kappa]^{<\kappa})^\kappa : (\forall i < \kappa)\ \phi(i) \in [\kappa]^{|h(i)|}\}.$$

For $\phi \in \mathcal{C}_h$, $f \in \kappa^\kappa$ we define

$$f \in^* \phi \iff (\forall^\infty i < \kappa)\ f(i) \in \phi(i).$$

Finally we let

$$\mathrm{add}(h\text{-slalom}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \kappa^\kappa,\ (\forall \phi \in \mathcal{C}_h)(\exists f \in \mathcal{F})\ f \not\in^* \phi\},$$
$$\mathrm{cf}(h\text{-slalom}) = \min\{|\Phi| : \Phi \subseteq \mathcal{C}_h,\ (\forall f \in \kappa^\kappa)(\exists \phi \in \Phi)\ f \in^* \phi\}.$$

Definition 7.1.2. We may also consider partial slaloms. Let $h \in \kappa^\kappa$ be unbounded and define

$$\mathsf{p}\mathcal{C}_h = \{\phi : (\exists \psi \in \mathcal{C}_h)\ \phi \subseteq \psi,\ |\mathrm{dom}(\phi)| = \kappa\}.$$

Again for $\phi \in \mathsf{p}\mathcal{C}_h$, $f \in \kappa^\kappa$ we define

$$f\ \mathsf{p}\!\in^* \phi \iff (\forall^\infty i \in \mathrm{dom}(\phi))\ f(i) \in \phi(i).$$

Finally, we let

$$\mathrm{add}^{\mathrm{partial}}(h\text{-slalom}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \kappa^\kappa,\ (\forall \phi \in \mathsf{p}\mathcal{C}_h)(\exists f \in \mathcal{F})\ f\ \mathsf{p}\!\not\in^* \phi\},$$
$$\mathrm{cf}^{\mathrm{partial}}(h\text{-slalom}) = \min\{|\Phi| : \Phi \subseteq \mathsf{p}\mathcal{C}_h,\ (\forall f \in \kappa^\kappa)(\exists \phi \in \Phi)\ f\ \mathsf{p}\!\in^* \phi\}.$$

Discussion 7.1.3. Note that in [BB+18] the notation

$$\mathrm{add}(h\text{-slalom}) = \mathfrak{b}_h(\in^*), \qquad \mathrm{cf}(h\text{-slalom}) = \mathfrak{d}_h(\in^*)$$

and similarly

$$\mathrm{add}^{\mathrm{partial}}(h\text{-slalom}) = \mathfrak{b}_h(\mathsf{p}\!\in^*), \qquad \mathrm{cf}^{\mathrm{partial}}(h\text{-slalom}) = \mathfrak{d}_h(\mathsf{p}\!\in^*)$$

is used.

Lemma 7.1.4. *Let $h \in \kappa^\kappa$ be unbounded. Then*

$$\mathrm{add}(h\text{-slalom}) \le \mathrm{add}^{\mathrm{partial}}(h\text{-slalom}) \le \mathrm{add}(\mathrm{Cohen}_\kappa),$$
$$\mathrm{cf}(h\text{-slalom}) \ge \mathrm{cf}^{\mathrm{partial}}(h\text{-slalom}) \ge \mathrm{cf}(\mathrm{Cohen}_\kappa).$$

Proof. Use [BB+18, Observation 36, Corollary 41]. ■

Lemma 7.1.5. *Let $h, g \in \kappa^\kappa$ be unbounded. Then*

$$\mathrm{add}^{\mathrm{partial}}(h\text{-slalom}) = \mathrm{add}^{\mathrm{partial}}(g\text{-}slalom),$$
$$\mathrm{cf}^{\mathrm{partial}}(h\text{-slalom}) = \mathrm{cf}^{\mathrm{partial}}(g\text{-slalom}).$$

Proof. Use [BB+18, Corollary 38]. ■

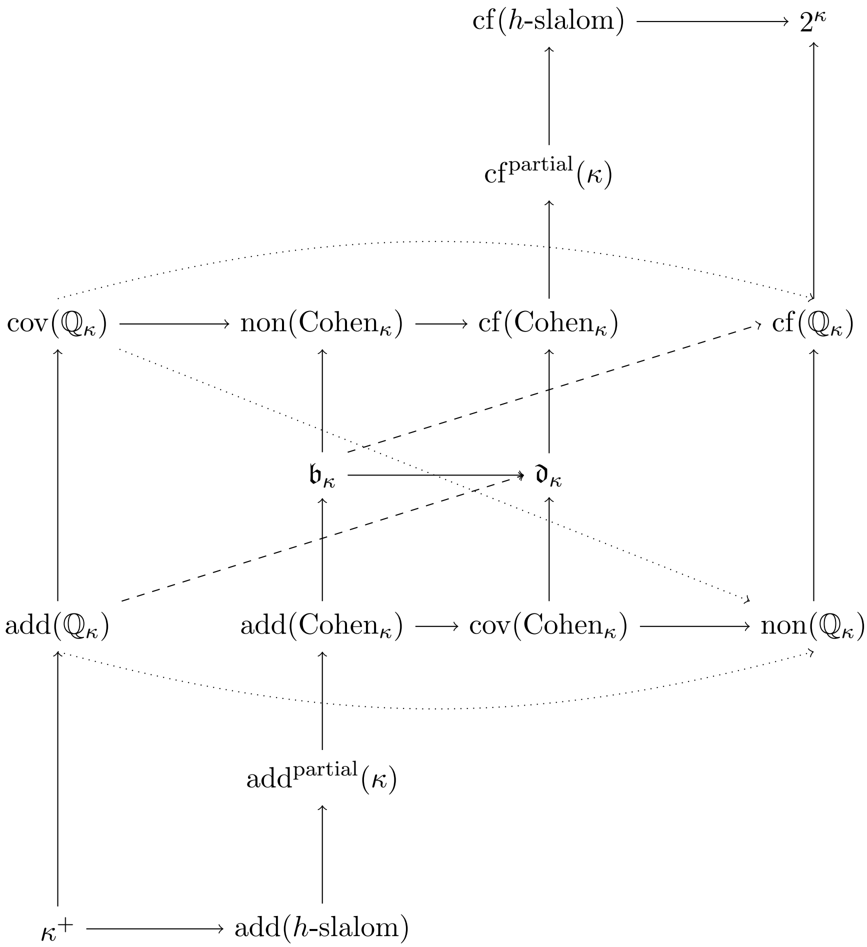

Fig. 12. The combined diagram: characteristics related to slaloms and $\mathrm{id}(\mathbb{Q}_\kappa)$. Remember that the dashed lines connected to $\mathfrak{b}_\kappa, \mathfrak{d}_\kappa$ require κ to be Mahlo.

Discussion 7.1.6. As a consequence, for partial slaloms we may ignore h and write $\mathrm{add}^{\mathrm{partial}}(\kappa)$ instead of $\mathrm{add}^{\mathrm{partial}}(h\text{-slalom})$ and similarly $\mathrm{cf}^{\mathrm{partial}}(\kappa)$ instead of $\mathrm{cf}^{\mathrm{partial}}(h\text{-slalom})$.

7.2. Separating partial slaloms from $\mathrm{id}(\mathbb{Q}_\kappa)$**.** The following forcing is used in [BB+18] to show $\mathrm{CON}(\mathrm{add}(h\text{-slalom}) < \mathrm{add}^{\mathrm{partial}}(\kappa))$. We are going to investigate its effect on $\mathrm{id}(\mathbb{Q}_\kappa)$.

Definition 7.2.1. Consider the forcing $\mathsf{p}\mathbb{L}_\kappa$ consisting of all pairs (ϕ, A) such that:

- ϕ is an initial segment of an element of $\mathsf{p}\mathcal{C}^\kappa_{\mathrm{id}}$, i.e. $|\mathrm{dom}(\phi)| < \kappa$.
- $A \subseteq \kappa^\kappa$, $|A| < \kappa$.

For $p = (\phi_p, A_p)$, $q = (\phi_q, A_q)$, $p, q \in \mathsf{p}\mathbb{L}_\kappa$ we define q to be stronger than p if:

- $\phi_q \supseteq \phi_p$.
- $(\mathrm{dom}(\phi_q) \setminus \mathrm{dom}(\phi_p)) \cap \sup(\mathrm{dom}(\phi_p)) = \emptyset$.
- $A_q \supseteq A_p$.
- $i \in (\mathrm{dom}(\phi_q) \setminus \mathrm{dom}(\phi_p))$, $f \in A_p \Rightarrow f(i) \in \phi_q(i)$.

If G is a $\mathsf{p}\mathbb{L}_\kappa$ generic filter then

$$\phi^* = \bigcup_{(\phi,A)\in G} \phi$$

is a partial slalom and we call ϕ^* a *generic partial slalom.* So the intended meaning of $(\phi, A) \in \mathsf{p}\mathbb{L}_\kappa$ is the promise that the generic partial slalom ϕ^* will satisfy

$$\phi \trianglelefteq \phi^*, \qquad f \,\mathsf{p}\!\in^* \phi^* \text{ for every } f \in A.$$

Definition 7.2.2. Let μ be an ordinal. Let $\mathsf{p}\mathbb{L}_{\kappa,\mu}$ be the limit of the $<\kappa$-support iteration $\langle \mathsf{p}\mathbb{L}_{\kappa,\alpha}, \dot{\mathbb{R}}_\alpha : \alpha < \mu \rangle$ where $\mathsf{p}\mathbb{L}_{\kappa,\alpha} \Vdash$ "$\dot{\mathbb{R}}_\alpha = \mathsf{p}\mathbb{L}_\kappa$" for every $\alpha < \mu$.

Note that 7.2.3(2), 7.2.4(2)–(5), and 7.2.6(2)–(4) were already shown in [BB+18].

Lemma 7.2.3.

(1) $\mathsf{p}\mathbb{L}_{\kappa,\mu}$ *satisfies* $(*)_\kappa$.
(2) *If* $\mu < (2^\kappa)^+$ *then* $\mathsf{p}\mathbb{L}_{\kappa,\mu}$ *is* κ*-centered*$_{<\kappa}$.

Proof. (1) Check that $\mathsf{p}\mathbb{L}_\kappa$ satisfies $(*)_\kappa$ and use 2.2.6.

(2) Check that

$$\mathsf{p}\mathbb{L}_\kappa = \bigcup_{\phi\in\mathsf{p}\mathcal{C}^\kappa} \{(\phi, A) : A \in [\kappa]^{<\kappa}\}$$

and use 2.3.7. ■

Theorem 7.2.4 (Partial slalom model). *Let* $\mathbf{V} \models 2^\kappa = \kappa^+$. *Then* $\mathbf{V}^{\mathsf{p}\mathbb{L}_{\kappa,\kappa^{++}}}$ *satisfies:*

(1) $\mathrm{cov}(\mathbb{Q}_\kappa) = \kappa^+$.
(2) $\mathrm{add}^{\mathrm{partial}}(\kappa) = \kappa^{++}$.
(3) $\mathrm{add}(h\text{-slalom}) = \kappa^+$.
(4) $\mathrm{add}(\mathrm{Cohen}_\kappa) = \kappa^{++}$.
(5) $2^\kappa = \kappa^{++}$.

Proof. (1) Argue as in 6.2.7.

(2) Assume $|\mathcal{F}|$ witnesses $\mathrm{add}^{\mathrm{partial}}(\kappa) = \kappa^+$. Then by the κ^+-c.c., $\mathcal{F}$ already appears in some $\mathbf{V}_\alpha$ and the generic partial slalom added by $\dot{\mathbb{R}}_\alpha$ covers every $f \in \mathcal{F}$, a contradiction.

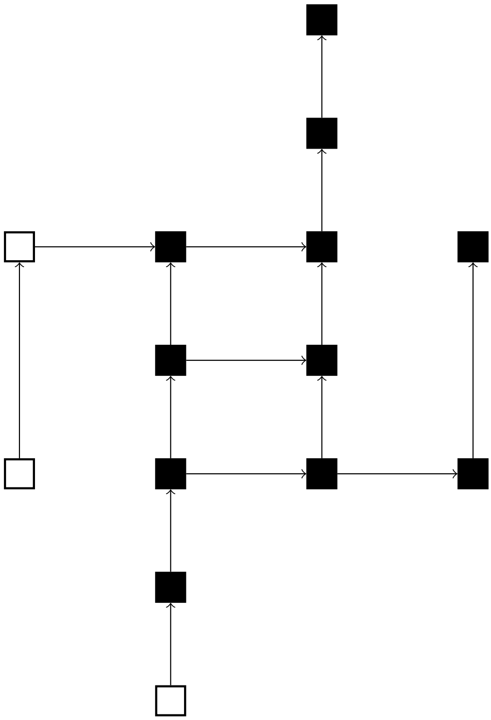

Fig. 13. The partial slalom model

(3) This is shown in [BB+18, Theorem 60(i)]. The argument there is similar to (1) in the sense that it is shown that κ-centered$_{<\kappa}$ forcings do not increase add(h-slalom) $= \kappa^+$.

(4) By (3) and 7.1.4.

(5) Should be clear. ■

Corollary 7.2.5.

(1) $\mathrm{CON}(\mathrm{add}(\mathbb{Q}_\kappa) < \mathrm{add}^{\mathrm{partial}}(\kappa))$.
(2) $\mathrm{add}(\mathbb{Q}_\kappa) = \mathrm{add}^{\mathrm{partial}}(\kappa)$ *is not a ZFC-theorem.*

Theorem 7.2.6 (Short partial slalom model). *Let* $\mathbf{V} \models 2^\kappa = \mathrm{cf}(h\text{-slalom}) = \mathrm{non}(\mathbb{Q}_\kappa) = \kappa^{++}$ *(e.g.* $\mathbf{V} = \mathbf{V}_0^{\mathbb{C}_{\kappa,\kappa^{++}}}$*) and* $\mathbf{V} \models \kappa$ *be weakly compact. Then* $\mathbf{V}^{\mathsf{p}\mathbb{L}_{\kappa,\kappa^+}}$ *satisfies:*

(1) $\mathrm{non}(\mathbb{Q}_\kappa) = \kappa^{++}$.
(2) $\mathrm{cf}^{\mathrm{partial}}(\kappa) = \kappa^+$.
(3) $\mathrm{cf}(h\text{-slalom}) = \kappa^{++}$.
(4) $2^\kappa = \kappa^{++}$.

Proof. (1) By 7.2.3(2) and 2.3.13.

(2) Let ϕ_α be the generic partial slalom added by $\dot{\mathbb{R}}_\alpha$. It is easily seen that $\langle \phi_\alpha : \alpha < \kappa^+ \rangle$ is a cofinal family.

(3) This is shown in [BB$^+$18, Theorem 60(ii)].

(4) Should be clear. ■

Corollary 7.2.7.

(1) $\mathrm{CON}(\mathrm{cf}(\mathbb{Q}_\kappa) > \mathrm{cf}^{\mathrm{partial}}(\kappa))$.
(2) $\mathrm{cf}(\mathbb{Q}_\kappa) = \mathrm{cf}^{\mathrm{partial}}(\kappa)$ *is not a ZFC-theorem.* ■

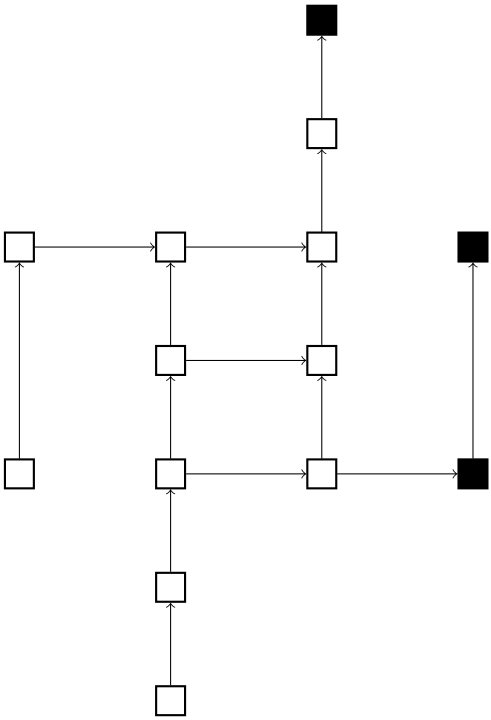

Fig. 14. The short partial slalom model

7.3. On total slaloms and $\mathrm{id}(\mathbb{Q}_\kappa)$**.** The next conjecture follows from Conjecture 5.2.12 (and may be easier to prove):

Conjecture 7.3.1.

(1) $\mathrm{CON}(\mathrm{add}(\mathbb{Q}_\kappa) > \mathrm{add}^{\mathrm{partial}}(\kappa))$.
(2) *In particular also* $\mathrm{CON}((\forall h \in \kappa^\kappa)\ \mathrm{add}(\mathbb{Q}_\kappa) > \mathrm{add}(h\text{-slalom}))$.
(3) $(\exists h \in \kappa^\kappa)\ \mathrm{add}(\mathbb{Q}_\kappa) = \mathrm{add}(h\text{-slalom})$ *is not a ZFC-theorem.* ■

Question 7.3.2. Is $\mathrm{add}(\mathbb{Q}_\kappa) < \mathrm{add}(h\text{-slalom})$ consistent? For a very partial answer see 7.3.4.

Lemma 7.3.3. *Let* $S \subseteq S^{\kappa}_{\mathrm{inc}}$ *be nowhere stationary. Then* add(h-slalom) $\leq$ add(id$^{-}(\mathbb{Q}^*_{\kappa,S})$) *if:*

(1) $\epsilon < \kappa \Rightarrow h(\epsilon) \leq \min(S \backslash (\epsilon+1))$, *or*
(2) *at least the above holds on club* $E \subseteq \kappa \backslash S$.

Proof. Let

$$\mathcal{A} \subseteq \{\langle A_\delta : \delta \in S\rangle : A_\delta \in \mathrm{id}(\mathbb{Q}_\delta)\}$$

be such that $|\mathcal{A}| <$ add(h-slalom). We are going to find an upper bound for $\mathcal{A}$. Let $\langle \epsilon_i : i < \kappa\rangle$, $\epsilon_0 = 0$, increasingly enumerate a club disjoint from S.

For $A \in \mathcal{A}$ we define $f_A : \kappa \to \kappa$ such that $f(\epsilon_i)$ codes $A \restriction (\epsilon_i, \epsilon_{i+1})$. Now by our assumption there exists an h-slalom ϕ which covers all f_A, i.e.

$$(\forall^\infty i < \kappa)\ f_A(\epsilon_i) \in \phi(\epsilon_i).$$

For $\delta \in (\epsilon_i, \epsilon_{i+1})$ with $\delta \in S$ define

$$A^*_\delta = \bigcup \{X : \text{there is a code of a sequence } \langle A_\sigma : \sigma \in S \cap (\epsilon_i, \epsilon_{i+1})\rangle \text{ appearing in } \phi(\epsilon_i) \text{ such that } X = A_\delta\}.$$

By our assumption on h we have $\epsilon_i < \min(S \backslash (\epsilon_i + 1)) \leq \delta$, so A^*_δ is the union of at most δ-many elements of id$(\mathbb{Q}_\delta)$, hence $A^*_\delta \in$ id$(\mathbb{Q}_\delta)$ and $\langle A^*_\delta : \delta \in S\rangle$ is an upper bound for $\mathcal{A}$. ■

Corollary 7.3.4. *Let* κ *be weakly compact and let* h *be the identity function on* κ. *Let*

$$\mu_1 = \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa), \qquad \mu_2 = \min\{\mathrm{add}(\mathrm{id}^-(\mathbb{Q}^*_{\kappa,S})) : S \in \mathbf{nst}^{\mathrm{pr}}_\kappa\}.$$

If either

(1) add$(\mathbb{Q}_\kappa) < \mu_1$, *or just*
(2) $\mu_2 \leq$ add$(\mathbb{Q}_\kappa)$,

then add$(\mathbb{Q}_\kappa) \geq$ add(h-slalom).

Proof. First we show (1) implies (2) by proving

$$\mathrm{add}(\mathbb{Q}_\kappa) \geq \min\{\mu_1, \mu_2\}.$$

This is done almost the same way as for the similar result in 3.3.9. The key point is that by weak compactness and 3.2.5 we get $A_i \in$ id$^-(\mathbb{Q}^*_{\kappa,S'_i})$ for some S'_i. Then find some $S^* \supseteq^* S'_i$ for all $i < i^*$ and clearly $A_i \in$ id$^-(\mathbb{Q}^*_{\kappa,S^*})$.

Now assume (2). Clearly $h =$ id satisfies the requirements of 7.3.3 for any $S \in \mathbf{nst}^{\mathrm{pr}}_\kappa$. Hence clearly add($h$-slalom) $\leq \mu_2$. ■

Acknowledgements. We thank James Cummings for helpful discussions on indestructibility, and Yair Hayut for alerting us to [Kön06].

We are grateful to the referee for a thorough report including many helpful suggestions for improvements.

The first and the second author were partially supported by the Austrian Science Fund through grant FWF P29575. Visits of the first and second author to the third author were supported by the National Science Foundation NSF grant no. 136974. All three authors were partially supported by the European Research Council under grant ERC-2013-ADG-338821. Publication 1144 on http://shelah.logic.at/.

References

[BJ95] T. Bartoszyński and H. Judah, *Set Theory: On the Structure of the Real Line*, A K Peters, Wellesley, MA, 1995.

[Bau19] T. Baumhauer, *Higher random indestructibility of MAD families*, arXiv:1904.04576 (2019).

[Bla10] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in: Handbook of Set Theory, Vol. 1, Springer, Dordrecht, 2010, 395–489.

[Bla11] A. Blass, *Finite support iterations of σ-centered forcing notions*, https://mathoverflow.net/q/84129, 2011.

[BB+18] J. Brendle, A. Brooke-Taylor, S.-D. Friedman, and D. C. Montoya, *Cichoń's diagram for uncountable cardinals*, Israel J. Math. 225 (2018), 959–1010.

[CS19] S. Cohen and S. Shelah, *Generalizing random real forcing for inaccessible cardinals*, Israel J. Math 234 (2019), 547–580.

[Eas70] W. B. Easton, *Powers of regular cardinals*, Ann. Math. Logic 1 (1970), 139–178.

[EK65] R. Engelking and M. Karłowicz, *Some theorems of set theory and their topological consequences*, Fund. Math. 57 (1965), 275–285.

[FKK16] S. D. Friedman, Y. Khomskii, and V. Kulikov, *Regularity properties on the generalized reals*, Ann. Pure Appl. Logic 167 (2016), 408–430.

[FL17] S.-D. Friedman and G. Laguzzi, *A null ideal for inaccessibles*, Arch. Math. Logic 56 (2017), 691–697.

[GK16] M. Goldstern and J. Kellner, *Review of U. Abraham's chapter "Proper Forcing" in "Handbook of Set Theory"*, Math. Rev. MR2768684, 2016.

[Kan80] A. Kanamori, *Perfect-set forcing for uncountable cardinals*, Ann. Math. Logic 19 (1980), 97–114.

[Kan94] A. Kanamori, *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings*, Perspectives Math. Logic, Springer, Berlin, 1994.

[KL+16] Y. Khomskii, G. Laguzzi, B. Löwe, and I. Sharankou, *Questions on generalised Baire spaces*, Math. Logic Quart. 62 (2016), 439–456.

[Kön06] B. König, *Chang's conjecture may fail at supercompact cardinals*, arXiv:math/0605128 (2006).

[Lav78] R. Laver, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, Israel J. Math. 29 (1978), 385–388.

[RZ99] I. Recław and P. Zakrzewski, *Fubini properties of ideals*, Real Anal. Exchange 25 (1999), 565–578.

[RS06] A. Rosłanowski and S. Shelah, *Reasonably complete forcing notions*, in: Set Theory: Recent Trends and Applications, Quad. Mat. 17, Dip. Mat., Seconda Univ. di Napoli, Caserta, 2006, 195–239.

[Shea] S. Shelah, *Corrected iteration*, Boll. Un. Mat. Ital., to appear.

[Sheb] S. Shelah, *On* CON($\mathfrak{d}_\lambda > \mathrm{cov}_\lambda(\mathrm{meagre})$), Trans. Amer. Math. Soc. 373 (2020), 5351–5360.

[She78] S. Shelah, *A weak generalization of MA to higher cardinals*, Israel J. Math. 30 (1978), 297–306.

[She98] S. Shelah, *Proper and Improper Forcing*, 2nd ed., Perspectives Math. Logic, Springer, Berlin, 1998.

[She00] S. Shelah, *The generalized continuum hypothesis revisited*, Israel J. Math. 116 (2000), 285–321.

[She17] S. Shelah, *A parallel to the null ideal for inaccessible λ: Part I*, Arch. Math. Logic 56 (2017), 319–383.

[ST71] R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. (2) 94 (1971), 201–245.

[Voj93] P. Vojtáš, *Generalized Galois–Tukey-connections between explicit relations on classical objects of real analysis*, in: Set Theory of the Reals (Ramat Gan, 1991), Israel Math. Conf. Proc. 6, Bar-Ilan Univ., Ramat Gan, 1993, 619–643.

Thomas Baumhauer, Martin Goldstern
Institute of Discrete Mathematics and Geometry
TU Wien
Wiedner Hauptstraße 8–10
1040 Wien, Austria
E-mail: Thomas.Baumhauer@gmail.com
Martin.Goldstern@tuwien.ac.at
http://www.tuwien.ac.at/goldstern

Saharon Shelah
Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Edmond J. Safra Campus, Givat Ram
Jerusalem, 91904, Israel
and
Department of Mathematics
The State University of New Jersey
Hill Center – Busch Campus, Rutgers
110 Frelinghuysen Road
Piscataway, NJ 08854-8019, U.S.A.
E-mail: shlhetal@math.huji.ac.il
http://shelah.logic.at